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Some Observable Effects of Heat Flow in Response to Thermal Potentials at the Boundary

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Abstract - When heat flow is subject to temperature dependent thermal potential at the boundary, the associated local temperature field responds significantly, while the neighboring field is marginally influenced. This response results into effects quite intriguing. This paper examines these effects over a pure metallic plate. By considering both linear and non-linear thermal potentials induced at the edge of the plate as test cases, governed by Poisson Equation in 2-dimensions, finite element algorithm is employed to compute the temperature profiles. A control model is set-up, which admits Laplace Equation in 2-dimensions, and the outputs from the test models and the control model are examined and compared. The MATLAB results show notable effects. These results are discussed which are invaluable design factors for optimum efficiency of thermally driven systems such as in nuclear power plants, thermo-chemical plants, thermo-mechanical industries, lasers, solid state plasma, e.t.c. This paper, when incorporated with our previous work [9], serves as good theoretical grounds for believing the notable physical anomalies in heat transfer processes, such as the paradox of moving medium detected in the non-Fourier DPL heat conduction model [10].

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Some Observable Effects of Heat Flow in Response to Thermal Potentials at the Boundary

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Abstract - When heat flow is subject to temperature dependent thermal potential at the boundary, the associated local temperature field responds significantly, while the neighboring field is marginally influenced. This response results into effects quite intriguing. This paper examines these effects over a pure metallic plate. By considering both linear and non-linear thermal potentials induced at the edge of the plate as test cases, governed by Poisson Equation in 2-dimensions, finite element algorithm is employed to compute the temperature profiles. A control model is set-up, which admits Laplace Equation in 2-dimensions, and the outputs from the test models and the control model are examined and compared. The MATLAB results show notable effects. These results are discussed which are invaluable design factors for optimum efficiency of thermally driven systems such as in nuclear power plants, thermo-chemical plants, thermo-mechanical industries, lasers, solid state plasma, e.t.c. This paper, when incorporated with our previous work [9], serves as good theoretical grounds for believing the notable physical anomalies in heat transfer processes, such as the paradox of moving medium detected in the non-Fourier DPL heat conduction model [10].

I. INTRODUCTION

The response of heat flow to any external thermal field is best understood at the molecular level. The original heat flow profile is significantly influenced by the particular form of induced potential at the boundary. Ideally these external thermal fields must cause significant changes to the system under study. Such changes yield certain effects which require qualitative treatments and analytic studies, either by laboratory experiments or by computer simulations.

Analysis of heat flow problems in the presence of external thermal fields finds applications in numerous systems. Thermal effects on Magnetohydrodynamics Rayleigh flow were studied [1] by varying the radiation-conduction parameter which significantly alters the heat flux and temperature. Heat reservoir for real transformer was shown to provide guidance for optimum design of absorption heat transformer [2] in which the resultant heat sink was found to decrease cost and noise and increasing reliability. The ground, as a source of heat pump systems and as a loop heat exchanger has been demonstrated to be efficient [3] that allows connection to both heating and chilled water plant loops. Induced Gaussian bump was shown to have yielded the shifting

surface kinks [4] that yield discontinuous changes in the interface orientation. Thermal resonances were observed in signal transmission [5], in neutron capture [6] and in energy propagation in oscillators [7]. Non-linear temperature dependent magnetization is used [8] in the study of Biomagnetic fluid flow. Thermal stability in response to non-linear potential was recently studied [9] in which the simulations pose strong stability due to the insufficiency of the applied potentials to permanently distort the saddle point. It is worth pointing out that all these researches conducted were centered on thermal radiation.

The primary goal of this paper is to complement our previous work [9] so as pave way for understanding some noticeable physical anomalies of heat flow. Closely similar to the earlier literatures, this paper includes both linear and logarithmic potentials, in addition to the radiation potential studied in our previous work, aimed at exploring more potential effects of heat flow in response to boundary formulations. The main appealing feature of this study is its mathematical simplicity and elegance, in that similar effects observed in the literatures have been achieved without necessarily employing any mathematical rigor. Based on previous experience, we have adopted similar computational data and numerical results obtained [9]. The remanding part of this paper is structured as follows: In section 2, the mathematical problem is formulated. The finite element theory is discussed and the basic mathematical equations employed are transformed into finite element numerical scheme, in section 3. In sections 4, the simulation test is discussed and the results presented in graphic form. The effects observed are also discussed explicitly.

II. THE MATHEMATICAL PROBLEM

Consider a solid plate bounded by a surface S as shown in fig 1 below. At any point in the plate the rate of heat flow per unit area in any given direction is proportional to the temperature gradient in that direction. The 2-dimensional steady state heat diffusion in the presence of heat sources is known to follow the familiar Poisson's equation in two dimensions given as;

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \rho(r), \quad (1a)$$

In the absence of heat source term, $\rho(r)$, the equation reduces into Laplace's equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad (1b)$$

The boundary fixed temperatures are as follows;

$$\begin{aligned} \theta(1,1) &= 750K, \theta(1,6) = 700K, \theta(1,12) = 700K, \theta(1,19) = 700K, \\ \theta(2,6) &= 700K, \theta(1,11) = 700K, \theta(2,1) = 800K, \theta(3,1) = 800K, \\ \theta(4,1) &= 800K, \theta(5,1) = 800K, \theta(2,33) = 500K, \theta(3,34) = 500K, \\ \theta(4,35) &= 500K, \theta(5,36) = 500, \theta(11,37) = 600K \end{aligned} \quad (2a)$$

The imposed linear, radiation and logarithmic differential boundary conditions are as follows:

$$\frac{\partial \theta}{\partial n} = \begin{cases} \frac{-M\theta + S}{\kappa}, \\ 4E\sigma\theta_r^3(\theta - \theta_r), \\ \frac{-h\theta}{b} \ln\left(1 + \frac{bg}{K_o\theta_o}\right) \end{cases} \quad (2b)$$

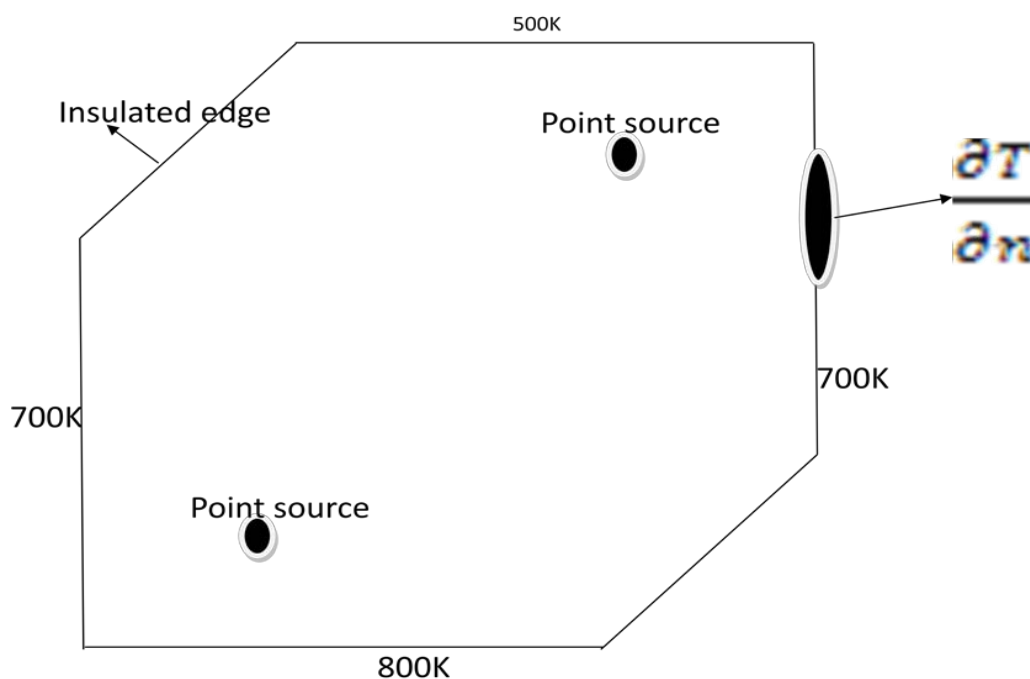


Figure 1 : The physical hexagonal pure metal for the simulation of heat flow

From equation (2a) above we have as follows:
 E is surface emissivity, h is thermal conductivity, σ is Stefan-Boltzmann constant, θ_r is the temperature of external radiation source, θ_o assumed lower limit temperature and M, S, b, g, K_o are constants.

III. FINITE ELEMENT FORMULATION

A class of physical problems arising in realistic systems can be expressed in terms of quantity minimization. These variational problems must be

stationary and must be of second order in their differential forms. Such a variational problem can be expressed as the functional

$$I(\Theta) = \int_a^b f(r, \Theta, \Theta') dr, \quad (3)$$

where Θ is the temperature field, Θ' is the temperature derivative, I is the integrable functional, f is a continuous function of temperature and position which minimizes the integral, a and b are extremities of the element.

The crux is to determine the solution of equation (3) in some closed bounded region. To achieve this, the finite element method is a very convenient tool. The use of finite element methods to simulate heat flow has gained attention over its finite difference counterpart. The popularity in the finite element methods comes from the fact that it is suitable in solving problems of higher dimensions with complex boundaries and little symmetry, contrary to the finite difference methods. In the Finite Element (FE) theory, it is usual to set-up the interpolation scheme and to choose the appropriate shape function, N , for the domain problem [11]. Also suitable element is used to span the entire domain.

To obtain the corresponding 2-D finite element scheme for the heat flow problem defined in Equations (1) and (2) we have as follows. As a strategy, we have simulated the finite element domain using triangular elements spanning the plate shown in Figure (1) thereby looping over the elements in a counterclockwise sense. From equation (3) we have the action,

$$dI = \int_a^b \left(\frac{\partial G}{\partial \Theta} d\Theta + \frac{\partial G}{\partial \Theta'} d\Theta' \right) dS = 0, a < S < b. \quad (4)$$

$$\begin{aligned} & \left[h \iint \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} \end{pmatrix}^T \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix} \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} \end{pmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} dxdy \right] \begin{pmatrix} \Theta_i \\ \Theta_j \\ \Theta_k \end{pmatrix} = \\ & h \iint \begin{pmatrix} h_1 & 0 & h_2 & 0 \\ 0 & h_1 & 0 & h_2 \end{pmatrix}^T Q \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} dxdy + \\ & \int_S \begin{pmatrix} h_1 & 0 & h_2 & 0 \\ 0 & h_1 & 0 & h_2 \end{pmatrix}^T \frac{\partial \Theta}{\partial n} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} ds + \\ & \sum_i \begin{pmatrix} h_1 & 0 & h_2 & 0 \\ 0 & h_1 & 0 & h_2 \end{pmatrix}^T Q_p \frac{\Theta_p}{k}. \quad (7) \end{aligned}$$

The above equilibrium equation can be written in compact form as

$$[h \iint B^{(e)T} C B^e |J^e| dA] \Theta = h \iint H^{(e)T} Q |J^e| dA + \int_S H^{(e)T} \frac{\partial \Theta}{\partial n} |J^e| ds + \sum_i H^{(e)T} Q_p \frac{\Theta_p}{k}. \quad (8)$$

To preserve the boundary conditions we have $\Delta\Theta(a) = \Delta\Theta(b) = 0$.

We employ the general functional for heat flow given [12] as;

$$G(x, y, \Theta, \Theta') = \frac{1}{2} \gamma \left[\left(\frac{d\Theta}{dy} \right)^2 \right] - \frac{1}{2} H \Theta^2 + \frac{1}{k} Q \Theta. \quad (5)$$

Putting Equation (5) into (3) and carrying out the integration gives,

$$I = \iint_A \frac{1}{2} \gamma \nabla^2 \Theta dA + \int_S \frac{1}{2} H \Theta^2 dS - \iint_A \frac{1}{k} Q \Theta dA, \quad (6)$$

for some edge S over which the plate losses heat to the surrounding, A is the plate area, γ is some constant coefficient, ∇^2 is Laplacian operator, k is thermal diffusivity.

The following relations follow: $x = \sum_{i=1}^3 N_i x_i$, $y = \sum_{i=1}^3 N_i y_i$ and in the natural coordinate we have $x = \sum_{i=1}^3 h_i x_i$, $y = \sum_{i=1}^3 h_i y_i$, $h_1 = 1 - r - s$, $h_2 = r$, $h_3 = s$, provide $\sum_{i=1}^3 h_i = 1$ holds.

Several algorithms have been discussed to assemble the resultant equations [12-17]. In the present work, it is convenient and consistent to strictly adhere to the principle of virtual temperature, discussed in the standard text book [18], and obtain the following equilibrium equation.

The term enclosed in the left hand side constitute the stiffness matrix. The terms in the right hand side are the contributions from the extended heat source, the applied thermal potential and the point sources, respectively. The matrices embedded in these terms are computed and defined [18] as follows: B^e is the temperature gradient interpolation matrix, C is the material property matrix, J^e is the Jacobian and H is the generalized element temperature matrix. The other

matrices are: Q the extended heat source, Q_p the point source, and $\frac{\partial \theta}{\partial n}$ the applied potential (defined in equation 2), H^T is the transpose of H .

Contrary to the test models, we ignore the applied external thermal potentials and the heat sources for the control mode, thus resulting into Laplace's equation as

$$h \iint \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} \end{pmatrix}^T \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix} \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} \end{pmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} dx dy \begin{pmatrix} \Theta_i \\ \Theta_j \\ \Theta_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (9)$$

Equations (7) and (9) give the contributions for the individual element. To obtain the resultant system of linear equations, we carry out the iterations for the entire elements, and thereafter assemble the equations. In practice, the finite element scheme results into large system of equations. Within this general arrangement, some additional steps must be taken to reduce the computational load. In particular, by judicious selection of the node numbers the stiffness matrix can be arranged into a symmetric band of finite width about the diagonal with zeros elsewhere. This can be used to reduce both the required memory and the computational load needed to solve the simultaneous equations. To solve the resultant set of linear equations, we have employed the well known Gauss-Jordan algorithm. However computer programs have been designed to ease the difficulty in handling finite element problems for large domain. Sample of these programs can be found in reference [19].

IV. SIMULATION RESULTS AND DISCUSSION

We employed the simulation data used in our earlier work following our previous experience. The radiation potential is approximated using Newton's law of cooling similar to that used in [16] and suggested in [19], rather than the traditional 4th power law, so as the preserve the linearity of the resultant system of equations. However, this approximation is precise only for specific range of temperature difference between the interacting thermal fields. In the standard text [19], it is specified that this temperature difference be at most 10%. To permit this approximation we have arbitrarily taken the value of the external radiation temperature to be 820K.

The results for the simulation are firstly obtained numerically and then we used Matrix Laboratory to obtain the graphs (Figure 3). While solving the resultant system of equations, we have simplified redundancies by eliminating any equation that resurfaced. For details

of the computational data and the numerical results see [9].

The finite element methods have been shown to give efficient, reliable, stable and converging solutions. The temperature profiles for the test cases have been computed and presented in tabular form shown in reference [9]. These results yield significant variations when compared with that of the control model as shown in the graphs (fig.3 data 4). The resultant variations are thus examined as effects manifesting due to the induction of the potential at the boundary. These effects are explicitly pointed out and discussed as follows.

Inducing the linear thermal potential at the boundary on the hexagonal plate has yielded results quite interesting. The temperature limits for the control model (Fig.3, data 4) have been significantly deviated as exhibited by the test models (Fig.3, data 1, 2 and 3). The linear potential (data 1) induces an 'artificial sink' at node 28 where the temperature drops to 483.45K. In principle since heat flow in the direction of lower temperature limit, this drop in the lower temperature limit for the control model could induce thermal cold reservoir. In the realm of Statistical Mechanics, the associated local fields are considered as system while the other particles and their degrees of freedom are considered as heat reservoirs, thus resulting into heat sinks.

Contrary to the graph (data 4) for the control model, the graphs for the test cases (data 1, 2 and 3) show deviations in the upper limit temperature at nodes 7, 8, 13, 14 and 19 due to the induced potentials. Ideally the deviations could have been expected at only the nodes where the potential is concentrated. However the deviations posed by the internal nodes, as well, confirms that the induced external thermal field also influences the neighborhood particles. These results are in agreement with the theoretical treatments discussed in the text book [20]. Depending on physical properties of the material solid (such as thermal resistivity, elongation temperature and melting point), these

deviations can pose overheating of the material surface which in turn pose the tendency of formation of surface bumps or deformation on cooling. Cad well and Kwan [21] predicted that such cooling or solidification primarily results to boundary perturbation.

Heat in certain continuous processes is studied [22] and it was noted that the difficulty is due to interaction of fluid flow and heat conduction. The non-linear potentials exhibit strong non-linearity. These are simply likened to the oscillatory behavior in thermally interacting packets in the direction of the heat flux, influenced by the induced potentials which results into locally flute-clarinet-like nascent marginally unstable heat flux (figure 3 data 2 and 3). Our results are strongly in confirmation of earlier results obtained [22-24].

More interestingly is the double peaks exhibited by the induction of the non-linear potentials. In particular the logarithmic potential exhibits the peaks at nodes 22 and 28, where the upper temperature limits are largely deviated giving the magnitudes of 1035.88 K and 1150K, respectively. The peaks result due to high pulse heating thermal resonances, a notable effect very useful in lacers and thin films, in which the thermal waves travel with finite speed and extreme temperature gradient at the lowest possible spatial mode. Obviously every oscillating system is capable of exhibiting resonance. The resonance occurs in the event that thermal energy

of the applied potential equals the spontaneous internal energy of the system, thereby temporarily eliminating nascent flute-clarinet like modes manifesting due to marginally unstable vibration at the nodal points closer to the boundary.

V. CONCLUSION

The finite element method is employed to compute the temperature profiles. We have assessed the response of heat flow profiles to boundary formulations. By comparing the results from the test cases and a control model, significant effects are observed. The associated local fields have been significantly influenced while the neighboring fields are marginally influenced. It could be deduced that the influenced packet is excited thereby interacting with other neighboring packets. We propose the results of this study as invaluable design consideration for optimum efficiency of thermally driven systems such as in nuclear power plants, thermo-chemical plants, thermo-mechanical industries, lacers, plasma e.t.c. This paper, when incorporated with our previous work [9], serves as good theoretical grounds for believing the notable physical anomalies in heat transfer processes, such as the paradox of moving medium detected in the non-Fourier DPL heat conduction model [10].

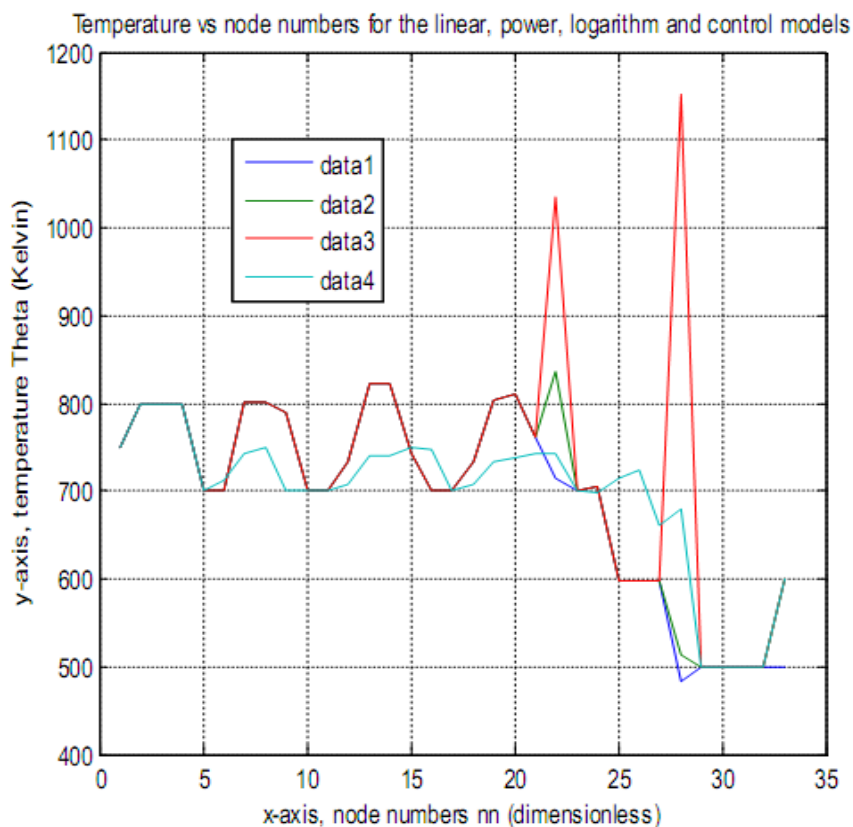


Figure 3 : MATLAB simulation results of the temperature Vs node numbers. Data 1 is the graph for the linear potential; Data 2 is the graph for the radiation potential; Data 3 is the graph for the logarithmic potential; Data 4 is the graph for the control model

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