Analytic and Numeric Solution of Linear Partial Differential Equation of Fractional Order

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Abstract - The existence and uniqueness solution of the Cauchy problem are discussed and proved in a Banach space E due to Bielecki method and Picard method depending on the properties we expect a solution to possess. Moreover, some properties concerning the stability of solution are obtained. The product Nyström method is used as a numerical method to obtain a linear system of algebraic equations. Also, many important theorems related to the existence and uniqueness solution of the algebraic system are derived. Finally, an application is given and numerical results are obtained.

Keywords : linear partial differential equation of fractional order, semigroup, linear algebraic system, product Nyström method.

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Analytic and Numeric Solution of Linear Partial Differential Equation of Fractional Order

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Abstract - The existence and uniqueness solution of the Cauchy problem are discussed and proved in a Banach space $E$ due to Bielecki method and Picard method depending on the properties we expect a solution to possess. Moreover, some properties concerning the stability of solution are obtained. The product Nyström method is used as a numerical method to obtain a linear system of algebraic equations. Also, many important theorems related to the existence and uniqueness solution of the algebraic system are derived. Finally, an application is given and numerical results are obtained.

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1. Introduction

The use of semigroups methods for partial differential equations has had a long history starting with the works of Feller [1], Hille [2], and Yosida [3]. The basic results of the semigroup theory may be considered as a natural generalization of theorem of M. Stone on one-parameter group of unitary operators in a Hilbert space (see Yosida [4]). Also, the semigroups play a special role in applications, for example they describe how densities of initial states evolve in time. Moreover, there are equations which generate semigroups. These equations appear in such diverse areas as astrophysics-fluctuations in the brightness of the Milky Way [5], population dynamics [6,7], and in the theory of jump processes.

In [8], Mijatovie and Pilipovie introduced and analyzed $\alpha$-times integrated semigroups for $\alpha \in (1/2, 1)$. In [9], El-Borai studied the Cauchy problem in a Banach space $E$ for a linear fractional evolution equation. In his paper, the existence and uniqueness of the solution of the Cauchy problem were discussed and proved. Also, the solution was obtained in terms of some probability densities. In [10], El-Borai discussed the existence and uniqueness solution of the nonlinear Cauchy problem.

In this work, we treat the following Cauchy problem of the fractional evolution equation:

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\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = Au(x,t) + B(t)u(x,t) \quad ; \quad 0 < \alpha \leq 1,
\]
with the initial condition: \( u(x,0) = u_0(x) \).

In a Banach space \( E \). Here \( u(x,t) \) is an \( E \)-valued function on \( E \times [0,T] \), \( T < \infty \). \( A \) is a linear closed operator defined on a dense set \( S_1 \) in \( E \) into \( E \), \( \{ B(t), t \in [0,T] \} \) is a family of linear closed operators defined on a dense set \( S_2 \supset S_1 \) in \( E \) into \( E \), and \( u_0(x) \in E \) and \( 0 < \alpha \leq 1 \).

II. Linear Fractional Evolution Equation

Consider the Cauchy problem of the fractional evolution equation

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = Au(x,t) + B(t)u(x,t) \quad ; \quad 0 < \alpha \leq 1,
\]
with the initial condition: \( u(x,0) = u_0(x) \),

in a Banach space \( E \), where \( u(x,t) \) is an \( E \)-valued function on \( E \times [0,T] \). \( T < \infty \), \( A \) is a linear closed operator defined on a dense set \( S_1 \) in \( E \) into \( E \), \( \{ B(t), t \in [0,T] \} \) is a family of linear closed operators defined on a dense set \( S_2 \supset S_1 \) in \( E \) into \( E \), and \( u_0(x) \in E \).

It is assumed that \( A \) generates an analytic semigroup \( Q(t) \). This condition implies:

\[
\|Q(t)\| \leq k \text{ for } t \geq 0 \text{ and } \|AQ(t)\| \leq \frac{k}{t} \text{ for } t > 0,
\]
where \( \| \cdot \| \) is the norm in \( E \) and \( k \) is a positive constant (Zaidman [11]).

Let us suppose that \( B(t)g \) is uniformly Hölder continuous in \( t \in [0,T] \), for every \( g \in S_1 \) ; that is

\[
\|B(t_2)g - B(t_1)g\| \leq k_1(t_2 - t_1)^\beta,
\]
for all \( t_2 > t_1, t_1, t_2 \in [0,T] \), where \( k_1 \) and \( \beta \) are positive constants, \( \beta \leq 1 \).

We suppose also that there exists a number \( \gamma \in (0,1) \), such that

\[
\|B(t_2)Q(t_1)h\| \leq \frac{k_2}{t_1^\gamma} \|h\|,
\]
where \( t_1 > 0, t_2 \in [0,T], h \in E \) and \( k_2 \) is a positive constant (El-Borai [9,12,13]).
(Notice that \( Q(t)h \in S_1 \) for each \( h \in E \) and each \( t > 0 \)).

Following Gelfand and Shilov ([14],[15]), we can define the integral of order \( \alpha > 0 \) by

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta) d\theta.
\]

If \( 0 < \alpha < 1 \), we can define the derivative of order \( \alpha \) by

\[
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^\alpha} d\theta \quad , \quad \left( f'(\theta) = \frac{df(\theta)}{d\theta} \right)
\]
where \( f \) is an abstract function with values in \( E \).
Let $C_E(E \times [0,T])$ be the set of all continuous functions $u(x,t) \in E$. We define on $C_E(E \times [0,T])$ a norm by $\|u(x,t)\|_{C(E \times [0,T])} = \max_{x,t} \|u(x,t)\|_E$, $\forall t \in [0,T], x \in E$. By a solution of the Cauchy problem (2.1), (2.2), we mean an abstract function $u(x,t)$ such that the following conditions are satisfied:

(a) $u(x,t) \in C_E(E \times [0,T])$ and $u(x,t) \in S_1$ for all $t \in [0,T], x \in E$.

(b) $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ exists and is continuous on $E \times [0,T]$, where $0 < \alpha < 1$.

(c) $u(x,t)$ satisfies (2.1) with the initial condition (2.2) on $E \times [0,T]$.

Now, it is suitable to rewrite the Cauchy problem (2.1), (2.2), in the form

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} Au(x,\theta) d\theta$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} B(\theta) u(x,\theta) d\theta,$$

where the solution of (2.6) is equivalent to the solution of Cauchy problem (2.1), (2.2)

### III. THE EXISTENCE AND UNIQUENESS SOLUTION OF LINEAR FRACTIONAL EVOLUTION EQUATION

In this section, the existence and uniqueness solution of (2.6) and consequently its equivalent Cauchy problem (2.1), (2.2), will be discussed and proved in a Banach $E$ by two different ways. The first way is due to "Bielecki method", and the second is "Picard method" depending on the properties we expect a solution to possess.

#### a) Modified Bielecki Method

Here, we will generalize the technique of Bielecki method to obtain the existence and uniqueness solution of (2.6) in $E$, for $0 < \alpha \leq 1$.

It’s suitable to consider the following lemma.

**Lemma 1:**

If $\lambda > 1$ and $0 < \delta < 1$, then

$$\int_0^t (t-\eta)^{\delta-1} d\eta \leq \left(\frac{1}{\lambda}\right)^{\delta-1} t,$$  \hspace{1cm} (3.1)

and

$$\int_0^t e^{\lambda t}(t-\eta)^{\delta-1} d\eta \leq (\frac{1}{\delta})^{\delta}[1 + \frac{1}{\delta}] e^{\lambda t}.  \hspace{1cm} (3.2)$$

**Theorem 1:**

If $A$ and $B$ are linear bounded operators in a Banach space $E$, and $0 < \alpha \leq 1$, then (2.6) has a unique solution in $E$.

**Proof:**

Let $K$ be an operator defined by
\[ K u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} A u(x,\theta) d\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} B(\theta) u(x,\theta) d\theta. \]  

(3.3)

Hence, we have

\[
\| K u(x,t) \| \leq \| u_0(x) \| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} \| A u(x,\theta) \| d\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} \| B(\theta) u(x,\theta) \| d\theta. 
\]

(3.4)

Since \( A \) and \( B \) are bounded operators, there exists positive constants \( L \) and \( M \) such that

\[
\| A u(x,t) \| \leq L \| u(x,t) \|, \text{ and } \| B(t) u(x,t) \| \leq M \| u(x,t) \|. 
\]

(3.5)

In the light of (3.5), (3.4) takes the form

\[
\| K u(x,t) \| \leq \| u_0(x) \| + \frac{(L+M)}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} \| u(x,\theta) \| d\theta. 
\]

(3.6)

Using (3.1) in (3.6), we get

\[
\| K u(x,t) \| \leq \| u_0(x) \| + \frac{(L+M)}{\Gamma(\alpha)} \left( \frac{1}{\lambda} \right)^{\alpha-1} T \| u(x,t) \|, \text{ where } T = \max_{0 \leq t \leq T}. 
\]

(3.7)

Inequality (3.7) shows that, the operator \( K \) maps the ball \( B_r \subset E \) into itself, where

\[
r = \frac{\sigma}{1-\delta}, \quad \sigma = \| u_0(x) \|, \delta = \frac{(L+M)}{\Gamma(\alpha)} \left( \frac{1}{\lambda} \right)^{\alpha-1} T.
\]

Since \( r > 0, \sigma > 0 \), therefore \( \delta < 1 \). Also, the inequality (3.7) involves the boundedness of the operator \( K \).

For the two functions \( u(x,t) \) and \( v(x,t) \) in \( E \), the formula (3.3) leads to

\[
\| K u(x,t) - K v(x,t) \| \leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^t (t-\theta)^{\alpha-1} \| A(u(x,\theta) - v(x,\theta)) \| d\theta + \int_0^t (t-\theta)^{\alpha-1} \| B(\theta)(u(x,\theta) - v(x,\theta)) \| d\theta \right\}. 
\]

(3.8)

Using (3.5) in (3.8), we have for \( \lambda > 1 \)

\[
\| K u(x,t) - K v(x,t) \| \leq \frac{(L+M)}{\Gamma(\alpha)} \max_{x,t} \left\{ e^{-\lambda(t+x)} \| u(x,t) - v(x,t) \| \left\| \int_0^t (t-\theta)^{\alpha-1} e^{\lambda(\theta+x)} d\theta \right\| \right\}. 
\]
Using (3.2), the above inequality becomes
\[
\max_{x,t} \left\{ e^{-\lambda(t+s)} \| K u(x,t) - K v(x,t) \| \right\} \leq 
\sigma \max_{x,t} \left\{ e^{-\lambda(t+s)} \| u(x,t) - v(x,t) \| \right\},
\]
where
\[
\sigma = \frac{(L + M)}{\Gamma(\alpha)} \left[ \frac{1}{\lambda} \right]^{\alpha} \left[ 1 + \frac{1}{\alpha} \right].
\]

Inequality (3.9) can be adapted in the form
\[
d(K u(x,t), K v(x,t)) \leq \sigma d(u(x,t), v(x,t)).
\]

If we choose \( \lambda \) sufficiently large, then \( \sigma < 1 \), and \( d \) is a contraction mapping. By Banach fixed point theorem, \( K \) has a unique fixed point which is the unique solution of (2.6).

**b) Semigroup Method**

To obtain the solution of the Cauchy problem (2.1), (2.2) in the dense set \( S \) in \( E \), we can follow the work of El-Borai [9]. Hence, the Cauchy problem (2.1), (2.2), and (2.6) are equivalent to the following integral equation
\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\theta \alpha} \zeta_{\alpha}(\theta) Q(t \alpha \theta) w(x,\eta) d\theta d\eta + \int_{0}^{\infty} \zeta_{\alpha}(\theta) Q(t \alpha \theta) u_0(x) d\theta,
\]
where \( \zeta_{\alpha}(\theta) \) is a probability density function defined on \( (0, \infty) \), and \( w(x,t) = B(t) u(x,t), \ (x,t) \in (E \times [0,T]) \).

The integral equation (3.10) represents a Volterra equation of the second kind with Abel kernel, where the first term of the R.H.S is known and continuous. The integral equation will be solved numerically in the next section.

Now, we will prove that (3.10) has a unique solution which represents the required solution of the Cauchy problem (2.1), (2.2).

**Theorem 2:**
The Cauchy problem (2.1), (2.2) has a unique solution in \( C_E(E \times [0,T]) \).
The proof of this theorem depends on the following lemmas.

**Lemma 2:**
Under the condition (2.5), (3.10) has a solution in the space \( C_E(E \times [0,T]) \).

**Proof:**
Consider the following integral equation
\[
w(x,t) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\theta \alpha} \zeta_{\alpha}(\theta) B(t) Q((t \alpha \theta) w(x,\eta) d\theta d\eta + \int_{0}^{\infty} \zeta_{\alpha}(\theta) B(t) Q(t \alpha \theta) u_0(x) d\theta.
\]

Using the method of successive approximations, we set

\[ w_{n+1}(x,t) = \int_0^\infty \xi_\alpha(\theta) B(t) Q(\theta^\alpha) u_0(x) d\theta + \]

\[ \alpha \int_0^\infty \theta (t-\theta)^{\alpha-1} \xi_\alpha(\theta) B(t) Q((t-\theta)^\alpha) w_n(x,\eta) d\theta d\eta . \]

Thus, we have

\[ \|w_1(x,t) - w_0(x,t)\| \leq \]

\[ \alpha \int_0^\infty \theta (t-\theta)^{\alpha-1} \xi_\alpha(\theta) \left\| B(t) Q((t-\theta)^\alpha) \right\| [w_1(x,\eta) - w_0(x,\eta)] d\theta d\eta , \]

where \( w_0(x,t) \) is the zero element in \( E \).

In view of the condition (2.5), we get

\[ \|w_2(x,t) - w_1(x,t)\| \leq \]

\[ \alpha \int_0^\infty \theta (t-\theta)^{\alpha-1} \xi_\alpha(\theta) \left\| B(t) Q((t-\theta)^\alpha) \right\| [w_1(x,\eta) - w_0(x,\eta)] d\theta d\eta . \]

The above inequality for \( u_0(x) \in S_1 \) can be adapted in the form

\[ \|w_2(x,t) - w_1(x,t)\| \leq \frac{\mu t^\nu}{\nu} , \]

where,

\[ \mu = \alpha k_2 \int_0^\infty \theta^{\alpha-\gamma} \xi_\alpha(\theta) \sup_{t,\theta} \left\| \xi_\alpha(\theta) B(\eta) Q(\eta^\alpha) u_0(x) d\theta \right\| d\theta , \]

(3.13)

\[ \nu = \alpha (1-\gamma) . \]

By induction, we obtain

\[ \|w_{n+1}(x,t) - w_n(x,t)\| \leq \frac{\mu t^\nu (\Gamma(\nu))^{n}}{\Gamma(n\nu + 1)} . \]

Thus, the series \( \sum_{i=0}^\infty \|w_{i+1}(x,t) - w_i(x,t)\| \) converges uniformly on \( E \times [0,T] \), under the condition, \( \mu t^\nu \Gamma(\nu) < 1 \).

Since \( w_{n+1}(x,t) = \sum_{i=0}^n [w_{i+1}(x,t) - w_i(x,t)] \), it follows that the sequence \( \{w_n(x,t)\} \) converges uniformly in the space \( C_E(E \times [0,T]) \) to a continuous function \( w(x,t) \) which satisfies (3.11), consequently \( u(x,t) \in C_E(E \times [0,T]) \).

Lemma 3:

Under the condition (2.5), (3.10) has a unique solution in the space \( C_E(E \times [0,T]) \).

Proof:

For the two functions \( w_1(x,t) \) and \( w_2(x,t) \) in the space \( C_E(E \times [0,T]) \), the formula (3.11) with the aid of condition (2.5), leads to
\[ \|w_2(x,t) - w_1(x,t)\| \leq \mu \frac{\rho}{0} (t-\eta)^{\nu-1}\|w_2(x,t) - w_1(x,t)\| d\eta . \]

Consequently,
\[ \|w_2(x,t) - w_1(x,t)\| \leq \mu \rho \int_{0}^{\eta} e^{\lambda(t-\eta)} (t-\eta)^{\nu-1} d\eta , \] (3.14)

where, \( \rho = \max_{x,t} [e^{-\lambda(t+x)}\|w_2(x,t) - w_1(x,t)\|], \) and \( \lambda > 1. \)

Using (3.2) in (3.14), we get
\[ \|w_2(x,t) - w_1(x,t)\| \leq \mu \rho \left(\frac{1}{\lambda}\right)^\nu [1 + \frac{1}{\nu}] e^{\lambda(t+x)}. \]

Thus, we have
\[ \max_{x,t} [e^{-\lambda(t+x)}\|w_2(x,t) - w_1(x,t)\|] \leq \mu \rho \left(\frac{1}{\lambda}\right)^\nu [1 + \frac{1}{\nu}]. \]

We can choose \( \lambda \) sufficiently large such that
\[ \mu \left(\frac{1}{\lambda}\right)^\nu [1 + \frac{1}{\nu}] = \mu_1 < 1. \]

Thus,
\[ \rho = \max_{x,t} [e^{-\lambda(t+x)}\|w_2(x,t) - w_1(x,t)\|] = 0. \]

This completes the proof of the lemma.

Lemma 4:
Under the conditions (2.4) and (2.5), the solution \( w(x,t) \) of (3.11) satisfies a uniform Hölder condition. (El-Borai [9])

Proof of Theorem 2:
By virtue of lemmas (3) and (4), we deduce that, the function \( u(x,t) \in S_t \) and represents the unique solution of Cauchy problem (2.1), (2.2) in the space \( C_E (E \times [0,T]) \).

Corollary 1:
The integral equation (3.10) has a unique solution in the Banach space \( C_\delta (\mathbb{R} \times [0,T]). \)

Now, we will prove the stability of the solutions of the Cauchy problem (2.1), (2.2). In other words, we will show that the Cauchy problem (2.1), (2.2) is correctly formulated.

Theorem 3:
Let \( \{u_n(x,t)\} \) be a sequence of functions, each of which is a solution of (2.1) with the initial condition \( u_n(x,0) = g_n(x), \) where \( g_n(x) \in S_t (n = 1,2,\ldots) \). If the sequence \( \{g_n(x)\} \) converges to an element \( u_0(x) \in S_t, \) the sequence \( \{A g_n(x)\} \) converges and the sequence \( \{B(t)g_n(x)\} \) converges uniformly on \( E \times [0,T]. \) Then, the sequence of solutions \( \{u_n(x,t)\} \) converges uniformly on \( E \times [0,T] \) to a limit function \( u(x,t), \) which is the solution of the Cauchy problem (2.1), (2.2).

Proof:
Consider the sequences \( \{z_n(x,t)\} \) and \( \{u_n^*(x,t)\}, \) where
\[ \frac{\partial^{\alpha} u_n^*(x,t)}{\partial t^{\alpha}} - Au_n^*(x,t) = z_n(x,t), \]
In view of the conditions (2.5) and (3.13), we get
\[
\left\| z_n(t,x) - z_m(t,x) \right\| \leq \mu \int_0^t (t-\eta)^{\nu-1} \left\| z_n(\eta,x) - z_m(\eta,x) \right\| d\eta
\]
\[
+ \left\| B(t)u_n(x) - B(t)u_m(x) \right\| + \left\| A g_n(x) - A g_m(x) \right\|
\]

Given \( \varepsilon > 0 \), we can find a positive integer \( N = N(\varepsilon) \), such that
\[
\left\| z_n(t,x) - z_m(t,x) \right\| \leq \mu \int_0^t (t-\eta)^{\nu-1} \left\| z_n(\eta,x) - z_m(\eta,x) \right\| d\eta + (1-\mu_1)\varepsilon
\]
for all \( n \geq N, m \geq N \) and \((x,t) \in E \times [0,T]\).

Using (3.2), the above inequality takes the form
\[
(1-\mu_1)e^{-\lambda(t+\varepsilon)} \left\| z_n(t,x) - z_m(t,x) \right\| \leq (1-\mu_1)e^{-\lambda(\varepsilon)}\varepsilon
\]

Thus, for sufficiently large \( \lambda \), we get
\[
\max_{x,t} \left\{ e^{-\lambda(t+\varepsilon)} \left\| z_n(t,x) - z_m(t,x) \right\| \right\} \leq \varepsilon
\]

Since \( E \) is a complete space, it follows that the sequence \( \{z_n(t,x)\} \) converges uniformly on \( E \times [0,T] \) to a continuous function \( z(x,t) \), so the sequence \( \{u_n(t,x)\} \) converges uniformly on \( E \times [0,T] \) to a continuous function \( u^*(x,t) \). It can be proved that \( z(x,t) \) satisfies a uniform Hölder condition on \([0,T] \), thus \( u^*(x,t) \in S_1 \).

IV. The Numerical Solution of Linear Fractional Evolution Equation

In this section, we will use the product Nyström method (Linz [17], and Dzhuraev [18]), to obtain numerically, the solution of the Cauchy problem (2.1), (2.2), in the Banach space \( \mathcal{C}^1(\mathbb{R} \times [0,T]) \), where \( \| u(x,t) \|_{\mathcal{C}^1(\mathbb{R} \times [0,T])} = \max_{x,t} |u(x,t)|, \forall t \in [0,T], -\infty < x < \infty \). For this, the integral equation (3.10) can be written in the form
\[
u(x,t) = f^*(x,t) + \alpha \int_0^t p(t,\eta) Q^*(t,\eta) B(\eta) u(x,\eta) \, d\eta,
\]

(4.1)
where,

\[ f^*(x,t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0(x) d\theta, \quad (4.2) \]

and the bad kernel

\[ p(t,\eta) = (\eta-t)^{\alpha-1}, \quad (0 < \alpha < 1, 0 \leq \eta \leq t \leq T; \ T < \infty). \quad (4.4) \]

Here, the unknown function \( u(x,t) \in C_\mathbb{R}(\mathbb{R} \times [0,T]) \), while \( f^*(x,t), Q^*(t,\eta) \) and \( p(t,\eta) \) are known functions and satisfy the following conditions:

1. \( f^*(x,t) \) is a continuous function in \( (\mathbb{R} \times [0,T]) \).
2. \( Q^*(t,\eta) \) with its partial derivatives are continuous functions in \([0,T]\).
3. \( p(t,\eta) \) is a badly behaved function of its arguments such that:
   a. for each continuous function \( u(x,t) \) and \( 0 \leq t_1 \leq t_2 \leq t \), the integrals
      \[ \int_{t_1}^{t_2} p(t,\eta)Q^*(t,\eta) B(\eta)u(x,\eta) d\eta, \]
      and
      \[ \int_0^{t} p(t,\eta)Q^*(t,\eta) B(\eta)u(x,\eta) d\eta, \]
      are continuous functions in \( (\mathbb{R} \times [0,T]) \).
   b. \( p(t,\eta) \) is absolutely integrable with respect to \( \eta \) for all \( 0 \leq t \leq T \).

Remark 1:

By virtue of corollary (1), the integral equation (4.1) has a unique solution in the Banach space \( C_\mathbb{R}(\mathbb{R} \times [0,T]) \).

Now, we will apply the product Nyström method, to obtain numerically, the solution of (4.1). Therefore, putting \( t = t_i = \eta_i = x_i = x, \ t_i = ih, \ h = t_{i+1} - t_i \quad (i = 0,1,\ldots,N \text{ and } N \text{ is even}) \), and using the following notations

\[
\begin{align*}
    u_{ij} &= u(t_i, x_i), \quad Q_{ij}^* = Q^*(t_i, \eta_j), \quad f_{ij}^* = f^*(t_i, x_i), \\
    B_j &= B(\eta_j),
\end{align*}
\]

we get the following linear algebraic system

\[ u_{ij} = f_{ij}^* + \sum_{j=0}^N w_{ij,j} Q_{ij,j}^* B_j u_{ij,j}, \quad (i = 0,1,2,\ldots,N) \quad (4.6) \]

where,

\[
\begin{align*}
    w_{i,0} &= \beta_i(t_i), \quad w_{i,2j+1} = 2\gamma_{j+1}(t_i) \\
    w_{i,2j} &= \alpha_j^* (t_i) + \beta_{j+1}(t_i), \quad w_{i,N} = \alpha_N^* (t_i).
\end{align*}
\quad (4.7)
\]
And,

\[ \alpha_j^*(t_i) = \frac{1}{2h^2} \int_{t_{j-1}}^{t_j} \left[ p(t_i, \eta)(\eta - \eta_{2j-2})(\eta - \eta_{2j-1}) \right] d\eta, \]

\[ \beta_j(t_i) = \frac{1}{2h^2} \int_{t_{j-1}}^{t_j} \left[ p(t_i, \eta)(\eta_{2j-1} - \eta)(\eta_{2j} - \eta) \right] d\eta, \]

\[ \gamma_j(t_i) = \frac{1}{2h^2} \int_{t_{j-1}}^{t_j} \left[ p(t_i, \eta)(\eta - \eta_{2j-2})(\eta_{2j} - \eta) \right] d\eta. \]  

(4.8)

Evaluating the integrals of (4.8), where \( p(t, \eta) = (t - \eta)^{\alpha-1} \), and introducing the results in the values of \( w_s \), we get

\[ w_{i,0} = \frac{-h^\alpha}{2\alpha(\alpha+1)(\alpha+2)} \left\{ \left[ 2| i - 2 | + \alpha + 2 \right] | i - 2 |^{\alpha+1} - \left[ -2 | i |^2 - 3(2 + \alpha) | i | + 2(\alpha + 1)(\alpha + 2) \left| i \right|^\alpha \right] \right\}, \]

\[ w_{i,j} = \frac{2h^\alpha}{\alpha(\alpha+1)(\alpha+2)} \left\{ (\alpha + 2) \left[ | i - 2j - 2 |^{\alpha+1} + | i - 2j - 2 |^{\alpha+2} - | i - 2j |^{\alpha+1} \right] \right\}, \]

\[ w_{i,2j} = \frac{-h^\alpha}{2\alpha(\alpha+1)(\alpha+2)} \left\{ (\alpha + 2) \left[ | i - 2j + 2 |^{\alpha+1} + (\alpha + 2) | i - 2j - 2 |^{\alpha+2} - 6(\alpha + 2) | i - 2j |^{\alpha+1} + 2 | i - 2j - 2 |^{\alpha+2} - 2 | i - 2j + 2 |^{\alpha+2} \right] \right\}, \]

and

\[ w_{i,n} = \frac{-h^\alpha}{2\alpha(\alpha+1)(\alpha+2)} \left\{ 2(\alpha+1)(\alpha+2) | i - n |^{\alpha+1} + 3(\alpha + 2) | i - n |^{\alpha+2} + (\alpha + 2) | i - n + 2 |^{\alpha+1} - 2 | i - n + 2 |^{\alpha+2} \right\}. \]  

(4.9)

The linear algebraic system (4.6) represents \((N+1)\) equations in \( u_{i,j} \). Therefore, the approximate solution of \( u(x,t) \) can be written in the vector form

\[ (I - \alpha W) U = F^*, \]  

(4.10)

where,

\[
W = \begin{bmatrix}
-\alpha w_{0,0} Q_{0,0} & -\alpha w_{0,1} Q_{0,1} & \cdots & -\alpha w_{0,N} Q_{0,N} \\
-\alpha w_{1,0} Q_{1,0} & -\alpha w_{1,1} Q_{1,1} & \cdots & -\alpha w_{1,N} Q_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha w_{N,0} Q_{N,0} & -\alpha w_{N,1} Q_{N,1} & \cdots & -\alpha w_{N,N} Q_{N,N}
\end{bmatrix},
\]
When \( \text{det}(W) \neq 0 \), the algebraic system (4.6) has a unique solution in the form

\[
U = (I - \alpha W)^{-1} F^*,
\]

where \( I \) is the identity matrix.

**Theorem 4:**

The algebraic system (4.6) has a unique solution in the Banach space \( \ell^\infty \), under the following conditions

\[
\sup_i |f^*_{i,j}| \leq q, \quad (q \text{ is a constant}). \tag{4.12}
\]

\[
\sup \sum_{j=0}^{N} |w_{i,j} Q^*_{i,j}| \leq q^*, \quad (q^* \text{ is a constant}). \tag{4.13}
\]

\[
\sup_i |B_i u_j| \leq M \sup_i |u_j|, \quad (M \text{ is a constant}). \tag{4.14}
\]

**Proof:**

Let \( Y \) be the set of all functions \( U = \{u_{i,i}\} \) in \( \ell^\infty \) such that \( \|U\|_\infty \leq \rho^* \), \( \rho^* \) is a constant. Define the operator \( \tilde{T} \) by

\[
\tilde{T} U = F^* + \alpha W U,
\]

where \( \|\tilde{T} U\|_\infty = \sup_i |\tilde{T} u_{i,i}|, \quad \forall \ i = 0, 1, 2, \ldots \).

The formulas (4.6) and (4.15) lead to

\[
|\tilde{T} u_{i,i}| \leq \sup_i |f^*_{i,i}| + \alpha \sup \sum_{j=0}^{N} |w_{i,j} Q^*_{i,j}| \sup_j |B_j u_j|, \quad \forall \ i = 0, 1, 2, \ldots.
\]

In view of the conditions (4.12) and (4.14), the above inequality takes the form

\[
\|\tilde{T} U\|_\infty \leq q + \lambda^*, \quad (\lambda^* = \alpha q^* M). \tag{4.16}
\]

Inequality (4.16) shows that, the operator \( \tilde{T} \) maps the set \( Y \) into itself, where

\[
\rho^* = \frac{q}{1 - \lambda^*}.
\]

Since \( \rho^* > 0 \), \( q > 0 \), therefore \( \lambda^* < 1 \). Also, the inequality (4.16) involves the boundedness of operator \( \tilde{T} \).
For the two functions $U$ and $V$ in $\ell^\infty$, the formulas (4.6) and (4.15) lead to

$$|\tilde{T}u_{i,i} - \tilde{T}v_{i,i}| \leq \alpha \sup_i \sum_{j=0}^{N} |w_{i,j} Q_{i,j}^*| \sup_j |B_j(u_{j,j} - v_{j,j})| .$$

The above inequality, with the aid of conditions (4.13) and (4.14), can be adapted in the form

$$\left\| \tilde{T}u - \tilde{T}v \right\|_{\ell^\infty} \leq \lambda^* \left\| u - v \right\|_{\ell^\infty} .$$

Therefore, $\tilde{T}$ is a continuous operator in $\ell^\infty$, then under the condition $\lambda^* < 1$, $\tilde{T}$ is contractive. Hence, by Banach fixed point theorem, $\tilde{T}$ has a unique fixed point which is the unique solution of the linear algebraic system in the Banach space $\ell^\infty$.

**Theorem 5:**

If the conditions (4.13) and (4.14) are verified, and the sequence of functions $\{F_m^*\} = \{f_{i,j}^*\}_m$ converges uniformly to the function $F^* = \{f_{i,j}^*\}$ in the Banach space $\ell^\infty$. Then, the sequence of approximate solutions $\{U_m\} = \{u_{i,i}\}_m$ converges uniformly to the exact solution $U = \{u_{i,i}\}$ of the linear algebraic system (4.6) in $\ell^\infty$.

**Proof:**

In the light of (4.6), we get

$$|u_{i,i} - (u_{i,i})_m| \leq \alpha \sup_i \sum_{j=0}^{N} |w_{i,j} Q_{i,j}^*| \sup_j |B_j(u_{j,j} - (u_{j,j})_m)| + \sup_i |f_{i,i} - (f_{i,i})_m| , \forall i = 0,1,2,... .$$

Using the conditions (4.13) and (4.14), we have

$$\left\| U - U_m \right\|_{\ell^\infty} \leq \frac{1}{1 - \lambda^*} \left\| F^* - F_m^* \right\|_{\ell^\infty} ; \quad (\lambda^* < 1).$$

Since $\left\| F^* - F_m^* \right\|_{\ell^\infty} \to 0$ as $m \to \infty$, so that $\left\| U - U_m \right\|_{\ell^\infty} \to 0$.

This complete the prove of the theorem.

When $N \to \infty$, it is natural to expect that the sum $\sum_{j=0}^{N} w_{i,j} Q_{i,j}^* B_j u_{j,j}$; $0 \leq i,j \leq N$, becomes $\int_0^{\omega(t,\eta)} Q^*(t,\eta) B(\eta)u(x,\eta) d\eta$. Consequently, the solution of the algebraic system (4.6) is the same solution of the integral equation (4.1).

**Theorem 6:**

If the sequence of continuous functions $\{f_{i,j}^m(x,t)\}$ converges uniformly to the function $f^*(x,t)$, and the functions $Q^*(t,\eta)$, $p(t,\eta)$ satisfy, respectively, the conditions (2) and (3-b). Then, the sequence of approximate solutions $\{u_n(x,t)\}$ converges uniformly to the exact solution of (4.1) in the Banach space $C_\mathbb{R} (\mathbb{R} \times [0,T])$. 

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Proof: The formula (4.1) with its approximate solution give
\[
\max_{x,t}\left|u(x,t)-u_n(x,t)\right| \leq \max_{x,t}\left|f^*(x,t)-f_n^*(x,t)\right| + \alpha \int_0^t \left|p(t,\eta)\right| \left|Q^*(t,\eta)\right| \max_{x,\eta} \left|B(\eta) (u(x,\eta)-u_n(x,\eta))\right| d\eta ,
\] (4.17)
\[
v \leq \alpha c_1, \forall 0 \leq \eta \leq T , \quad -\infty < x < \infty .
\]

In view of the conditions (2) and (3-b), there exist two constants \( c_1 \) and \( c_2 \), such that
\[
\left|Q^*(t,\eta)\right| \leq c_1, \quad \text{and} \quad \int_0^T \left|p(t,\eta)\right| d\eta \leq c_2 .
\] (4.18)

Hence, the inequality (4.17) with the aid of (4.18) and (3.5), takes the form
\[
\left\|u(x,t)-u_n(x,t)\right\|_{\mathcal{C}^r([0,T])} \leq \frac{1}{(1-c')} \left\|f^*(x,t)-f_n^*(x,t)\right\|_{\mathcal{C}^r([0,T])} ,
\]
\[
\left(c' = \alpha c_1 c_2 M\right) .
\]

Since
\[
\left\|f^*(x,t)-f_n^*(x,t)\right\|_{\mathcal{C}^r([0,T])} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty ,
\]
hence
\[
\left\|u(x,t)-u_n(x,t)\right\|_{\mathcal{C}^r([0,T])} \rightarrow 0 .
\]

Definition 1:
The product Nyström method is said to convergent of order \( r \) in \([a,b]\) if and only if for sufficiently large \( N \), there exists a constant \( c > 0 \) independent of \( N \), such that
\[
\left\|\phi(x)-\phi_N(x)\right\|_{\mathcal{C}^r} \leq c N^{-r} .
\]

Definition 2:
The consistency error \( R_N \) of the product Nyström method is determined by the following equation
\[
R_N = \int_0^t \left|p(t,\eta)Q^*(t,\eta)B(\eta)u(x,\eta)d\eta - \sum_{j=0}^N w_{i,j} Q^*_{i,j} B_j u_{i,j}\right| .
\] (4.19)

Also, (4.19) gives
\[
u(x,t)-u_N(x,t) = \sum_{j=0}^N w_{i,j} Q^*_{i,j} \left[B_j (u(x,\eta_j) - u_N(x,\eta_j))\right] + R_N ,
\] (4.20)

where \( u_N(x,t) \) is the approximate solution of (4.1).

Theorem 7:
Assume that, the hypothesis of Theorem (5) are verified, then
\[
\lim_{N \rightarrow \infty} R_N = 0 .
\] (4.21)
Proof:
The formula (4.20) leads to
\[
|R_N| \leq \sup_i |u_{i,i} - (u_{i,i})_N| \\
+ \sup \sum_{j=0}^{N} \left| w_{i,j} Q_{i,j}^* \right| \sup_i \left| B_j (u_{j,j} - (u_{j,j})_N) \right|
\]

In view of the conditions (4.13) and (4.14), the above inequality takes the form
\[
\|R_N\| \leq \|U - U_N\| + q^* M \|U - U_N\|, \ \forall N = 1,2,\ldots,
\]

Since \(\|U - U_N\| \to 0\) as \(N \to \infty\) (see Theorem (5)), it follows that \(\|R_N\| \to 0\).

Application I:

In (4.1), let \(0 < \alpha < 1\), \(Q^*(t, \eta) = 1\), \(B(\eta) = I\), where \(I\) is the identity operator. Hence, we get a linear Volterra integral equation of the second kind with Abel kernel
\[
u(x, t) = x (1 - t^\alpha) + t \left(1 - \frac{t^\alpha}{\alpha + 1}\right) + \alpha \int_0^t (t - \eta)^{\alpha - 1} u(x, \eta) \, d\eta,
\]

where the exact solution \(u(x, t) = x + t\).

The results are obtained numerically in the following Table which lists various values of \(x, t \in [0,0.8]\) together with the values of the exact and approximate solutions and the error of (4.22). Also, we can see from this table that:

1. The exact and approximate solutions are coincident for \(x = t = 0\).

2. As \(x\) and \(t\) are increasing through \([0,0.8]\), the error is also increasing for \(\alpha = 0.98\), \(\alpha = 0.8\) and \(\alpha = 0.4\).

3. The maximum value of the error is 0.421056 which occurs at \(x = t = 0.8\) for \(\alpha = 0.8\).

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References Références Referencias