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Coefficient Problem for Certain Subclass of Analytic Functions Using Quasi-Subordination

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Abstract - An analytic function f is quasi-subordinate to an analytic function g, in the open unit disk if there exist analytic function φ and w, with $|\varphi(z)| \leq 1$, w (0) = 0 and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szego coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses are derived.

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COEFFICIENT PROBLEM FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS USING QUASI-SUBORDINATION

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Coefficient Problem for Certain Subclass of Analytic Functions Using Quasi-Subordination

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Abstract - An analytic function f is quasi-subordinate to an analytic function g, in the open unit disk if there exist analytic function φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szego coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses are derived.

I. Introduction and Motivation

Let A be the class of analytic function f in the open unit disk $D = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For two analytic functions f and g, the function f is subordinate to g, written as follows:

$$f(z) \prec g(z), \tag{1.1}$$

if there exists an analytic function w, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in D, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$. For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{[f(z)]} \prec \phi(z) \right\}, \tag{1.2}$$

where ϕ is an analytic function with positive real part in D, $\phi(D)$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called Ma-Minda starlike (with respect to ϕ). The class $C(\phi)$ is the class of functions $f \in A$ for which

Ref

[1] P. Duren, Subordination, in Complex Analysis, Lecture Notes in Mathematics, Springer, Berlin, Germany, 599 (1977), 22–29.

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$1 + zf''(z)/f'(z) \prec \phi(z)$. The class $S^*(\phi)$ and $C(\phi)$ include several well-known subclasses of starlike functions as special case.

In the year 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is quasi-subordinate to g, written as follows:

$$f(z) \prec_q g(z), \tag{1.3}$$

if there exist analytic function φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Observe that when $\varphi(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in D. Also notice that if $w(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that f is majorized by g and written $f(z) \ll g(z)$ in D. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4,5,6] for works related to quasi-subordination.

Throughout this paper it is assumed that ϕ is analytic in D with $\phi(0) = 1$. Motivated by [2,3], we define the following classes.

Definition 1.1. *Let the class $R_q^*(\alpha, \phi)$ consists of functions $f \in A$ satisfying the quasi-subordination*

$$\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1 \prec_q \phi(z) - 1, \quad \alpha \geq 0 \tag{1.4}$$

Example 1.2. *The function $f : D \to C$ defined by the following*

$$\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1 = z[\phi(z) - 1], \quad \alpha \geq 0 \tag{1.5}$$

belongs to the class $R_q^*(\alpha, \phi)$.

It is well known (see [10]) that the n^{th} coefficient of a univalent function $f \in A$ is bounded by n. The bounds for coefficient give information about various geometric properties of the function. Many authors have also investigated the bounds for the Fekete-Szego coefficient for various classes [11, 12, 13,14, 15,16,17,18,19, 20,21, 22,23,24,25]. In this paper, we obtain coefficient estimates for the functions in the above defined classes.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. We need the following lemma to prove our results.

Lemma 1.3 (see [26]). *If $w \in \Omega$, then for any complex number f*

$$|w_2 - tw_1^2| \leq \max\{1; |t|\}. \tag{1.6}$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Ref.

[3] M.S. Robertson, Quasi-subordination and coefficient conjectures, Bulletin of the American Mathematical Society, 76 (1970), 1–9.

II. Main Results

Throughout, let $f(z) = z+a_2z^2+a_3z^3+\cdots$, $\phi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots$, $\varphi(z) = c_0 + c_1z + c_2z^2 + c_3z^3 + \cdots$, $B_1 \in R$ and $B_1 > 0$.

Theorem 2.1. *If $f \in A$ belongs to $R_q^*(\alpha, \phi)$, then*

$$|a_2| \leq \frac{B_1}{1+\alpha},$$
$$|a_3| \leq \frac{B_1}{2}\left(1+\max\left\{1, B_1\left|\frac{1-\alpha}{1+\alpha}+\frac{\alpha}{2B_1}\right|+\left|\frac{B_2}{B_1}\right|\right\}\right) \tag{2.1}$$

and for any complex number μ,

$$|a_3-\mu a_2^2| \leq \frac{B_1}{2}\left(1+\max\left\{1, B_1\left|\frac{1-\alpha}{1+\alpha}-\frac{2\mu}{(1+\alpha)^2}+\frac{\alpha}{2B_1}\right|+\left|\frac{B_2}{B_1}\right|\right\}\right). \tag{2.2}$$

Proof. If $f \in R_q^*(\alpha, \phi)$, then there exist analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z^{1-\alpha}f'(z)}{[f(z)]^{1-\alpha}} - 1 = \varphi(z)(\phi(w(z)) - 1). \tag{2.3}$$

Since

$$\phi(w(z)) - 1 = B_1w_1z + (B_1w_2 + B_2w_1^2)z^2 + \cdots$$

$$\varphi(z)(\phi(w(z)) - 1) = B_1c_0w_1z + (B_1c_1w_1 + c_0(B_1w_2 + B_2w_1^2))z^2 + \cdots \tag{2.4}$$

it follows from (2,3) that

$$a_2 = \frac{B_1c_0w_1}{(1+\alpha)}$$

$$a_3 = \frac{1}{2+\alpha}\left[\frac{\alpha}{2}B_1c_0w_1 + B_1c_1w_1 + B_1c_0w_2 + c_0\left(\left(\frac{1-\alpha}{1+\alpha}\right)B_1^2c_0 + B_2\right)w_1^2\right] \tag{2.5}$$

Since $\varphi(z)$ is analytic and bounded in D, we have [27, page 172]

$$|c_n| \leq 1 - |c_0|^2 \leq 1 \quad (n > 0). \tag{2.6}$$

By using this fact and the well-known inequality, $|w_1| \leq 1$, we get

$$|a_2| \leq \frac{B_1}{1+\alpha}. \tag{2.7}$$

Ref

[27] Z. Nehari, Conformal mapping, Dover, New York, NY, USA, 1975, Reprinting of the 1952 edition.

Further,

$$a_3 - \mu a_2^2 = \frac{1}{2+\alpha}\Big[B_1c_1w_1 + c_0(B_1w_2 + \frac{\alpha}{2}B_1w_1 + \left(B_2 + \left(\frac{1-\alpha}{1+\alpha}\right)B_1^2c_0 - \frac{2\mu}{(1+\alpha)^2}B_1^2c_0\right)w_1^2\Big]. \tag{2.8}$$

Then

$$|a_3 - \mu a_2^2| \leq \frac{1}{2+\alpha}\Bigg(|B_1c_1w_1| + \left|B_1c_0\left(w_2 - \left(\frac{2\mu}{(1+\alpha)^2}B_1c_0 - \left(\frac{1-\alpha}{1+\alpha}\right)B_1c_0 + \frac{\alpha}{2}\frac{w_1}{c_0} - \frac{B_2}{B_1}\right)w_1^2\right)\right|\Bigg). \tag{2.9}$$

Again applying $|c_n| \leq 1$ and $|w_1| \leq 1$, we have

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2+\alpha}\left(1 + \left|w_2 - \left(\frac{\alpha}{2} - \left(\frac{1-\alpha}{1+\alpha} - \frac{2\mu}{(1+\alpha)^2}\right)B_1c_0 - \frac{B_2}{B_1}\right)w_1^2\right|\right). \tag{2.10}$$

Applying Lemma 1.3 to

$$\left|w_2 - \left(\frac{\alpha}{2} - \left(\frac{1-\alpha}{1+\alpha} - \frac{2\mu}{(1+\alpha)^2}\right)B_1c_0 - \frac{B_2}{B_1}\right)w_1^2\right| \tag{2.11}$$

yields

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2+\alpha}\left(1 + \max\left\{1, \left|\frac{\alpha}{2} - \left(\frac{1-\alpha}{1+\alpha} - \frac{2\mu}{(1+\alpha)^2}\right)B_1c_0 - \frac{B_2}{B_1}\right|\right\}\right). \tag{2.12}$$

Observe that

$$\left|\frac{\alpha}{2} - \left(\frac{1-\alpha}{1+\alpha} - \frac{2\mu}{(1+\alpha)^2}\right)B_1c_0 - \frac{B_2}{B_1}\right| \leq B_1|c_0|\left|\frac{1-\alpha}{1+\alpha} - \frac{2\mu}{(1+\alpha)^2} + \frac{\alpha}{2B_1}\right| + \left|\frac{B_2}{B_1}\right|, \tag{2.13}$$

and hence we can conclude that

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2}\left(1 + \max\left\{1, B_1\left|\frac{1-\alpha}{1+\alpha} - \frac{2\mu}{(1+\alpha)^2} + \frac{\alpha}{2B_1}\right| + \left|\frac{B_2}{B_1}\right|\right\}\right). \tag{2.14}$$

For $\mu = 0$, the above will reduce to estimate of $|a_3|$.

Theorem 2.2. *If $f \in A$ satisfies*

$$\frac{z^{1-\alpha}f'(z)}{[f(z)]^{1-\alpha}} - 1 \ll \phi(z) - 1, \tag{2.15}$$

Notes

then the following inequalities hold:

$$|a_2| \leq \frac{B_1}{1+\alpha},$$

$$|a_3| \leq \frac{1}{2+\alpha}(B_1 + B_1^2 + |B_2|), \tag{2.16}$$

Notes

and, for any complex number μ,

$$|a_3 - \mu a_2^2| \leq \frac{1}{(2+\alpha)(1+\alpha)^2}((1+\alpha)^2 B_1 + |(1+\alpha)^2 - (2+\alpha)\mu|B_1^2 + (1+\alpha)^2|B_2|). \tag{2.17}$$

Proof. The result follows by taking $w(z) = z$ in the proof of Theorem 2.1.

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