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Generalized and Perturbed Lamé System

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GENERALIZED AND PERTURBED LAM SYSTEM

Strictly as per the compliance and regulations of :

Generalized and Perturbed Lamé System

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Abstract - In this work, we study the existence, the uniqueness and the regularity of the solution for some boundary value problems gouverned by perturbed and generalized dynamical Lamé system operator.

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I. Introduction

1. Notations.- Ω is a bounded and connected open set of $\mathbb{R}^n$ $(n = 2,3)$ with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, a lipschitzian manifold of dimension $n-1$, were $\Gamma_i \subset \Gamma$, $i = 1,2$, with $\text{mes}(\Gamma_1) > 0$ and $\Gamma_1 \cap \Gamma_2 = \phi$.

2. Position of the Problem.- We consider firstly the mathematical model of the perturbed Lamé system :

$$-L_p u + F(u),$$

were $F(u)$ is the perturbation and

$$L_p u = \mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + (\lambda+\mu)\nabla(div(u)),$$

p, q are two real numbers such that $p \in]1,\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$,

λ and μ are the Lamé coefficients subjected to the constraint $\lambda + \mu \geq 0$ and $\lambda > 0$,

ν denotes the outgoing normal vector to Γ_2.

For $p = 2$, we recover the classical dynamical Lamé system.

Given f and $\varphi = (\varphi_{i,j})_{1<i,j<n}$,such that $\varphi_{i,j} = \varphi_{j,i} \in C^{0,1}(\overline{\Omega})$ and $\varphi_{i,j}(x) > 0, \forall\ x \in \Gamma_2$.We study the existence, the uniqueness and the regularity of the complex-valued solution $u = u(x)$, $x \in \Omega$, for the following problem :

$$(P)\begin{cases} -L_p u + F(u) = f, \ \text{ in } Q & (2.1) \\ u = 0, \ \text{ on } \Gamma_1 & (2.2) \\ \sigma(u).\nu + \varphi(x)\ u = 0, \ \text{ on } \Gamma_2 & (2.3) \end{cases}$$

Here $\sigma(u) = (\sigma_{ij}(u))_{1<i,j<n}$ is the matrix of the constraints tensor $\sigma_{ij}(u) = \lambda div(u)\delta_{ij} + 2\mu\ \varepsilon_{ij}(u)$, were $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), 1 \leq i,j \leq n$, are the components of the deformation tensor.

In this work, we consider the cases $F(u) = 0$, $F(u) = |u|^{\rho}\, u$ with $\rho = p-2 > 0$, $F(u) = u^3$ and $F(u) \equiv a(x,t)$.

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We distinguish the cases:

when $\Gamma_2 = \phi$, (P) becomes a Dirichlet problem,

when $\Gamma_1 = \phi$, and $\varphi \equiv 0$ on Γ_2, (P) becomes a Neumann problem,

when , $\Gamma_2 \neq \phi$ and $\varphi \equiv 0$ on Γ_2, (P) becomes a mixed problem : Dirichlet-Neumann,

when Γ_1 , $\Gamma_2 \neq \phi$ and $\varphi(x) \neq 0$ on Γ_2, (P) becomes a mixed problem : Dirichlet-(2.3).

Of course, when it is question of a Neumann problem ($\Gamma_1 = \phi$ and $\varphi \equiv 0$ on Γ_2), we suppose verified the necessary condition of existence that is, the data are orthogonal to the rigid displacements :

$$\int_\Omega f.vdx = \int_\Gamma 0.vds = 0,$$

for any v of the form

$$v(x,y) = \left\{ \begin{array}{c} a + cy \\ b - cx \end{array} \right\},$$

with a, b, c arbitrary real numbers.

In the remaining part of this paper we study with details the last cas with $F(u) = |u|^\rho u$.

The main result is

Theorem 2.1.- We suppose that

$$f \in (W^{-1,q}(\Omega))^n).$$

Then, there exist a function $u = u(x)$ solution of the problem (P) with :

$$u \in (W^{1,p}(\Omega))^n),$$

Before giving the proof, we make the following remarks :

Remark 2.1.- The space $V = (H_0^1(\Omega))^n \cap (L^p(\Omega))^n$, were $p = \rho + 2$, is separable (i.e. admits a countable dense subset).

In fact, V is identified, by the application $v \to \left\{ v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_n} \right\}$, to a closed subspace of

$(L^p(\Omega))^n \times (L^2(\Omega))^n \times ... \times (L^2(\Omega))^n$,separable and uniformly convex, in such way that it possible to project a countable dense set on this subspace.

Remarque 2.3.- The application defined on $(L^p(\Omega))^n$ by $u \longrightarrow |u|^{p-2} u$, is $(L^q(\Omega))^n$-valued, moreover it is continuous. To see that, if $u \in (L^p(\Omega))^n$, $|u|^{p-2} u$ est mesurable and

$$\int_\Omega \left| |u|^{p-2} u \right|^q dx = \int_\Omega |u|^p \, dx < \infty \Longrightarrow u \in (L^q(\Omega))^n.$$

We deduce that $\forall u \in (W^{1,p}(\Omega))^n$, $\forall\, i$, $1 \leq i \leq n$,

$$\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \in (L^q(\Omega))^n.$$

So, it is possible to define the real-valued application :

$$\left((W^{1,p}(\Omega))^n \right)^2 \longrightarrow \mathbb{R}, \ (u,v) \longmapsto a_p(u,v).$$

$$a_p(u,v) = \ \mu \sum_{i,j=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \, dx + (\lambda + \mu) \int_\Omega div(u) \ div(v) \ dx.$$

Notes

For any u in $(W^{1,p}(\Omega))^n$, the application $(W^{1,p}(\Omega))^n \longrightarrow \mathbb{R}$, $v \longrightarrow a_p(u,v)$, is a continuous linear form. then $c.f.$ [5] there exist a unique element $A(u)$ of $(W^{-1,q}(\Omega))^n$, such that

$$a(u,v)_p = \langle A(u), v\rangle \,, \forall v \in (W_0^{1,p}(\Omega))^n.$$

The application $(W^{1,p}(\Omega))^n \longrightarrow (W^{-1,q}(\Omega))^n$, $u \longrightarrow A(u)$, is noted :

$$-L_p u = \; -\mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - (\lambda+\mu)\nabla(div(u)),$$

and is called a **p-Lamé application.**

The following proposition gives some properties of $-L_p$:

Proposition 2.1.- The operator $-L_p : (W^{1,p}(\Omega))^n \rightarrow (W^{-1,q}(\Omega))^n$ is bounded, hemicontinuous, monotone and coercitive.

Demonstration: Using the expression of the norm in dual space espace dual and Lebesgue's dominated convergence theorem, we prove that $-L_p$ is bounded and hemicontinuous. From the convexity of the real application $t \longrightarrow |t|^p$, we deduce the monotonicity of $-L_p$.

Proposition 2.2.- The problem (P) and the variational problem $(P.V)$:

$$a_p(u,v) + (|u|^{p-2}\, u, v) = (f,v) + (-\varphi(x)\;(u,v), \forall v \in (W^{1,p}(\Omega))^n,$$

are equivalent.

Demonstration: Indeed, it suffices to observe that $u = 0$ on $\Gamma_1 = 0 \Leftrightarrow\in (W_0^{1,p}(\Omega))^n$,and the variationnal equality is then equivalent to

$$-L_p u + |u|^{p-2}\, u = f \quad \text{in } \Omega,$$

because $(D(\Omega))^n$ is dense in $(W_0^{1,p}(\Omega))^n$.

Let us return to the demonstration of **Theorème2.1.**

(i) Construction of approximated solutions :

We look for $u_m = \displaystyle\sum_{i=1}^{n} \lambda_i v_i$ solution of the following problem (P_m) :

$\forall j,\, 1 \leq j \leq m$:

$$a_p(u_m, v_j) + (|u_m|^{p-2}\, u_m, v_j) = (f, v_j) + (-\varphi(x)\;(u_m, v_j)$$

We obtain a second order nonlinear differential system. Let be the function

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

$$F(\lambda_1, ..., \lambda_m) = \left(\left\langle A(\sum_{i=1}^{n} \lambda_i v_i), v_j\right\rangle - ((f,v_j) + (-\varphi(x)(u_m, v_j)).\right)_{1\leq j\leq m}$$

(ii) Establishment of priori estimates.-

- Of the coercivité of to one deducts that $||u_m||$ is a bounded;
- The operator has a bounded $\Longrightarrow$ $(A(u_m))_{\ m\in\mathbb{N}}$ is a bounded in $V\prime$;
- $\exists u \in V, \exists\chi \in V' \Longrightarrow \begin{cases} u_p \rightharpoonup u, \;\sigma(V,V'), \\ A(u_m) \rightharpoonup \chi, \;\sigma(V',V). \end{cases}$

(iii) Passage to the limit via compactness.

- The monotony and the hemicontinuous $\Longrightarrow \chi = A(u)$.

What finishes the demonstration of the **Theorem 2.1.**

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Notes