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A New Approach of Iteration Method for Solving Some Nonlinear Jerk Equations

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Abstract- A new approach of the Mickens' iteration method has been presented to obtain approximate analytic solutions of some nonlinear jerk equations. It has been shown that the partial derivatives of integral functions are valid for iteration method in each step of iteration. Also the solutions give more accurate result than other existing methods.

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A New Approach of Iteration Method for Solving Some Nonlinear Jerk Equations

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I. Introduction

The subject of differential equation not only is one of the most beautiful parts of mathematics, but it is also an essential tool for modeling many physical situations such as mechanical vibration, nonlinear circuits, chemical oscillation, space dynamics and so forth. These equations have also demonstrated their usefulness in ecology, business cycle and biology. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. The differential equations are linear or nonlinear, autonomous or non-autonomous. Practically, numerous differential equations involving physical phenomena are nonlinear. Methods of solutions of linear differential equations are comparatively easy and highly developed. Whereas, very little of a general character is known about nonlinear equations.

Generally, the nonlinear problems are solved by converting into linear equations imposing some conditions; but such linearization is always not possible. In this situation there are several analytical approaches to find approximate solutions to nonlinear problems, such as: Perturbation [11,19,20], Standard and modified Linstedt-Poincaré [20-22], Harmonic Balance [1-4,6-9,13,14,16,24-29], Homotopy [5,15], Iterative [10, 12,18] methods, etc.

Among them the most widely used method is the perturbation method where the nonlinear term is small. Another recent technique is developed by Mickens [17] and farther work has

Ref

[11] Hu, H.: *Perturbation method for periodic solutions of nonlinear jerk equations*, Phys. Lett. A, vol. 372, pp. 4205-4209, 2008.

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been done by Wu [29], Gottlieb [7,8], Alam [1] and so forth for handling the strong nonlinear problems named HB method. Recently, some authors used an iteration procedure which is valid for small as well as large amplitude of oscillation to obtain the approximate frequency and the corresponding periodic solution of such nonlinear problems.

The term "jerk" i.e. the third-order derivative of the displacement (which might also be termed "triceleration") was first introduced by Schot in 1978 [27]. We know most of the efforts on dynamical systems are related to second-order differential equations, but some dynamical systems can be described by nonlinear jerk (third-order) differential equations. As for example oscillations in a nonlinear vacuum tube circuit [24] and third order mechanical oscillators [6]. Originally jerk equations

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) . \tag{1}$$

being of some interest in mechanics [4,26], these equations are finding increasing importance in the study of chaos [4]. Many third-order nonlinear systems (three simultaneous first order differential equations), both mathematically and physically motivated, such as the now–classical Rössler system [25], may be recast into a single nonlinear third-order differential (jerk) equation involving only one of the dependent variables [7,14]. Some early investigations into nonlinear jerk equations (although not termed as such) include oscillations in a nonlinear vacuum tube circuit [24], and third order mechanical oscillators [6]. Other physical situations in which nonlinear jerk-type equations have been investigated with more emphasis in chaotic solutions (called a periodic in earlier works), include a thermo–mechanical oscillator model with thermal dissipation [16], fluid dynamical convection [2], and stellar ionization zone oscillations [3]. Jerk equations, even if not nearly as common as acceleration (or force) equations $\ddot{x} = f(x, \ddot{x})$, are therefore of direct physical interest. Moreover, simple forms of the jerk function J which lead to maybe the simplest manifestation if chaoses have been found by Sprott [28].

In this article we have investigated not chaotic solutions to jerk equations (as many of the above references do), but the analytical approximate periodic solutions and corresponding frequencies of some nonlinear jerk equations using the iteration method.

Ref

[7] Gottlieb, H.P.W.: *Question #38. What is the simplest jerk function that gives chaos?* American Journal of Physics, vol.64, pp. 525, 1996.

Gottlieb [8] used the lowest order harmonic balance method to calculate approximations to the periodic solution and the angular frequencies obtained by Gottlieb were not accurate enough. But it is very difficult to construct the higher order approximation by harmonic balance method for the reason that the method requires analytical solutions of a set of complicated nonlinear algebraic equations. Wu et al. [29] and Leung et al. [13] applied, respectively, an improved harmonic balance approach and a residue harmonic balance approach to solve nonlinear jerk equations, and their higher order approximations give accurate results to a large range of initial velocity amplitudes. Ma et al. [15] and Hu [11] used, respectively, Homotopy perturbation method and parameter perturbation method to determine the high order approximate solutions of nonlinear jerk equations, and their results obtained are more accurate than those obtained by the low order harmonic balance method. Recently, Ramos [21-23] presented an order reduction method, two-level iterative procedure and a volterra integral formulation, respectively, to solve nonlinear jerk equations. He found that the second reduction method provides accurate solutions only for initial velocities close to unity, the third reduction method produces very accurate for the first and second differential equation in [21], the fourth reduction method provides as accurate results as or more accurate results obtained by parameter perturbation method. Hu et al. [12] presented the modified Mickens iteration procedure for a nonlinear jerk equation, and the second order approximate angular frequency was obtained by Newton's method. Leung and Guo [13] has established a residue harmonic balance approach for determining limit cycles to parity- and time-reversal invariant general non-linear jerk equations with cubic nonlinearities.

To obtain periodic solutions for jerk-oscillators certain restrictions must be placed on the mathematical structure of the oscillator. We only consider the following cubic nonlinear functions investigated by Gottlieb [8]:

$$x\,\dot{x}\,\ddot{x}\,. \tag{2}$$

The most general jerk function with invariance of time- and space-reversal and which has only cubic nonlinearities may be written in the form [8]

$$\dddot{x} = \alpha\, x\,\dot{x}\,\ddot{x} - \beta\,\dot{x}\,\ddot{x}^2 - \gamma\,\dot{x} - \delta\, x^2\,\dot{x} - \varepsilon\,\dot{x}^3\,. \tag{3}$$

Ref

[8] Gottlieb, H.P.W.: *Harmonic balance approach to periodic solutions of nonlinear jerk equation*, J. Sound Vib., vol. 271, pp. 671-683, 2004.

where an over-dot denotes the time derivative and the parameters α, β, γ, δ and ε are given real constants. The corresponding initial conditions are

$$x(0)=0,\ \dot{x}(0)=A \text{ and } \ddot{x}(0)=0. \tag{4}$$

These three initial conditions in Eq. (4) are to satisfy the periodic requirement. Here, at least one of α, β, δ and ε should be non-zero.

For $\alpha=1$ with cubic nonlinearity includes $x\dot{x}\ddot{x}$ only. The resulting standard jerk equation, after rescaling of both x and t, is taken to be

$$\dddot{x}+\dot{x}-x\dot{x}\ddot{x}=0, x(0)=0,\ \dot{x}(0)=A,\ \ddot{x}(0)=0. \tag{5}$$

In this article, we present a new approach of the Mickens' iteration method [16] for the determination of approximate solutions of some nonlinear jerk equations. Here we have defined the function which contains dependent variable, its derivatives of first and second order also the integrals of the dependent variable. Previously the function contains only dependent variable and its derivatives of first and second order. It is mentioned that our method is valid for second and higher order period of oscillations and show a good agreement compared to other existing solutions.

II. The Method

Let us consider a nonlinear oscillator modeled by

$$\ddot{x}+f(\ddot{x},\dot{x},x',x)=0, \quad x(0)=A, \quad \dot{x}(0)=0, \tag{6}$$

where over dots denote differentiation with respect to time, t over dash denotes integration with respect to time, t.

We choose the frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (6), we obtain

$$\ddot{x}+\Omega^2 x=\Omega^2 x-f(x,x',\dot{x},\ddot{x})\equiv G(x,x',\dot{x},\ddot{x}). \tag{7}$$

Following [18], we formulate the iteration scheme as

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=G(x_k,x'_k,\dot{x}_k,\ddot{x}_k); \quad k=0,1,2,\cdots, \tag{8}$$

Ref

[16] Moore, D.W. and Spigel, E.A.: A *thermally excited nonlinear oscillator*, *The Astrophysical Journal*, vol. 143, pp. 871-887, 1966.

together with

$$x_0(t) = A\cos(\Omega_0 t) \ . \tag{9}$$

Herein x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0 \, . \tag{10}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms (see [19] for details) should not occur in the solution. This procedure gives the sequence of solutions: $x_0(t), x_1(t), \cdots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [18].

III. Solution Procedure

Here we have considered the particular Jerk function $x\,\dot{x}\,\ddot{x} - \dot{x}$

i.e. Jerk function containing displacement times velocity time's acceleration, and velocity:

Let us consider the nonlinear jerk oscillator [8]

$$\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x} \ . \tag{11}$$

Introducing the phase space variable (y,t) by the relation $\dot{x} = y$ then Eq. (11) becomes

$$\ddot{y} + y = y\,\dot{y}\,y' \, . \tag{12}$$

Obviously, Eq. (12) can be written as

$$\ddot{y} + \Omega^2 y = (\Omega^2 - 1 + \dot{y}\,y')\,y \, . \tag{13}$$

Now the iteration scheme is (according to Eq. (8))

$$\ddot{y}_{k+1} + \Omega_k^2\, y_{k+1} = (\Omega_k^2 \, - 1 - \dot{y}_k\, y_k')\, y_k \, . \tag{14}$$

Equation (9) is rewritten as

$$y_0 = y_0(t) = A\cos\theta \ , \tag{15}$$

[19] Nayfeh, A.H.: *Perturbation Method*, John Wiley & Sons, New York, 1973.

where $\theta = \Omega t$. For $k = 0$, the Eq. (14) becomes

$$\ddot{y}_1 + \Omega_0^{\,2}\, y_1 = (\Omega_0^{\,2} - 1 - \dot{y}_0\, y_0')\, y_0 \,. \tag{16}$$

Substituting the initial function Eq. (15) into the right hand side of Eq. (16) and expanding in a Cosine series, we obtain

$$\ddot{y}_1 + \Omega_0^{\,2}\, y_1 = a_1 \cos\theta + a_3 \cos 3\theta \,. \tag{17}$$

where,

$$a_1 = \frac{A}{4}(-4 - A^2 + 4\,\Omega_0^{\,2}), \quad a_3 = \frac{1}{4} A^3 \quad . \tag{18}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (17). Thus we have

$$\Omega_0^{\,2} = 1 + \frac{A^2}{4} \quad . \tag{19}$$

Then solving Eq. (17) and satisfying the initial condition $y_1(0) = A$, we obtain

$$y_1(t) = \left(A + \frac{a_3}{8\,\Omega_0^{\,2}} \right) \cos\theta - \frac{a_3}{8\,\Omega_0^{\,2}} \cos 3\theta \quad . \tag{20}$$

This is the first approximate solution of Eq. (12) and the related Ω_1 is to be determined. The value of Ω_1 will be obtained from the solution of

$$\ddot{y}_2 + \Omega_1^{\,2}\, y_2 = (\Omega_1^{\,2} - 1 - \dot{y}_1\, y_1')\, y_1 \,. \tag{21}$$

Substituting $y_1(t)$ from Eq. (20) into the right hand side of Eq. (21) and then expanding in a truncated Cosine series, we obtain

$$\ddot{y}_2 + \Omega_1^2\, y_2 = \sum_{r=1}^{4} b_{2r-1} \cos(2r-1)\theta \,. \tag{22}$$

where,

$$b_1 = -\frac{A(32 + 9A^2)}{3072\,(4 + A^2)^3} \{ 6144 + 4608A^2 + 1136A^4 + 93A^6 - 384\,(4 + A^2)^2\, \Omega_1^2 \}$$

Notes

$$b_3 = \frac{A^3(32+9A^2)}{512(4+A^2)^3}\{9216+7936A^2+2296A^4+223A^6-64(4+A^2)^2\,\Omega_1^2\}$$

$$b_5 = -\frac{A^5}{1536(4+A^2)^3}\{3328+1928A^2+279A^4\}$$

Notes

$$b_7 = \frac{13A^7}{6144(4+A^2)^3}(32+9A^2) \tag{23}$$

Again avoiding secular terms in the solution of Eq. (22), we obtain

$$\Omega_1^2 = \frac{(6144+4608A^2+1136A^4+93A^6)}{384(4+A^2)^2}. \tag{24}$$

Then solving Eq. (22) and satisfying initial condition, we obtain the second approximate solution,

$$y_2(t) = \left(A + \frac{1}{\Omega_1{}^2}\left(\frac{b_3}{8}+\frac{b_5}{24}+\frac{b_7}{48}\right)\right)\cos\theta - \frac{1}{\Omega_1{}^2}\left(\frac{b_3}{8}\cos 3\theta + \frac{b_5}{24}\cos 5\theta + \frac{b_7}{48}\cos 7\theta\right). \tag{25}$$

This is the second approximate solution of Eq. (12).

The third approximation y_3 and the value of Ω_2 will be obtained from the solution of

$$\ddot{y}_3 + \Omega_2{}^2\, y_3 = (\Omega_2{}^2 - 1 - \dot{y}_2\, y_2')\, y_2\,. \tag{26}$$

After substituting the second approximate solution $y_2(t)$ of Eq. (12) from Eq. (25) into the right hand side of Eq. (26) and avoiding secular terms in the solution, we obtain

$$\begin{aligned}\Omega_2^2 = \{&768(14843406974976+48704929136640A^2+73635603677184A^4\\&+67914279419904A^6+213141799305216A^8/5+2884018883133344A^{10}/15\\&+6409791774720A^{12}+100727452289312A^{14}/63+40272237000484A^{16}/135\\&+20716379322491A^{18}/504+2621464123311569A^{20}/645120\\&+5631142163784439A^{22}/20643840+817336388391143A^{24}/73400320\\&+489874066020999A^{26}/2348810240)\}/\{(24576+24576A^2+9152A^4\\&+1508A^6+93A^8)^2(18874368+19464192A^2+7518208A^4+1294816A^6\\&+84249A^6)\}\end{aligned} \tag{27}$$

In a similar way the method can be proceeded higher order approximations and Ω_0, Ω_1, Ω_2, … respectively obtained by Eqs. (19), (24), (27), … represent the approximation of frequencies of oscillator (11). However due to growing algebraic complexity, most of the approximate techniques are applied to second or third approximations.

IV. Results and Discussions

An iteration method is developed based on Mickens [16] iteration method to solve a class of nonlinear jerk equations. In this section, we express the accuracy of the modified approach of iteration method by comparing with the existing results from different methods and with the exact results of the nonlinear jerk equations. To show the accuracy, we have calculated the percentage errors (denoted by Er(%)) by the definitions $|100(T_e - T_i)/T_e|$, where $T_i = 2\pi/\Omega_i; i = 0, 1, 2, \cdots$ represents the approximate periods obtained by the present method and T_e represents the corresponding exact period of the oscillator.

Now we demonstrate the comparison of results for the oscillator.

i.e., For the Jerk function containing displacement time's velocity time's acceleration, and velocity $x\dot{x}\ddot{x} - \dot{x}$.

Recently, Gottlieb [8], Ma *et al.* [15], Ramos [21] and Leung & Guo [13] has found approximate solutions, frequencies and as well as approximate periods of nonlinear jerk oscillator (given by Eq. (11)) by different methods other than Iteration method.

We have used a modified iteration procedure to obtaining approximate solutions of the oscillator. The procedure is very simple. It has been shown that, in most of the cases our solution gives significantly better result than other existing results and sometimes it is almost similar to other existing results.

Herein we have calculated the first second and third approximate frequencies which are denoted by Ω_1, Ω_2 and Ω_3 respectively and corresponding periods are T_1, T_2 and T_3. All the results are given in Table 5.a and Table 5.b, to compare the approximate frequencies we have also given the existing results determined separately by Gottlieb [8], Ma *et al.* [15], Ramos [21] and Leung & Guo [13] respectively.

Ref

[15] Ma, X., Wei, L. and Guo, Z.: *He's homotopy perturbation method to periodic solutions of nonlinear jerk equations*, J. Sound Vib., vol. 314, pp. 217-227, 2008.

Notes

Table 5.a

Comparison of the approximate periods with exact periods T_e [8] of $\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}$:

A	T_e	Modified T_0 Er(%)	Modified T_1 Er(%)	Modified T_2 Er(%)
0.1	6.275347	6.275346 $1.56\ e^{-5}$	6.275347 $2.65\ e^{-6}$	6.275347 $2.60\ e^{-6}$
0.2	6.252016	6.252003 $2.07\ e^{-4}$	6.252016 $3.35\ e^{-6}$	6.252016 $1.42\ e^{-7}$
0.5	6.096061	6.095585 $7.8\ e^{-3}$	6.096018 $7.0\ e^{-4}$	6.096060 $2.1\ e^{-5}$
1	5.626007	5.619852 $1.09\ e^{-1}$	5.624306 $3.0\ e^{-2}$	5.625880 $2.3\ e^{-3}$
2	4.491214	4.442883	4.463270	4.484913
		1.08	$6.2\ e^{-1}$	$1.4\ e^{-1}$

T_0 , T_1 and T_2 respectively denote initial, first and second modified approximate periods. Er(%) denotes percentage error.

Table 5.b

Comparison of the approximate periods obtained by our method and other existing result with exact periods T_e [8] of $\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}$:

A	T_e	Modified T_2 Er(%)	Gottlieb T_{G2} (2004) [8] Er(%)	Ma et al. T_{M2} (2008) [15] Er(%)	Ramos T_{R2} (2010) [21] Er(%)	Leung-Guo T_{LG2} (2011) [13] Er(%)
0.1	6.275347	6.275347 $2.60\ e^{-6}$	6.275346 $1.3\ e^{-5}$	6.275347 $2.5\ e^{-6}$	6.275347 $9.0\ e^{-8}$	6.275347 $2.6\ e^{-6}$
0.2	6.252016	6.252016 $1.42\ e^{-7}$	6.252003 $2.11\ e^{-4}$	6.252016 $1.6\ e^{-7}$	6.252016 $6.4\ e^{-6}$	6.252016 $1.6\ e^{-7}$
0.5	6.096061	6.096060 $2.1\ e^{-5}$	6.095585 $7.8\ e^{-3}$	6.096059 $3.21\ e^{-5}$	6.096025 $6.06\ e^{-4}$	6.096061 $8.2\ e^{-6}$
1	5.626007	5.625880 $2.3\ e^{-3}$	5.619852 $1.09\ e^{-1}$	5.625795 $3.8\ e^{-3}$	5.624539 $2.6\ e^{-2}$	5.625993 $2.5\ e^{-4}$
2	4.491214	4.484913 $1.4\ e^{-1}$	4.442883 1.08	4.482081 2.03	4.466144 $5.6\ e^{-1}$	4.490125 $2.4\ e^{-2}$

T_2 denotes second modified approximate periods; T_{G2} , T_{M2} , T_{R2} and T_{LG2} respectively denote second approximate periods obtained by Gottlieb, Ma et al., Ramos and Leung-Guo. Er(%) denotes percentage error.

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