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On Certain Summation Formulae Involving Gauss Theorem

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ON CERTAIN SUMMATION FORMULAE INVOLVING GAUSS THEOREM

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On Certain Summation Formulae Involving Gauss Theorem

Salahuddin [α] & Intazar Husain [σ]

Abstract - The main object of present paper is to obtain some summation formuale involving Contiguous relation, Recurrence relation, Gauss second summation theorem and Legendre duplication formula.

Keywords and Phrases : contiguous relation, recurrence relation, gauss second summation theorem, legendre duplication formula.

I. Introduction

The Pochhammer's symbol is defined by

$$(\alpha,\ k) = (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 \quad ; & \text{if } k = 0 \\ k! \quad ; & \text{if } \alpha = 1 \end{cases} \tag{1}$$

Generalized Gaussian Hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ b_1, b_2, \cdots, b_B & ; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{ll} (a_A) & ; \\ (b_B) & ; \end{array} z\right] \equiv {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} & ; \\ (b_j)_{j=1}^{B} & ; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{2}$$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation is defined by

[Andrews p.367(8), E. D. p.52(19)]

$$c\,(1-z)\ {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ c & ; \end{array} z\right] = c\ {}_2F_1\left[\begin{array}{ll} a-1,\ b & ; \\ c & ; \end{array} z\right] - (c-b)\ z\ {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ c+1 & ; \end{array} z\right] \tag{3}$$

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Recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z) \tag{4}$$

Legendre's duplication formula

$$\sqrt{\pi}\,\Gamma(2z) = 2^{(2z-1)}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \tag{5}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{6}$$

$$= \frac{2^{(a-1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{7}$$

Gauss second summation theorem [Prudnikov., 491(7.3.7.8)]

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b+1}{2}\ ; \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma(\frac{a+b+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} \tag{8}$$

$$= \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\,\Gamma(\frac{a+1}{2})} \tag{9}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b-1}{2}\ ; \end{matrix}\ \frac{1}{2}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\,\Gamma(b)}\right] \tag{10}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b-1}{2}\ ; \end{matrix}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{11}$$

II. Main Results of Summation Formulae

For $a < 1$ and $a > 9$

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b-9}{2}\ ; \end{matrix}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\left\{\frac{(128a+384b-1152)}{(a-9)} - \right.\right.$$

$$-\frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)} + \frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)} -$$

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$$-\frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-$$

$$-\frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)}+\frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg] \tag{12}$$

For $a<1$ and $a>10$

$${}_2F_1\left[\begin{matrix} a, & b & ; \\ \frac{a+b-10}{2} & & ; \end{matrix} \quad \frac{1}{2}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-10}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-10}{2})}\ \Bigg\{\frac{(256a+768b-2560)}{(a-10)}-$$

$$-\frac{(a-b-10)(576a+1472b-4608)}{(a-10)(a-8)}+\frac{(a-b-10)(a-b-8)(432a+912b-2592)}{(a-10)(a-8)(a-6)}-$$

$$-\frac{(a-b-10)(a-b-8)(a-b-6)(120a+200b-480)}{(a-10)(a-8)(a-6)(a-4)}+$$

$$+\frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(9a+11b-18)}{(a-10)(a-8)(a-6)(a-4)(a-2)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(512a+512b-4096)}{(a-9)}-\frac{(a-b-10)(960a+832b-5888)}{(a-9)(a-7)}+$$

$$+\frac{(a-b-10)(a-b-8)(560a+400b-2400)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-10)(a-b-8)(a-b-6)(104a+56b-256)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(3a+b-2)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}\Bigg] \tag{13}$$

For $a<1$ and $a>11$

$${}_2F_1\left[\begin{matrix} a, & b & ; \\ \frac{a+b-11}{2} & & ; \end{matrix} \quad \frac{1}{2}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-11}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}\ \Bigg\{\frac{(512a+1536b-5632)}{(a-11)}-$$

$$-\frac{(a-b-11)(1280a+3328b-11520)}{(a-11)(a-9)}+\frac{(a-b-11)(a-b-9)(1120a+2464b-7840)}{(a-11)(a-9)(a-7)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(400a+720b-2000)}{(a-11)(a-9)(a-7)(a-5)}+$$

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$$+\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(50a+70b-150)}{(a-11)(a-9)(a-7)(a-5)(a-3)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})}\Bigg\{\frac{(1024a+1024b-9216)}{(a-10)}-\frac{(a-b-11)(2176a+1920b-15488)}{(a-10)(a-8)}+$$

$$+\frac{(a-b-11)(a-b-9)(1536a+1152b-8064)}{(a-10)(a-8)(a-6)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(400a+240b-1360)}{(a-10)(a-8)(a-6)(a-4)}+$$

$$+\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(28a+12b-44)}{(a-10)(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg] \tag{14}$$

The above summation theorems can be easily verified by using computer algebra system programming languages, like Maple, Matlab, or Mathematica.

III. Derivations of Summation Formulae (12) to (14)

Derivation of (12): Substituting $c=\frac{a+b-9}{2}$ and $z=\frac{1}{2}$ in equation (3), we get

$$\left(\frac{a+b-9}{4}\right)\,{}_2F_1\left[\begin{matrix} a,\ b & ; \\ \frac{a+b-9}{2} & ; \end{matrix}\ \frac{1}{2}\right]=\left(\frac{a+b-9}{2}\right)\,{}_2F_1\left[\begin{matrix} a-1,\ b & ; \\ \frac{a+b-9}{2} & ; \end{matrix}\ \frac{1}{2}\right]-$$

$$-\left(\frac{a-b-9}{4}\right)\,{}_2F_1\left[\begin{matrix} a,\ b & ; \\ \frac{a+b-7}{2} & ; \end{matrix}\ \frac{1}{2}\right]$$

$${}_2F_1\left[\begin{matrix} a,\ b & ; \\ \frac{a+b-9}{2} & ; \end{matrix}\ \frac{1}{2}\right]=2.\ {}_2F_1\left[\begin{matrix} a-1,\ b & ; \\ \frac{a+b-9}{2} & ; \end{matrix}\ \frac{1}{2}\right]-\left(\frac{a-b-9}{a+b-9}\right)\,{}_2F_1\left[\begin{matrix} a,\ b & ; \\ \frac{a+b-7}{2} & ; \end{matrix}\ \frac{1}{2}\right]$$

Now involving the derived from Asish et all , we get

$${}_2F_1\left[\begin{matrix} a,\ b & ; \\ \frac{a+b-9}{2} & ; \end{matrix}\ \frac{1}{2}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(128a+384b-1152)}{(a-9)}-$$

$$-\frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)}+\frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)}\Bigg\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-$$

$$-\frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)}+\frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)}-$$

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$$-\frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]-$$

$$-\left(\frac{a-b-9}{a+b-9}\right)\frac{2^{(b-1)}\,\Gamma(\frac{a+b-7}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-7}{2})}\Bigg\{\frac{(32a+96b-224)}{(a-7)}-\frac{(a-b-7)(48a+112b-240)}{(a-7)(a-5)}+$$

$$+\frac{(a-b-7)(a-b-5)(18a+30b-54)}{(a-7)(a-5)(a-3)}-\frac{(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-6}{2})}\Bigg\{\frac{(64a+64b-320)}{(a-6)}-\frac{(a-b-7)(72a+56b-232)}{(a-6)(a-4)}+$$

$$+\frac{(a-b-7)(a-b-5)(16a+8b-24)}{(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

$$=\frac{2^{(b-1)}\,\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(128a+384b-1152)}{(a-9)}-\frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)}+$$

$$+\frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-\frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)}+$$

$$+\frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]-$$

$$-\frac{2^{(b-1)}\,\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(a-b-9)(32a+96b-224)}{(a-9)(a-7)}-$$

$$-\frac{(a-b-9)(a-b-7)(48a+112b-240)}{(a-9)(a-7)(a-5)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(18a+30b-54)}{(a-9)(a-7)(a-5)(a-3)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(a-b-9)(64a+64b-320)}{(a-8)(a-6)}-\frac{(a-b-9)(a-b-7)(72a+56b-232)}{(a-8)(a-6)(a-4)}+$$

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$$+\frac{(a-b-9)(a-b-7)(a-b-5)(16a+8b-24)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

$$=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\ \Bigg\{\frac{(128a+384b-1152)}{(a-9)}-$$

$$-\frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)}+\frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-\frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)}+$$

$$+\frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

Thus , we prove the result (12)

Similarly, we can prove the other results.

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