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Some Indefinite Integrals in the Light of Hypergeometric Function

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Abstract - In this paper we have evaluated some indefinite integrals associated to Hypergeometric function. The results represent here are assume to be new.

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SOME INDEFINITE INTEGRALS IN THE LIGHT OF HYPERGEOMETRIC FUNCTION

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Some Indefinite Integrals in the Light of Hypergeometric Function

Salahuddin [α] & Intazar Husain [σ]

Abstract - In this paper we have evaluated some indefinite integrals associated to Hypergeometric function. The results represent here are assume to be new.

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I. Introduction and Preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k)=(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if}\ \ k=1,2,3,\cdots \\ 1\ ; & \text{if}\ \ k=0 \\ k!\ ; & \text{if}\ \ b=1,\ k=1,2,3,\cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

a) Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{matrix} a_1,a_2,\cdots,a_A & ; \\ & & z \\ b_1,b_2,\cdots,b_B & ; \end{matrix}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_kz^k}{(b_1)_k(b_2)_k\cdots(b_B)_kk!}$$

or

$${}_AF_B\left[\begin{matrix} (a_A) & ; \\ & & z \\ (b_B) & ; \end{matrix}\right]\equiv {}_AF_B\left[\begin{matrix} (a_j)_{j=1}^A & ; \\ & & z \\ (b_j)_{j=1}^B & ; \end{matrix}\right]=\sum_{k=0}^{\infty}\frac{((a_A))_kz^k}{((b_B))_kk!} \tag{1.1}$$

where denominator parameters $b_1,b_2,\cdots,b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

b) Kampé de Fjeriet's General Double Hypergeometric Function

In 1921, Appell's four double hypergeometric functions F_1,F_2,F_3,F_4 and their confluent forms $\Phi_1,\Phi_2,\Phi_3,\Psi_1,\Psi_2,\Xi_1,\Xi_2$ were unified and generalized by Kampé de Fériet.

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We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$\mathrm{F}^{A:B;D}_{E:G;H}\left[\begin{array}{ll}(a_A):(b_B);(d_D) & ; \\ & x,\ y \\ (e_E):(g_G);(h_H) & ;\end{array}\right] = \sum_{m,n=0}^{\infty} \frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\; x^m\, y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\; m!\, n!} \qquad (1.2)$$

where for convergence

(i) $A+B<E+G+1,\ A+D<E+H+1\quad ;|x|<\infty,\quad |y|<\infty,$ or

(ii) $A+B=E+G+1,\ A+D=E+H+1,$ and

$$\begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } \ E<A \\ \max\ \{|x|,|y|\}<1 & ,\text{if } \ E\geq A \end{cases}$$

c) Srivastava's General Triple Hypergeometric Function

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$F^{(3)}\left[\begin{array}{l}(a_A) :: (b_B);\, (d_D);\, (e_E) : (g_G);\, (h_H);\, (l_L); \\ \qquad\qquad\qquad\qquad\qquad\qquad x,\ y,\ z \\ (m_M) :: (n_N);\, (p_P);\, (q_Q) : (r_R);\, (s_S);\, (t_T);\end{array}\right]$$

$$= \sum_{i,j,k=0}^{\infty} \frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\ y^j\ z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\ j!\ k!} \qquad (1.3)$$

d) Wright's Generalized Hypergeometric Function

$${}_p\Psi_q\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ; \\ & x \\ (\lambda_1,B_1),\cdots,(\lambda_q,B_q) & ;\end{array}\right] = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1+mA_1)\Gamma(\alpha_2+mA_2)\cdots\Gamma(\alpha_p+mA_p)x^m}{\Gamma(\lambda_1+mB_1)\Gamma(\lambda_2+mB_2)\cdots\Gamma(\lambda_q+mA_q)m!} \qquad (1.4)$$

$${}_p\Psi^*_q\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ; \\ & x \\ (\lambda_1,B_1),\cdots,(\lambda_q,B_q) & ;\end{array}\right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_{mA_1}(\alpha_2)_{mA_2}\cdots(\alpha_p)_{mA_p}x^m}{(\lambda_1)_{mB_1}(\lambda_2)_{mB_2}\cdots(\lambda_q)_{mB_q}m!} \qquad (1.5)$$

II. Main Integrals

$$\int \frac{\mathrm{d}y}{\sqrt{[1-(\frac{1+x}{2})\sin^3 y]}} =$$

$$= -\cos y \sin^{3m+1} y\ (\sin^2 y)^{\frac{-1-3m}{2}}\ F^{1;2}_{0;1}\left[\begin{array}{ll}\frac{1}{2}\ ;\frac{1}{2},\ \frac{1-3m}{2} & ; \\ & \frac{1+x}{2},\ \cos^2 y \\ -;\ \frac{3}{2} & ;\end{array}\right] + Constant \qquad (2.1)$$

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$$\int \frac{\mathrm{d}y}{\sqrt{[1-(\frac{1+x}{2})\cos^3 y]}} =$$

$$= \frac{\sqrt{-\sin^2 y}\ \ \mathrm{cosec}\ y \cos^{3m+1} y}{3m+1}\, F^{1;2}_{0;1}\left[\begin{array}{ll} \frac{1}{2}\ ;\frac{1}{2},\frac{3m+1}{2} & ; \\ -;\ \frac{3m+3}{2} & ; \end{array} \frac{1+x}{2},\cos^2 y\right] + Constant \qquad (2.2)$$

$$\int \frac{\mathrm{d}y}{\sqrt{[1-(\frac{1+x}{2})\tan^3 y]}} = \frac{\tan^{3m+1} y}{(3m+1)}\, F^{1;2}_{0;1}\left[\begin{array}{ll} \frac{1}{2}\ ;1,\frac{3m+1}{2} & ; \\ -;\ \frac{3m+3}{2} & ; \end{array} \frac{1+x}{2},-\tan^2 y\right] + Constant \qquad (2.3)$$

$$\int \frac{\mathrm{d}y}{\sqrt{[1-(\frac{1+x}{2})\cot^3 y]}} = -\frac{\cot^{3m+1} y}{(3m+1)}\, F^{1;2}_{0;1}\left[\begin{array}{ll} \frac{1}{2}\ ;1,\frac{3m+1}{2} & ; \\ -;\ \frac{3m+3}{2} & ; \end{array} \frac{1+x}{2},-\cot^2 y\right] + Constant \qquad (2.4)$$

$$\int \frac{\mathrm{d}y}{\sqrt{[1-(\frac{1+x}{2})\sec^3 y]}} =$$

$$= \sin(y)\cos^2(y)^{\frac{3m+1}{2}}\sec^{3m+1}(y)\ F^{1;2}_{0;1}\left[\begin{array}{ll} \frac{1}{2}\ ;\frac{1}{2},\frac{1+3m}{2} & ; \\ -;\ \frac{3}{2} & ; \end{array} \frac{1+x}{2},\sin^2 y\right] + Constant \qquad (2.5)$$

$$\int \frac{\mathrm{d}x}{\sqrt{(1-(\frac{1+x}{2})\ \mathrm{cosech}^3\ y)}} =$$

$$= -\cos y(\sin^2(y))^{\frac{3m+1}{2}}\mathrm{cosec}^{3m+1} y\ F^{1;2}_{0;1}\left[\begin{array}{ll} \frac{1}{2}\ ;\frac{1}{2},\frac{1+3m}{2} & ; \\ -;\ \frac{3}{2} & ; \end{array} \frac{1+x}{2},\cos^2 y\right] + Constant \qquad (2.6)$$

III. Derivation of Integrals

Derivation of integral (2.1)

$$\int \frac{\mathrm{d}y}{\sqrt{[1-(\frac{1+x}{2})\sin^3 y]}} = \int \left[1-\left(\frac{1+x}{2}\right)\sin^3 y\right]^{-\frac{1}{2}}\mathrm{d}y$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\sin^{3m} y\ \mathrm{d}y = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(\frac{1+\mathrm{x}}{2})^{\mathrm{m}}}{\mathrm{m}!}\int \sin^{3\mathrm{m}} \mathrm{y}\ \mathrm{dy}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}(-\cos y)\ \sin^{3m+1} y(\sin^2 y)^{\frac{1-3m}{2}}\,{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\frac{1-3m}{2} & ; \\ \frac{3}{2} & ; \end{array} \cos^2 y\right] + Constant$$

$$= -\cos y\sin^{3m+1} y\ (\sin^2 y)^{\frac{-1-3m}{2}}\ F^{1;2}_{0;1}\left[\begin{array}{ll} \frac{1}{2}\ ;\frac{1}{2},\frac{1-3m}{2} & ; \\ -;\ \frac{3}{2} & ; \end{array} \frac{1+x}{2},\cos^2 y\right] + Constant \qquad (3.1)$$

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Derivation of integral (2.2)

$$\int \frac{dy}{\sqrt{[1-(\frac{1+x}{2})\cos^3 y]}} = \int \left[1-\left(\frac{1+x}{2}\right)\cos^3 y\right]^{-\frac{1}{2}} dy$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\cos^{3m} y\ dy = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\int \cos^{3m} y\ dy$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\frac{\sqrt{-\sin^2 y}\cos^{3m+1} y\ \mathrm{cosec}\ y}{(3m+1)}{}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{3m+1}{2} & ; \\ \frac{3m+3}{2} & ;\end{matrix}\ \cos^2 y\right] + Constant$$

$$= \frac{\sqrt{-\sin^2 y}\ \ \mathrm{cosec}\ y\cos^{3m+1} y}{3m+1}\ F^{1;2}_{0;1}\left[\begin{matrix}\frac{1}{2}\ ;\frac{1}{2}, \frac{3m+1}{2} & ; \\ —;\ \frac{3m+3}{2} & ;\end{matrix}\ \frac{1+x}{2}, \cos^2 y\right] + Constant \qquad (3.2)$$

Derivation of integral (2.3)

$$\int \frac{dy}{\sqrt{[1-(\frac{1+x}{2})\tan^3 y]}} = \int \left[1-\left(\frac{1+x}{2}\right)\tan^3 y\right]^{-\frac{1}{2}} dy$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\tan^{3m} y\ dy = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\int \tan^{3m} y\ dy$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\frac{\tan^{3m+1} y}{(3m+1)}{}_2F_1\left[\begin{matrix}1, \frac{3m+1}{2} & ; \\ \frac{3m+3}{2} & ;\end{matrix}\ -\tan^2 y\right] + Constant$$

$$= \frac{\tan^{3m+1} y}{(3m+1)}\ F^{1;2}_{0;1}\left[\begin{matrix}\frac{1}{2}\ ;1, \frac{3m+1}{2} & ; \\ —;\ \frac{3m+3}{2} & ;\end{matrix}\ \frac{1+x}{2}, -\tan^2 y\right] + Constant \qquad (3.3)$$

Derivation of integral (2.4)

$$\int \frac{dy}{\sqrt{[1-(\frac{1+x}{2})\cot^3 y]}} = \int \left[1-\left(\frac{1+x}{2}\right)\cot^3 y\right]^{-\frac{1}{2}} dy$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\cot^{3m} y\ dy = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\int \cot^{3m} y\ dy$$

$$= -\sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!}\frac{\cot^{3m+1} y}{(3m+1)}{}_2F_1\left[\begin{matrix}1, \frac{3m+1}{2} & ; \\ \frac{3m+3}{2} & ;\end{matrix}\ -\cot^2 y\right] + Constant$$

$$= -\frac{\cot^{3m+1} y}{(3m+1)}\ F^{1;2}_{0;1}\left[\begin{matrix}\frac{1}{2}\ ;1, \frac{3m+1}{2} & ; \\ —;\ \frac{3m+3}{2} & ;\end{matrix}\ \frac{1+x}{2}, -\cot^2 y\right] + Constant \qquad (3.4)$$

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Derivation of integral (2.5)

$$\int \frac{dy}{\sqrt{(1-(\frac{1+x}{2})\sec^3 y)}} = \int \left[1-\left(\frac{1+x}{2}\right)\sec^3 y\right]^{-\frac{1}{2}} dy$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!} \sec^{3m} y\ dy = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!} \int \sec^{3m} y\ dy$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!} \sin y\ \cos^2(y)^{\frac{3m+1}{2}} \sec^{3m+1} y\ {}_2F_1\left[\begin{array}{ccc} \frac{1}{2}, \frac{3m+1}{2} & ; & \\ \frac{3}{2} & ; & \end{array} \sin^2 y\right] + Constant$$

$$= \sin(y)\cos^2(y)^{\frac{3m+1}{2}} \sec^{3m+1}(y)\ F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2}\ ; \frac{1}{2}, \frac{1+3m}{2} & ; & \\ -;\ \frac{3}{2} & ; & \end{array} \frac{1+x}{2}, \sin^2 y\right] + Constant \qquad (3.5)$$

Derivation of integral (2.6)

$$\int \frac{dy}{\sqrt{(1-(\frac{1+x}{2})\operatorname{cosec}^3 y)}} = \int \left[1-\left(\frac{1+x}{2}\right)\operatorname{cosec}^3 y\right]^{-\frac{1}{2}} dy$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!} \operatorname{cosec}^{3m} y\ dy = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!} \int \operatorname{cosec}^{3m} y\ dy$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(\frac{1+x}{2})^m}{m!} (-\cos y)\ (\sin^2(y))^{\frac{3m+1}{2}} \operatorname{cosec}^{3m+1} y\ {}_2F_1\left[\begin{array}{ccc} \frac{1}{2}, \frac{3m+1}{2} & ; & \\ \frac{3}{2} & ; & \end{array} \cos^2 y\right] + Constant$$

$$= -\cos y(\sin^2(y))^{\frac{3m+1}{2}} \operatorname{cosec}^{3m+1} y\ F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2}\ ; \frac{1}{2}, \frac{1+3m}{2} & ; & \\ -;\ \frac{3}{2} & ; & \end{array} \frac{1+x}{2}, \cos^2 y\right] + Constant \qquad (3.6)$$

IV. Conclusion

In our work we have established hypergeometric form of some indefinite integrals . We can only expect that the development presented in this work will stimulate further interest and research in this important area of classical special functions. Just as the mathematical properties of the Gauss hypergeometric function are already of immense and significant utility in mathematical sciences and numerous other areas of pure and applied mathematics, the elucidation and discovery of the formula of hypergeometric functions considered herein should certainly eventually prove useful to further developments in the broad areas alluded to above.

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