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On Some Problems in Fuzzy Sets Theory

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ON SOME PROBLEMS IN FUZZY SETS THEORY

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On Some Problems in Fuzzy Sets Theory

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Abstract - The aim of this paper is to sketch some connections between concepts and results of the classical mathematics and, respectively the mathematics of fuzzy sets.

I. Introduction ([3,6])

Let us remember some known definitions and results useful for the next sections.

Let U be a nonempty set. A pair (U, μ), where $\mu : U \to [0,1]$ is a mapping is called a fuzzy set. We shall also call $\mu : U \to [0,1]$ a fuzzy subset of U and denote

$$\mathcal{F}(U) = [0,1]^U = \{\mu | \mu : U \to [0,1]\}.$$

For (U, μ) and $\alpha \in [0,1]$ the set ${}_\mu U_\alpha = \{x \in U | \mu(x) \geq \alpha\}$ (also denoted μ_α) is called the α-level set of (U, μ).

In this context we have:

1.1 For any $x \in U$, $\mu(x) = \sup\{k \in [0,1] | x \in {}_\mu U_k\}$.

1.2 Let $(U_\alpha)_{\alpha \in [0,1]} \subseteq \mathcal{P}(U)$ be a family of subsets of U. Then $(U_\alpha)_{\alpha \in [0,1]}$ is the family of level sets of a fuzzy subset $\mu : U \to [0,1]$ if and only if it satisfies the conditions:

i) $U_0 = U$;

ii) $\alpha \leq \beta \Rightarrow U_\beta \subseteq U_\alpha$ $(\alpha, \beta \in [0,1])$;

iii) for any increasing sequence $(\alpha_i)_{i \in \mathbb{N}}$, $\alpha_i \in [0,1]$, $\forall i \in \mathbb{N}$, having limit α we have $U_\alpha = \bigcap_{i \in \mathbb{N}} U\alpha_i$.

It is clear that a fuzzy set is completely determined by the family of its level sets.

1.3 In relation with $\operatorname{Im}\mu$ $(\mu : U \to [0,1])$ it is sayed that μ satisfies the sup property if any nonempty subset of $\operatorname{Im}\mu$ has the greatest element. In other words, μ has the sup property if and only if for every nonempty subset A of $\operatorname{Im}\mu$, these exists $x \in \{y \in U | \mu(y) \in A\}$ such that $\mu(x) = \sup A$.

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Ref

[1] Alkhamees Y., Mordeson J., Reduced fields, primitive anf fuzzy Galois theory, J. Fuzzy Math. v. 8 (2000), 157-173.

1.4 Using the couples (t, t^*) where t is a t-norm, and t^* is its dual t-conorm the operations with fuzzy subsets are defined:

for $\mu, \eta \in \mathcal{F}(U)$ we define $\mu \cap_t \eta$ and $\mu \cup_t \eta$ by

$$\mu \cap_t \eta : U \to [0,1],\ \mu \cap_t \eta(x) = t(\mu(x), \eta(x));$$

$$\mu \cup_t \eta : U \to [0,1],\ \mu \cup_t \eta(x) = t^*(\mu(x), \eta(x)).$$

The most useful couples (t, t^*) are given by:

i) $t_1(, y) = \min\{x, y\},\ t_1^*(x, y) = \max\{x, y\};$

ii) $t_2(x, y) = xy,\ t_2^*(x, y) = x + y - xy;$

iii) $t_3(x, y) = \max\{x + y - 1, 0\},\ t_3^*(x, y) = \min\{x + y, 1\}.$

The empty fuzzy set is given by $\widetilde{\emptyset} : U \to [0,1]$, $\widetilde{\emptyset}(x) = 0$, $\forall x \in U$. By $\widetilde{U}$ one intend the application $\widetilde{U} : U \to [0,1]$ $\widetilde{U}(x) = 1$, $\forall x \in U$. We shall denote $\mu \subseteq \eta$, if $\mu(x) \leq \eta(x)$, $\forall x \in U$. The complement $\overline{\mu}$ of μ is given by $\overline{\mu} : U \to [0,1]$, $\overline{\mu}(x) = 1 - \mu(x)$, $\forall x \in U$.

1.5 A fuzzy subset $\mu : \mathbb{R} \to [0,1]$ (where $\mathbb{R}$ is the field of real numbers is called a fuzzy number if satisfies the following conditions:

i) there exists $x_\mu \in \mathbb{R}$ with $\mu(x_\mu) = 1$;

ii) the set $\{x | \mu(x) \neq 0\}$ is bounded;

iii) the level sets ${}_\mu\mathbb{R}_\alpha$, $\alpha \in [0,1]$, are closed intervals.

One usually takes the fuzzy numbers of the following type:

$$\mu(x) = \begin{cases} 0, & x < a \\ \pi_1(x), & x \in [a, b) \\ 1, & x \in [b, c] \\ \pi_2(x), & x \in [c, d] \\ 0, & x > d, \end{cases}$$

where $a \leq b \leq c \leq d$ are reals and π_1 and π_2 satisfy the conditions that turn μ in a fuzzy number.

For $\pi_1(x) = \frac{x-a}{b-a}$, $\pi_2(x) = \frac{d-x}{d-c}$ $(a \neq b, c \neq d)$ one gets trapezoidal fuzzy numbers. If also $b = c$, triangular fuzzy numbers are obtained. It is clear that a triangular fuzzy number is perfectly determined be a triple (x, y, z), where x, y, z are reals such that $x \leq y \leq z$ (for $x = y = z$, practically one obtains the reals).

1.6 Finaly, we recall that by fuzzy relation on U one intends a fuzzy subset of $U \times U$.

Ref

[1] Alkhamees Y., Mordeson J., Reduced fields, primitive anf fuzzy Galois theory, J. Fuzzy Math. v. 8 (2000), 157-173.

II. Level Sets ([1,7,8,9])

Using the notion of level set of a fuzzy set one can define:

2.1 Let $(G, \cdot)$ be a group and let $\mu : G \to [0,1]$ be a fuzzy subset of G. We say that μ is a (normal) fuzzy subgroup of G if the level sets of G, ${}_\mu G_\alpha$, are (normal) subgroups of G, for all $\alpha \in \operatorname{Im} \mu$.

One obtains that $\mu : G \to [0,1]$ is a fuzzy subgroup of G if and only if:

i) $\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}$, for any $x, y \in G$;

ii) $\mu(x^{-1}) \geq \mu(x)$, $\forall x \in G$.

In the case of normal fuzzy subgroups we must add the condition iii) $\mu(x \cdot y) = \mu(y \cdot x)$, for any $x, y \in G$.

2.2 In a similar way as above it is possible to define the notions of fuzzy subring or fuzzy ideal of a ring. In the following, we shall construct the fuzzy ring of quotients (a detailed study is given in [8]).

Let $(R, +, \cdot)$ be a unitary commutative ring and let $\sigma : R \to [0,1]$ be a fuzzy subset of R. We say that σ is a fuzzy multiplicative subset of R if every level set σ_t, $t \geq \sigma(0)$ is a multiplicative system (in the classical sense).

One obtain that $\sigma : R \to [0,1]$ is a fuzzy multiplicative subset of R if and only if the following conditions are satisfied:

i) $\sigma(x \cdot y) \geq \min\{\sigma(x), \sigma(y)\}$, for any $x, y \in R$;

ii) $\sigma(0) = \min\{\sigma(x) | x \in R\}$;

iii) $\sigma(1) = \max\{\sigma(x) | x \in R\}$.

If σ is a fuzzy multiplicative subset of R, then for every $t > \sigma(0)$, we may construct the classical ring of fractions $\sigma_t^{-1} R = S_t$ with respect to the multiplicative system σ_t. Let φ_t denote the canonical ring homomorphism $R \to S_t$. If $\sigma(0) \leq s \leq t$, since $\sigma_t \leq \sigma_s$, the universality property of the ring of fractions yields to the existence of a unique ring homomorphism $\varphi_{ts} : S_t \to S_s$ such that $\varphi_{ts} \circ \varphi_t = \varphi_s$.

The system of rings and homomorphisms (S_t, φ_{ts}) is an inductive system ($[\sigma(0), 1]$ being endowed with the reverse of the usual order).

Let $\sigma^{-1} R$ denote the inductive limit of this system. It is natural to call $\sigma^{-1} R$ the ring of quotients relative to the fuzzy multiplicative system σ (a universality property is satisfied [8,9]).

2.3 In the case of field extensions let F/k be a field extension and $\mu : F \to [0,1]$ be a fuzzy subset of F. μ is called a fuzzy intermediate field of F/k if $\forall \alpha \in \operatorname{Im} \mu$, the level set μ_α is an intermediate field of F/k.

We obtain that μ is a fuzzy intermediate field of F/k if and only if the following conditions are satisfied:

i) $\mu(x - y) \geq \min\{\mu(x), \mu y\}$, for any $x, y \in F$;

ii) $\mu(x \cdot y^{-1}) \geq \min\{\mu(x), \mu(y)\}$, if $y \neq 0$;

Ref

[8] Tofan I., Volf A.C., Fuzzy rings of quotients, Italian Journal of Pure and Applied Mathematics, 8 (2000), 83-90.

iii) for every $x \in F$ $\mu(x) \leq \mu(z)$, $\forall z \in K$.

2.4 Let X be a nonempty set and $\rho : X \times X \to [0,1]$ be a fuzzy relation on X. ρ will be called a similarity relation on X if $\forall \alpha \in [0,1]$, ρ_α is an equivalence relation on X.

We obtain that ρ is a similarity relation on X if and only if the following conditions are satisfied:

i) $\rho(x,x) = 1, \forall x \in X$;

ii) $\rho(x,y) = \rho(y,x)$, for any $x, y \in X$;

iii) $\rho(x,z) \geq \sup_{y \in X} \min\{\rho(x,y), \rho(y,z)\}$, for any $x, z \in X$.

2.5 The above described context suggests the following open problem: what kind of properties from classical mathematics have a fuzzy counterpart?

III. Sup Property ([1,8,9])

In relation with sup property we have the following remarks:

3.1 An unitary commutative ring R is artinian if and only if every fuzzy ideal of R has the sup property.

3.2 If F/k is a field extension then every fuzzy intermediate field of F/k has the property if and only if there are no infinite strictly decreasing sequences of intermediate fields of F/k.

3.3 Similarly, let G be a group. Then every fuzzy subgroups of G has the sup property if and only if there are no infinite strictly decreasing sequences of subgroups of G.

3.4 It is clear that this kind of results can be obtained for any algebraic structure for which is defined a notion of fuzzy substructure.
The investigation of the cosequences of the sup property in, for example, lattice theory can give interesting results.

IV. Fuzzy Numbers ([5,10])

In relation with fuzzy numbers at least two problems are of interest: firstly, the investigation of some possible arithmetic operations with fuzzy numbers as well as of the derived algebraic structures based on the proposed operations - and - secondly, the building of the sets of fuzzy numbers using, for example, 2.2.

The standard operations on the set of fuzzy numbers are tipically defined using Zadeh's extension principle or using the level subsets of the operands, but they do not have, for example, the property of distributivity among other lacking desirable properties.

Some new operations, in the case of triangular fuzzy numbers, are proposed in [10].

Ref

[10] Tofan I., Some remarks about fuzzy numbers, International Journal of Risk Theory, 1 (2011), 87-92.

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4.1 Usually a triangular fuzzy number $(x, y, z) \in \mathbb{R}$, $x \leq y \leq z$ can be uniquely represented by a triple (λ, y, ρ), where $\lambda = y - x$, $\rho = z - y$ are positive reals and are called left, respectively, right tolerance.

We will use the notation (y, λ, ρ) instead of (λ, y, ρ), with $y \in \mathbb{R}$, $\lambda, \rho \in \mathbb{R}$, $\lambda, \rho \geq 0$.

4.2 We consider the operations

$$(a, \lambda, \rho) \oplus (b, \lambda', \rho') = (a + b, \max\{\lambda, \lambda'\}, \max\{\rho, \rho'\});$$

$$(a, \lambda, \rho) \odot (b, \lambda', \rho') = (a \cdot b, \max\{\lambda, \lambda'\}, \max\{\rho, \rho'\}),$$

and the relation " $\sim$ " (on the set of triangular fuzzy numbers) given by

$$(a, \lambda, \rho) \sim (b, \lambda', \rho') \text{ if } \begin{cases} a = b \\ \lambda - \lambda' = \rho - \rho' \end{cases}$$

4.3 One obtains:

i) "$\sim$" is an equivalence relation;

ii) $\oplus, \odot$ are commutative and associative;

iii) "$\odot$" is distributive with respect to $\oplus$;

iv) $(0, 0, 0,)$ is neutral element for $\oplus$; $(1, 0, 0)$ is neutral element for $\odot$;

v) $(a, \lambda, \rho) \oplus (-a, \rho, \lambda) \sim (0, 0, 0)$, $\forall a \in \mathbb{R}$;

for $a \neq 0$ $(a, \lambda, \rho) \odot (a^{-1}, \rho, \lambda) \sim (1, 0, 0)$.

More than that, these operations are unique, such that the tolerance (left and right) of the result is not less than the tolerance of any of operands and satisfy a property of monotonicity.

4.4 As an open problem one can propose the investigation of the case in which the left and the right tolerance of the result can be obtained as functions of the all tolerances of the operands and as having proper form for any of the operations $\oplus, \odot$.

V. Fuzzy Probability ([2,4,11])

Finally, another type of transfer between classical and, respectively, fuzzy mathematics consist in the substitution of the set $\mathcal{P}(U)$ with $\mathcal{F}(U)$ (with keeping fit of the properties).

5.1 Let $\Omega \neq \Phi$ and $\mathcal{F}(\Omega)$. By fuzzy field of events we intend $K \subseteq \mathcal{F}(\Omega)$ such that:

i) $\widetilde{\Omega} \in K$;

ii) $\mu, \eta \in K \Rightarrow \mu \bigcup_t \eta \in K$;

iii) $\mu \in K \Rightarrow \overline{\mu} \in K$.

We obtain:

iv) $\widetilde{\emptyset} \in K$;

v) $\mu \bigcap_t \eta \in K$;

5.2 Let K be a fuzzy field of events. By probability on K one intend $P : K \to [0,1]$ such that

i) $P(\widetilde{\Omega}) = 1$;

ii) $\mu \bigcap_t \eta = \widetilde{\phi} \Rightarrow P(\mu \bigcup_t \eta) = P(\mu) + P(\eta)$.

We obtain:

iii) $P(\widetilde{\emptyset}) = 0$;

iv) $P(\overline{\mu}) = 1 - P(\mu)$;

v) $P(\mu \bigcup_t \eta) + P(\mu \bigcap_t \eta) = P(\mu) + P(\eta)$;

5.3 In the case $t = t_3$ $(t^* = t_3^*)$ supposing in 2.1 that $\mu, \eta \in K \Rightarrow \mu \odot \eta \in K$, where $\mu \otimes \eta : \Omega \to [0,1]$, $\mu \otimes \eta(x) = \mu(x) \cdot \eta(x)$ we shall denote $P(\mu/\eta) = P(\mu \otimes \eta)/P(\eta)$,$(P(\eta) \neq 0)$. In the above condition we have

$$P(\mu/\eta) = \frac{P(\mu) \cdot P(\eta/\mu)}{P(\mu) \cdot P(n/\eta) + P(\eta) \cdot P(\mu/n)}$$

and

if $\mu_1, \ldots, \mu_n \in K$ are such that $\mu_i \bigcap_t \mu_j = \widetilde{\emptyset}$ for $i \neq j$, then $P(\mu, \bigcup_t \mu_2 \bigcup_t \ldots \bigcup_t \mu_n) = P(\mu_1) + P(\mu_2) + \ldots + P(\mu_n)$.

5.4 In this area many open problems can appear:

- the connection with other operations with subsets (difference, implication);

- the next step can be to substitute $[0,1]$ with the, for example, $I_t = \{(a, \lambda, \rho) | a \in [0,1], 0 \leq \lambda \leq a, 0 \leq \rho \leq 1 - a\}$ (some new operations are needed).

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