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Non Split Geodetic Number of a Line Graph

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NON SPLIT GEODETIC NUMBER OF A LINE GRAPH

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Non Split Geodetic Number of a Line Graph

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Abstract- A set $S \subseteq V[L(G)]$ is a non split geodetic set of $L(G)$, if S is a geodetic set and $\langle V-S \rangle$ is connected. The non split geodetic number of a line graph $L(G)$, is denoted by $g_{ns}[L(G)]$, is the minimum cardinality of a non split geodetic set of $L(G)$. In this paper we obtain the non split geodetic number of line graph of any graph. Also obtain many bounds on non split geodetic number in terms of elements of G and covering number of G. We investigate the relationship between non split geodetic number and geodetic number.

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I. Introduction

In this paper we follow the notations of [3]. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively.

The graphs considered here have at least one component which is not complete or at least two non trivial components.

For any graph $G(V, E)$, the line graph $L(G)$ whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A $u-v$ path of length d(u, v) is called a $u-v$ geodesic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u-v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by $g(G)$.

Non split geodetic number of a graph was studied by in [5]. A geodetic set S of a graph $G = (V, E)$ is a non split geodetic set if the induced subgraph

[3] F.Harary, Graph Theory, Addison-Wesely, Reading, MA,(1969)

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$< V - S >$ is connected. The non split geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a non split geodetic set. Geodetic number of a line graph was studied by in [4]. Geodetic number of a line graph $L(G)$ of G is a set $S^{'}$ of vertices of $L(G) = H$ is called the geodetic set in H if $I(S^{'}) = V(H)$ and a geodetic set of minimum cardinality is the geodetic number of $L(G)$ and is denoted by $g[L(G)]$. Now we define non split geodetic number of a line graph. A set $S^{'}$ of vertices of $L(G) = H$ is called the non split geodetic set in H if the induced subgraph $V(H) - S^{'}$ is connected and a non split geodetic set of minimum cardinality is the non split geodetic number of $L(G)$ and is denoted by $g_{ns}[L(G)]$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbors is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G.

For any undefined term in this paper, see [2] and [3].

II. Preliminary Notes

We need the following results to prove further results.

Theorem 2.1 (1) *Every geodetic set of a graph contains its extreme vertices.*

Proposition 2.2 *For any graph G, $g(G) \leq g_{ns}(G)$.*

Proposition 2.3 *For any tree T of order n and number of cut vertices c_i then the number of end edges is $n - c_i$.*

III. Main Results

Theorem 3.1 *For any tree T with k end edges and c_i be the number of cut vertices, then $g_{ns}[L(T)] = n - c_i$.*

Proof. Let S be the set of all extreme vertices of a line graph $L(T)$ of a tree T. By Theorem 2.1 $g_{ns}[L(T)] \geq |S|$. On the other hand, for an internal vertex v of $L(T)$, there exists x, y of $L(T)$ such that v lies on the unique $x - y$ geodesic in $L(T)$. The end edges of T are the extreme vertices of $L(T)$ and the induced subgraph $V - S$ is connected. Thus $g_{ns}[L(T)] \leq |S|$. Also every split geodetic set S_1 of $L(T)$ must contain S which is the unique minimum split geodetic set. Thus $|S| = |S_1| = k$, by proposition 2.3 $|S_1| = n - c_i$. Hence $g_{ns}[L(T)] = n - c_i$.

Corollary 3.2 *For any path P_n, $n \geq 6$, $g_{ns}[L(P_n)] = 2$.*

Proof. Clearly the set of two end vertices of a path P_n is its unique geodetic set. From Theorem 3.1 the results follows.

Ref

[4] Venkanagouda.M.Goudar, K.S.Ashalatha, Venkatesha, M.H.Muddebihal, On the Geodetic Number of Line Graph, Int.J.Contemp.Math.Sciences, vol.7, no.46, 2289-2295 (2012)

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Theorem 3.3 *For the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 6)$*,*

$$g_{ns}[L(W_n)] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}(n \geq 6)$ and let $V(W_n) = \{x, v_1, v_2, ..., v_{n-1}\}$, where $deg(x) = n - 1 > 3$ and $deg(v_i) = 3$ for each $i \in \{1, 2, ..., n-1\}$. Now $U = \{u_1, u_2, ..., u_j\}$ are the vertices of $L(W_n)$ formed from edges of C_{n-1}, i.e $U \subseteq V[L(W_n)]$ and $Y = \{y_1, y_2, ..., y_j\}$ are the vertices of $L(W_n)$ formed from internal edges of W_n, i.e $Y \subseteq V[L(W_n)]$.

We have the following cases

Case 1. For n is even.

Let $H \subseteq U$, now $S = H \cup \{y_j\}$ forms a minimum geodetic set of $L(W_n)$ and $V - S$ is connected. Thus S itself is the minimum non split geodetic set of $L(W_n)$. Clearly $|H \cup \{y_j\}| = \frac{n}{2}$. Therefore $g_{ns}[L(W_n)] = \frac{n}{2}$.

Case 2. For n is odd.

Let $H \subseteq U$, now $S = H \cup \{y_j, y_{j-1}\}$ forms a minimum geodetic set of $L(W_n)$ and $V - S$ is connected. Thus S itself is the minimum non split geodetic set of $L(W_n)$. Clearly $|H \cup \{y_j, y_{j-1}\}| = \frac{n+1}{2}$. Therefore $g_{ns}[L(W_n)] = \frac{n+1}{2}$.

As an immediate consequence of the above theorem we have the following.

Corollary 3.4 *For the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 6)$*,*

$$g_{ns}[L(W_n)] = \begin{cases} \frac{\Delta}{2} & \text{if } n \text{ is even} \\ \frac{\Delta+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Maximum degree(Δ) of $L(W_n)$ is equal to n. i,e number of vertices in W_n.

Case 1. For n is even.

We have from case 1. of Theorem 3.3 $g_{ns}[L(W_n)] = \frac{n}{2}$.

$g_{ns}[L(W_n)] = \frac{\Delta}{2}$.

Case 2. For n is odd.

We have from case 2. of Theorem 3.3 $g_{ns}[L(W_n)] = \frac{n+1}{2}$.

$g_{ns}[L(W_n)] = \frac{\Delta+1}{2}$.

Theorem 3.5 *For any tree* T*, with* m *edges,* $g_{ns}[L(T)] \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$*. Where* α_1 *is the edge covering number.*

Proof. Suppose $S = \{e_1, e_2, ..., e_k\}$ be the set of all end edges in T. Then $S \cup J$ where $J \subseteq E(T) - S$, be the minimal set of edges which covers all the vertices of T and is not covered by S, such that $|S \cup J| = \alpha_1(T)$. Now

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without loss of generality in $L(T)$, let $S' = \{u_1, u_2, ..., u_n\} \subseteq V[L(T)]$ be the set of vertices in $L(T)$ formed by the end edges in T and $V - S'$ is connected which is the minimal non split geodetic set of $L(T)$. Clearly it follows that $g_{ns}[L(T)] \leq |E(T)| - |\lceil \frac{S \cup J}{2} \rceil| + 2 \Rightarrow g_{ns}[L(T)] \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$.

Theorem 3.6 *For any connected graph G of order n, $g_{ns}(G) + g_{ns}[L(G)] \leq 2n$.*

Proof. Let $S = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ be the minimum non split geodetic set of G. Now without loss of generality in $L(G)$, if $F = \{u_1, u_2, ..., u_k\}$ be the set of all end vertices in $L(G)$. Then $F \cup H$ where $H \subseteq V[L(G)] - F$ forms a minimum non split geodetic set of $L(G)$. Since each vertex in $L(G)$ corresponds to two adjacent vertices of G, it follows that $|S| \cup |F \cup H| \leq 2n$. Therefore $g_{ns}(G) + g_{ns}[L(G)] \leq 2n$.

Theorem 3.7 *Let G be a connected graph of order n and diameter d. Then $g_{ns}[L(G)] \leq n - d + 1$.*

Proof. Let u and v be vertices of $L(G)$ for which $d(u, v) = d$ and let $u = v_0, v_1, ..., v_d = v$ be the $u - v$ path of length d. Now let $S = V[L(G)] - \{v_1, v_2, ..., v_{d-1}\}$. Then $I(S) = V[L(G)]$, $V[L(G)] - S$ is connected and $g_{ns}[L(G)] \leq |S| = n - d + 1$.

Observation 3.8 *For cycle C_n of order n $g_{ns}[L(C_n)] = n - d + 1$.*

Theorem 3.9 *For cycle C_n of order $n > 3$*

$$g_{ns}[L(C_n)] = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Line graph of a cycle is again a cycle of same order and d be the diameter. We have the following cases.

Case 1. If n is even. Let n be the number of vertices of $L(C_n)$ and diameter of $L(C_{2n})$ is $\frac{n}{2}$. Hence by Theorem 3.7 and Observation 3.8, $g_{ns}[L(C_n)] = n - d + 1$.

Now we have

$g_{ns}[L(C_n)] = n - \frac{n}{2} + 1$.

$\Rightarrow g_{ns}[L(C_n)] = \frac{n}{2} + 1$.

$\Rightarrow g_{ns}[L(C_n)] = \frac{n+2}{2}$.

Case 2. If n is odd. Let n be the number of vertices of $L(C_n)$ and diameter of $L(C_{2n+1})$ is $\frac{n-1}{2}$. Hence by Theorem 3.7 and Observation 3.8, $g_{ns}[L(C_n)] = n - d + 1$. We have

$g_{ns}[L(C_n)] = n - \frac{n-1}{2} + 1$.

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$\Rightarrow g_{ns}[L(C_n)] = \frac{n}{2} + \frac{1}{2} + 1.$

$\Rightarrow g_{ns}[L(C_n)] = \frac{n+3}{2}.$

Theorem 3.10 *For any integers $r, s > 2$, $g_{ns}[L(K_{r,s})] \leq rs - 1$.*

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Proof. Let $r + s$ and rs be the number of vertices and edges of the given graph $K_{r,s}$ and d be the diameter. Since diameter of $L(K_{r,s}) = 2$, the number of vertices in $L(K_{r,s})$ is rs. Hence by Theorem 3.7 $g_{ns}[L(G)] \leq n-d+1$. Now we have $g_{ns}[L(K_{r,s})] \leq rs - 2 + 1$. $\Rightarrow g_{ns}[L(K_{r,s})] \leq rs - 1$.

Theorem 3.11 *For any integer $n \geq 4$, $g_{ns}[L(K_n)] \leq \frac{(n+1)(n-2)}{2}$.*

Proof. Let $n \geq 4$ be the vertices of the given graph K_n and d be the diameter. Since diameter of $L(K_n)$ is 2 and the number of vertices in $L(K_n)$ is $\frac{n(n-1)}{2}$, hence by Theorem 3.7 $g_{ns}[L(G)] \leq n - d + 1$. We have

$g_{ns}[L(K_n)] \leq \frac{n(n-1)}{2} - 2 + 1.$

$\Rightarrow g_{ns}[L(K_n)] \leq \frac{n(n-1)}{2} - 1.$

$\Rightarrow g_{ns}[L(K_n)] \leq \frac{n^2-n-2}{2}.$

$\Rightarrow g_{ns}[L(K_n)] \leq \frac{(n+1)(n-2)}{2}.$

IV. Adding an End Edge

For an edge $e = (u, v)$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$, we call e an end-edge and u an end-vertex.

Theorem 4.1 *G' be the graph obtained by adding k end edges $\{(u, v_1), (u, v_2), ..., (u, v_k)\}$ to a cycle $C_n = G$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, ..., v_k\} \notin G$. Then*

$$g_{ns}[L(G')] = \begin{cases} k + \frac{n}{2} + 1 & \text{if } n \text{ is even} \\ k + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\{e_1, e_2, ..., e_n, e_1\}$ be a cycle with n vertices and let G' be the graph obtained from $G = C_n$ by adding end edges (u, v_i), $i = 1, 2, ..., k$. Such that $u \in G$ and $v_i \notin G$.

Case 1. If n is even.

By the definition of line graph, $L(G')$ has $\langle K_{k+2} \rangle$ as an induced subgraph. Also the edges (u, v_i), $i = 1, 2, ..., k$ becomes vertices of $L(G')$ and it belongs to some geodetic set of $L(G')$. Hence $\{e_1, e_2, ..., e_k, e_l, e_m\}$ are the vertices of $L(G')$ where e_l, e_m are the edges incident on the antipodal vertex of u in G' and these vertices belongs to some geodetic set of $L(G')$. $L(G') = C_n \cup K_{k+2}$.

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Let$S = \{e_1, e_2, ..., e_k, e_l, e_m\}$ be the minimum geodetic set. Consider $H \subseteq V - S$, now $S' = S \cup H_1$ where $H_1 \subseteq H$ forms minimum non split geodetic number of $L(G')$. Therefore $g_{ns}[L(G')] = k + \frac{n}{2} + 1$.

Case 2. If n is odd.

By the definition of line graph, $L(G')$ has $\langle K_{k+2} \rangle$ as an induced subgraph, also the edges $(u, v_i) = \{e_1, e_2, ..., e_k\}$ becomes vertices of $L(G')$. Let $e_l = (a, b) \in G$ such that $d(u, a) = d(u, b)$ in the graph $L(G')$. Let $S = \{e_1, e_2, ..., e_k, e_l\}$ be the minimum geodetic set. Since $V - S$ is connected S is the minimum non split geodetic set. Therefore $g_{ns}[L(G')] = k + 1$.

Theorem 4.2 *Let G' be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, ..., n$, $j = 1, 2, ..., n$ to each vertex of $G = C_n$ of order $n > 3$ such that $u_i \in G$, $v_j \notin G$. Then $g_{ns}[L(G')] = n$.*

Proof. Let $\{e_1, e_2, ..., e_n, e_1\}$ be a cycle with n vertices and $G = C_n$. Let G' be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, ..., n$, $j = 1, 2, ..., n$ to each vertex of G, such that $u_i \in G$, $v_j \notin G$. Clearly n be the number of end vertices of G'. By the definition of line graph $L(G')$ have n copies of K_3 as an induced subgraph. The edges $(u_i, v_j) = e_j$ for all j, becomes n vertices of $L(G')$ and those lies on geodetic set of $L(G')$. Since they forms the extreme vertices of $L(G')$. By Theorem 2.1 $S = \{e_1, e_2, ..., e_n\}$ forms minimum geodetic set. Since $V - S$ is connected S is the minimum non split geodetic set. Therefore $g_{ns}[L(G')] = n$.

V. Cartesian Product

The cartesian product of the graphs H_1 and H_2, written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 5.1 *For any path P_n of order n,*

$$g_{ns}[L(K_2 \times P_n)] = \begin{cases} 3 & \text{for } n = 2 \\ 4 & \text{for } n = 3 \\ 5 & \text{for } n > 3. \end{cases}$$

Proof. Let $K_2 \times P_n$ be formed from two copies of G_1 and G_2 of P_n. Now $L(K_2 \times P_n)$ formed from two copies of G'_1, G'_2 of $L(P_n)$. And let $U = \{u_1, u_2, ..., u_n\} \in V(G'_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G'_2)$. We have the following cases.

Case 1. If $n = 2$.

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Notes

Let $S = \{u_1, w_2\}$ forms minimum geodetic set of $L(K_2 \times P_2)$. Consider $H = \{u_2, w_1\} \subseteq V - S$. Now $S' = S \cup \{w_1\}$ or $\{u_2\}$, where w_1, u_2 are isolated vertices in $V - S$ forms minimum non split geodetic set of $L(K_2 \times P_2)$. Therefore $g_{ns}[L(K_2 \times P_2)] = 3$.
Case 2. If $n = 3$.

Let $S = \{u_2, w_1, w_3\}$ forms minimum geodetic set of $L(K_2 \times P_3)$. Consider $H = \{u_1, u_3, u_4, w_2\} \subseteq V - S$. Now $S' = S \cup \{u_1\}$, where u_1 is a isolated vertex in $V - S$ forms minimum non split geodetic set of $L(K_2 \times P_3)$. Therefore $g_{ns}[L(K_2 \times P_2)] = 4$.
Case 3. Suppose $n > 3$. Let S be the non split geodetic set of $L(K_2 \times P_n)$.

We claim that $S = \{u_1, u_n, w_1, w_{n-1}, w_n\}$ and $V - S$ is connected. Since $I(S) = V[L(K_2 \times P_n)]$, it follows that $g_{ns}[L(K_2 \times P_n)] = 5$. If S' is a four element subset of $V[L(K_2 \times P_n)]$ then $V - S$ is disconnected. It remains to show that if S' is a three element subset of $V[L(K_2 \times P_n)]$ then $I(S') \neq V[L(K_2 \times P_n)]$. First assume that S' is a subset U or W, say the farmer. Then $I(S') = S' \cup W \neq V$. Therefore, we may take that $S' \cap U = \{u_i, u_j\}$ and $S' \cap W = \{w_k\}$. Then $I(S') = \{u_i, u_j\} \cup W \neq V[L(K_2 \times P_n)]$.

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