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On Markovian Queueing Model as Birth-Death Process

By Okoro Otonritse Joshua
Landmark University, Nigeria

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Strictly as per the compliance and regulations of:

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I. Introduction

This article is a review on Markovian queuing model. The general expression for an explicit markovian queueing model by definition is given as $M/M/./.$ the first M is a Poisson rate of arrival with an exponential time distribution and the second M represent the exponential service time. The other dots represent other attributes similar to general queueing model.

The need for queueing models cannot be overemphasize because in any service station, the owner may be interested to know when to increase service points or number of queues, putting cost into consideration, In a bank, or a selling out feet, how long will one have to wait, and how can we decompose the waiting time during rush period. A production manager will want to know the lead time production for an order or for the production in mounting vertical components on printed circuit boards, how can this lead time be reduced and what will be the effect in the production system also when order are prioritized. The information or computer technologist will want to estimate the number of cell delay at the switches, the fraction of cell lost, and the size of the buffer that will be good enough to accommodate more cells. In air and sea port or any other out feet, it is important to maximize the available parking space. Managers of Call centers will want to minimize the waiting time of customers, by increasing call centers, operators, pooling teams for better efficiency and also traffic light regulators

Author: Industrial Mathematics, Physical Science Department College of Science and Engineering Landmark University, Omu-Aran, Kwara, Nigeria

and tollbooth managers will need to give acceptable waiting time and acceptable amount to pay to motorist respectively. There may be need for Server managers to increase efficiency and capacity of their servers in order to handle more transactions. All these and more can be modeled using Markov queueing processing. Ivo Adan and Jacques in 2002 give some application of queue model.

Birth and death process has been regarded as an important subclass of Markov Chains and is frequently used to model growth of biological population, Zhong Li 2013. He also compute the expected extinction time of birth-death chain. In 2010 on the application queueing theory to epidemic model Carlos M. H. and others give an expression that relates basic reproductive number, R_0 and the server utilization, ρ. also they derive new approximations to quasistationary distribution (QSD) of SIS (Susceptible- Infected- Susceptible) and SEIS (Susceptible- Latent- Infected-Susceptible) stochastic epidemic models. In their work they considered all individual in a close population to be server of which this individual may either be busy (infected) or idle (susceptible). Research work on epidemiology continuous markov chain in queueing model is just too few. In 1971 n −phase generalization of the typical $M/M/1$ queueing model, were considered, where the queueing-type birth-and-death process is defined on a continuous-time n-state Markov chain. It was conclude that the n −phase generalization of the steady-state $M/M/1$ queue will not yield, in general closed-form solution. Hence there will be need to employ numerical method to solve any specific case. Some applications to classical birth-death Markov process are given by Carlos M. and Carlos C. 1999. John Willey in 2006 and son give a thorough treatment of queue system and queueing network; among other method used, continuous Markov chain was employed. Forrest and Marc, 2011 used the continuous-time Markov chain that counts particles in a system over a time as a birth and death proceces to obtain expressions for Laplace transforms of transition probabilities in a general birth-death process with arbitrary birth and death rates and make explicit important derivation connecting transition probabilities and continued fraction. Markovian model from Markov chain where used in application and examples to illustrate key points. Solution techniques of Markova regeneration processes where investigated.

It surprising to note that, no so much research has been done using $M/M/./.$ queue theory and model in *Epidemiology* analysis which has to do with study of disease origin and spread pattern of disease development. In this article our focus is on Markovian queueing model as a birth-death process with emphasis on epidemiological analysis.

II. Definitions from Queue Model

a) *Memoryless property of the exponential distribution*

Memoryless means that the probability of time of occurrence of the event no matter how long since the last event occur is the same. That is, $Pr(x \leq T + t \mid x > T) = Pr(x \leq t)$ in real world situation this not always true. In most cases it is applicable to

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phenomena that follow random variable and random processes. For example, the longer a real traffic light has been red, the greater the probability that it will turn green in the next, say, 10 seconds, this situation is not a random process. If the probability of a traffic light turning green in the next 10 seconds does not change independent of how long it has been red, then the distribution of the red light is memoryless. Only two distributions are memoryless - the exponential (continuous) and geometric (discrete). Here is the memoryless proof for the exponential distribution…

$$\Pr(x \leq T+t|x>T) = \frac{Pr[(x \leq T+t) \cap (x>T)]}{\Pr(x>T)} = \frac{Pr[(x \leq T+t) - (x \leq T)]}{\Pr(x>T)}$$

$$= \frac{\left(1-e^{-\lambda(T+t)}\right) - \left(1-e^{-\lambda T}\right)}{1-(1-e^{-\lambda T})} = \frac{e^{-\lambda T}\left(1-e^{-\lambda t}\right)}{e^{-\lambda T}} = 1-e^{-\lambda t}$$

$P(x > T+\ t|x >\ T) =\ P(X\ >\ x) = 1 - Pr(x \leq T+t|x>T) = 1 -\ \left(1-e^{-\lambda t}\right) =$

$e^{-\lambda t}$. This memoryless property state that

b) The Little's Formula

Assume that entering customers are required to pay an entrance fee (according to some rule) to the system. Then we have Average rate at which the system earns $= \lambda_a \times$ average amount an entering customer pays where λ_a is the average arrival rate of entering customers

$\lambda_a = \lim_{t\to\infty} \frac{X(t)}{t}$ and $X(t)$ denotes the number of customer arrivals by time t.(see [1] and [2])

c) Birth-Death Process of Markov Chain

The birth-Death process is a case of Markov time continuous process. The current size of the population represent the state. For a birth-death Markov time continuous process the movement from one state to another; known as transition is limited to birth and death. Let i represent each state such that the state can move from i to $i+1$ by birth and $i-1$ by death, we assume that the movement from one state to another is independent from each other. Let λ_i and μ_i for $i = 1, 2, \ldots$ represent birth and death process respectively. We define pure death process as $\mu_i$such that $\lambda_i = 0$ and pure birth process as $\lambda_i$such that $\mu_i = 0$ for all i. the probability transition from state i to $i+1$ and to $i-1$, is $\frac{\lambda_i}{\lambda_i+\mu_i} = P[B(i) < D(i)]$ and $\frac{\mu_i}{\lambda_i+\mu_i} = P[D(i) < B(i)]$, respectively where $P[B(i) < D(i)]$ and $P[D(i) < B(i)]$ are the probability of the time until a birth $B(i)$ is less than the time until a death $D(i)$ and probability of the time until a birth $B(i)$ is less than the time until a death$D(i)$ respectively. The process remain in state i with exponential distribution $\lambda_i + \mu_i$. For a death to occur there must be a birth, for there to be any first noticeable change in the system, the process must move from state i to state $i+1$ which implies one birth and no death, of which its probability is given by

$$P_{i,i+1}(h) = P(X(t+h) - X(t) = 1|X(t) = i) = \frac{(\lambda_i h)^1 e^{-\lambda_i h}}{1!}\frac{(\mu_i h)e^{-\mu_i h}}{0!} + 0(h)$$

$$= (\lambda_i h)e^{-\lambda_i h}e^{-\mu_i h} + o(h) = (\lambda_i h)e^{-h(\lambda_i+\mu_i)} = (\lambda_i h)\sum_{n=0}^{\infty}\frac{(-h(\lambda_i+\mu_i))^n}{n!}$$

$$= (\lambda_i h)\left(1 - h(\lambda_i+\mu_i) - \frac{1}{2!}h^2(\lambda_i+\mu_i)^2 \ - \cdots\right) + 0(h)$$

$$= \lambda_i h + o(h)$$

The probability for moving from state i to $i-1$ is given by $P_{i,i-1}(h) = P(X(t+h) - X(t) = -1|X(t) = i) = \mu_i h + o(h)$. The probability of having any other moves other than this two is non-zero instead is given by $P(X(t+h) - X(t) > 1|X(t) = i) = o(h)$, for $\mu_0 = 0, \lambda_0 > 0;\ \mu_i, \lambda_i > 0; for\ i = 1, 2, 3, ...$ This also implies that $P(X(t+h) - X(t) = 0|X(t) = i) = 1 - h(\lambda_i + \mu_i) + o(h)$. Generally we can represent the birth and death process by

$$p_{i,j}(h) = \begin{cases} \lambda_i h + o(h) & if\ j = i+1 \\ \mu_i h + o(h) & if\ j = i-1 \\ \quad 1 - h(\lambda_i+\mu_i) + o(h) & if\ j = i \\ o(h) & otherwise \end{cases}$$

This can be $p_{i,j}(h) = \delta_{i,j} + r_{i,j}(h) + h(0)$

$\delta_{i,j} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$ the Kronecker's delta

$$g_{i,j}(h) = \begin{cases} \lambda_i & if\ j = i+1 \\ \mu_i & if\ j = i-1 \\ -(\lambda_i+\mu_i) & if\ j = i \\ o & otherwise \end{cases}$$

Then the matrix G is the infinitesimal generator of the process $X(t)$ define by $[g_{i,j}]$, where $g_{i,j}$ are called transition rate.

$$G = \left\| \begin{matrix} -\lambda_0 & \lambda_0 & 0 & 0 & . & . & . \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & . & . & . \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & . & . & . \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{matrix} \right\|$$

Note that $\delta_{i,j} = p_{i,j}(0)$ since the process remain in the same state in zero step with probability one and move to another state in zero step with probability one. We have

Notes

$g_{i,j} = \frac{p_{i,j}(h)-\delta_{i,j}}{h} = \frac{p_{i,j}(h)- p_{i,j}(0)}{h} = p'_{i,j}(0)$ hence differentiating term by term and setting $t = 0$ $\sum_j g_{i,j}(0) = 0$ this implies that $g_{i,j}(t) = p'_{i,j}(0) \geq 0 \quad for\ i \neq j$

$g_{i,j}(t) = p'_{i,j}(0) \leq 0$ otherwise

Notes

Also $\sum_j p_{i,j}(t) = 1$

d) Probability Transition of Birth-Death Process and Differential Equation from Kolmogorov

Kolmogorov backward differential equation describe the transition probabilities in their dependence on the initial point i

Basically

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(t)$$

$$= P_{i,i-1}(h)P_{i-1,j}(t) + P_{i,i+1}(h)P_{i+1,j}(t) + P_{i,i}(h)P_{i,j}(t) + \sum_{k}' P_{ik}(h)P_{kj}(t)$$

The last summation is for $k \neq i-1, i, i+1$.

$$P_{ij}(t+h)$$
$$= (\mu_i h + o(h))P_{i-1,j}(t) + (\lambda_i h + o(h))P_{i+1,j}(t) + (1 - h(\lambda_i + \mu_i) + o(h))P_{i,j}(t) + \sum_k' P_{ik}(h)P_{kj}(t)$$

But $\sum_k' P_{ik}(h)P_{kj}(t) \leq \sum_k' P_{ik}(h) = 1 - \left(P_{i,i}(h) + P_{i,i-1}(h) + P_{i,i+1}(h)\right)$

$$= 1 - (1 - h(\lambda_i + \mu_i) + o(h) + (\mu_i h + o(h) + (\lambda_i h + o(h))$$

$$= o(h)$$

hence we have

$$P_{ij}(t+h) = \mu_i h P_{i-1,j}(t) + \lambda_i h P_{i+1,j}(t) + (1 - h(\lambda_i + \mu_i))P_{i,j}(t)$$
$$+ o(h)\left(P_{i-1,j}(t) + P_{i+1,j}(t) + P_{i,j}(t) + 1\right)$$

and so;

$$P_{ij}(t+h) = \mu_i h P_{i-1,j}(t) + \lambda_i h P_{i+1,j}(t) + (1 - h(\lambda_i + \mu_i)P_{i,j}(t) + o(h)$$

$$= \mu_i h P_{i-1,j}(t) + \lambda_i h P_{i+1,j}(t) + P_{i,j}(t) - P_{i,j}(t)h(\lambda_i + \mu_i) + o(h)$$

also we have

$$\frac{P_{ij}(t+h)-P_{i,j}(t)}{h}=\frac{\mu_i hP_{i-1,j}(t)+\lambda_i hP_{i+1,j}(t)-P_{i,j}(t)h(\lambda_i+\mu_i)+o(h)}{h}$$

$$=\mu_i P_{i-1,j}(t)+\lambda_i P_{i+1,j}(t)-P_{i,j}(t)(\lambda_i+\mu_i)+o(1)$$

$$P'_{ij}(t)=\mu_i P_{i-1,j}(t)+\lambda_i P_{i+1,j}(t)-P_{i,j}(t)(\lambda_i+\mu_i),$$

we then derive the differential equation knowing there is no birth without death that $\mu_0=0$

$$P'_{0j}(t)=\mu_0 P_{0-1,j}(t)+\lambda_0 P_{0+1,j}(t)-P_{0,j}(t)(\lambda_0+\mu_0)$$

$$P'_{0j}(t)=\lambda_0 P_{1,j}(t)-\lambda_0 P_{0,j}(t)$$

and again

$$P'_{ij}(t)=\mu_i P_{i-1,j}(t)+\lambda_i P_{i+1,j}(t)-P_{i,j}(t)(\lambda_i+\mu_i)$$

we know that

$$P_{ij}(s+t)=\sum_k P_{ik}(s)P_{kj}(t)$$

by Chapman-Kolmogorov

Differentiating with respect to s we have

$$P'_{ij}(s+t)=\sum_k P'_{ik}(s)P_{kj}(t)$$

Setting $s=0$ gives

$$P'_{ij}(s+t)=\sum_k P'_{ik}(0)P_{kj}(t)$$

$$=\sum_k g_{ik}(0)P_{kj}(t)$$

Therefore

$$\boldsymbol{P'(t)=GP(t)}$$

While the forward Kolmogorov differential equation describes the probability distribution of a state in time t keeping the initial point fixed, decomposing the interval $(0,t+h)$ into $(0,t)$ and $(t,t+h)$

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$$P_{ij}(t+h)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(h)$$

$$=P_{i,j-1}(t)P_{j-1,j}(h)+P_{i,j+1}(t)P_{j+1,j}(h)+P_{i,j}(t)P_{j,j}(h)+\sum_{k}^{\prime}P_{ik}(t)P_{kj}(h)$$

The last summation is for $k\neq j-1,j,j+1$.

$$P_{ij}(t+h)=P_{i,j-1}(t)\lambda_{j-1}h+P_{i,j+1}(t)\mu_{j+1}h+P_{i,j}(t)\big(1-h(\lambda_j+\mu_j)\big)+o(h)$$

Similar apply here as in Kolmogorov back differential equation

$$P_{ij}'(t)=P_{i,j-1}(t)\lambda_{j-1}\ +P_{i,j+1}(t)\mu_{j+1}-P_{i,j}(t)(\lambda_j+\mu_j)$$

$$P_{i0}'(t)=P_{i,1}(t)\mu_1-P_{i,0}(t)\lambda_0$$

With the same initial condition $P_{ij}(0)=\delta_{ij}$.

We know that

$$P_{ij}(s+t)=\sum_{k}P_{ik}(s)P_{kj}(t)$$

by Chapman-Kolmogorov

Differentiating with respect to t we have

$$P_{ij}'(s+t)=\sum_{k}P_{ik}(s)P_{kj}'(t)$$

Setting $t=0$ gives

$$P_{ij}'(s)=\sum_{k}P_{ik}(s)P_{kj}'(0)$$

$$=\sum_{k}P_{ik}(t)g_{kj}$$

therefore

$$\boldsymbol{P}'(\boldsymbol{t})=\boldsymbol{P}(\boldsymbol{t})\boldsymbol{G}$$

For $$\boldsymbol{p}(\boldsymbol{0})=\boldsymbol{1}$$

$$\boldsymbol{P}(\boldsymbol{t})=\mathrm{e}^{\mathbf{Gt}}$$

Where $$\mathrm{e}^{\mathbf{Gt}}=\sum_{n=0}^{\infty}\frac{\boldsymbol{G}^n\boldsymbol{t}^n}{n!}=1+\sum_{n=1}^{\infty}\frac{\boldsymbol{G}^n\boldsymbol{t}^n}{n!}$$

Also changes or number of event $j-i$ over a period of time of let T for a Poisson pure birth process with the infinitesimal transition rate

$$\begin{cases} p_{i,j} = \lambda_i & if\ j = i+1 \\ p_{i,j} = \lambda_i & if\ j = i \\ 0 & otherwise \end{cases}$$

follows a Bernoulli trial performed n times with the probability of success λ hence we have

$$p_{ij}^{(n)} = \begin{cases} \binom{n}{j-i} \lambda^{j-i}(-\lambda)^{n-(j-i)} & if\ 0 \le j-i \le n \\ 0 & otherwise \end{cases}$$

$$\binom{n}{j-i} \lambda^{j-i}(-\lambda)^{n-(j-i)} = \binom{n}{j-i} \lambda^{n}(-1)^{n-(j-i)}$$

From the formula

$$\boldsymbol{P}(t) = \mathrm{e}^{\mathrm{G}t} = \sum_{n=0}^{\infty} \frac{\boldsymbol{G}^n \boldsymbol{t}^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\boldsymbol{G}^n \boldsymbol{t}^n}{n!}$$

$$p_{ij}(t) = \delta_{ij} + \frac{\boldsymbol{t}^1}{1!} g_{ij} + \frac{\boldsymbol{t}^2}{2!} g_{ij}^2 + \cdots$$

$$p_{ij}(t) = \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^n}{n!} g_{ij}^n = \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^n}{n!} \binom{n}{j-i} \lambda^{j-i}(-\lambda)^{n-(j-i)}$$

$$= \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^n}{n!} \frac{n!}{(n-(j-i))!\,(j-i)!} \lambda^{j-i}(-\lambda)^{n-(j-i)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(n-(j-i))!\,(j-i)!} \lambda^{j-i}(-\lambda)^{n-(j-i)}$$

$$= \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^{(j-i)}}{(j-i)!} \lambda^{j-i} \frac{t^{(n-(j-i))}}{(n-(j-i))!} (-\lambda)^{n-(j-i)}$$

$$= \frac{(\lambda t)^{(j-i)}}{(j-i)!} \sum_{n=0}^{\infty} \frac{(-\lambda t)^{(n-(j-i))}}{(n-(j-i))!}$$

But $j-i \neq 0$ hence n cannot start from zero, but from $j-i$ and letting $k = n-(j-i)$ we have

$$p_{ij}(t) = \frac{(\lambda t)^{(j-i)}}{(j-i)!} \sum_{n=j-i}^{\infty} \frac{(-\lambda t)^{(n-(j-i))}}{(n-(j-i))!}$$

Notes

$$= \frac{(\lambda t)^{(j-i)}}{(j-i)!}\sum_{n=0}^{\infty}\frac{(-\lambda t)^k}{k!} = \frac{(\lambda t)^{(j-i)}}{(j-i)!}e^{-\lambda t}$$

Which is Poisson process

Notes

e) The law of Rare Events

This law holds when the probability of success p occurrence from large number N of independent Bernoulli trails is small and constant from one occurrence to another. Let $X_{N,p}$ follows the binomial distribution, such that $X_{N,p}$ is the total number of success in trails for $k = 0,1,2,\ldots,N$.

$$P = \binom{N}{k}p^k(1-p)^{N-k}$$

$$P(X_{N,p} = k) = \binom{N}{k}p^k(1-p)^{N-k} = \frac{N(N-1)(N-2)\ldots(1-k+1)}{k!}p^k(1-p)^{N-k} \quad k = 0,1,\ldots,N$$

Multiplying and dividing the right-hand side by N^k, we have

$$p_X(k) = \binom{N}{k}p^k(1-p)^{N-k} = \frac{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{k-1}{N}\right)}{k!}(Np)^k p^k\left(1-\frac{Np}{n}\right)^{N-k}$$

If we let $n \to \infty$ in such a way that $Np = \lambda$ remains constant, then

$$\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{k-1}{N}\right)\xrightarrow[N\to\infty]{} 1$$

$$\left(1-\frac{Np}{N}\right)^{N-k} = \left(1-\frac{\mu}{N}\right)^n\left(1-\frac{\mu}{N}\right)^{-k}\xrightarrow[N\to\infty]{} e^{-\mu}(1) = e^{-\mu}$$

Where we need the fact that

$$\lim_{n\to\infty}\left(1-\frac{\mu}{N}\right)^N = e^{-\mu}$$

Hence, in the limit as $N \to \infty$ with $Np = \mu$ (and as $p = \frac{\mu}{N} \to 0$),

$$\binom{N}{k}p^k(1-p)^{N-k}\xrightarrow[n\to\infty]{} e^{-\mu}\frac{\mu^k}{k!} \qquad Np = \mu$$

Thus, in the case of large N and small p

$$\binom{N}{k}p^k(1-p)^{N-k} \approx e^{-\mu}\frac{\mu^k}{k!} \qquad Np = \mu$$

Which, indicate that the binomial distribution can be approximated by the Poisson under some circumstance in Stochastic modeling. The Poisson distribution has independent increment, that is for $t_0 \leq t_1 < t_2 < \cdots < t_n$, and $t_0 = 0$ such that $\{X(t); t \geq 0\}$ which is the Stochastic process of event happening we have $X(t_1) - X(t_0)$, $X(t_2) - X(t_1), X(t_3) - X(t_2) \ldots$, $X(t_n) - X(t_{n-1})$ this is the independent increment property Poisson process such that for each increment for s to $s + t$, the probability associated to the stochastic process $\{X(t); t \geq 0\}$ for $\lambda > 0$ as the intensity of the process is given by $P\big(X(s+t) - X(s) = k\big) = \frac{(\lambda t)e^{-\lambda t}}{k!}$. for exactly one event happening over a period h we have $P\big(X(s+t) - X(s) = 1\big) = \frac{(\lambda h)e^{-\lambda h}}{1!}$

$$= (\lambda h) \sum_{n=0}^{\infty} \frac{(-\lambda h)^n}{k!} = (\lambda h)\left(\frac{(-\lambda h)^0}{0!} + \frac{(-\lambda h)^1}{1!} + \frac{(-\lambda h)^2}{2!} + \cdots\right)$$

$$= (\lambda h)\left(1 - \lambda h - \tfrac{1}{2!}\lambda^2 h^2 - \cdots\right) = \lambda h + 0(h)$$

Note: Poisson process count the number of events entering into the system while the time in between these events, also known as sojourn time, follows an exponential probability distribution. Suppose at this time interval no arrival $P(X = 0) = e^{-\lambda t}$ this can interpreted that the time T of the first occurrence greater than t, hence we have $P(T > t) = P(X = 0|\lambda t) = e^{-\lambda t}$ hence this implies that $P(T \leq t) = 1 - P(X = 0|\lambda t) = 1 - e^{-\lambda t}$, differentiating we have $f(t) = \lambda e^{-\lambda t}$ which is an exponential distribution. This give the Connection between Poisson And Exponential Distribution. The distribution of the Sojourn time of the random variable $X(t)$ in state i can adequately be describe by the random variable J_i. Hence the distribution of the time j_i the process $X(t)$ first leaves state i can be determine; that is $Z_i(t) = P(J_i \geq t)$, by the independency of Markovian property $h \to 0$ we have $Z_i(t+h) = Z_i(t) + Z_i(h) = Z_i(t)\big[P_{i,i}(h) + o(h)\big]$

$$= Z_i(t)[1 - h(\lambda_i + \mu_i)] + o(h)$$

$$= Z_i(t) - Z_i(t)h(\lambda_i + \mu_i) + o(h)$$

Subtracting $Z_i(t)$ from both sides and dividing by h gives

$$\frac{Z_i(t+h) - Z_i(t)}{h} = \frac{Z_i(t) - Z_i(t)h(\lambda_i + \mu_i) + o(h) - Z_i(t)}{h}$$

$$= \frac{Z_i(t+h) - Z_i(t)}{h} = -Z_i(t)(\lambda_i + \mu_i) + o(h)$$

$Z_i'(t) = -Z_i(t)(\lambda_i + \mu_i)$ hence we have $Z_i(t) = e^{-t(\lambda_i + \mu_i)}$

Notes

For the fact that $Z_i(t) = P(J_i \geq t) = 1 - P(J_i \leq t) = 1 - e^{-t(\lambda_i+\mu_i)}$, it follows that J_i follows exponential distribution with parameter $\frac{1}{\lambda_i+\mu_i}$ as mean.

f) Steady-State Probability of Birth-Death Process

Steady state is reach under the condition $\frac{\lambda_n}{\mu_n} < 1$, where λ_n and μ_n are birth and death rate respectively. So from

Notes

$$P'_{i0}(t) = P_{i,1}(t)\mu_1 - P_{i,0}(t)\lambda_0$$

$$P'_{ij}(t) = P_{i,j-1}(t)\lambda_{j-1} + P_{i,j+1}(t)\mu_{j+1} - P_{i,j}(t)(\lambda_j + \mu_j),$$

With the same initial condition $P_{ij}(0) = \delta_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$ we assume steady such that

$$0 = P_{i,1}(t)\mu_1 - P_{i,0}(t)\lambda_0$$

$$0 = P_{i,j-1}(t)\lambda_{j-1} + P_{i,j+1}(t)\mu_{j+1} - P_{i,j}(t)(\lambda_j + \mu_j),$$

$$P_{i,1}(t) = P_{i,0}(t)\frac{\lambda_0}{\mu_1}$$

$$P_{i,2}(t) = P_{i,1}(t)\frac{\lambda_1}{\mu_2} = P_{i,0}(t)\frac{\lambda_0\lambda_1}{\mu_1\mu_2} \qquad e.t.c$$

$$P_{i,n}(t) = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2 \dots \mu_n} P_{i,0}(t) \qquad e.t.c$$

We know that $\sum_{n=0}^{\infty} P_{i,n}(t) = 1$ for the ith system

$$P_{i,0}(t) = \frac{1}{\left\{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \cdots\right\}}$$

The necessary condition for the existence of a steady-state solution is that $\sum_{n=0}^{\infty} \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2 \dots \mu_n}$ must converge to ensure that $P_{i,0}(t) \neq 0$

III. The Epidemiology M / M /. /. Queue Model

In a system, entry into the system, can be regarded as arrival, the Epidemiologist might interested to know the rate of entry into the system, the number in the system at time t how long will someone be in the system to be infected, when will infection begin, when will the signs or symptom start manifesting, at what time will infection elapse, at what stage or rate will the process of removal change from real death to recovering. Other case may arise where entry in the system may not mean infection, may be because of vaccination or high resistance of that individual, the researcher will want to know how effective is the vaccine, and how significant is the resistance. Sometime removal may not necessarily mean recovery or removal, but rendering the patient incapacitated, the epidemiologist will want to know the time, duration for such to happen. The $M/M/./.$ queue model can be used to give exact or approximated answers to the above questions. On this light we will discuss well known $M/M/./.$ queue model in Epidemiologist point of view.

Notes

a) $M/M/1$ Queueing Model

This can be divided into two. There might be a situation when entry into the system, sometime may elapse before one is infected, the other case is the situation one get infected immediately he/she enter the system, the former will be discuss later.

$\lambda_n = \lambda$ $\quad n \geq 0$ the rate of infection

$\mu_n = \mu$ $\quad n \geq 1$ the rate of removal (recovering or death)

$p_{i0}(t) = P_i\{N(t) = 0\} = 1 - \frac{\lambda}{\mu} = 1 - \rho$ The probability of infection, such that non is infected.

$$p_{in}(t) = P_i\{N(t) = n\} = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n$$

The probability of infection, such that n, has been infected.

where $\rho = \frac{\lambda}{\mu} < 1$, and $i = 1$ number of server which implies that the rate of infection which implies rate of removal, on the average, must process faster than their average entry rate into the system.

The average number in the system that are infected is given by

$$N = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

in this case there is no waiting time for infection to take place once you find yourself in the system you became infected.

The average waiting time for a removal to occur in the system is given by

$$T = \frac{1}{\mu - \lambda}$$

On the other hand there might be a situation where, entering the system does not q your infection, this case there is a waiting time. We give the following expression for the above as follow;

Average waiting time in the system before infection is given by

$$= W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

here it is assume they do not get infected once they enter the system, but have to wait for some time.

The expected number of person in system before infection

$$= L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

if there exist a great deviation from this, the epidemiologist will be interested to why it is so. This is for just one type of infection, however this is just the foundation for other model on Epidemiology and it can also be decompose.

b) The $M / M / S$ Queueing Model

Situation may arise, where there are more than one system infected with diseases, such that diseases are not the same from system to system. Ideally system real or abstract, depend on the purpose of research, in some cases this model have to be modify. However fundamentally, this may be model after the well-known $M/M/s$ queue model, where s is the number of systems, here i take the value from 1 to s or any value that may suit the analysis.

$\lambda_n = \begin{cases} \lambda, & 0 \leq n < K \\ 0, & n \geq K \end{cases}$ here we assume that the rates infection are the same for each system of infection

$\mu_n = \begin{cases} n\mu, & 0 < n < s \\ s\mu, & n \geq s \end{cases}$ here we assume that the rate of removal depend on the number of infected system.

Note that the removal parameter μ_n is state dependent.

Balance equations

$$\lambda p_{i-1} = \begin{cases} i\mu p_i, & for\ \ i < s \\ s\mu p_i, & for\ \ i = s \end{cases}$$

$p_{i0}(t) = P_i\{N(t) = 0\} = \left[\sum_{n=0}^{s-1} \frac{(s\rho)}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1}$ for each i in the number of s system, here we make use of some past record for the computation of $p_{i0}(t)$

$$p_{in}(t) = P_i\{N(t) = n\} = \begin{cases} \frac{(s\rho)^n}{n!} p_{i0}, & n < s \\ \frac{\rho^n s^s}{s!} p_{i0}, & n \geq s \end{cases}$$

where $\rho = \frac{\lambda}{(s\mu)} < 1$. Note that the ratio $\rho = \frac{\lambda}{(s\mu)}$ is the infection intensity of the $M/M/s$ queueing model.

the expected number in the system before infection is given by;

$$E(N_q) = L_q = \lambda W_q = L - \frac{\lambda}{\mu} = \frac{\rho(s\rho)^s}{s!\,(1-\rho)^2} p_{i0}$$

The expected number in the system before and during infection,

$$E(N) = L = \lambda W = \frac{\lambda}{\mu} + \lambda W_q = \frac{\lambda}{\mu} + \frac{\rho(s\rho)^s}{s!(1-\rho)^2} p_{i0}$$

because after removal it is expected they will no longer be in the system, however some model will not allow this.

While the expected time spent before infection in the system

$$= W_q = \frac{L_q}{\lambda} = w - \frac{1}{\mu} = \frac{\rho(s\rho)^s}{\lambda s!\,(1-\rho)^2} p_{i0}$$

The expected time lapse spent in the system before and during infection

$$= W = \frac{L}{\lambda} = \frac{1}{\mu} + W_q = \frac{1}{\mu} + \frac{\rho(s\rho)^s}{\lambda s!(1-\rho)^2} p_{i0}$$

When analytical computation of μ is very difficult or almost impossible, a Monte Carlo simulation is applied in order to get estimations. A standard Monte Carlo simulation algorithm fix a regenerative state and generate a sample of regenerative cycles, and then use this sample to construct a likelihood estimator of state (Nasroallah, 2004). Monte Carlo simulation uses the mathematical models to generate random variables for the artificial events and collect observations. (Banks, 2001).

c) The $M/M/1/K$ Queueing Model

This model may be built for the purpose of research. Here the population or the capacity of the system to be considered is known which is assume to be K. The $M/M/1/K$ queueing system can be modeled as a birth-death process with the following assumed parameters:

$$\lambda_n = \begin{cases} \lambda, & 0 \le n < K \\ 0, & n \ge K \end{cases}$$

$$\mu_n = \mu \qquad n \ge 1$$

$$p_{i0} = \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1-\rho}{1-\rho^{k+1}} \qquad \rho \neq 1$$

Notes

$$p_{in} = \begin{cases} \rho^n p_{i0} = \left(\frac{\lambda}{\mu}\right)^n p_0 = \frac{(1-\rho)\rho^n}{1-\rho^{k+1}}, \; for\ 0 \le n \le K & n = 1, \dots, K \\ 0, \qquad\qquad\qquad\qquad for\ \ n > K \end{cases}$$

where $\rho = \frac{\lambda}{\mu}$

Notes

It is important to note that it is no longer necessary for infection intensity $\rho = \frac{\lambda}{\mu}$ *to* be less than 1. Patient will not be infected when the system is in state K

the expected number in the system before infection is given by;

$$L_q = L - (1 - p_{i0})$$

The expected number in the system before and during infection is given by,

$$L = \rho \frac{1-(K+1)\rho^K + Kp^{K+1}}{(1-\rho)(1-\rho^{K+1})}, \qquad \rho = \frac{\lambda}{\mu}$$

While the expected time spent before infection in the system is given by;

$$W_q = \frac{1}{\mu} L$$

The expected time lapse spent in the system before and during infection is given by;

$$W = \frac{1}{\mu}(L+1)$$

Since the fraction of infection that actually enter into the is $1 — p_{iK}$, the infective arrival rate is given by

$$\lambda_e = \lambda(1 - p_{ik})$$

the expected number in the system before infection is equivalent to;

$$L_q = \lambda_e W_q = \lambda(1 - p_{ik}) W_q$$

The expected time lapse spent in the system before and during infection is equivalent to;

$$W = \frac{L}{\lambda_e} = \frac{L}{\lambda(1-p_K)}$$

While the expected time spent before infection in the system is equivalent to;

$$W_q = W - \frac{1}{\mu}$$

d) The $M/M/s/K$ Queueing Model

This general model contains only limited number K in the system. However, if there are unlimited number then $K = \infty$, then our model will be labeled as $M/M/s$ (Hillier & Lieberman, 2001.). The $M/M/s/K$ queueing system can be modeled as a birth-death process with the following parameter:

$$\lambda_n = \begin{cases} \lambda, & 0 \le n < K \\ 0, & n \ge K \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 < n < s \\ s\mu, & n \ge s \end{cases}$$

$$p_{\mathrm{i}0} = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right)\right]^{-1}$$

$$p_{in} = \begin{cases} \frac{(s\rho)^n}{n!} p_{\mathrm{i}0}, & n < \mathrm{s} \\ \frac{\rho^n s^s}{s!} p_{\mathrm{i}0}, & \mathrm{s} \le n \ge \mathrm{K} \end{cases}$$

where $$\rho = \frac{\lambda}{(s\mu)}.$$

The expected number in the system before infection is given by;

$$L_q = \rho_0 \frac{\rho(s\rho)^s}{s!\,(1-\rho)^2}\{1 - [1 + (1-\rho)(K-s)]\rho^{K-s}\}$$

The expected number in the system before and during infection is given by,

$$L = L_q + \frac{\lambda_e}{\mu} = L_q + \frac{\lambda}{\mu}(1 - p_K)$$

The quantities W and $\mathrm{W_q}$ which is the expected time spent before infection in the system and The expected time lapse spent in the system before and during infection respectively are given by;

$$W = \frac{L}{\lambda_e} = L_q + \frac{1}{\mu}$$

$$W_q = \frac{L_q}{\lambda_e} = \frac{L_q}{\lambda(1 - p_K)}$$

e) Tabulated Result

However, for easy understanding these results has been tabulated as follow;

Notes

	$p_0(t)$	$p_n(t)$	L	L_q	W	W_q
$M/M/1$	$1-\frac{\lambda}{\mu}$, $1-\rho$	$\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$, $(1-\rho)\rho^n$ $\rho=\frac{\lambda}{\mu}<1$	λW $\frac{\lambda}{\mu-\lambda}$ $\frac{\rho}{1-\rho}$	$\frac{\lambda^2}{\mu(\mu-\lambda)}$ $\frac{\rho^2}{1-\rho}$	$\frac{1}{\mu-\lambda}$ $\frac{1}{\mu(1-\rho)}$	$\frac{\lambda}{\mu(\mu-\lambda)}$ $\frac{\rho}{\mu(1-\rho)}$
$M/M/s$	$\left[\sum_{n=0}^{s-1}\frac{(s\rho)}{n!}\frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1}$ $\rho=\frac{\lambda}{(s\mu)}$	$\begin{cases}\frac{(s\rho)^n}{n!}p_0, & n<s \\ \frac{\rho^n s^s}{s!}p_0, & n\geq s\end{cases}$ $\rho=\frac{\lambda}{(sp)}<1$	λW $\frac{\lambda}{\mu}+\lambda W_q$ $\frac{\lambda}{\mu}+\frac{\rho(s\rho)^s}{s!(1-\rho)^2}p_0$	λW_q $L-\frac{\lambda}{\mu}$ $\frac{\rho(s\rho)^s}{s!\,(1-\rho)^2}p_0$	$\frac{L}{\lambda}$ $\frac{1}{\mu}+W_q$ $\frac{1}{\mu}+\frac{\rho(s\rho)^s}{\lambda s!\,(1-\rho)^2}p_0$	$\frac{L_q}{\lambda}$ $w-\frac{1}{\mu}$ $\frac{\rho(s\rho)^s}{\lambda s!(1-\rho)^2}p_0$
$M/M/1/K$	$\frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$ $\frac{1-\rho}{1-\rho^{k+1}}$ $\rho\neq 1$	$\left(\frac{\lambda}{\mu}\right)^n p_0$ $\frac{(1-\rho)\rho^n}{1-\rho^{k+1}}$ $n=1,\ldots,K$	$\rho\frac{1-(K+1)\rho^K+Kp^{K+1}}{(1-\rho)(1-\rho^{K+1})}$ $\rho=\frac{\lambda}{\mu}$	$L-(1-p_0)$ Or $\lambda_e W_q$ $\lambda(1-p_k)W_q$ $\lambda_e=\lambda(1-p_k)$	$\frac{1}{\mu}(L+1)$ Or $\frac{L}{\lambda_e}$ $\frac{L}{\lambda(1-p_K)}$	$\frac{1}{\mu}L$ $W-\frac{1}{\mu}$
$(M/M)/s/K$	$\left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!}+\frac{(s\rho)^s}{s!}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right)\right]^{-1}$	$\begin{cases}\frac{(s\rho)^n}{n!}p_0, & n<s \\ \frac{\rho^n s^s}{s!}p_0, & s\leq n\geq K\end{cases}$ $\rho=\frac{\lambda}{(s\mu)}$	$\rho_0\frac{\rho(s\rho)^s}{s!(1-\rho)^2}\{1-[1+(1-\rho)(K-s)]\rho^{K-s}\}$	$L_q+\frac{\lambda_e}{\mu}=L_q+\frac{\lambda}{\mu}(1-p_K)$	$\frac{L}{\lambda_e}$ $L_q+\frac{1}{\mu}$	$\frac{L_q}{\lambda_e}$ $\frac{L_q}{\lambda(1-p_K)}$

Notes

IV. The Epidemiology $M/M/./././$ Queue Model

Applications abound in $M/M/./.$ these rages from theoretical and analytical applications and numerical, (Ivo Adan and Jacques Resing, (2002))with the advent of computer programming it is much easier to simulate using these model. In Epidemiology, no much work can be seen,

3.1 Let W_E denote the amount of time lapse before infection in the $M/M/1$ queueing system. the distribution of W_E is given by

$$F_{W_E}=P\{W_a\leq a\}=\sum_{n=0}^{\infty}\left[\int_0^a \mu e^{-\mu t}\frac{(\mu t)^n}{n!}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n dt\right]$$

$$=\int_0^a(\mu-\lambda)e^{-\mu t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}dt$$

$$=\int_0^a(\mu-\lambda)e^{-(\mu-\lambda)t}dt=1-e^{-(\mu-\lambda)a}$$

Note; $(W_E)=\frac{1}{\mu-\lambda}$, since $W=E(W_E)$. (Hwei Hsu, (2011))

3.2. The rate λ of entry into infected system follow a Poisson process and the service time is an exponential r.v. with mean μ. the expected number in the system is L, the w, and the waiting time before infection W_q. When the rate λ, of entry into infected system increases by any percent. It increases the expected number of infection in the system. The average time spent in the system waiting to be infected also increased.

3.3. For an $M/M/1$ queueing model/system rate of infection most be at least $\sqrt[k]{\frac{100\lambda^k}{1-n}}$ with entry rate λ, when there are k number yet to be infected in the system up n percent of the capacity of the system accommodating number yet to be infected but in the system.

3.4. The probability that an arrival may enter the infected system s is given by

$$P(that\ an\ arrival\ may\ enter\ the\ infected\ system\ s) = \sum_{n=s}^{\infty} p_n = p_0 \frac{s^s}{s!} \sum_{n=s}^{\infty} \rho^n$$

$$= p_0 \frac{(s\rho)^s}{s!\,(1-\rho)} = \frac{\frac{(s\rho)^s}{s!\,(1-\rho)}}{\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!\,(1-\rho)}}$$

This is the modify Erlang's delay $(or\ C)$ formula and denoted by $C(s, \frac{\lambda}{\mu})$. (Hwei Hsu, (2011).

3.5 In an $M/M/1$ queue model one can compute the expected number of a arrival during an infected period and the probability that no arrival during that period, the expected number of arrival= $E[E\{number\ of\ arrival|infected\ period\ T\}] = E[\lambda T] = \frac{\lambda}{\mu}$. while 0 arrival probability $P\{zero\ arrival\} = E[P\{zero\ arrival|infected\ period\ T\}] =$ $E[P\{N(T) = 0\}] = E\left[e^{-\lambda T}\right] = \int_0^x e^{-\lambda T} \mu e^{-\lambda s} ds = \frac{\mu}{\lambda+\mu}$

IV. Conclusion

We have considered result on $M/M/./.$/ Queue model. In some of the section model was form without showing its derivation, since these derivation can be found in any queue model text book [e.g., Hwei Hsu, 2011] and in order for the article not be too lighten. Markovian queueing model as a birth-death process is very vital in epidemiology study. For a more detail work, one may need to discuss on the following the rate of entry, infection and recovery, waiting time of entry, infection, and recovery. Also one may also discuss on number in the system, infected, and recovered. Heuristics data may be needed to validate this claim, simulation can also be employ in complex cases. All these result are observable at steady-state. It should be noted that result obtain here can actually be different from simulated result, why?. Queueing model has been so important in performance analysis in manufacturing system, which is another open area of research. We are next

Notes

Notes

concerned about how to obtain solution for a queuing model with a network of queues? Such questions require running Queuing Simulation. Simulation can be used for more refined analysis to represent complex systems. The queuing system is when classified as M/M/c with multiple queues where number in the system and in a queue is infinite, the solution for such models are difficult to compute. When analytical computation of T is very difficult or almost impossible, a Monte Carlo simulation may be applied in order to get estimations.

References Références Referencias

[1] Sheldon M. Ross (2010): Introduction to Probability Models Tenth Edition Academic Press is an imprint of Elsevier 30 Corporate Drive, Suite 400, Burlington, MA 01803, USA.

[2] Hwei Hsu, (2011); Probability, Random Variables, and Random Processes, Schaum's Series Outline, Mc Graw Hill Companies.

[3] Brian D. Bunday, (1996); An Introduction to Queueing Theory, ARNOLD, London, Sydney, Aukland.

[4] Brian D. Bunday, (1986); Basic Queueing Theory, Edward Arnold (Australia) Ltd,

[5] Sheldon M. Ross (2010): Student's Manual to Accompany Introduction to Probability Models Tenth Edition. Academic Press is an imprint of Elsevier 30 Corporate Drive, Suite 400, Burlington, MA01803, USA

[6] Carlos M. Hernandez-Suarez, Carlos Castillo-Chavez, Osval Montesinos Lopez, Karla Hern andez-Cuevas, (2010). An Application Of Queuing Theory To SIS And SEIS Epidemic Models.

[7] J.Virtamo 38.3143 Queueing Theory / Birth-death processes

[8] Carlos M. Hernandez-Suarez, Carlos Castillo-Chavez, (1999); A basic result on the integral for birth-death Markov processes, Mathematical Biosciences 161 95-104, Elsevier Science Inc.

[9] Alan Krinik, Carrie Mortensen, Gerardo Rubino, (2003); Connections Between Birth-Death Processes. Advances in Stochastic Processes and Functional Analysis, Mathematics Subject Classifications: 60K25, 60J27.

[10] Donald Gross, John F. Shortle, James M. Thompson, Carl M. Harris, (2008); Fundamental of Queue Theory-Fourth Edition, A John Wiley and Sons, Inc., Publication.

[11] Ivo Adan and Jacques Resing, (2002); Queueing Theory, Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

[12] Nasroallah, A. (2004), "Monte Carlo Simulation of Markov Chain Steady-state Distribution," Extracta Mathematicae, Vol. 19, No. 2, p279-288.

[13] Banks, J., Carson, J. S., Nelson, B. L., Nicol, D. M. (2001), Discrete-Event System Simulation, Prentice Hall international series, 3rd edition, p24–37.

[14] Hillier, F. S., Lieberman, G. J. (2001), Introduction to Operations Research, McGraw-Hill higher education, 7th edition, p834–8.

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