Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 9 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Results on Compactness in Bicomplex Space

By Sukhdev Singh & Rajiv K. Srivastava
Lovely Professional University, India

Abstract - In this paper, we have studied the continuity and compactness of the bicomplex space and its subsets. We have studied the compactness of some subsets of the bicomplex space in the idempotent order topology. We have also given a result regarding homeomorphism in the idempotent order topology and the complex order topology on the bicomplex space.

Keywords : Bicomplex net, idempotent order topology, complex order topology, compactness, homeomorphism.

GJSFR-F Classification : *FOR Code: 820305, 079999*

SOME RESULTS ON COMPACTNESS IN BICOMPLEX SPACE

Strictly as per the compliance and regulations of :

Ref

1. Price, G. B.: An introduction to multicomplex spaces and functions, Marcel Dekker Inc., New York, (1991).

Some Results on Compactness in Bicomplex Space

Sukhdev Singh [α] & Rajiv K. Srivastava [σ]

Abstract - In this paper, we have studied the continuity and compactness of the bicomplex space and its subsets. We have studied the compactness of some subsets of the bicomplex space in the idempotent order topology. We have also given a result regarding homeomorphism in the idempotent order topology and the complex order topology on the bicomplex space.

Keywords : Bicomplex net, idempotent order topology, complex order topology, compactness, homeomorphism.

I. Introduction

Throughout the paper, C_0, C_1 and C_2 denote sets of real numbers, complex numbers and bicomplex numbers, respectively. A bicomplex number is defined as (cf. [1], [3])

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2, \text{ where } x_p \in C_0, 1 \le p \le 4, \; z_1, z_2 \in C_1, \; i_1^2 = i_2^2 = -1$$

and $i_1 i_2 = i_2 i_1$.

With usual binary compositions, C_2 becomes a commutative algebra with identity. Besides the additive and multiplicative identities 0 and 1, there exist exactly two non-trivial idempotent elements denoted by e_1 and e_2 defined as $e_1 = (1 + i_1 i_2)/2$ and $e_2 = (1 - i_1 i_2)/2$. Note that $e_1 + e_2 = 1$ and $e_1 . e_2 = 0$.

A bicomplex number $\xi = z_1 + i_2 z_2$ can be uniquely expressed as a complex combination of e_1 and e_2 as (cf. [3])

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2$$
$$= {}^1\xi e_1 + {}^2\xi e_2$$

where ${}^1\xi = z_1 - i_1 z_2$ and ${}^2\xi = z_1 + i_1 z_2$.

Author α : Department of Mathematics, Lovely Professional University, Phagwara-144411, Punjab, India.
E-mail : ssukhdev1209@yahoo.com
Author σ : Department of Mathematics, Institute of Basic Science, Dr. B. R. Ambedkar University, Agra-282002, India.
E-mail : rajivmathsibs@yahoo.com

The complex coefficients ${}^1\xi$ and ${}^2\xi$ are called the *idempotent components* and the combination ${}^1\xi e_1 + {}^2\xi e_2$ is known as *idempotent representation* of bicomplex number ξ. The auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{z_1 - i_1 z_2 \; ; z_1, z_2 \in C_1\} = \{{}^1\xi : \xi \in C_2\}$$

and

$$A_2 = \{z_1 + i_1 z_2 \; ; z_1, z_2 \in C_1\} = \{{}^2\xi : \xi \in C_2\}.$$

The idempotent representation $(z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2) e_2 = {}^1\xi e_1 + {}^2\xi e_2$ associates with each point $\xi = z_1 + i_2 z_2$ in C_2, the points ${}^1\xi = z_1 - i_1 z_2$ and ${}^2\xi = z_1 + i_1 z_2$ in A_1 and A_2, respectively and to each pair of points $(z, w) \in A_1 \times_e A_2$, there corresponds a unique bicomplex point $\xi = z e_1 + w e_2$.

Srivastava [3] initiated the topological study of C_2. He defined three topologies on C_2, viz., norm topology τ_1, complex topology τ_2 and idempotent topology τ_3 and has proved some results on these topological structures.

In the present paper, we shall confine ourselves mainly to C_2 equipped with τ_5 and τ_6. For the sake of ready reference, we give below relevant literature of τ_5 and τ_6 (for details cf. [4]).

Denote by $\prec_C$, the dictionary ordering of the bicomplex numbers expressed in the complex form. The order topology induced by this ordering is called as *Complex Order Topology*. Complex order topology τ_5 is generated by the basis B_5 comprising members of the following families of subsets of C_2:

(i) $K_1 = \{(z_1 + i_2 z_2,\, w_1 + i_2 w_2)_C : z_1 \prec w_1\}$

(ii) $K_2 = \{(z_1 + i_2 z_2,\, z_1 + i_2 w_2)_C : z_2 \prec w_2\}$.

Remark 1.1: Note that, since $z_1 \prec w_1$ and $z_2 \prec w_2$ in the dictionary order topology in C_1, K_1 and K_2 can also be described as $K_1 = M_1 \cup M_2$ and $K_2 = M_3 \cup M_4$, where

(i) $M_1 = \{(z_1 + i_2 z_2\,,\, w_1 + i_2 w_2)_C : \operatorname{Re} z_1 < \operatorname{Re} w_1\}$

(ii) $M_2 = \{(z_1 + i_2 z_2\,,\, w_1 + i_2 w_2)_C : \operatorname{Re} z_1 = \operatorname{Re} w_1,\, \operatorname{Im} z_1 < \operatorname{Im} w_1\}$

(iii) $M_3 = \{(z_1 + i_2 z_2\,,\, z_1 + i_2 w_2)_C : \operatorname{Re} z_2 < \operatorname{Re} w_2\}$

(iv) $M_4 = \{(z_1 + i_2 z_2\,,\, w_1 + i_2 w_2)_C : \operatorname{Re} z_2 = \operatorname{Re} w_2\,,\, \operatorname{Im} z_2 < \operatorname{Im} w_2\}$.

Note further that M_1, M_2, M_3 and M_4 are in fact, families of space segments, frame segments, plane segments and line segments, respectively.

Ref

3. Srivastava, Rajiv K.: Certain Topological Aspects of Bicomplex Space, Bull. Pure & Appl. Math., 2 (2008), 222–234.

Notes

Similarly, denote by $\prec_{ID}$ the dictionary ordering of the bicomplex numbers expressed in the idempotent from. The order topology induced by this ordering is called as *Idempotent Order Topology*. Hence, idempotent order topology τ_6 is generated by the basis B_6 comprising of members of the following families of subsets of C_2:

(i) $L_1 = \left\{ \left({}^1\xi e_1 + {}^2\xi e_2 , {}^1\eta e_1 + {}^2\eta e_2 \right)_{ID} : {}^1\xi \prec {}^1\eta \right\}$

(ii) $L_2 = \left\{ \left({}^1\xi e_1 + {}^2\xi e_2 , {}^1\xi e_1 + {}^2\eta e_2 \right)_{ID} : {}^2\xi \prec {}^2\eta \right\}$

the set $(\xi , \eta)_{ID}$ denoting the open interval with respect to the ordering $\prec_{ID}$ and $\prec$ denoting the dictionary order relation in A_1 and A_2.

The set of the type $\{\xi : a < \operatorname{Re} {}^1\xi < b\}$ is called an ID– space segment. A set of the type $\{\xi : \operatorname{Re} {}^1\xi = a\}$ is called an ID – frame and is denoted as $(\operatorname{Re} {}^1\xi = a)$. A set of the type $\{\xi : \operatorname{Re} {}^1\xi = a, b < \operatorname{Im} {}^1\xi < c\}$ is called as an open ID – frame segment. The terms ID – plane, ID –plane segment, ID – line and ID – line segment are define analogously (for details cf. [5]).

Note that L_1 and L_2 can also be described as $L_1 = N_1 \cup N_2$ and $L_2 = N_3 \cup N_4$, where

(i) $N_1 = \left\{ \left({}^1\xi e_1 + {}^2\xi e_2 , {}^1\eta e_1 + {}^2\eta e_2 \right)_{ID} : \operatorname{Re} {}^1\xi < \operatorname{Re} {}^1\eta \right\}$

(ii) $N_2 = \left\{ \left({}^1\xi e_1 + {}^2\xi e_2 , {}^1\eta e_1 + {}^2\eta e_2 \right)_{ID} : \operatorname{Re} {}^1\xi = \operatorname{Re} {}^1\eta, \operatorname{Im} {}^1\xi < \operatorname{Im} {}^1\eta \right\}$

(iii) $N_3 = \left\{ \left({}^1\xi e_1 + {}^2\xi e_2 , {}^1\xi e_1 + {}^2\eta e_2 \right)_{ID} : \operatorname{Re} {}^2\xi < \operatorname{Re} {}^2\eta \right\}$

(iv) $N_4 = \left\{ \left({}^1\xi e_1 + {}^2\xi e_2 , {}^1\xi e_1 + {}^2\eta e_2 \right)_{ID} : \operatorname{Re} {}^2\xi = \operatorname{Re} {}^2\eta, \operatorname{Im} {}^2\xi < \operatorname{Im} {}^2\eta \right\}$.

Note further that N_1, N_2, N_3 and N_4 are families of open ID–space segments, open ID–frame segments, open ID–plane segments and open ID–line segments, respectively.

In other words,

$$B_5 = \bigcup_{p=1}^{2} K_p = \bigcup_{p=1}^{4} M_p$$

and

$$B_6 = \bigcup_{p=1}^{2} L_p = \bigcup_{p=1}^{4} N_p .$$

Theorem 1.1 [7]**:** Every order topology is Hausdorff.

Remarks 1.2: The auxiliary complex spaces A_1 and A_2 are Hausdorff space with respect to the order topology generated by dictionary order relation $\prec$ on them.

II. Compactness in Bicomplex Space

In this section, we have given some results regarding compactness on some subsets of the bicomplex space.

Lemma 2.1: Suppose that $S = \{z e_1 + w e_2 : z \in A_1, w = z^{-1}, 0 \prec z \prec 1\}$.

Any subset $(z_0 e_1 + w_1 e_2, z_0 e_1 + w_2 e_2)_{ID}$, where $0 \prec z_0 \prec 1$, of C_2 contains at most one point of the set S.

Proof: We have $S = \left\{ z e_1 + w e_2 : z \in A_1, w = z^{-1}, 0 \prec z \prec 1 \right\}$.

Let $z_0 \in C_1$, where $0 \prec z \prec 1$.

As $z_0 \neq 0$.

$\Rightarrow \exists$ an element $w \in C_1$ such that $z_0^{-1} \neq w$.

Therefore, $\exists$ $z_0 e_1 + w e_2 \in C_2$ such that

$$z_0 e_1 + w e_2 \in (z_0 e_1 + w_1 e_2, z_0 e_1 + w_2 e_2)_{ID},$$

where $w_1, w_2 \in A_2$ and $w_1 \prec w \prec w_2$.

Further, if $(z_0 e_1 + w_1 e_2, z e_1 + w_2 e_2)_{ID}$ is a subset of C_2 such that $w \prec w_1$ where $w = z_0^{-1}$, then

$$(z_0 e_1 + w_1 e_2, z e_1 + w_2 e_2)_{ID} \cap S = \phi.$$

Theorem 2.1: The set $S = \{z e_1 + w e_2 : z \in A_1, w = z^{-1}, 0 \prec z \prec 1\}$ is a compact subset of C_2.

Proof: To prove S is a compact, we show that the set S is bounded and closed with respect to the idempotent order topology in C_2. Since, the coefficients of e_1 of the elements of S are bounded by the lower and upper bounds 0 and 1, respectively.

Therefore, the elements of the set S are bounded by the bicomplex numbers $w_1 e_2$ and $e_1 + w_1 e_2$, $w_1 \in A_2$ as lower and upper bounds respectively, i.e., $z e_1 + w e_2 \in (w_1 e_2, e_1 + w_1 e_2)_{ID}$, where $0 \prec z \prec 1$.

Hence, the set S is bounded.

To prove that S is closed, we shall show that S^c is open in C_2.

Let $\xi = z_0 e_1 + w_0 e_2 \in S^c$.

Then there are three possibilities:

1) $z_0^{-1} \neq w_0$ and $z_0 \in (0, 1)$
2) $z_0^{-1} = w_0$ and $z_0 \notin (0, 1)$
3) $z_0^{-1} \neq w_0$ and $z_0 \notin (0, 1)$.

Ref

5. Srivastava, Rajiv K. and Sukhdev Singh: on Bicomplex Nets and their Confinements, Amer. J. of Math. and Stat., Vol. 1(1), (2011), 8 – 16.

Notes

Case 1): When $z_0^{-1} \neq w_0$ and $z_0 \in (0,1)$.

Since both z_0^{-1} and w_0 are members of the auxiliary complex space A_2 and A_2 is a Hausdorff space under the dictionary order topology.
Therefore, $\exists\ u_1, u_2 \in A_2$ such that

$$z_0^{-1} \in (u_1, u_2).$$

Similarly, $\exists\ v_1, v_2 \in A_2$ such that

$$w_0 \in (v_1, v_2) \text{ and } (u_1, u_2) \cap (v_1, v_2) = \phi.$$

So, we have

$$\xi = z_0 e_1 + w_0 e_2 \in (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{ID}$$

Now if the interval $(z_0 e_1 + v_1 e_2,\ z_0 e_1 + v_2 e_2)_{ID}$ has non $-$ empty intersection with the set S then it will contain only one point namely, $z_0 e_1 + z_0^{-1} e_1$.
But

$$z_0 e_1 + z_0^{-1} e_1 \notin (z_0 e_1 + v_1 e_2, z_0 e_1 + v_2 e_2)_{ID}.$$

Hence, $$(z_0 e_1 + v_1 e_2, z_0 e_1 + v\_ e\_)_{ID} \cap S = \phi.$$

$$\Rightarrow \xi = z_0 e_1 + w_0 e_2 \in (z_0 e_1 + v_1 e_2,\ z_0 e_1 + v_2 e_2)_{ID} \subset S^c$$

Case 2): When $z_0^{-1} = w_0$ and $z_0 \notin (0,1)$.

Two sub cases will arise as follows:

Either $z_0 \preceq 0$ or $z_0 \succeq 1$.

Sub – case 2) (i): If $z_0 \preceq 0$.

If $z_0 = 0$. Then $\xi = z_0 e_1 + w_0 e_2$ does not exist. Therefore we can say that $z_0 \neq 0$.

If $z_0 \prec 0$, then as $z_0\ (\neq 0) \in A_1$ and as A_1 is a Hausdorff space under dictionary order topology.
Therefore, there exist $a_1, a_2 \in A_1$ such that

$$0 \in (a_1, a_2)$$

and similarly there exist

$$w_1, w_2 \in A_1 \text{ such that } z_0 \in (w_1, w_2).$$

and $$(a_1, a_2) \cap (w_1, w_2) = \phi.$$

Also, $$(0,1) \cap (w_1, w_2) = \phi.$$

Therefore, $w_1 \prec 0$ as well as $w_2 \prec 0$.

Global Journal of Science Frontier Research (F) Volume XIII Issue IX Version I Year 2013

Hence, for any $b_1, b_2 \in A_2$ we have

$$z_0 e_1 + w_0 e_2 \in (w_1 e_1 + b_1 e_2,\ w_1 e_1 + b_2 e_2)_{ID} \subset S^c$$

Sub – case 2(ii): If $z_0 \succeq 1$.

Either $z_0 = 1$ or $z_0 \succ 1$.

If $z_0 = 1$, then $\xi = 1$.

So there exists two bicomplex numbers $\eta = e_1 + \left(\frac{1}{2}\right)e_2$ and $\zeta = e_1 + \left(\frac{3}{2}\right)e_2$

such that $\xi \in (\eta, \zeta)_{ID} \subset S^c$.

If $z_0 \succ 1$, then $z_0^{-1}(= w_0) \prec 1$.

Since $z_0\ (\neq 1) \in A_1$ and as w_0 and 1 are two points of A_2.

Also A_2 is Hausdorff under dictionary order topology.

$\Rightarrow \exists\, u_1, u_2 \in A_2$ such that $w_0 \in (u_1, u_2)$

and similarly, $\exists\, v_1, v_2 \in A_2$ such that

$$1 \in (v_1, v_2)$$

and

$$(u_1, u_2) \cap (v_1, v_2) = \phi.$$

Therefore,

$$z_0 e_1 + w_0 e_2 \in (z_0 e_1 + u_1 e_2,\ z_0 e_1 + u_2 e_2)_{ID}$$

and
$$(z_0 e_1 + u_1 e_2,\ z_0 e_1 + u_2 e_2)_{ID} \cap S \neq \phi$$

$\Rightarrow (z_0 e_1 + u_1 e_2,\ z_0 e_1 + u_2 e_2)_{ID} \subset S^c$.

Case 3): When $z_0^{-1} \neq w_0$ and $z_0 \notin (0, 1)$.

As $z_0^{-1} \neq w_0$ and $z_0^{-1}, w_0 \in A_2$ and A_2 is Hausdorff space in the dictionary order topology.

Therefore, there exist $c_1, c_2 \in A_2$ such that

$$z_0^{-1} \in (c_1, c_2)$$

Similarly there exist $d_1, d_2 \in A_2$ such that

$$w_0 \in (d_1, d_2) \text{ and } (c_1, c_2) \cap (d_1, d_2) = \phi.$$

Therefore we have obtained an interval

$$(z_0 e_1 + d_1 e_2,\ z_0 e_1 + d_2 e_2)_{ID}$$

Notes

such that

$$z_0 e_1 + w_0 e_2 \in (z_0 e_1 + d_1 e_2 , \ z_0 e_1 + d_2 e_2)_{ID}$$

and

$$(z_0 e_1 + d_1 e_2 , z_0 e_1 + d_2 e_2)_{ID} \subset S^c .$$

We conclude that set S is closed as well as bounded.

Hence, S is compact subset of C_2.

N

Theorem 2.2: The set

$S = \{(z , \sin z^{-1}) \ : 0 \prec z \prec 1\}$ is a compact subset of C_2.

Proof : W have $S = \{(z , \sin z^{-1}) \ : 0 \prec z \prec 1\}$

$\Rightarrow \ S = \{(z , \sin w) \ : 0 \prec z \prec 1 , \ w = z^{-1}\}$

$S = \{z e_1 + (\sin w) e_2 \ : \ 0 \prec z \prec 1 , \ w = z^{-1}\}$.

$\Rightarrow \xi = z_0 e_1 + w_0 e_2 \in (z_0 e_1 + v_1 e_2 , \ z_0 e_1 + v_2 e_2)_{ID} \subset S^c$

To prove S is a compact subset of C_2, we show that S is closed and bounded.

Since, $0 \prec z \prec 1$,

$0 \prec z$ as well as $z \prec 1$.

Therefore, z is bounded with lower and upper bounds 0 and 1, respectively.

Hence, S is bounded subset of C_2 under the idempotent order relation with lower and upper bounds $w_1 e_2$ and $e_1 + w_2 e_2$, respectively.

Now to show S is closed in C_2.

Let $\xi = u e_1 + v e_2 \in S^c$.

Then there are three possibilities as follows:

(i) $u \notin (0, 1)$ and $v = \sin u^{-1}$

(ii) $u \in (0, 1)$ and $v \neq \sin u^{-1}$

(iii) $u \notin (0, 1)$ and $v \neq \sin u^{-1}$.

Case (i): If $u \notin (0, 1)$ and $v = \sin u^{-1}$

Now as $u \notin (0, 1)$.

Therefore, either $u \preceq 0$ or $1 \preceq u$.

If u = 0, then $\sin u^{-1}$ is not defined. So that, $u \neq 0$.

Now suppose that $u < 0$.

Since u and 0 are two distinct points of A_1 and A_1 is Hausdorff space with respect to the dictionary order topology.

Therefore, there exists $u_1, u_2 \in A_1$ such that

$$u \in (u_1, u_2).$$

Similarly there exists $z_1, z_2 \in A_1$ such that

$0 \in (z_1, z_2)$ and also $(u_1, u_2) \cap (z_1, z_2) = \phi$.

So that

$$ue_1 + ve_2 \in (u_1 e_1 + v_1 e_2, u_2 e_1 + v_2 e_2)_{ID}$$

and $$(u_1 e_1 + v_1 e_2, u_2 e_1 + v_2 e_2)_{ID} \cap S = \phi.$$

$$ue_1 + ve_2 \in (u_1 e_1 + v_1 e_2, u_2 e_1 + v_2 e_2)_{ID} \subset S^c$$

Therefore, S^c is an open set. Hence S is a closed set.

Similarly we can prove that S is closed if $1 \preceq u$.

Case b): If $u \in (0 + i_1 0, 1 + i_1 0)$ and $v \neq \sin u^{-1}$.

Then v and $\sin u^{-1}$ are two distinct points of A_2 and A_2 is Hausdorff space with respect to the dictionary order topology.

Therefore there exists $v_1, v_2 \in A_2$ such that

$$v \in (v_1, v_2).$$

Similarly, there exists $w_1, w_2 \in A_2$ such that $\sin u^{-1} \in (w_1, w_2)$

and

$$(v_1, v_2) \cap (w_1, w_2) = \phi.$$

Therefore, $$ue_1 + ve_2 \in (ue_1 + v_1 e_2, ue_1 + v_2 e_2)_{ID}$$

and $$(ue_1 + v_1 e_2, ue_1 + v_2 e_2)_{ID} \cap S = \phi$$

$\Rightarrow ue_1 + ve_2$

$\in (ue_1 + v_1 e_2, ue_1 + v_2 e_2)_{ID} \subset S^c$

So that S^c is an open set.

Hence, S is a closed set.

Case c): If $u \notin (0 + i_1 0, 1 + i_1 0)$ and $v \neq \sin u^{-1}$.

By the similar procedure as in case (a), we have S is a closed subset of C_2.

Hence we conclude that S is a closed and bounded subset of C_2.

Therefore, S is a compact subset of C_2.

Notes

Notes

References Références Referencias

1. Price, G. B.: An introduction to multicomplex spaces and functions, Marcel Dekker Inc., New York, (1991).
2. Srivastava, Rajiv K.: Bicomplex Numbers: Analysis and Applications, Math. Student, 72(1–4), (2003), 63–87.
3. Srivastava, Rajiv K.: Certain Topological Aspects of Bicomplex Space, Bull. Pure & Appl. Math., 2 (2008), 222–234.
4. Srivastava, Rajiv K. and Sukhdev Singh: Certain Bicomplex Dictionary Order Topologies, Inter. J. of Math. Sci. and Engg. Appls. (4), III (2010), 245 – 258.
5. Srivastava, Rajiv K. and Sukhdev Singh: on Bicomplex Nets and their Confinements, Amer. J. of Math. and Stat., Vol. 1(1), (2011), 8 – 16.
6. Srivastava, Rajiv K. and Sukhdev Singh: On Various Types of Convergences of Bicomplex Nets, Proceedings ISMAMS, Vol. 6, (2011), 69 – 82.
7. Willard, S.: General topology, Addison Wesley Massachusetts, (1970)