Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 10 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Computation of Some Wonderful Results Involving Certain Polynomials

By Salahuddin & R.K. Khola
Mewar University, India

Abstract- In this paper we have developed some indefinite integrals involving certain polynomials in the form of Hypergeometric function. The results represent here are assume to be new.

Keywords: hypergeometric function, lucas polynomials, gegenbaur polynomials, harmonic number, bernoulli polynomial, hermite polynomials.

GJSFR-F Classification : MSC 2010 11B39, 11B68, 33C05, 33C45, 33D50, 33D60

COMPUTATION OF SOME WONDERFUL RESULTS INVOLVING CERTAIN POLYNOMIALS

Strictly as per the compliance and regulations of :

Notes

Computation of Some Wonderful Results Involving Certain Polynomials

Salahuddin[α] & R.K. Khola[σ]

Abstract- In this paper we have developed some indefinite integrals involving certain polynomials in the form of Hypergeometric function. The results represent here are assume to be new.

Keywords: hypergeometric function, lucas polynomials, gegenbaur polynomials, harmonic number, bernoulli polynomial, hermite polynomials.

I. Introduction

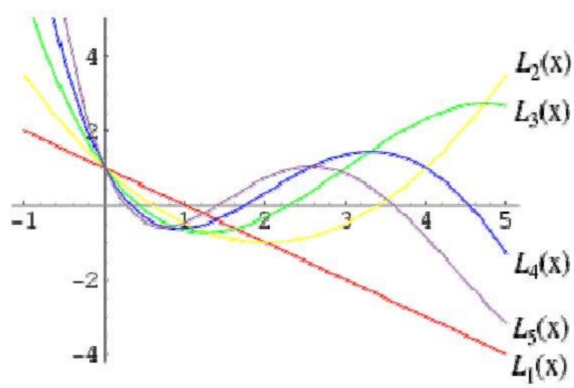
a) We have the generalized Gaussian hypergeometric function of one variable

$$ {}_AF_B(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots b_B;z\,) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k.....(a_A)_k\, z}{(b_1)_k(b_2)_k.....(b_B)_k\, k!} \qquad (1)$$

where the parameters b_1 , b_2 , ...,b_B are neither zero nor negative integers and A , B are non negative integers.

The series converges for all finite z if $A \le B$, converges for $|z|<1$ if $A=B+1$, diverges for all z, $z \neq 0$ if $A > B+1$.

b) Lucas Polynomials



The Lucas polynomials are the w-polynomials obtained by setting p(x) = x and q(x) = 1 in the Lucas polynomials sequence. It is given explicitly by

$$ L_n(x) = 2^{-n}[(x-\sqrt{x^2+4}\ \ )^n + (x+\sqrt{x^2+4}\ \ )^n\ ] \qquad (2)$$

Authors α σ : Mewar University, Gangrar, Chittorgarh (Rajasthan), India. e-mails: vsludn@gmail.com, sludn@yahoo.com

The first few are

$$\left.\begin{aligned} L_1(x) &= x \\ L_2(x) &= x^2+2 \\ L_3(x) &= x^3+3x \\ L_4(x) &= x^4+4x^2+2 \end{aligned}\right\} \quad (3)$$

c) Generalized Harmonic Number

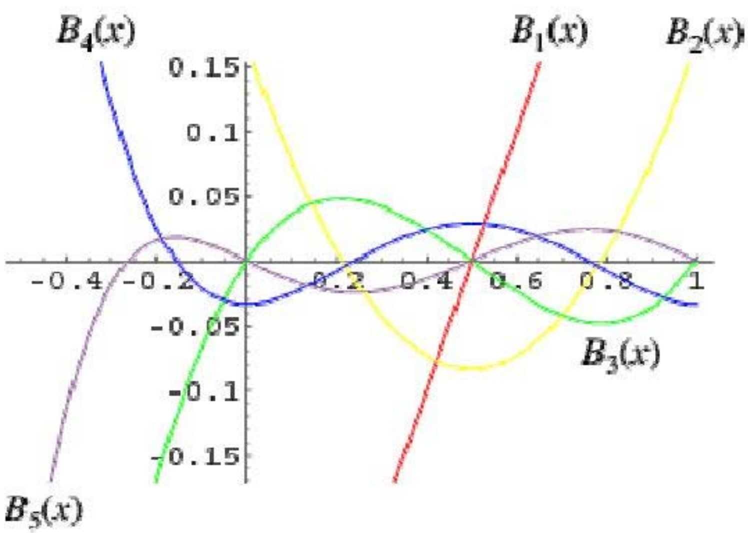
The generalized harmonic number of order n of m is given by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \quad (4)$$

In the limit of $n \to \infty$ the generalized harmonic number converges to the Riemann zeta function

$$\lim_{n\to\infty} H_n^{(m)} = \varsigma(m) \quad (5)$$

d) Bernoulli Polynomial



In mathematics, the Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and Hurwitz zeta function. This is in large part because they are an Appell sequence, i.e. a Sheffer sequence for the ordinary derivative operator. Unlike orthogonal polynomials, the Bernoulli polynomials are remarkable in that the number of crossing of the x-axis in the unit interval does not go up as the degree of the polynomials goes up. In the limit of large degree, the Bernoulli polynomials, appropriately scaled, approach the sine and cosine functions.
Explicit formula of Bernoulli polynomials is

$B_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} b_{n-k} x^k$, for n≥0, where b_k are the Bernoulli numbers.

The generating function for the Bernoulli polynomials is

Notes

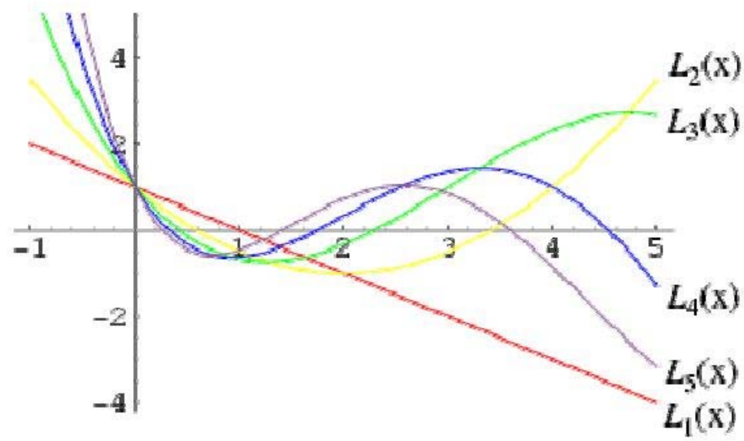
Notes

e) Gegenbauer polynomials

In Mathematics, Gegenbauer polynomials or ultraspherical polynomials $C_n^{(\alpha)}(x)$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. They are named after Leopold Gegenbauer.
Explicitly ,

$$C_n^{(\alpha)}(x)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k\frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!}(2z)^{(n-2k)} \tag{7}$$

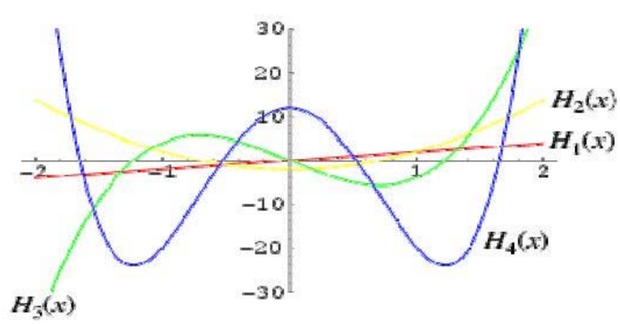
f) Laguerre polynomials



The Laguerre polynomials are solutions $L_n(x)$ to the Laguerre differential equation $xy''+(1-x)y'+\lambda y=0$, which is a special case of the more general associated Laguerre differential equation, defined by
$xy''+(\nu+1-x)y'+\lambda y=0$, where λ and ν are real numbers with ν=0.
The Laguerre polynomials are given by the sum

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{n!}{k!(n-k)!}x^k \tag{8}$$

g) Hermite polynomials



Global Journal of Science Frontier Research (F) Volume XIII Issue X Version I Year 2013

The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain (-∞,∞) with weighting function e^{-x^2}.

The Hermite polynomials $H_n(x)$ can be defined by the contour integral

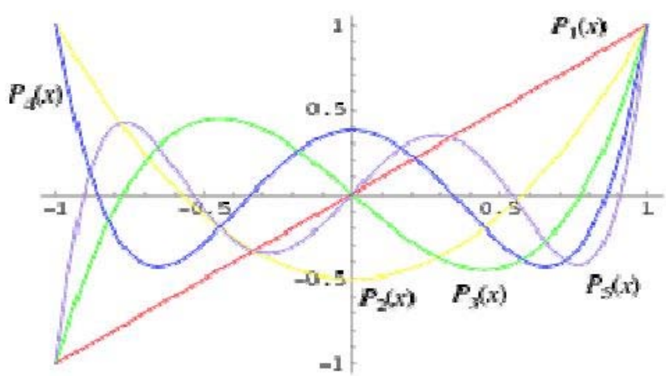
$$H_n(z) = \frac{n!}{2\pi i} \oint e^{-t^2+2tz} t^{-n-1} dt,$$

Where the contour incloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The first few Hermite polynomials are

$$\left.\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \end{aligned}\right\} \tag{9}$$

h) Legendre function of the first kind



The Legendre polynomials, sometimes called Legendre functions of the first kind, Legendre coefficients, or zonal harmonics (Whittaker and Watson 1990, p. 302), are solutions to the Legendre differential equation. If l is an integer, they are polynomials. The Legendre polynomials $P_n(x)$ are illustrated above for $x \in [-1, 1]$ and n=1, 2, ..., 5.

The Legendre polynomials $P_n(x)$ can be defined by the contour integral

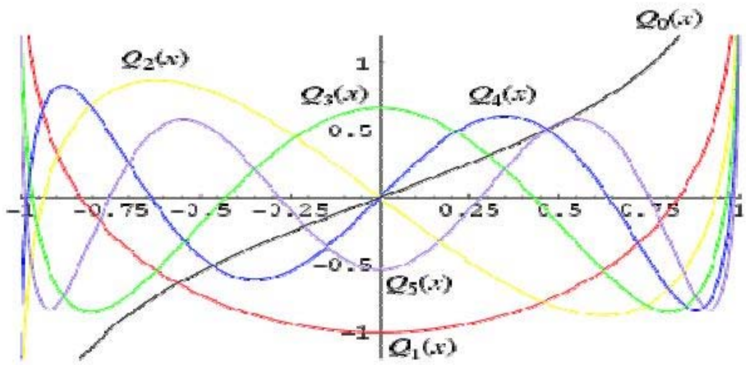
$$P_n(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{-\frac{1}{2}} t^{-n-1} dt, \tag{10}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

Notes

Notes

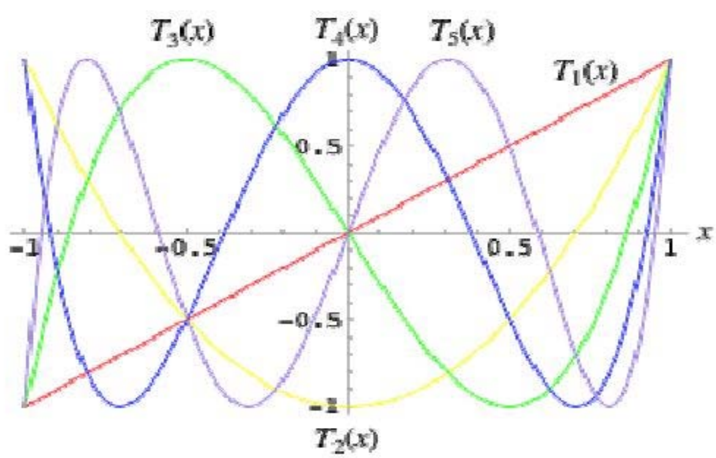
i) Legendre function of the second kind



The second solution $Q_1(x)$ to the Legendre differential equation. The Legendre functions of the second kind satisfy the same recurrence relation as the Legendre polynomials.
The first few are

$$\left.\begin{aligned} Q_0(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\ Q_1(x) &= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1 \\ Q_2(x) &= \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{3x}{2} \end{aligned}\right\} \tag{11}$$

j) Chebyshev polynomial of the first kind



The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_n(x)$. They are used as an approximation to a least squares fit, and are a special case of the Gegenbauer polynomial with $\alpha=0$. They are also intimately connected with trigonometric multiple-angle formulas.

The Chebyshev polynomial of the first kind $T_n(z)$ can be defined by the contour integral

$$T_n(z)=\frac{1}{4\pi i}\oint\frac{(1-t^2)t^{-n-1}}{1-2tz+t^2}dt\ , \tag{12}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The first few Chebyshev polynomials of the first kind are

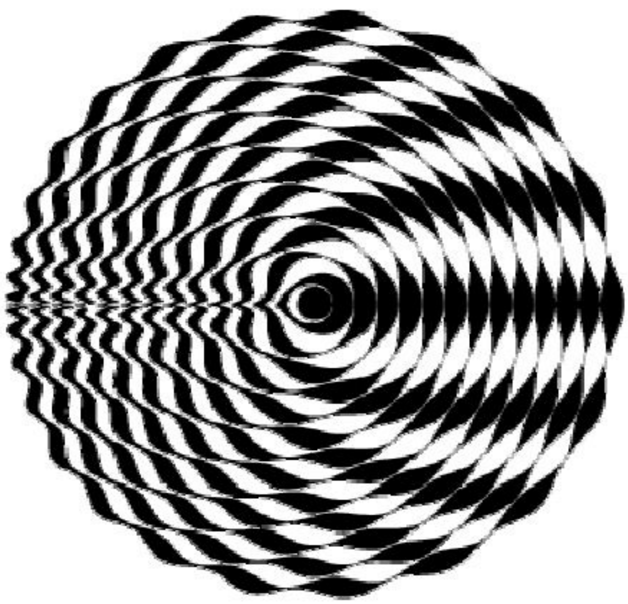
$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

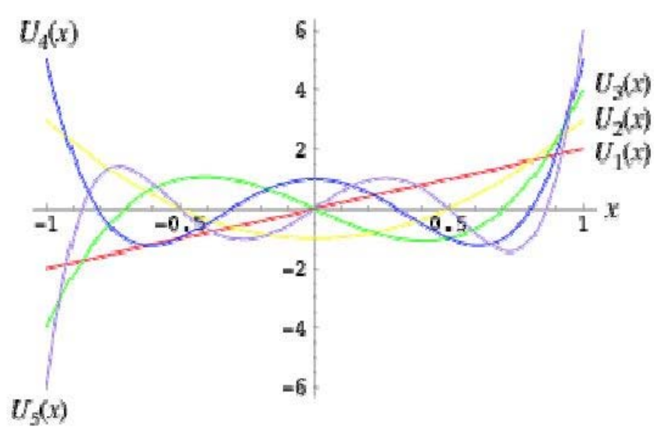
$T_3(x) = 4x^3 - 3x$

A beautiful plot can be obtained by plotting $T_n(x)$ radially, increasing the radius for each value of n, and filling in the areas between the curves (Trott 1999, pp. 10 and 84).



The Chebyshev polynomials of the first kind are defined through the identity $T_n(\cos\theta) = \cos n\theta$.

k) Chebyshev polynomial of the second kind



A modified set of Chebyshev polynomials defined by a slightly different generating function. They arise in the development of four-dimensionalspherical harmonics in angular momentum

Notes

Global Journal of Science Frontier Research (F) Volume XIII Issue X Version I Year 2013

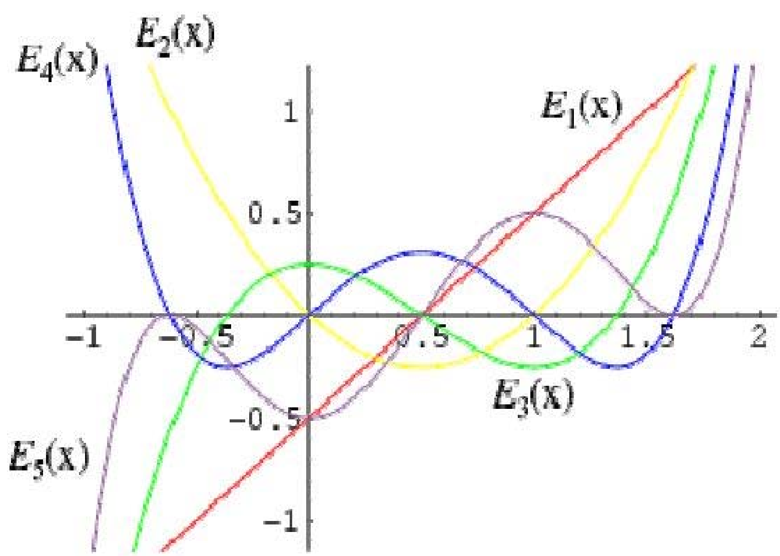
theory. They are a special case of the Gegenbauer polynomial with $\alpha = 1$. They are also intimately connected with trigonometric multiple-angle formulas.

The first few Chebyshev polynomials of the second kind are

$$\left.\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \end{aligned}\right\} \quad (13)$$

Notes

l) Euler polynomial



The Euler polynomial $E_n(x)$ is given by the Appell sequence with

$$g(t) = \frac{1}{2}(e^t + 1) \ ,$$

giving the generating function

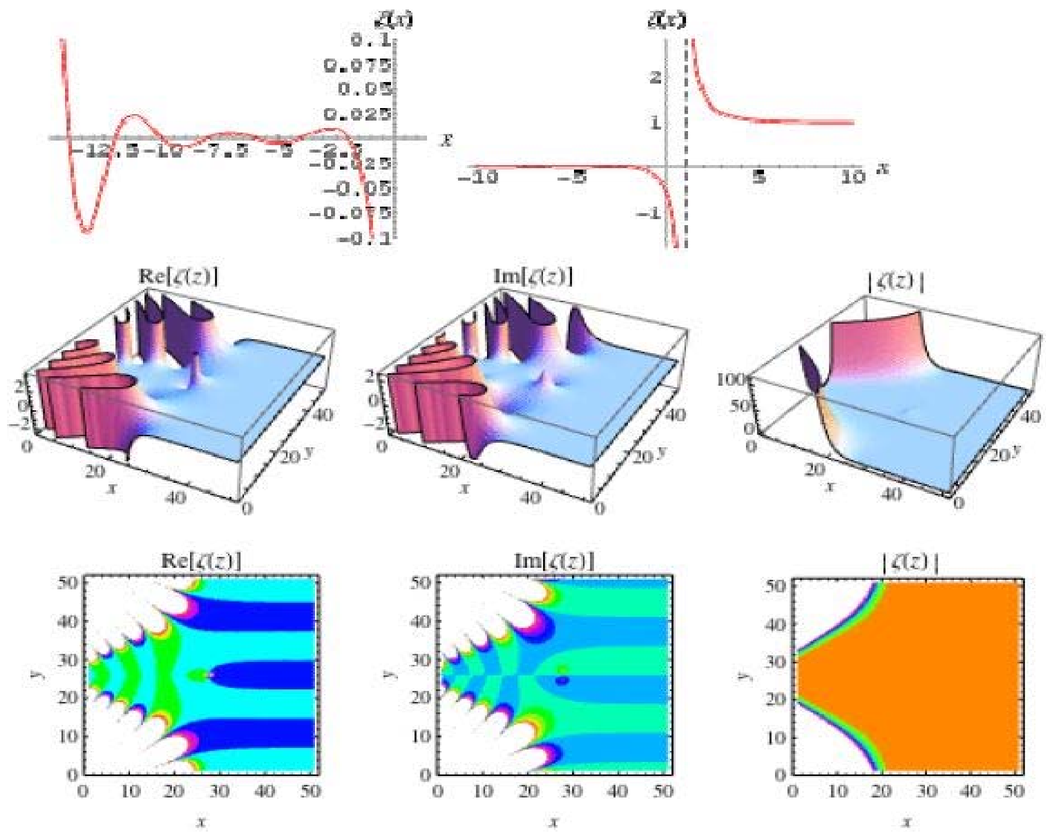
$$\frac{2e^{xt}}{e^t + 1} \equiv \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \ . \quad (14)$$

The first few Euler polynomials are

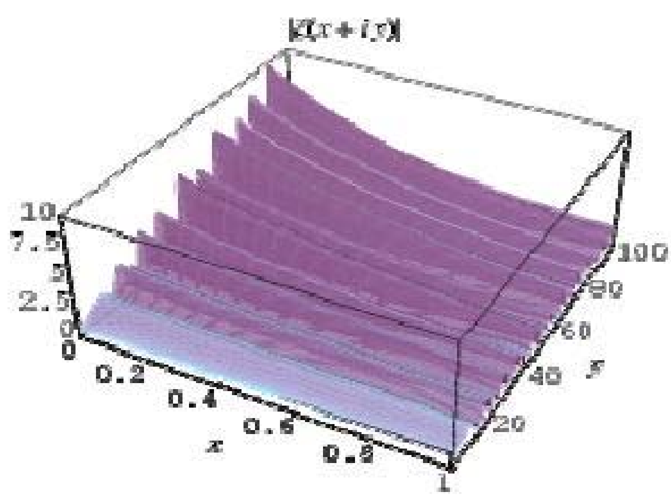
$$\left.\begin{aligned} E_0(x) &= 1 \\ E_1(x) &= x - \frac{1}{2} \\ E_2(x) &= x^2 - x \\ E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4} \end{aligned}\right\} \quad (15)$$

Notes

m) Generalized Riemann zeta function



The Riemann zeta function is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the Riemann hypothesis) that remain unproved to this day. The Riemann zeta function $\zeta(s)$ is defined over the complex plane for one complex variable, which is conventionally denoted s (instead of the usual z) in deference to the notation used by Riemann in his 1859 paper that founded the study of this function (Riemann 1859).



Notes

The plot above shows the "ridges" of $|\zeta(x+iy)|$ for $0<x<1$ and $1<y<100$. The fact that the ridges appear to decrease monotonically for $0\le x\le 1/2$ is not a coincidence since it turns out that monotonic decrease implies the Riemann hypothesis (Zvengrowski and Saidak 2003; Borwein and Bailey 2003, pp. 95-96).

On the real line with $x>1$, the Riemann zeta function can be defined by the integral

$\varsigma(x)\equiv\frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u-1}du$, where $\Gamma(x)$ is the gamma function.

n) Complex infinity

Complex infinity is an infinite number in the complex plane whose complex argument is unknown or undefined. Complex infinity may be returned by *Mathematica*, where it is represented symbolically by ComplexInfinity. The Wolfram Functions Site uses the notation $\tilde{\infty}$ to represent complex infinity.

II. MAIN RESULTS

$$\int\frac{L_1(x)}{\sqrt{1-x^n}}\,dx=\frac{1}{2}\,x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)+C \tag{17}$$

$$\int\frac{H_1(x)}{\sqrt{1-x^n}}\,dx=x\,{}_2F_1\left(\frac{1}{2},\frac{1}{n};\frac{n+1}{n};x^n\right)+C \tag{18}$$

$$\int\frac{B_1(x)}{\sqrt{1-x^n}}\,dx=\frac{1}{2}\,x\left[x\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)-{}_2F_1\left(\frac{1}{2},\frac{1}{n};\frac{n+1}{n};x^n\right)\right]+C \tag{19}$$

$$\int\frac{C_1(x)}{\sqrt{1-x^n}}\,dx=x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)+C \tag{20}$$

$$\int\frac{F_1(x)}{\sqrt{1-x^n}}\,dx=\frac{1}{2}\,x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)+C \tag{21}$$

$$\int\frac{L_1[x]}{\sqrt{1-x^n}}\,dx=x\,{}_2F_1\left(\frac{1}{2},\frac{1}{n};\frac{n+1}{n};x^n\right)-\frac{1}{2}\,x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)+C \tag{22}$$

$$\int\frac{H_1(x)}{\sqrt{1-x^n}}\,dx=x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)+C \tag{23}$$

$$\int\frac{P_1(x)}{\sqrt{1-x^n}}\,dx=\frac{1}{2}\,x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right)+C \tag{24}$$

$$\int \frac{T_1(x)}{\sqrt{1-x^n}}\,dx = \frac{1}{2}\,x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right) + C \tag{25}$$

$$\int \frac{U_1(x)}{\sqrt{1-x^n}}\,dx = x^2\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right) + C \tag{26}$$

$$\int \frac{E_1(x)}{\sqrt{1-x^n}}\,dx = \frac{1}{2}\,x\left[x\,{}_2F_1\left(\frac{1}{2},\frac{2}{n};\frac{n+2}{n};x^n\right) - {}_2F_1\left(\frac{1}{2},\frac{1}{n};\frac{n+1}{n};x^n\right)\right] + C \tag{27}$$

$$\int \frac{\zeta(1,x)}{\sqrt{1-x^n}} = \tilde{\infty} + C \tag{28}$$

Notes

References Références Referencias

1. Abramowitz, Milton; Stegun, Irene A.(1965), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover.
2. Andrews, L.C.(1992) ; *Special Function of mathematics for Engineers,Second Edition*, McGraw-Hill Co Inc., New York.
3. Arfken, G. ; *Mathematical Methods for Physicists*, Orlando, FL: Academic Press.
4. Bells, Richard, Wong, Roderick ; *Special Functions , A Graduate Tex*t, Cambridge Studies in Advanced Mathematics, 2010.
5. Prudnikov, A. P., Marichev, O. I., and Brychkov, Yu. A. ; *Integrals and Series, Vol. 3: More Special Functions.*, NJ:Gordon and Breach, 1990.
6. Rivlin, T. J. ;*Chebyshev Polynomials.,* New York: Wiley, 1990.
7. Wikipedia(2013,October), Bernoulli Polynomials. Retrived from website: http://en.wikipedia.org/wiki/Bernoulli_polynomials.
8. Wikipedia(2013,October), Gegenbaur Polynomials. Retrived from website: http://en.wikipedia.org/wiki/Gegenbauer_polynomials.
9. WolframMathworld(2013,October), Riemann Zeta function. Retrived from website: http://mathworld.wolfram.com/RiemannZetaFunction.html.
10. WolframMathworld(2013,October), Harmonic Number . Retrived from website: http://mathworld.wolfram.com/HarmonicNumber.html.
11. WolframMathworld(2013,October), LaguerrePolynomials. Retrived from website: http://mathworld.wolfram.com/LaguerrePolynomial.html.
12. WolframMathworld(2013,October), Hermite Polynomials. Retrived from website: http://mathworld.wolfram.com/HermitePolynomial.html.
13. WolframMathworld(2013,October), Legendre Polynomial. Retrived from website: http://mathworld.wolfram.com/LegendrePolynomial.html.
14. WolframMathworld(2013,October), Chebyshev Polynomial of the first kind. Retrived from website: http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html.
15. WolframMathworld(2013,October), Lucas polynomials. Retrived from website: http://mathworld.wolfram.com/LucasPolynomials.html.