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A Priori Estimate and Fourier's Method for Nonlocal Boundary Conditions of Mixed Problem for Singular Parabolic Equations in Sobolev Function Spaces

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A PRIORI ESTIMATE AND FOURIERS METHOD FOR NONLOCAL BOUNDARY CONDITIONS OF MIXED PROBLEM FOR SINGULAR PARABOLIC EQUATIONS IN SOBOLEV FUNCTION SPACES

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A Priori Estimate and Fourier's Method for Nonlocal Boundary Conditions of Mixed Problem for Singular Parabolic Equations in Sobolev Function Spaces

Djibibe Moussa Zakari [α] & Tcharie Kokou [σ]

Abstract - The aims of this paper is to prove existence and uniqueness of following integral boundary conditions mixed problem for parabolic equation:

$$\begin{cases} \dfrac{\partial\theta}{\partial t} - \dfrac{a(t)}{x^2}\dfrac{\partial}{\partial x}\left(x^2\dfrac{\partial\theta}{\partial x}\right) + b(t)\theta = \vartheta(x,t) \\ \theta(x,0) = \lambda(x),\ 0 \le x \le \ell \\ \displaystyle\int_0^\ell x\theta(x,t)\,dx = E(t),\ 0 \le t \le T \\ \displaystyle\int_0^\ell x^2\theta(x,t)\,dx = G(t),\ 0 \le t \le \ell, \end{cases} \tag{0.1}$$

The proofs are based on a priori estimates established in Sobolev function spaces and Fourier's method.

Keywords : fourier's method, a priori estimate, nonlocal conditions, mixed problem, parabolic, sobolev espace.

I. Introduction

This paper deals with existence and uniqueness of a class of parabolic equation with time and space-variable characteristics, with a nonlocal boundary condition. The precise statement of the problem is a follows: let $\ell > 0$, $T > 0$, and $\Omega = \{(x,t) \in \mathbb{R}^2 : 0 < x < \ell,\ 0 < t < T\}$. We shall determine a solution θ, in Ω of the differential equation

$$\frac{\partial\theta}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) + b(t)\theta = \vartheta(x,t), \quad (x,t) \in \Omega, \tag{1.1}$$

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satisfying the initial condition

$$\theta(x,0)=\lambda(x),\ 0\le x\le \ell. \tag{1.2}$$

and the integral conditions

$$\int_0^\ell x\theta(x,t)\,dx=E(t),\ 0\le t\le T \tag{1.3}$$

$$\int_0^\ell x^2\theta(x,t)\,dx=G(t),\ 0\le t\le T. \tag{1.4}$$

where $\lambda,\ E,\ G,\ a,b$ and ϑ are known functions

Assumption 1.1

For all $(x,t)\in\overline{\Omega}$, *we assume that*

$$a_0\le a(t)\le a_1,\quad a_2\le \frac{da(t)}{dt}\le a_3$$

$$b_0\le b(t)\le b_1,\quad b_2\le \frac{db(t)}{dt}\le b_3,$$

$$\frac{b(t)}{a(t)}=s.$$

where $a_0,\ a_1,\ a_2,\ a_3,\ b_0,\ b_1,\ b_2$ *are positive constants.*

The data satisfies the following compatibility conditions : For consistency, we have

$$\int_0^\ell x\lambda(x)=E(0),\quad \text{and } \int_0^\ell x^2\lambda(x)=G(0),$$

The importance of problems with integral conditions has been pointed out by Samarskii[11]. Mathematical modelling by evolution problems with a nonlocal constraint of the form $\dfrac{1}{1-\alpha}\displaystyle\int_\alpha^1 u(x,t)\,dx=\chi(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. Cannon [7] used the potentiel method, combining a Dirichlet and an intégral condition for a parabolic equation. L. A. Mouravey and V. Philinovoski [10] used the maximum principle, combining a Neumann and an integral condition for heat equation. Ionkin [8] and L. Bougoffa[4] used the Fourier method for same purpose.

Recently, mixed problems with integral conditions for generalization of equation (1.1) have been treated using the energy-integral method. See N. E. Benouar and N. I. Yurchuk [1], N. E. Benouar and A. Bouziani [2],[3], A. Bouziani[5], [6], M. Z. Djibibe el al. [12],[13], N. I. Yurchuk[14],[15], M. Mesloub, A. Bouziani and N. Kechkar[9]. Differently to these works, in the present paper we combine a priori estimate and Fourier's method to prove existence and uniqueness solution of the problem (1.1)- (1.4).

The result of the paper are new. It is interesting to note that the application of Fourier method to this nonlocal problem is made possible thanks, essentially, to the use of a Sobolev function space.

Ref.

[1] N. E. Benouar and N. I. Yurchuk; Mixed problem with an integral condition for parabolique equation with the Bessel operator, *Differentsial'nye Uravneniya*, 27 (1991), p. 2094 - 2098.

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To this, we reduce the inhomogeneous boundary conditions (1.3) and (1.4) to homogeneous conditions, by introducing a new unknown function z by $z(x,t)=\theta(x,t)-\eta(x,t)$, where

$$\begin{aligned}\eta(x,t)=&\frac{10l^4x^2-12(l^5+20)x+3l(l^5+60)}{10\ell^3}E(t)\\&+\frac{10\ell^5x^2-12(\ell^6-30)x+3\ell(l^6-80)}{10\ell^4}G(t).\end{aligned}\tag{1.5}$$

Then, problem (1.1), (1.2), (1.3) and (1.4) is transformed into the following homogeneous boundary value problem

$$\frac{\partial z}{\partial t}-\frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right)+b(t)z=\xi(x,t),\quad (x,t)\in\Omega,\tag{1.6}$$

$$z(x,0)=\Lambda(x),\ 0\le x\le \ell,\tag{1.7}$$

$$\int_0^\ell xz(x,t)\,dx=0,\ 0\le t\le T,\tag{1.8}$$

$$\int_0^\ell x^2z(x,t)\,dx=0,\ 0\le t\le T,\tag{1.9}$$

where

$$\begin{aligned}\xi(x,t)=&\vartheta(x,t)-\frac{10l^4x^2-12(l^5+20)x+3l(l^5+60)}{10\ell^3}E'(t)\\&-\frac{10\ell^5x^2-12(\ell^6-30)x+3\ell(l^6-80)}{10\ell^4}G'(t)\\&+a(t)\left(6\ell-\frac{6(\ell^5+20)}{5\ell^3x^2}+\right)E(t)+a(t)\left(4\ell-\frac{6(\ell^6-30)}{5\ell^4x^2}\right)G(t)\\&-\frac{10l^4x^2-12(l^5+20)x+3l(l^5+60)}{10\ell^3}b(t)E(t)\\&-\frac{10\ell^5x^2-12(\ell^6-30)x+3\ell(l^6-80)}{10\ell^4}b(t)G(t),\end{aligned}\tag{1.10}$$

$$\begin{aligned}\Lambda(x)=&\lambda(x)-\frac{10l^4x^2-12(l^5+20)x+3l(l^5+60)}{10\ell^3}E(0)\\&+\frac{10\ell^5x^2-12(\ell^6-30)x+3\ell(l^6-80)}{10\ell^4}G(0).\end{aligned}\tag{1.11}$$

Here, taking account assumption 1.1, we assume that the function Λ satisfy conditions of (1.8) and (1.9), that is

$$\int_0^\ell x\Lambda(x)\,dx=\int_0^\ell x^2\Lambda(x)\,dx=0.\tag{1.12}$$

Instead of searching for the function θ, we search for the function z. So the solution of problem (1.1), (1.2), (1.3) and (1.4) will be given by $\theta(x,t)=z(x,t)+\eta(x,t)$.

The general difficult which arises to us is the presence of integral conditions which complicates the application of standard methods. It may, however, be worth while if this type of problem can be transformed into another equivalent problem which involves no integral conditions. For this, we convert problem (1.6), (1.7), (1.8) and (1.9) to the following classical problem.

Theorem 1.1

The problem (1.6), (1.7), (1.8) *and* (1.9) *is equivalent to the following classical problem :*

$$\frac{\partial z}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) + b(t)z = \xi(x,t), \quad (x,t) \in \Omega, \tag{1.13}$$

$$z(x,0) = \Lambda(x), \; 0 \leq x \leq \ell, \tag{1.14}$$

$$z(t,\ell) - u(0,t) = \frac{1}{\ell a(t)}\int_0^\ell (x^2 - \ell x)\xi(x,t)\,dx, \; 0 \leq t \leq T, \tag{1.15}$$

$$\frac{\partial z}{\partial x}(\ell,t) = -\frac{1}{\ell^2 a(t)}\int_0^\ell x^2\xi(x,t)\,dx, \; 0 \leq t \leq T. \tag{1.16}$$

Proof

Firstly, multiplying (1.6) with x, and integrating the obtained result with respect x over $(0,\ell)$, we obtain

$$\frac{\partial}{\partial t}\int_0^\ell xz\,dx - a(t)\int_0^\ell \frac{1}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right)\,dx + b\int_0^\ell xz\,dx = \int_0^\ell x\xi(x,t)\,dx. \tag{1.17}$$

Integrating by parts the integrals on the left-hand side of (1.17), and taking into account condition (1.8), we get

$$\ell\frac{\partial z}{\partial x}(\ell,t) + z(\ell,t) - z(0,t) = -\frac{1}{a(t)}\int_0^\ell x\xi(x,t)\,dx. \tag{1.18}$$

Secondly, multiplying (1.6) with x^2 and integrating the result obtained over $(0,\ell)$, he have

$$-a(t)\int_0^\ell \frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right)\,dx = \int_0^\ell x^2\xi(x,t)\,dx. \tag{1.19}$$

Integrating by parts the integrals on the left-hand side of (1.19).

$$\frac{\partial z}{\partial x}(\ell,t) = -\frac{1}{\ell^2 a(t)}\int_0^\ell x^2\xi(x,t)\,dx \tag{1.20}$$

Combining the equalities (1.19) and (1.20), we have

$$z(\ell,t) - z(0,t) = \frac{1}{\ell a(t)}\int_0^\ell (x^2 - \ell x)\xi(x,t)\,dx. \tag{1.21}$$

Finaly, it remains to prove that $\int_0^\ell xz(x,t)\,dx = 0$ and $\int_0^\ell x^2 z(x,t)\,dx = 0$.

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By using (1.6) and taking into account (1.18) and (1.20) we get

$$\frac{d}{dt}\int_0^\ell xz(x,t)\,dx + b\int_0^\ell xz(x,t)\,dx = 0,\ \ 0 \le t \le T$$

$$\frac{d}{dt}\int_0^\ell x^2z(x,t)\,dx + b\int_0^\ell x^2z(x,t)\,dx = 0,\ \ 0 \le t \le T$$

By virtue of the compatibility of the conditions, it follows that

$$\int_0^\ell xz(x,t)\,dx = \int_0^\ell x^2z(x,t)\,dx = 0.$$

This complete the proof of Theorem (1.1).

By introducing the new unknown function

$$u(x,t) = z(x,t) - \alpha(x,t)\int_0^\ell x^2\xi(x,t)\,dx - \beta(x,t)\int_0^\ell x\xi(x,t)\,dx,$$

where

$$\alpha(x,t) = \frac{-3x^3 + 3\ell x^2 + \ell^5 a(t)}{\ell^4 a(t)}$$

and

$$\beta(x,t) = \frac{2x^3 - 3\ell x^2 - \ell^3 a(t)}{\ell^3 a(t)},$$

the problem (1.13), (1.14), (1.15) and (1.16) is transformed into the following local boundary conditions problem,

$$\frac{\partial u}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) + bu = f(x,t), \quad (x,t) \in \Omega, \tag{1.22}$$

$$u(x,0) = \varphi(x),\ \ 0 \le x \le \ell, \tag{1.23}$$

$$z(\ell,t) = u(0,t),\ \ 0 \le t \le T, \tag{1.24}$$

$$\frac{\partial u}{\partial x}(\ell,t) = 0,\ \ 0 \le t \le T, \tag{1.25}$$

where

$$f(x,t) = \xi(x,t) + \frac{3x^3 - 3\ell x^2 - a'(t)\ell^5}{\ell^4 a'(t)}\int_0^\ell x^2\frac{\partial \xi}{\partial t}(x,t)\,dx - \frac{18(2x-\ell)}{\ell^4}\int_0^\ell x^2\xi(x,t)\,dx$$

$$+\frac{(3x^3 - 3\ell x^2 - a(t)\ell^5)b(t)}{\ell^4 a(t)}\int_0^\ell x^2\xi(x,t)\,dx + \frac{6(4x-3\ell)}{\ell^3}\int_0^\ell x\xi(x,t)\,dx$$

$$-\frac{(2x^3-3\ell x^2-\ell^3 a(t))b(t)}{\ell^3 a(t)}\int_0^\ell x\xi(x,t)\,dx-\frac{2x^3-3\ell x^2-\ell^3 a'(t)}{\ell^3 a'(t)}\int_0^\ell x\frac{\partial\xi}{\partial t}(x,t)\,dx,$$

$$\varphi(x)=\Lambda(x)+\frac{3x^3-3\ell x^2-a(0)\ell^5}{\ell^4 a(0)}\int_0^\ell x^2\xi(x,0)\,dx-\frac{2x^3-3\ell x^2-\ell^3 a(0)}{\ell^3 a(0)}\int_0^\ell x\xi(x,o)\,dx.$$

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II. Main Results

a) *An Energy Inequality*

The problem (1.17), (1.18), (1.19) and (1.20) can be considered as solving the following operator equation :

$$Lu=(\varphi,f)=\mathcal{F},$$

where L is an operator defined on E into F. E is the banach space of functions $u\in L^2(\Omega)$, satisfying conditions (1.19) and (1.20) with the norm

$$\|u\|_E^2=\int_{\Omega_\tau}\left(x^2u^2+\left|x\frac{\partial u}{\partial x}\right|^2+\left|x\frac{\partial u}{\partial t}\right|^2\right)dxdt+\sup_{0\le t\le T}\int_0^\ell\left(x^2u^2+\left|x\frac{\partial u}{\partial x}\right|^2\right)dx$$

and F is the Hilbert space $L^2(\Omega)\times L^2(0,\ell)$ which consists of elements $\mathcal{F}=(\varphi,f)$ with the norm

$$\|\mathcal{F}\|_F^2=\int_0^\ell\left(\varphi^2(x)+\left(\frac{d\varphi}{dx}\right)^2\right)dx+\int_{\Omega_\tau}f^2(x,t)\,dxdt.$$

Let $D(L)$ be the set of all function u, for which $u,\ xu,\ x^2\frac{\partial u}{\partial x}\in L^2(0,\ell)$ and $u,\ x\frac{\partial u}{\partial x},\ x\frac{\partial u}{\partial t},$ $x^2\frac{\partial u}{\partial x},\ \frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\in L^2(\Omega)$.

Theorem 2.1

There exists a positive constant c*, such that for each function* $u\in D(L)$ *we have*

$$\|u\|_E\le c\|Lu\|_F. \tag{2.1}$$

Proof

We consider the scalar product in $L^2(\Omega_\tau)$ with $0\le t\le\tau$ of equation (1.22) and $xu(x,t)+x^2\frac{\partial u}{\partial t}$, yields

$$\int_{\Omega_\tau}xu\frac{\partial u}{\partial t}\,dxdt+\int_\Omega x^2\left(\frac{\partial u}{\partial t}\right)^2dxdt-\int_{\Omega_\tau}\frac{a(t)u}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)dxdt$$
$$-\int_{\Omega_\tau}a(t)\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,dxdt+\int_{\Omega_\tau}b(t)xu^2\,dxdt+\int_{\Omega_\tau}b(t)x^2u\frac{\partial u}{\partial t}\,dxdt$$
$$=\int_{\Omega_\tau}xuf(x,t)\,dxdt+\int_{\Omega_\tau}x^2f(x,t)\frac{\partial u}{\partial t}\,dxdt. \tag{2.2}$$

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Integrating by parts certains integrals of left-hand side of (2.2), we get

$$\int_{\Omega_\tau} xu\frac{\partial u}{\partial t}\,dxdt = \int_0^\ell xu^2(x,\tau)\,dx - \frac{1}{2}\int_0^\ell x\varphi^2(x)\,dx, \tag{2.3}$$

$$-\int_{\Omega_\tau} \frac{a(t)u}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\,dxdt = \int_{\Omega_\tau} a(t)\left(x\frac{\partial u}{\partial x}\right)^2\,dxdt, \tag{2.4}$$

$$\begin{aligned}\int_{\Omega_\tau} b(t)x^2u\frac{\partial u}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell b(0)x^2u(x,\tau)\,dx - \frac{1}{2}\int_0^\ell b(0)x^2\varphi^2(x)\,dx\\ -\frac{1}{2}\int_{\Omega_\tau} b'(t)x^2u^2\,dxdt,\end{aligned} \tag{2.5}$$

$$-\int_{\Omega_\tau} a(t)\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell a(t)\left(x\frac{\partial u}{\partial x}\right)^2\,dx - \frac{1}{2}\int_0^\ell a'(t)\left(x\frac{d\varphi}{dx}\right)^2\,dx. \tag{2.6}$$

Substituting the equalities (2.3), (2.4), (2.5) and (2.6) into (2.2), it follows that

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\tau}(2b(t)x - b'(t)x^2)u^2\,dxdt + \int_{\Omega_\tau} a(t)\left(x\frac{\partial u}{\partial x}\right)^2\,dxdt + \int_{\Omega}\left(x\frac{\partial u}{\partial t}\right)^2\,dxdt\\ &+\int_0^\ell xu^2(x,\tau)\,dx + \frac{b(0)}{2}\int_0^\ell x^2u(x,\tau)\,dx + \frac{1}{2}\int_0^\ell a(t)\left(x\frac{\partial u}{\partial x}\right)^2\,dx = \int_{\Omega_\tau} xuf(x,t)\,dxdt\\ &+\int_{\Omega_\tau} x^2f(x,t)\frac{\partial u}{\partial t}\,dxdt + \frac{1}{2}\int_0^\ell x\varphi^2(x)\,dx + \frac{b(0)}{2}\int_0^\ell x^2\varphi^2(x)\,dx + \frac{1}{2}\int_0^\ell a'(t)\left(x\frac{d\varphi}{dx}\right)^2\,dx.\end{aligned} \tag{2.7}$$

Estimating the first and the two first integrals of the right-hand side of (2.7), by applying elementary inequalities, we get

$$\int_{\Omega_\tau} xuf(x,t)\,dxdt \le \frac{\varepsilon_1}{2}\int_{\Omega_\tau} xu^2\,dxdt + \frac{1}{2\varepsilon_1}\int_{\Omega_\tau} xf(x,t)\,dxdt, \tag{2.8}$$

$$\int_{\Omega_\tau} x^2f(x,t)\frac{\partial u}{\partial t}\,dxdt \le \frac{1}{2}\int_{\Omega_\tau}\left(x\frac{\partial u}{\partial t}\right)^2\,dxdt + \frac{1}{2}\int_{\Omega_\tau} x^2f^2(x,t)\,dxdt. \tag{2.9}$$

Therefore, by formulas (2.7), (2.8) and (2.9), we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\tau}(2b(t) - \varepsilon_1 - b'(t)x)xu^2\,dxdt + \frac{1}{2}\int_{\Omega}\left(x\frac{\partial u}{\partial t}\right)^2\,dxdt + \int_0^\ell xu^2(x,\tau)\,dx\\ &+\frac{b(0)}{2}\int_0^\ell x^2u(x,\tau)\,dx + \frac{1}{2}\int_0^\ell a(t)\left(x\frac{\partial u}{\partial x}\right)^2\,dx \le \frac{1+\ell^2}{2\varepsilon_1}\int_{\Omega_\tau} f^2(x,t)\,dxdt\\ &+\frac{1}{2}\int_0^\ell x\varphi^2(x)\,dx + \frac{b(0)}{2}\int_0^\ell x^2\varphi^2(x)\,dx + \frac{1}{2}\int_0^\ell a'(t)\left(x\frac{d\varphi}{dx}\right)^2\,dx.\end{aligned} \tag{2.10}$$

Hence, if $\varepsilon_1 > 0$ satisfies $\varepsilon_1 \le \min_{\Omega_\tau} \left\{ 2b(t) - x\frac{db(t)}{dt} \right\}$, and choosing $\varepsilon_2 = 2b_0 - \varepsilon_1 - \ell b_3$, then inequality (2.10) implies

$$\frac{\varepsilon_2}{2} \int_{\Omega_\tau} xu^2 \, dxdt + a_0 \int_{\Omega_\tau} \left(x\frac{\partial u}{\partial x} \right)^2 dxdt + \frac{1}{2} \int_{\Omega} \left(x\frac{\partial u}{\partial t} \right)^2 dxdt$$
$$+ \frac{2+b_0\ell}{2\ell} \int_0^\ell x^2 u^2(x,\tau)\, dx + \frac{a_0}{2} \int_0^\ell \left(x\frac{\partial u}{\partial x} \right)^2 dx \le \frac{1+\ell^2}{2\varepsilon_1} \int_{\Omega_\tau} f^2(x,t)\, dxdt$$
$$+ \frac{1+b_1\ell}{2} \int_0^\ell x\varphi^2(x)\, dx + \frac{a_3}{2} \int_0^\ell \left(x\frac{d\varphi}{dx} \right)^2 dx. \qquad (2.11)$$

Therefore, by formulas (2.11) and of assumption (1.1), we obtain

$$\int_{\Omega_\tau} xu^2 \, dxdt + \int_{\Omega_\tau} \left(x\frac{\partial u}{\partial x} \right)^2 dxdt + \int_{\Omega} \left(x\frac{\partial u}{\partial t} \right)^2 dxdt + \int_0^\ell x^2 u^2(x,\tau)\, dx +$$
$$+ \int_0^\ell \left(x\frac{\partial u}{\partial x} \right)^2 dx \le \frac{A}{B} \left[\iint_{\Omega_\tau} f^2(x,t)\, dxdt + \int_0^\ell x\varphi^2(x)\, dx + \int_0^\ell \left(x\frac{d\varphi}{dx} \right)^2 dx \right], \qquad (2.12)$$

where

$$A = \min \left(\frac{2+b_0\ell}{2\ell}, \frac{a_0}{2}, \frac{1}{2}, \frac{\varepsilon_2}{2} \right) \quad B = \max \left(\frac{1+\ell^2}{2\varepsilon_1}, \frac{1+b_1\ell}{2}, \frac{a_3}{2} \right),$$

The right-hand side of (2.12) is independent of τ, replacing the left-hand side by the upper with respect to τ. Thus inequality (2.1) holds, where

$$c = \sqrt{\frac{\min \left(\frac{2+b_0\ell}{2\ell}, \frac{a_0}{2}, \frac{1}{2}, \frac{\varepsilon_2}{2} \right)}{\max \left(\frac{1+\ell^2}{2\varepsilon_1}, \frac{1+b_1\ell}{2}, \frac{a_3}{2} \right)}}$$

This completes the proof of Theorem (1.1).

b) Solvability of the Problem

Now we shall start to prove the existence of the boundary value problem (1.13), (1.14), (1.15) and (1.16). We use the Fourier's method.
Consider the fouction $u_n(x,t) = v_n(x)w_n(t)$, where $v_n(t)$ is a eigenfunction of the following boundary value problem

$$\begin{cases} -\dfrac{1}{x^2}\dfrac{d}{dx}\left(x^2 \dfrac{dv_n(x)}{dx} \right) + sv_n(x) = \alpha_n v_n(x) \\ v_n(0) = v_n(\ell) \\ \dfrac{dv_n(\ell)}{dx} = 0 \end{cases}$$

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where α_n is the eigenvalue corresponding to the eigenfunction $v_n(x)$, and $w_n(t)$ satisfying the initial problem

$$\begin{cases} \dfrac{dw_n(t)}{a(t)dt} - \alpha_n w_n(t) = f_n(t) \\ \\ w_n(0) = \varphi_n. \end{cases}$$

Here

$$\varphi(x) = \sum_{n=1}^{+\infty} \varphi_n v_n(x)$$
$$\varphi'(x) = \sum_{n=0}^{+\infty} \rho_n v_n(x)$$
$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t) v_n(x).$$

Lemma 2.1
Using the Parseval-Steklov *equality, we have*

$$\|(f,\varphi)\|_F = \sum_{n=1}^{+\infty} \left(\int_0^T f_n^2(t)\,dt + \varphi_n^2 + \rho_n^2 \right).$$

The direct computation, the solution of the initial problem is giving by

$$w_n(t) = \varphi_n e^{\alpha_n t} + \int_0^t f_n(t) e^{\alpha_n (t-\tau)}\,dt.$$

By virtue principle of superposition, the solution of the boundary value problem (1.13), (1.14), (1.15) *and* (1.16) *is giving by the series*

$$u(x,t) = \sum_{1}^{+\infty} v_n(x) w_n(t). \tag{2.13}$$

Theorem 2.2
Let assumption 1.1 be fulfilled. Then for any $f \in L^2(\Omega)$ *and* $\varphi \in L_2(0,\ell)$ *which* $\dfrac{d\varphi}{dx} \in L^2(0,\ell)$, *problem*(1.13), (1.14), (1.15) *and* (1.16) *admits a unique solution and its represented by series* (2.13) *which converge in* E.

Proof

Consider the partial sum $S_n(x,t) = \sum_{k=1}^{n} v_k(x) w_k(t)$ of the series (2.13).

By applaying the Theorem 1.1, then it follows that

$$\left\| \sum_{i=1}^{n} v_k(x) w_k(t) \right\| \le C \sum_{n=1}^{+\infty} \left(\int_0^T f_n^2(t)\,dt + \varphi_n^2 + \rho_n^2 \right) \tag{2.14}$$

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The series $\sum_{n=1}^{+\infty} \int_0^T f_n^2(t)\,dt = \int_\Omega f^2(x,t)\,dxdt$, $\sum_{n=1}^{+\infty} \varphi_n^2$ and $\sum_{n=1}^{+\infty} \rho_n^2$ converge.

Therefore, from (2.14) it follows that the series (2.13) converge in E.
This completes the proof of the Theorem 2.2.

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