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Some Properties of Finite Boolean Algebra

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Abstract - In this paper we examine the relationship between the Ideal and Boolean Algebra of Lattice. Here the main result is that principal ideal (atom), principal dual ideal (filter) and also their product are Boolean algebra.

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SOME PROPERTIES OF FINITE BOOLEAN ALGEBRA

Notes

Some Properties of Finite Boolean Algebra

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Abstract - In this paper we examine the relationship between the Ideal and Boolean Algebra of Lattice. Here the main result is that principal ideal (atom), principal dual ideal (filter) and also their product are Boolean algebra.

Keywords : principal ideal, principal dual ideal, boolean algebra.

I. Introduction

Boolean algebra, as developed in 1854 by George Boole in his book An Investigation of the Laws of Thought, is a variant of ordinary elementary algebra differing in its values, operations, and laws. Instead of the usual algebra of numbers, Boolean algebra is the algebra of truth values 0 and 1, or equivalently of subsets of a given set. The operations are usually taken to be conjunction $\wedge$, disjunction $\vee$, and negation $\neg$, with constants 0 and 1. And the laws are definable as those equations that hold for all values of their variables, for example $x \vee (y \wedge x) = x$. Applications include mathematical logic, digital logic, computer programming, set theory, and statistics.

Boole's algebra predated the modern developments in abstract algebra and mathematical logic; it is however seen as connected to the origins of both fields. In an abstract setting, Boolean algebra was perfected in the late 19th century by Jevons, Schröder, Huntington, and others until it reached the modern conception of an (abstract) mathematical structure. For example, the empirical observation that one can manipulate expressions in the algebra of sets by translating them into expressions in Boole's algebra is explained in modern terms by saying that the algebra of sets is *a* Boolean algebra (note the indefinite article). In fact, M. H. Stone proved in 1936 that every Boolean algebra is isomorphic to a field of sets.

In the 1930s, while studying switching circuits, Claude Shannon observed that one could also apply the rules of Boole's algebra in this setting, and he introduced **switching algebra** as a way to analyze and design circuits by algebraic means in terms of logic gates. Shannon already had at his disposal the abstract mathematical apparatus, thus he cast his switching algebra as the two-element Boolean algebra. In circuit engineering settings today, there is little need to consider other Boolean algebras, thus "switching algebra" and "Boolean algebra" are often used interchangeably. Efficient implementation of Boolean functions is a fundamental problem in the design of combinatorial logic circuits.

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II. Material and Method

2.1 Definition: A sub-lattice H of a lattice L is called a **convex sub-lattice** if for any $a, b \in H$, $a < c < b$ implies $c \in H$.

Example: in the fig below (fig-1) $\{0, a, c\}, \{0, b, c\}$ are convex lattices.

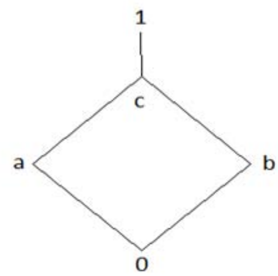

Fig-1 : Convex Lattice

2.2 Definition: Let L be a lattice and I a non-empty subset of L. then I is called an **ideal** of L if

i. I is a sub -lattice of L   ii. for any $a \in I, x \in L, x \wedge a \in I$.

Example: in fig below (fig-2) $\{0, a\}, \{0, b\}, \{0, a, b, c\}$ are ideals:

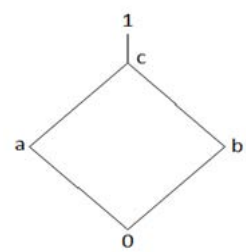

Fig-2 : Ideal

2.3 Definition: An ideal of a lattice L is generated by a single element is called a **principal ideal**. If $a \in L$ then the ideal generated by a principal ideal denoted by $(a]$. Infact $(a] = \{x \in L : x \leq a)$. The set of all ideals of a lattice L is denoted by $I(L)$.

2.4 Definition: A non-empty subset D of a lattice L is called a **dual ideal** or **filters** if

i. For all $a, b \in D, a \wedge b \in D$ ii. For $a \in D, a \leq t, t \in L$ implies $t \in D$.

Example: In the lattice below (fig-3) $\{1\}, \{1, c\}, \{1, c, a\}, \{1, c, b\}, \{1, c, a, b, 0\}$ are dual lattice:

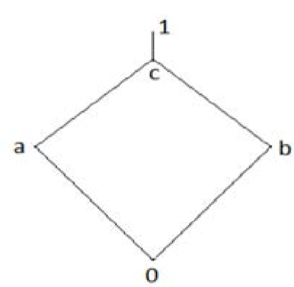

Fig-3 : Dual Lattice

2.5 Definition: Let L be a lattice and $a \in L$. Then $[a) = \{x \in L : a \leq x\}$ form a dual ideal. This ideal is called a **principal dual ideal**.

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2.6 Definition: Let L_1 and L_2 be two lattices then the mapping $f: L_1 \to L_2$ is called a **meet homomorphism** if for all $a, b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$ and is called **join homomorphism** if for all $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$ and f called a **homomorphism** if f is both meet and join homomorphism.

2.7 Definition: A mapping $f: L_1 \to L_2$ is called an **isomorphism** if it is homomorphism, one-one and onto. It is denoted by $L_1 \cong L_2$.

2.8 Definition: Let L be a bounded lattice that is a lattice L with 0 and 1. for an element $a \in L$, an element $b \in L$ is called a **compliment** of a if $a \wedge b = 0$ and $a \vee b = 1$ and is denoted by $b = \neg a$. Obviously $\neg 0 = 1$ and $\neg 1 = 0$.

2.9 Definition: In a lattice if every element has a complimented, then it is called a **complimented lattice**.

2.10 Definition: A complimented distributive lattice is called a **Boolean lattice**.

2.11 Algebraic definition of Boolean algebra: An algebra $B = < L, \wedge, \vee, \neg, 0, 1 >$, where L is a non-empty set together with two binary operations $\wedge$ and $\vee$ and a unary operation $\neg$ and null operations 0 and 1, is called a **Boolean algebra** if it satisfies the following conditions:

a) $\wedge, \vee$ are independent

b) $\wedge, \vee$ are associative

c) $\wedge, \vee$ are commutative

d) $\wedge, \vee$ are satisfy absorption law

e) For all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

f) for all $a \in L$, there exists $\neg a \in L$ such that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.

Example : Let $B = \{0, a, b, 1\}$. Define $\wedge, \vee, \neg$ by following:

$\vee$	0	a	b	1		$\wedge$	0	a	b		$\neg$	
0	0	a	b	1		0	0	0	0		0	1
a	a	a	1	1		a	0	a	0		a	b
b	b	1	b	1		b	0	0	b		b	a
1	1	1	1	1		1	0	a	b		1	0

Then B form a Boolean algebra under those operations.

III. Results and Discussion

3.1 Theorem: A principal ideal of a Boolean algebra is again a Boolean algebra.

Proof: Let $B = < B, \le, \neg, \wedge, \vee, 0, 1 >$ be a Boolean algebra and $a \in B$.

A principal ideal of B generated by a, $(a]$, is: $(a] = \{x \in B : x \le a\}$

Now let an algebra $((a] =< (a], \le_{(a]}, \neg_{(a]}, \wedge_{(a]}, \vee_{(a]}, 0_{(a]}, 1_{(a]} >)$ with the relation of B as

1. $\le_{(a]} = \le \partial(a]$
2. $\neg_{(a]} = \{< x, \neg x \wedge a >: x \in (a]\}$
3. $\wedge_{(a]} = \wedge\partial(a]$
4. $\vee_{(a]} = \vee\partial(a]$
5. $0_{(a]} = 0$
6. $1_{(a]} = a$

Now if $x, y \in (a]$, then $x, y \le a$ hence $(x \wedge y) \le a$ and $(x \vee y) \le a$, hence $(x \wedge y) \in (a]$ and $(x \vee y) \in (a]$. Hence $< (a], \le_{(a]}, \wedge_{(a]}, \vee_{(a]}, 0_{(a]} >$ is a substructure of $< B, \le, \wedge, \vee, 0 >$ and $< (a], \le_{(a]}, \wedge_{(a]}, \vee_{(a]} >$ is a sub-lattice of $< B, \le, \wedge, \vee >$.

Again $0_{(a]}$ is the minimum of (a]. $1_{(a]} = a$, which is, obviously, the maximum of (a].

Hence $< (a], \le_{(a]}, \wedge_{(a]}, \vee_{(a]}, 0_{(a]}, 1_{(a]} >$ is bounded by $0_{(a]}$ and $1_{(a]}$.

Now as $< B, \le, \wedge, \vee >$ is distributive, it follows that $< (a], \le_{(a]}, \wedge_{(a]}, \vee_{(a]} >$ is distributive. Hence (a] is a bounded distributive lattice. So we only need to prove that (a] is complemented, i.e. that $\neg_{(a]}$ is complementation on (a].

First: (a] is closed under $\neg_{(a]}$. Because $\neg x \wedge a \le a$, for any $x \in B$, hence also for any $x \le a$.

Secondly: $\neg_{(a]}$ holds the laws of $0_{(a]}$ and $1_{(a]}$: Let $x \in (a]$ then

$$x \wedge_{(a]} \neg_{(a]}(x) = x \wedge (\neg x \wedge a) = (x \wedge \neg x) \wedge (x \wedge a) = 0 \wedge (x \wedge a) = 0 = 0_{(a]}$$

And $$x \vee_{(a]} \neg_{(a]}(x) = x \vee (\neg x \wedge a) = (x \vee \neg x) \wedge (x \vee a) = 1 \wedge (x \vee a) = x \vee a = a = 1_{(a]}$$

Thus, indeed $\neg_{(a]}$ is complementation on (a]. (Proved)

3.2 Theorem: Principal dual ideal of a Boolean algebra is again a Boolean algebra.

Proof: Let $B =< B, \le, \neg, \wedge, \vee, 0, 1 >$ be a Boolean algebra and $a \in B$. A principal dual ideal of B generated by a, $[a)$, is: $[a) = \{x \in B : a \le x\}$. Now an algebra

$[a) =< [a), \le_{[a)}, \neg_{[a)}, \wedge_{[a)}, \vee_{[a)}, 0_{[a)}, 1_{[a)} >)$ with the relation of B as

1. $\le_{[a)} = \le \partial[a)$
2. $\neg_{[a)} = \{< x, \neg x \vee a >: x \in [a)\}$
3. $\wedge_{[a)} = \wedge\partial[a)$

4. $\vee_{[a)} = \vee\partial[a)$

5. $0_{[a)} = a$

6. $1_{[a)} = 1$

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Let $x, y \in [a)$, then $a \le x, y$, hence $a \le (x \wedge y)$ and $a \le (x \vee y)$, hence $(x \wedge y) \in [a)$ and $(x \vee y) \in [c)$. Hence $< [a), \le_{[a)}, \wedge_{[a)}, \vee_{[a)}, 1_{[a)} >$ is a substructure of $< B, \le, \wedge, \vee, 1 >$ and also $< [a), \le_{[a)}, \wedge_{[a)}, \vee_{[a)} >$ is a sub-lattice of $< B, \le, \wedge, \vee >$. Now $1_{[a)}$ is the maximum of [a) and $0_{[a)} = a$, which is, obviously, the minimum of [a). Hence $< [a), \le_{[a)}, \wedge_{[a)}, \vee_{[a)}, 0_{[a)}, 1_{[a)} >$ is bounded by $0_{[a)}$ and $1_{[a)}$.

As $< B, \le, \wedge, \vee >$ is distributive, it follows that $< [a), \le_{[a)}, \wedge_{[a)}, \vee_{[a)} >$ is distributive.

Hence [a) is a bounded distributive lattice. So we only need to prove that [a) is complemented, i.e. that $\neg_{[a)}$ is complementation on [a).

First: [a) is closed under $\neg_{[a)}$. This is obvious, since obviously $a \le \neg x \vee a$ for any $x \in B$, hence also for any x such that $a \le x$.

Secondly, $\neg_{[a)}$ respects the laws of $0_{[a)}$ and $1_{[a)}$:

Let $x \in [a)$ then,

$x \wedge_{[a)} \neg_{[a)}(x) = x \wedge (\neg x \vee a) = (x \wedge \neg x) \vee (x \wedge a) = 0 \vee (x \wedge a) = x \wedge a = a = 0_{[a)}$

$x \vee_{[a)} \neg_{[a)}(x) = x \vee (\neg x \vee a) \;= (x \vee x\neg) \vee (x \vee a) = 1 \vee (x \vee a) = 1 = 1_{[a)}$

Thus, indeed $\neg_{[a)}$ is complementation on [a). (proved).

3.3 Lemma: If $a \ne 1$ **then** $(a] \cap [\neg a) = \phi$

Proof: Let $x \in (a] \cap [\neg a)$. Then

$x \le a$ and $\neg a \le x$. Then $x \le a$ and $\neg x \le a$.

Then $x \vee \neg x \le a \Rightarrow 1 \le a \Rightarrow a = 1$

Hence if $a \ne 1$ then $(a] \cap [\neg a) = \phi$.

3.4 Theorem: (a] **and** $[\neg a)$ **are isomorphic**.

Proof: If $a = 1$, then $(a] = [\neg a) = B$. So clearly they are isomorphic. So let $a \ne 1$. We define: $f : (a] \to [\neg a)$ by: for every $x \in (a] : f(x) = x \vee \neg a$.

1. f is a function:

Since for every x ∈ B, $\neg a \le x \vee a$, also for every $x \in (a] : \neg a \le x \vee a$. Hence for every $x \in (a] : f(x) \in [\neg a)$, hence f is indeed a function from (a] into $[\neg a)$.

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2. f is onto:

Let $y \in [\neg a)$.

Then $\neg a \leq y \Rightarrow \neg y \leq a$ .... (1). Hence $\neg y \in (a]$.

Now $\neg_{[a)} y = \neg(\neg y) \wedge a = y \wedge a$. Hence $(y \wedge a) \in (a]$.

Again $f(y \wedge a) = (y \wedge a) \vee \neg a = (y \vee \neg_{[a)} a) \wedge (a \vee \neg a) = (y \vee \neg a) \wedge 1 = y \vee \neg a = y$ [from (1)]

Hence f is onto.

3. f is one-one:

Let $f(x_1) = f(x_2)$.

Then $x_1 \vee \neg a = x_2 \vee \neg a \Rightarrow \neg(x_1 \vee \neg a) = \neg(x_2 \vee \neg a)$

$\Rightarrow \neg x_1 \wedge a = \neg x_2 \wedge a$ [by De-Morgan's law]

$\Rightarrow \neg_{(a]} x_1 = \neg_{(a]} x_2$

Hence these are the relative complements of x_1 and x_2 in Boolean algebra $(a]$.

Hence $x_1 = x_2$.

Hence h is one-one.

4. f is homomorphism:

for $0_{(a]} = 0$, $f(0) = 0 \vee \neg a = \neg a = 0_{(\neg a]}$

for $1_{(a]} = a$, $f(a) = a \vee \neg a = 1 = 1_{[\neg a)}$

$f(x \wedge y) = (x \wedge y) \vee \neg a = (x \vee \neg a) \wedge (y \vee \neg a) = f(x) \wedge f(y)$

$f(x \vee y) = (x \vee y) \vee \neg a = (x \vee \neg a) \vee (y \vee \neg a) = f(x) \vee f(y)$

And $f(\neg_{(a]} x) = f(\neg x \wedge a) = (\neg x \wedge a) \vee \neg a = \neg(x \vee \neg a) \vee \neg a = \neg_{(\neg a]}(x \vee \neg a) = \neg_{(\neg a]}(f(x))$

Hence f is homomorphism.

Thus, indeed, h is an isomorphism. (Proved).

3.5 Lemma: If a is an atom in B, then for every $x \in B - \{0\} : a \leq x$ or $a \leq \neg x$

Proof: Let a be an atom in B. Suppose that $\neg(a \leq x)$. Then $(a \wedge x) \neq a$. But $a \wedge x \leq a$. Since a is an atom, that means that $a \wedge x = 0$. But that means that $a \leq \neg x$.

3.6 Corollary: If a is an atom in B, then $(\neg a] \cup [a) = B$.

Proof: This follows from lemma 5: Let $x \in B$ and $x \notin [a)$. Then $\neg(a) \leq x$. Hence by lemma 5 $a \leq \neg x$, and that means that $x \leq \neg a$, hence $x \in (\neg a]$.

3.7 Theorem: Let A and B be two Boolean algebra, then the product of A and B, $A \times B$, is a Boolean algebra.

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Proof: Since **A** and **B** are Boolean algebras. The **product** of **A** and **B**, $A \times B$, is given by:

$A \times B =< B_\times, \leq_\times, \neg_\times, \wedge_\times, \vee_\times, 0_\times, 1_\times >$ where

1. $B_\times = A \times B$

2. $\leq_\times = \{<< a_1, b_1 >, < a_2, b_2 >>: a_1 \leq_A a_2 \text{ and } b_1 \leq_B b_2\}$

3. For every

$< a, b > \in A \times B : \neg_\times (< a, b >) =< \neg_A a, \neg_B b >$

4. For every

$<< a_1, b_1 >, < a_2, b_2 >> \in A \times B : << a_1, b_1 > \wedge_\times < a_2, b_2 >> =< a_1 \wedge_A a_2, b_1 \wedge_B b_2 >$

5. For every $<< a_1, b_1 >, < a_2, b_2 >> \in A \times B : << a_1, b_1 > \vee_\times < a_2, b_2 >> =< a_1 \vee_A a_2, b_1 \vee_B b_2 >$

6. $0_\times =< 0_A, 0_B >$

7. $1_\times =< 1_A, 1_B >$

(a) $\leq_\times$ is a partial order.

Reflexive: Since for every $a \in A : a \leq_A a$ and for every $b \in B : b \leq_B b$, for every

$< a, b > \in A \times B :< a, b > \leq_\times < a, b >$

Anti-symmetric: Let $< a_1, b_1 > \leq_\times < a_2, b_2 >$ and . $< a_2, b_2 > \leq_\times < a_1, b_1 >$

Then $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$ and $a_2 \leq_A a_1$ and $b_2 \leq_B b_1$, Hence $a_1 = a_2$ and $b_1 = b_2$, hence $< a_1, b_1 > = < a_2, b_2 >$

Transitive: Let $< a_1, b_1 > \leq_\times < a_2, b_2 >$ and. $< a_2, b_2 > \leq_\times < a_3, b_3 >$. Then $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$ and $a_2 \leq_A a_3$ and $b_2 \leq_B b_3$, hence $a_1 \leq_A a_3$ and $b_1 \leq_B b_3$, hence $< a_1, b_1 > \leq_\times < a_3, b_3 >$

(b) $< a_1, b_1 > \wedge_\times < a_2, b_2 > = < a_1 \wedge_A a_2, b_1 \wedge_B b_2 >$:

$a_1 \wedge_A a_2 \leq_A a_1$, $a_1 \wedge_A a_2 \leq_A a_2$, $b_1 \wedge_B b_2 \leq_B b_1$, $b_1 \wedge_B b_2 \leq_B b_2$

Hence

$< a_1, b_1 > \wedge_\times < a_2, b_2 > \leq_\times < a_1, b_1 >$

$< a_1, b_1 > \wedge_\times < a_2, b_2 > \leq_\times < a_2, b_2 >$

Let $< a, b > \leq_\times < a_1, b_1 >$ and, $< a, b > \leq_\times < a_2, b_2 >$

Then $a \leq_A a_1$ and $b \leq_B b_1$ and $a \leq_A a_2$ and $b \leq_B b_2$,

hence $a \leq_A a_1 \wedge_A a_2$ and $b \leq_B b_1 \wedge_B b_2$, hence $< a, b > \leq_\times < a_1, b_1 > \wedge_\times < a_2, b_2 >$

Hence $\wedge_\times$ is meet in $\leq_\times$.

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Similarly $\vee_\times$ is join in $\leq_\times$.

(c). $0_\times =< 0_A , 0_B >$:

Since for every $a \in A$, $0_A \leq_A a$ and for every $b \in B$, $0_B \leq_B b$, for every $<a,b> \in A \times B$, $<0_A , 0_B> \leq_\times <a,b>$. Hence $0_\times$ is the minimum under $\leq_\times$.

Similarly $1_\times$ is the maximum under $\leq_A$.

So $\mathbf{A} \times \mathbf{B}$ is a bounded lattice.

(d). $<a_1,b_1> \wedge_\times (<a_2,b_2> \vee_\times <a_3,b_3>) = <a_1 \wedge_A (a_2 \vee_A a_3), b_1 \wedge_B (b_2 \vee_B b_3)) = <(a_1 \wedge_A a_2) \vee_A (a_1 \wedge_A a_3), (b_1 \wedge_B b_2) \vee_B (b_1 \wedge_B b_3)> = <a_1 \wedge_A a_2, b_1 \wedge_B b_2> \vee_\times <a_1 \wedge_A a_3, b_1 \wedge_B b_3> = (<a_1,b_1> \wedge_\times <a_2,b_2>) \vee_\times (<a_1,b_1> \wedge_\times <a_3,b_3>)$

So $\mathbf{A} \times \mathbf{B}$ is distributive.

(e). $\neg_\times$ satisfies the laws of $0_\times$ and $1_\times$:

$<a,b> \wedge_\times \neg_\times(<a,b>) = <a,b> \wedge_\times <\neg_A a, \neg_B b> = <a \wedge_A \neg_A a, b \wedge_B \neg_B b> = <0_A , 0_B> = 0_\times$.

$<a,b> \vee_\times \neg_\times(<a,b>) = <a,b> \vee_\times <\neg_A a, \neg_B b> = <a \vee_A \neg_A a, b \vee_B \neg_B b> = <1_A , 1_B> = 1_\times$.

So $\mathbf{A} \times \mathbf{B}$ is a Boolean algebra.

3.8 Theorem: Let B_1 and B_2 be isomorphic Boolean algebras such that $B_1 \cap B_2 = \phi$ and let h be an isomorphism between B_1 and B_2. The product of B_1 and B_2 under h, $B^h{}_{1+2}$, is a Boolean algebra.

Proof: Let us definite the **product of B_1 and B_2 under h**, $B^h{}_{1+2}$ as,

$B^h{}_{1+2} =< B^h{}_{1+2}, \leq_{1+2}, \neg_{1+2}, \wedge_{1+2}, \vee_{1+2}, 0_{1+2}, 1_{1+2} >$ where:

1. $B^h{}_{1+2} = B_1 \cup B_2$

2. $\leq_{1+2} = \leq_1 \cup \leq_2 \cup \{< b_1, b_2 >: h(b_1) \leq_2 b_2\}$

3. $\neg_{1+2}$ is defined by:

$$b \in B_{1+2} : \neg_{1+2}(b) = \begin{cases} \neg_1(h(b)), \text{if } b \in B_1 \\ \\ \neg_1(h^{-1}(b)), \text{if } b \in B_2 \end{cases}$$

4. $\wedge_{1+2}$ is defined by:

$$a,b \in B_{1+2} : a \wedge_{1+2} b = \begin{cases} a \wedge_1 b, & \text{if } a,b \in B_1 \\ a \wedge_2 b, & \text{if } a,b \in B_2 \\ a \wedge_1 h^{-1}(b), & \text{if } a \in B_1, b \in B_2 \end{cases}$$

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5. $\vee_{1+2}$ is defined by:

$$a, b \in B_{1+2} : a \vee_{1+2} b = \begin{cases} a \vee_1 b, & \text{if } a, b \in B_1 \\ a \vee_2 b, & \text{if } a, b \in B_2 \\ h(a) \vee_1 b, & \text{if } a \in B_1, b \in B_2 \end{cases}$$

6. $0_{1+2} = 0_1$.

7. $1_{1+2} = 1_1$.

1. $\leq_{1+2}$ is a partial order:

$\leq_{1+2}$ is reflexive:

If $a \in B_1 : a \leq_1 a$, hence, $a \leq_{1+2} a$

If $a \in B_2 : a \leq_2 a$, hence, $a \leq_{1+2} a$

$\leq_{1+2}$ is anti-symmetric.

Let $a \leq_{1+2} b$ and $b \leq_{1+2} a$. This is only possible if $a, b \in B_1$ or $a, b \in B_2$.

In the first case $a \leq_1 b$ and $b \leq_1 a$, hence a=b. In the second case $a \leq_2 b$ and $b \leq_2 a$, hence a=b.

$\leq_{1+2}$ is transitive.

Let $a \leq_{1+2} b$ and $b \leq_{1+2} c$

If $a, b, c \in B_1$, then $a \leq_1 b$ and $b \leq_1 c$, hence $a \leq_1 c$, and $a \leq_{1+2} c$.

If $a, b, c \in B_2$, then $a \leq_2 b$ and $b \leq_2 c$, hence $a \leq_2 c$, and $a \leq_{1+2} c$.

If $a \in B_1$ and $b, c \in B_2$, then $h(a) \leq_2 b$ and $b \leq_2 c$. Then $h(a) \leq_2 c$ and $a \leq_{1+2} c$.

If $a, b \in B_1$ and $c \in B_2$, then $a \leq_1 b$ and $h(b) \leq_2 c$.

Since h is an isomorphism, this means that $h(a) \leq_2 h(b)$, and hence $h(a) \leq_2 c$. Hence $a \leq_{1+2} c$.

2. $\wedge_{1+2}$ is meet under $\leq_{1+2}$.

If $a, b \in B_1 : a \wedge_{1+2} b = a \wedge_1 b$, which is meet under $\leq_1$, and $\leq_1 = \leq_{1+2} \partial B_1$. If $a, b \in B_2 : a \wedge_{1+2} b = a \wedge_2 b$, which is meet under $\leq_2$, and $\leq_2 = \leq_{1+2} \partial B_2$.

If $a \in B_1, b \in B_2$, then $a \wedge_{1+2} b = a \wedge_1 h^{-1}(b)$, $a \wedge_1 h^{-1}(b) \leq_1 a$ and $a \wedge_1 h^{-1}(b) \leq_1 h^{-1}(b)$.

By definition of $\leq_{1+2}$, $h^{-1}(b) \leq_{1+2} h(h^{-1}(b))$. So $h^{-1}(b) \leq_{1+2} b$. Then $a \wedge_1 h^{-1}(b) \leq_{1+2} b$. This means that $a \wedge_{1+2} b \leq_{1+2} a$ and $a \wedge_{1+2} b \leq_{1+2} b$.

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If $x \le_{1+2} a$ and $x \le_{1+2} b$, then $x \le_1 a$ and $h(x) \le_2 b$. Since h is an isomorphism, then $h^{-1}(h(x)) \le_1 h^{-1}(b)$ i.e. $x \le_1 h^{-1}(b)$, then $x \le_1 a \wedge_1 h^{-1}(b)$. Hence $x \le_{1+2} a \wedge_{1+2} b$. So indeed $\wedge_{1+2}$ is meet under $\le_{1+2}$.

3. $\vee_{1+2}$ is join under $\le_{1+2}$.

If $a,b \in B_1 : a \vee_{1+2} b = a \vee_1 b$, which is join under $\le_1$, and $\le_1 = \le_{1+2} \partial B_1$. If $a,b \in B_2 : a \vee_{1+2} b = a \vee_2 b$, which is join under $\le_2$, and $\le_2 = \le_{1+2} \partial B_2$. If $a \in B_1, b \in B_2$, then $a \vee_{1+2} b = h(a) \vee_2 b$.

$b \le_2 h(a) \vee_2 b$ and $h(a) \le_2 h(a) \vee_2 b$.

As we have seen $a \le_{1+2} h(a)$, hence $a \le_{1+2} h(a) \vee_2 b$. So $a \le_{1+2} a \vee_{1+2} b$ and $b \le_{1+2} a \vee_{1+2} b$. If $a \le_{1+2} x$ and $b \le_{1+2} x$, then $h(a) \le_{1+2} x$, hence $h(a) \vee_2 b \le_2 x$.

Hence $a \vee_{1+2} b \le_{1+2} x$. So indeed $\vee_{1+2}$ is join under $\le_{1+2}$.

4. $0_{1+2} = 0_1$

If $a \in B, 0_1 \le_1 a$. Hence $0_{1+2} \le_{1+2} a$. h is an isomorphism, so $h(0_1) = 0_2$. If $a \in B_2$, then $h(0_1) \le_2 a$, hence $0_{1+2} \le_{1+2} a$. So indeed 0_{1+2} is the minimum under $\le_{1+2}$.

Similarly, 1_{1+2} is the maximum under $\le_{1+2}$. We have proved so far that $B^h{}_{1+2}$ is a bounded lattice.

5. Distributivity: $a \wedge_{1+2} (b \vee_{1+2} c) = (a \wedge_{1+2} b) \vee_{1+2} (a \wedge_{1+2} c)$

a. Let $a,b,c \in B_1$ or $a,b,c \in B_2$, then distributivity follows from distributivity of $\wedge_1$ and $\vee_1$ and of $\wedge_2$ and $\vee_2$.

b. Let $a \in B_1$ and $b,c \in B_2$

$$a \wedge_{1+2} (b \vee_{1+2} c) = a \wedge_1 h^{-1}(b \vee_2 c) = a \wedge_1 (h^{-1}(b) \vee_1 h^{-1}(c)) = (a \wedge_1 (h^{-1}(b)) \vee_1 (a \wedge_1 h^{-1}(c))$$

$$= (a \wedge_{1+2} b) \vee_{1+2} (a \wedge_{1+2} c)$$

c. Let $b \in B_1$ and $a,c \in B_2$

$$a \wedge_{1+2} (b \vee_{1+2} c) = a \wedge_2 (h(b) \vee_2 c) = a \wedge_2 h(b)) \vee_2 (a \wedge_2 c)$$

If $a \in B_2$ and $b \in B_1$, then $a \wedge_{1+2} b = h^{-1}(a) \wedge_1 b$.

Then $h(a \wedge_{1+2} b) = h(h^{-1}(a) \wedge_1 b) = h(h^{-1}(a)) \wedge_2 h(b)) = a \wedge_2 h(b))$

Hence $(a \wedge_2 h(b)) \vee_2 (a \wedge_2 c) = h(a \wedge_{1+2} b) \vee_2 (a \wedge_2 c) = (a \wedge_{1+2} b) \vee_{1+2} (a \wedge_{1+2} c)$

d. Let $b,c \in B_1$ and $a \in B_2$

$$a \wedge_{1+2} (b \vee_{1+2} c) = a \wedge_{1+2} (b \vee_1 c) = h^{-1}(a) \wedge_1 (b \vee_1 c) = (b \wedge_1 h^{-1}(a)) \vee_1 (c \wedge_1 h^{-1}(a))$$

$$= (a \wedge_{1+2} b) \vee_{1+2} (a \wedge_{1+2} c)$$

Notes

e. Let $a,b \in B_1, c \in B_2$

$a \wedge_{1+2} (b \vee_{1+2} c) = a \wedge_1 (h^{-1}(b \vee_{1+2} c))$

If $b \in B_1, c \in B_2$, then $b \vee_{1+2} c = h(b) \vee_2 c$

Hence $h^{-1}(b \vee_{1+2} c) = h^{-1}(h(b) \vee_2 c) = h^{-1}(h(b)) \vee_1 h^{-1}(c) = b \vee_1 h^{-1}(c)$

So $a \wedge_1 (h^{-1}(b \vee_{1+2} c)) = a \wedge_1 (b \vee_1 h^{-1}(c)) = (a \wedge_1 b) \vee_1 (a \wedge_1 h^{-1}(c))$

$= (a \wedge_{1+2} b) \vee_{1+2} (a \wedge_{1+2} c)$

These are all the relevant cases, so ${B^h}_{1+2}$ is a distributive bounded lattice.

6. $\neg_{1+2}$ satisfies the laws of 0_{1+2} and 1_{1+2}.

If $a \in B_1$, $a \wedge_{1+2} \neg_{1+2}(a) = a \wedge_1 h^{-1}(\neg_{1+2}(a)) = a \wedge_1 h^{-1}(\neg_2(h(a))) = a \wedge_1 h^{-1}(h(\neg_1 a))$

$= a \wedge_1 \neg_1 a = 0_1 = 0_{1+2}$

$a \vee_{1+2} \neg_{1+2}(a) = h(a) \vee_2 \neg_{1+2}(a) = h(a) \vee_2 \neg_2(h(a)) = 1_2 = 1_{1+2}$

If $a \in B_2$,

$a \wedge_{1+2} \neg_{1+2}(a) = h^{-1}(a) \wedge_1 \neg_{1+2}(a) = h^{-1}(a) \wedge_1 \neg_1(h^{-1}(a)) = 0_1 = 0_{1+2}$

$a \vee_{1+2} \neg_{1+2}(a) = a \vee_2 h(\neg_{1+2}(a)) = a \vee_2 h(\neg_1(h^{-1}(a)) = a \vee_2 \neg_2(h(h^{-1}(a)) = a \vee_2 \neg_2 a$

$= 1_2 = 1_{1+2}$

Thus ${B^h}_{1+2}$ is a Boolean algebra.

3.9 Theorem: Let B_1 and B_2 be isomorphic Boolean algebras such that $B_1 \cap B_2 = \phi$, and let h be an isomorphism between B_1 and B_2. Let {0,1} be a Boolean algebra of cardinality 2. Prove that ${B^h}_{1+2}$ is isomorphic to $B_1 \times \{0,1\}$.

Proof: We define function k from $B_1 \cup B_2$ into $B_1 \times \{0,1\}$:

For all $x \in B_1 : k(x) = \langle x,0 \rangle$

For all $x \in B_2 : k(x) = \langle h^{-1}(x),1 \rangle$

1. Since h is an isomorphism between B_1 and B_2, and $B_1 \cap B_2 = \phi$, k is obviously a bijection between $B_1 \cup B_2$ and $B_1 \times \{0,1\}$.

2. If $x \in B_1$,

$k(\neg_{1+2}(x)) = \langle h^{-1}(\neg_{1+2}(x)),1 \rangle = \langle h^{-1}(\neg_2(h(x)), \neg_{\{0,1\}} 0 \rangle = h^{-1}(h(\neg_1(x))), \neg_{\{0,1\}} 0 \rangle$

$= \langle \neg_1(x), \neg_{\{0,1\}} 0$

$= \neg_\times \langle x,0 \rangle$

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Notes

3. $k(0_{1+2}) = k(0_1) = < 0_1, 0 > = 0_\times$

$k(1_{1+2}) = k(1_2) = < h^{-1}(1_2), 1) > \ = 1_\times$

4. k preserves meet:

If $a, b \in B_1$ then

$k(a \wedge_{1+2} b) = k(a \wedge_1 b) . = < a \wedge_1 b, 0 > = < a, 0 > \wedge_\times < b, 0 > = k(a) \wedge_\times k(b)$

If $a, b \in B_2$ then

$k(a \wedge_{1+2} b) = k(a \wedge_2 b) = < h^{-1}(a \wedge_2 b), 1 > = < h^{-1}(a) \wedge_1 h^{-1}(b), 1 >$

$= < h^{-1}(a), 1 > \wedge_\times h^{-1}(b), 1 > = k(a) \wedge_\times k(b)$

If $a \in B_1$ and $b \in B_2$ then

$k(a \wedge_{1+2} b) = k(a \wedge_1 h^{-1}(b)) = < a \wedge_1 h^{-1}(b), 0 > = < a \wedge_1 h^{-1}(b), 0 \wedge_{\{0,1\}} 1 >$

$= < a, 0 > \wedge_\times < h_{-1}(b), 1 >$

$= k(a) \wedge_\times k(b)$

5. k preserves join:

If $a, b \in B_1$ then If $a, b \in B_1$ then

$k(a \vee_{1+2} b) = k(a \vee_1 b) = < a \vee_1 b, 0 > = < a, 0 > \vee_\times < b, 0 > = k(a) \vee_\times k(b)$

If $a, b \in B_2$ then

$k(a \vee_{1+2} b) = k(a \vee_2 b) = < h^{-1}(a \vee_2 b), 1 > = < h^{-1}(a) \vee_1 h^{-1}(b), 1 >$

$= < h^{-1}(a), 1 > \vee_\times h^{-1}(b), 1 > = k(a) \vee_\times k(b)$

If $a \in B_1$ and $b \in B_2$ then

$k(a \vee_{1+2} b) = k(h(a) \vee_2 b) = < h^{-1}(h(a) \vee_2 b), 1 > = < a \vee_1 h^{-1}(b), 1 >$

$= < a \vee_1 h^{-1}(b), 0 \vee_{\{0,1\}} 1 > = < a, 0 > \vee_\times < h_{-1}(b), 1 > = k(a) \wedge_\times k(b)$

Thus indeed k is an isomorphism.

3.10 Theorem: Let B_1 and B_2 be Boolean algebras, $\neg a_1$ an atom in B_1 and $\neg a_2$ an atom in B_2, and let $(a_1]$ be isomorphic to $[a_2)$. Then B_1 and B_2 are isomorphic.

Proof: Let h_1 be the isomorphism between $(a_1]$ and $[\neg a_1)$ defined by:

for all $x \in (a_1] : h_1(x) = x \vee_{B_1} \neg_{B_1}(a_1)$

Let h_2 be the isomorphism between $(a_2]$ and $[\neg a2)$ defined by:

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for all $x \in (a_2] : h_2(x) = x \vee_{B_2} \neg_{B_2}(a_2)$

Let k the isomorphism between $(a_1]$ and $(a_2]$. Say $-0,1''$ be a two element Boolean algebra. Then $B_1 = B_{(a_1]+[\neg a_1)}^{h_1}$ and $B_2 = B_{(a_2]+[\neg a_2)}^{h_2}$, by Theorem11.

B_1 is isomorphic to $(a_1] \times \{0,1\}$ and B_2 is isomorphic to $(a_2] \times \{0,1\}$, by Theorem 10.

Define $g : (a_1] \times \{0,1\} \to (a_2] \times \{0,1\}$ by:

for every $< a, b > \in (a_1] \times \{0,1\} : g(< a, b >) = < k(a), b >$

It is straightforward to prove that g is an isomorphism between B_1 and B_2.

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