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Some Subclasses of P-Valent Analytic Functions

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Abstract - The object of the present paper is to derive the sufficient conditions for certain subclasses of p-valent analytic functions in the open unit disk. A number of known results would follow upon specializing the parameters involved in our main results. Also, sufficient conditions are found for function to be univalent.

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SOME SUBCLASSES OF P-VALENT ANALYTIC FUNCTIONS

Strictly as per the compliance and regulations of :

Ref.

[1] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer- Verlag, New York Berlin, Heidelberg, and Tokyo, (1983).

Some Subclasses of P-Valent Analytic Functions

M. P. Jeyaraman [α], T. K. Suresh [σ] & E. Keshava Reddy [ρ]

Abstract - The object of the present paper is to derive the sufficient conditions for certain subclasses of p-valent analytic functions in the open unit disk. A number of known results would follow upon specializing the parameters involved in our main results. Also, sufficient conditions are found for function to be univalent.

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I. Introduction and Preliminaries

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic and p-valent in the open unit disk $\mathbb{U} := \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of analytic and univalent functions in $\mathbb{U}$. We note that $\mathcal{A}(1) \equiv \mathcal{S}$.

A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}^*(p, \alpha)$ of p-valently starlike of order α in $\mathbb{U}$ if and only if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}; 0 \le \alpha < p).$$

On the other hand, a function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{K}(p, \alpha)$ of p-valently convex of order α in $\mathbb{U}$ if and only if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}; 0 \le \alpha < p).$$

In particular, we write $\mathcal{S}^*(1, 0) := \mathcal{S}^*, \mathcal{K}(1, 0) := \mathcal{K}$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the usual subclass of $\mathcal{A}$, consisting of functions which are starlike and convex, respectively (see [1, 2]).

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The object of the present paper is to investigate various properties of the following classes of analytic and p-valent function defined as follows.

A function $f \in \mathcal{A}(p)$ is said to be a member of the class $\mathcal{B}(\gamma, \beta, p, \alpha)$ if and only if it satisfies the inequality

$$\left| \left(\frac{\beta\gamma z^3 f'''(z) + (2\beta\gamma + \beta - \gamma) z^2 f''(z) + z f'(z)}{\beta\gamma z^2 f''(z) + (\beta - \gamma) z f'(z) + (1 - \beta + \gamma) f(z)} \right) - p \right| < p - \alpha \tag{1}$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

for some α, for all $z \in \mathbb{U}$.

Note that the condition (1) implies that

$$\mathrm{Re} \left\{ \frac{\beta\gamma z^3 f'''(z) + (2\beta\gamma + \beta - \gamma) z^2 f''(z) + z f'(z)}{\beta\gamma z^2 f''(z) + (\beta - \gamma) z f'(z) + (1 - \beta + \gamma) f(z)} \right\} > \alpha,$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}).$$

We note that $\mathcal{B}(0, \beta, p, \alpha) \equiv \mathcal{T}_\beta(p; \alpha)$ is the class studied by Irmak and Raina in [3]. The important subclasses such as $\mathcal{S}^*(p, \alpha), \mathcal{K}(p, \alpha)$, $\mathcal{S}^*$ and $\mathcal{K}$ are seen to be easily identifiable with the aforesaid class.

In recent times, Irmak et al. [3] and Prajapat [9] investigated certain subclasses of multivalent analytic functions and obtained some sufficient conditions for these classes. In this paper, motivated by the aforementioned works, we obtained sufficient conditions for functions to be a member of the class $\mathcal{B}(\gamma, \beta, p, \alpha)$. We also indicate some special cases and consequences of the main result. The other results investigated include certain inequalities for p-valent functions which characterize the properties of starlikeness and convexity in the open unit disk. Furthermore our result unifies the result for a functions belonging to the class of p-valently starlike function of order α and p-valently convex function of order α.

In order to derive our main results, we need the following Lemmas.

Lemma 1. [4] *Let $w(z)$ be the non-constant and analytic function in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0) \tag{2}$$

where $k \geq 1$ is a real number.

Lemma 2. [5] *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(z)$ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \leq -n(1 + x^2)/2$. If the function $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ such that $\Phi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then* $\mathrm{Re}\, q(z) > 0$.

Lemma 3. [7] *Let δ be the complex number,* $\mathrm{Re}\, \delta > 0$, *and λ be a complex number, $|\lambda| \leq 1, \lambda \neq -1$ and let $h(z) = z + a_2 z^2 + \cdots$ be a regular function on $\mathbb{U}$. If*

$$\left| \lambda |z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{z h''(z)}{\delta h'(z)} \right| \leq 1$$

Ref.

[3] H. Irmak and R. K. Raina, The starlikeness and convexity of multivalent functions involving certain inequalities, *Rev. Mat. Comput.* (16) (2) (2003), 391–398.

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for all $z \in \mathbb{U}$, then the function

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} h'(t)dt\right)^{1/\delta}$$

$$= z + \frac{2a_2}{\delta+1} z^2 + \left(\frac{3a_3}{\delta+2} + \frac{2\delta(1-\delta)a_2^2}{(\delta+1)^2}\right) z^3 + \cdots$$

is regular and univalent in $\mathbb{U}$.

Lemma 4. [8] *Let δ be a complex number, $\operatorname{Re}\delta > 0$, and λ a complex number, $|\lambda| < 1$, and $h \in \mathcal{A}$. If*

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le 1 - |\lambda|$$

for all $z \in \mathbb{U}$, then for any complex number $\eta, \operatorname{Re}\eta \ge \operatorname{Re}\delta$, the function

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} h'(t)dt\right)^{1/\eta}$$

is in the class $\mathcal{S}$.

Lemma 5. [6] *Let $p(z)$ be analytic in $\mathbb{U}, p(0) = 1, p(z) \ne 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad for\ |z| < |z_0|, \ \ |\arg(p(z_0))| = \frac{\pi}{2}\alpha,$$

where $0 < \alpha \le 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \ge \frac{1}{2}\left(a + \frac{1}{a}\right) \ge 1 \ \ when \quad \arg(p(z_0)) = \frac{\pi}{2}\alpha,$$

$$k \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \le -1 \ \ when \quad \arg(p(z_0)) = -\frac{\pi}{2}\alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).$$

II. Main Results

By using Lemma 2, we first prove the following theorem.

Theorem 6. *Let $f \in \mathcal{A}(p)$. Define a function $G_{\beta,\gamma}$ by*

$$G_{\beta,\gamma}(z) := \beta\gamma z^2 f''(z) + (\beta-\gamma)zf'(z) + (1-\beta+\gamma)f(z), \quad (0 \le \gamma \le \beta \le 1; z \in \mathbb{U}),$$

and if $G_{\beta,\gamma}(z)$ satisfies

$$\operatorname{Re}\left\{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(2 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right)\right\} > p\left(1 - \frac{n}{2}\right) + \frac{n}{2}\alpha$$

Ref.

[8] V. Pescar, Univalence criteria of certain integral operators, *Acta Cienc. Indica Math.*, (29) (1) (2003), 135–138.

$$(0 \le \gamma \le \beta \le 1; 0 \le \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

Proof. Let $f \in \mathcal{A}(p)$. Define a function $w(z)$ in $\mathbb{U}$ by

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = p + (p-\alpha)w(z), (0 \le \gamma \le \beta \le 1; 0 \le \alpha < p; p \in \mathbb{N}), \tag{3}$$

then the function $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$.
A computation using (3) shows that

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(2 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right) = (p-\alpha)[zw'(z) + w(z)] + p$$
$$= \Phi(w(z), zw'(z); z),$$

where $\Phi(r, s; z) = (p-\alpha)[s + r] + p$.
For all real x, y satisfying $y \le -n(1+x^2)/2$, we have

$$\begin{aligned}\text{Re}\ \Phi(ix, y; z) &= \text{Re}\ \{(p-\alpha)[y + ix] + p\} \\ &\le -\frac{n}{2}(p-\alpha)(1+x^2) + p \\ &\le -\frac{n}{2}(p-\alpha) + p \\ &= p\left(1 - \frac{n}{2}\right) + \frac{n}{2}\alpha.\end{aligned}$$

Let $\Omega = \left\{w : \text{Re}\ w > p(1-\frac{n}{2}) + \frac{n}{2}\alpha\right\}$. Then $\Phi(w(z), zw'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \le -n(1+x^2)/2,\ z \in \mathbb{U}$.

By using Lemma 2, we have Re $w(z) > 0$, which implies that

$$\text{Re}\left\{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right\} > \alpha, (0 \le \gamma \le \beta \le 1; 0 \le \alpha < p; p \in \mathbb{N}),$$

and hence $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

By setting $\gamma = \beta = 0$ in Theorem 6, we have following corollary.

Corollary 7. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > p\left(1 - \frac{n}{2}\right) + \frac{n}{2}\alpha$$
$$(0 \le \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in \mathcal{S}^*(p, \alpha)$.

Its further case when $\alpha = 0$ and $p = 1$, Corollary 7 reduces to Corollary 8.

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Corollary 8. *If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\} > 1-\frac{n}{2}$$
$$(n \in \mathbb{N}),$$

then $f(z) \in \mathcal{S}^*$.

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By taking $\gamma = 0, \beta = 1$ in Theorem 6, we have the following corollary.

Corollary 9. *If $f \in \mathcal{A}(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{(zf'(z))'}{f'(z)}\left(1+\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right)\right\} > p\left(1-\frac{n}{2}\right)+\frac{n}{2}\alpha$$
$$(0 \le \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in \mathcal{K}(p,\alpha)$.

A further case of Corollary 9, when $\alpha = 0, p = 1$ gives the following corollary.

Corollary 10. *If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left\{\frac{(zf'(z))'}{f'(z)}\left(1+\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right)\right\} > 1-\frac{n}{2}$$
$$(n \in \mathbb{N}),$$

then $f(z) \in \mathcal{K}$.

Theorem 11. *Let $-1 < b < a \le 1$, $0 \le \alpha < p$, $p \in \mathbb{N}$ such that $p(1+\alpha)+a \le 2p(p-b)+b$. If $G_{\beta,\gamma}(z)$ satisfies the inequality*

$$\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right| < \frac{p(a+b)}{(p+a)(p-b)} \quad (z \in \mathbb{U}), \tag{4}$$

then $f(z) \in \mathcal{B}(\gamma,\beta,p,\alpha)$.

Proof. Define a function $w(z)$ by

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{p+aw(z)}{p-bw(z)} \quad (z \in \mathbb{U}). \tag{5}$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. By the logarithmic differentiation of (5), we get

$$1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{p(a+b)zw'(z)}{(p+aw(z))(p-bw(z))}. \tag{6}$$

Now suppose that there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|}|w(z)| = |w(z_0)| = 1,$$

then from Lemma 1, we have (2). Letting $w(z_0) = e^{i\theta}$, from (6), we have

$$\left|1 + \frac{z_0 G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right| = \left|\frac{p(a+b)ke^{i\theta}}{(p+ae^{i\theta})(p-be^{i\theta})}\right| \geq \frac{p(a+b)}{(p+a)(p-b)}.$$

This contradicts our assumption (4). Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. Thus we conclude from (5) that

$$\begin{aligned}\left|\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p\right| &= \left|\frac{p+aw(z)}{p-bw(z)} - p\right| \\ &< \frac{p+a-p(p-b)}{p-b} \\ &\leq p-\alpha \quad (z \in \mathbb{U}),\end{aligned}$$

which implies that $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

Theorem 12. *Let $f \in \mathcal{A}(p)$. If $G_{\beta,\gamma}(z)$ satisfies anyone of the following conditions:*

$$\left|1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right| < \frac{p-\alpha}{2p-\alpha}, \tag{7}$$

$$\left|\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right)\right| < p-\alpha, \tag{8}$$

$$\left|\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}} - 1\right| < \frac{p-\alpha}{(2p-\alpha)^2}, \tag{9}$$

$$\left|\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1\right| < \frac{1}{(2p-\alpha)}, \tag{10}$$

$$\operatorname{Re}\left\{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1\right)\right\} < 1, \tag{11}$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

then $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

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Proof. Let $f \in \mathcal{A}(p)$. Define a function $w(z)$ in $\mathbb{U}$ by

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = p + (p-\alpha)w(z), (0 \le \gamma \le \beta \le 1; 0 \le \alpha < p; p \in \mathbb{N}), \tag{12}$$

then the function $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$.
It follows from (12) that

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$$1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{(p-\alpha)zw'(z)}{p + (p-\alpha)w(z)}. \tag{13}$$

Hence, from (12) and (13), we have

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left(1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right) = (p-\alpha)zw'(z), \tag{14}$$

$$\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}} - 1 = \frac{(p-\alpha)zw'(z)}{[p + (p-\alpha)w(z)]^2}, \tag{15}$$

$$\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1 = \frac{zw'(z)}{w(z)} \frac{1}{p + (p-\alpha)w(z)}, \tag{16}$$

and

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left(\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1\right) = \frac{zw'(z)}{w(z)}. \tag{17}$$

Now, suppose there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1,$$

then from Lemma 1, we have (2). Therefore, letting $w(z_0) = e^{i\theta}$ in each of (13)-(17), we obtain that

$$\left|1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right| = \left|\frac{(p-\alpha)ke^{i\theta}}{p + (p-\alpha)e^{i\theta}}\right| \ge \frac{p-\alpha}{2p-\alpha},$$

$$\left|\frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \left(1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right)\right| = \left|(p-\alpha)ke^{i\theta}\right| \ge (p-\alpha),$$

$$\left| \frac{1+\dfrac{z_0 G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)}}{\dfrac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}} - 1 \right| = \left| \frac{(p-\alpha)ke^{i\theta}}{(p+(p-\alpha)e^{i\theta})^2} \right| \geq \frac{p-\alpha}{(2p-\alpha)^2},$$

$$\left| \frac{1+\dfrac{z_0 G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - p}{\dfrac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} - p} - 1 \right| = \left| \frac{k}{p+(p-\alpha)e^{i\theta}} \right| \geq \frac{1}{(2p-\alpha)},$$

$$\mathrm{Re}\left\{ \frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \left(\frac{1+\dfrac{z_0 G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - p}{\dfrac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} - p} - 1 \right) \right\} = k \geq 1,$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

which contradict our assumption (7)-(11), respectively. Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. From (12), we have

$$\left| \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p \right| = |(p-\alpha)w(z)| < (p-\alpha), (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

which implies that

$$\mathrm{Re}\left\{ \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right\} = \mathrm{Re}\left\{ \frac{\beta\gamma z^3 f'''(z) + (2\beta\gamma+\beta-\gamma)z^2 f''(z) + zf'(z)}{\beta\gamma z^2 f''(z) + (\beta-\gamma)zf'(z) + (1-\beta+\gamma)f(z)} \right\} > \alpha,$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

and hence $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

Remark 1. *By taking* $\gamma = 0$; $\gamma = \beta = 0$; $\gamma = 0$ *and* $\beta = 1$; $\gamma = \beta = \alpha = 0$ *and* $p = 1$; $\gamma = \alpha = 0$ *and* $\beta = p = 1$ *in Theorem 12, we get the results of Irmak and Raina* [3, Theorem 1, Corollary 1-4].

Theorem 13. *Let* $0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < 1, \delta$ *be a complex number,* $\mathrm{Re}\,\delta \geq \dfrac{3-2\alpha}{2-\alpha}$ *and* λ *be a complex number which satisfies the inequality*

$$|\lambda| \leq 1 - \frac{3-2\alpha}{\mathrm{Re}\,\delta(2-\alpha)}. \tag{18}$$

If $F_{\beta,\gamma}(z) := \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}$ *is regular in* $\mathbb{U}$ *and*

Ref.

[3] H. Irmak and R. K. Raina, The starlikeness and convexity of multivalent functions involving certain inequalities, *Rev. Mat. Comput.* (16) (2) (2003), 391–398.

$$\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right|\leq\frac{1-\alpha}{2-\alpha}\quad(z\in\mathbb{U}),\tag{19}$$

then the function

$$F(z)=\left(\delta\int_0^z t^{\delta-1}\frac{G'_{\beta,\gamma}(t)}{G_{\beta,\gamma}(t)}dt\right)^{1/\delta}\tag{20}$$

is univalent in $\mathbb{U}$.

Proof. Define a function

$$h(z)=\int_0^z\frac{F_{\beta,\gamma}(t)}{t}dt,$$

then we have $h(0)=h'(0)-1=0$. Also a simple computation yields $h'(z)=\dfrac{F_{\beta,\gamma}(z)}{z}$ and

$$\frac{zh''(z)}{h'(z)}=\frac{zF'_{\beta,\gamma}(z)}{F_{\beta,\gamma}(z)}-1.\tag{21}$$

From (21), we have

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right|&\leq\left|\frac{zF'_{\beta,\gamma}(z)}{F_{\beta,\gamma}(z)}\right|+1\\&=\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right|+1.\end{aligned}\tag{22}$$

Hence, from (19) and (22), we have

$$\left|\frac{zh''(z)}{h'(z)}\right|\leq\frac{3-2\alpha}{2-\alpha}.\tag{23}$$

Using (23), we have

$$\begin{aligned}\left|\lambda|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\frac{zh''(z)}{\delta h'(z)}\right|&\leq|\lambda|+\left|\frac{zh''(z)}{\delta h'(z)}\right|\\&\leq|\lambda|+\frac{1}{\mathrm{Re}\delta}\frac{3-2\alpha}{2-\alpha}.\end{aligned}$$

Again using (18), we have

$$\left|\lambda|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\frac{zh''(z)}{\delta h'(z)}\right|\leq 1.$$

Applying Lemma 3, we obtain that the function $F(z)$ defined by (20) is univalent in $\mathbb{U}$.

We obtain Theorem 14 below, by using Lemma 4 and the same techniques as in the proof of Theorem 13.

Theorem 14. *Let δ be a complex number,* $\operatorname{Re}\delta > 0$, *$\lambda$ a complex number,* $|\lambda| < 1$, *and $f \in \mathcal{A}$. If*

$$\left|1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right| \leq \frac{1-\alpha}{2-\alpha} \quad (z \in \mathbb{U}; 0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < 1),$$

then for any complex number η,

$$\operatorname{Re}\eta \geq \operatorname{Re}\delta \geq \frac{3-2\alpha}{(1-|\lambda|)(2-\alpha)},$$

the integral operator

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} \frac{G'_{\beta,\gamma}(t)}{G_{\beta,\gamma}(t)} dt\right)^{1/\eta}$$

is in the class $\mathcal{S}$.

Theorem 15. *Let $p(z)$ be an analytic function in $\mathbb{U}$, $p(z) \neq 0$ in $\mathbb{U}$ and suppose that*

$$\left|\arg\left(p(z) + \frac{z^2 G'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} p'(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}), \tag{24}$$

where $0 < \alpha < p, 0 \leq \gamma \leq \beta \leq 1$ and $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$, then we have

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Proof. Suppose there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad for\ |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2}\alpha.$$

Then, applying Lemma 4, we have

$$\begin{aligned}\arg\left(p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} p'(z_0)\right) &= \arg\left(p(z_0)\left(1 + \frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\ &= \arg(p(z_0)) + \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right).\end{aligned} \tag{25}$$

When $\arg(p(z_0)) = \pi\alpha/2$, since

$$\operatorname{Re}\left(\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) > 0 \Rightarrow \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) > 0,$$

Eq. (25) becomes

$$\arg\left(p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} p'(z_0)\right) > \frac{\pi}{2}\alpha. \tag{26}$$

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Similarly, if $\arg(p(z_0)) = -\pi\alpha/2$, since

$$\mathrm{Re}\left(\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}k\alpha\right) < 0 \Rightarrow \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}k\alpha\right) < 0,$$

we obtain that

$$\arg\left(p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}p'(z_0)\right) = \arg(p(z_0)) + \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}k\alpha\right) < -\frac{\pi}{2}\alpha. \tag{27}$$

Thus, we see that (26) and (27) contradict our assumption (24). Consequently, we conclude that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

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