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Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

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Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

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STRICTLY PRACTICAL STABILIZATION OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS BY USING LYAPUNOV FUNCTIONS

Ref

1. Dilbaj Singh, S.K. Srivastava, Stability criteria of impulsive differential equations with comparison results, Advances in Differential Equations and Control Processes, Vol. 10, pp. 171-182, 2012.

Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

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Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

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I. Introduction

Various physical processes undergo abrupt changes of state at certain moments of time between intervals of continuous equation. The duration of these changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well approximated in terms of the instantaneous changes of state i.e. Impulses [7]. On the other hand Functional differential equations are important in scientific and technical professions and they are used to represent a rate of change of time varying phenomenon [5]. When both functional differential equations and Impulses are Involved, Impulsive functional differential system becomes a natural framework of mathematical modelling of varying physical phenomenon [5]. Impulsive functional differential systems are different from ordinary differential systems in the sense that the state undergo abrupt changes at certain moments and the derivation of the state variable depends not only on present state but also on past state. Stability is highly important in all physical application [1,2,3]. A stable equilibrium represents a behaviour usually which cannot be changed. Several stability criteria are obtained by many authors which shows that impulses do contribute to the stabilization of functional differential equations [2,3,5].

Theory of stability in the sense of Lyapunov is now well known and is widely used in concrete problems of real world. The desirable feature is to know the size of region of stability so that we can judge whether or not a given system is sufficiently stable to function properly and may be able to see how to improve its stability. On the other hand the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. So we need a notion of stability that is more suitable than stability and such concept is practical stability [4,6,8,9].

In this paper we establish stability result which provides sufficient conditions to maintain uniform strict practical stability of the trivial solution of a functional differential equation with impulse. The Lyapunov's second method of functions called Lyapunov function is employed in this work.

This paper is organized as: In section 2, we introduce some basic definitions and notations. In section 3, based upon Lyapunov functions and Razumikhin method, some conditions for strict uniform practical stability are obtained. Finally in section 4, some concluding remarks are given.

Notes

II. Preliminaries

Consider the following impulsive functional differential system:

$$\dot{x}(t)=f(t, x_t), \ t\geq t_0, \ t\neq\tau_k \tag{2.1}$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^-) = I_k(x(\tau_k^-)), \quad t=\tau_k, \ k= 1,2,....$$

Where $f\in C([R^+\times D, R^n]$, D is an open set in PC([-τ, 0], R^n), where τ>0, PC([-τ, 0], R^n) = {φ :([-τ, 0]→R^n), φ(t) is continuous everywhere except a finite number of points $\hat{t}$ at which φ($\hat{t}^+$) and φ($\hat{t}^-$) exist and φ($\hat{t}^+$) = φ($\hat{t}$)}. f(t,0) = 0 for all t∈ R^+,$I_k(0) = 0$, for all k∈Z, $0 = \tau_0 < \tau_1 < \tau_2 < \ldots\ldots\ldots\ldots \tau_k \to \infty$ for $k \to \infty$ and $x(t^+) = \lim_{s\to t^+} x(s)$, $x(t^-) = \lim_{s\to t^-} x(s)$. For each $t \geq t_0$, $x_t \in D$ is defined by $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$. For φ∈ PC ([-τ, 0], R^n), $|\varphi|_1$ is defined by $|\varphi|_1 = sup_{-\tau\leq s\leq 0}\|\varphi\|$, $|\varphi|_2$ is defined by $|\varphi|_2 = inf_{-\tau\leq s\leq 0}\|\varphi\|$. We can see that x (t) = 0 is a solution of (2.1) which we call the zero solution or trivial solution.

For a given σ ≥ 0 and φ∈ PC ([-τ, 0], R^n), the initial value problem of equation (2.1) is

$$\dot{x}(t)=f(t, x_t), \ \sigma\geq t_0, \ t\neq\tau_k \tag{2.2}$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^-)= I_k(x(\tau_k^-)), \ t = \tau_k, \ k = 1,2,... \text{ Where } x_\sigma = \varphi.$$

Throughout this paper we let the following hypothesis hold:

(H_1) For each function x(s): [σ − τ, ∞] →R^n, σ ≥ t_0, which continuous everywhere except a finite number of points τ_k at which $x(\tau_k^+)$ and $x(\tau_k^-)$ exist and $x(\tau_k^+) = x(\tau_k)$, where $f(t, x_t)$ is continuous for almost all t∈[σ,∞) and at the discontinuous f is the right continuous.

(H_2) f (t,φ) is lipschitzian in φ in each compact set in PC([-τ, 0], R^n).

(H_3) The functions $I_k: R^n\to R^n$, k = 1,2,3 ..... are such that for any H > 0, there exist a ρ > 0 such that if,

x∈ s(ρ) = {x ∈ R^n: ‖x‖ < ρ} implies that ‖x + I_k(x)‖ < H.

Under these hypothesis a unique solution of problem (2.2) exist throughout (σ, φ).

Let

K= {a ∈[R^+, R^+]: a(t) is monotone strictly increasing and a(0) = 0}

$$K_1 = \{w \in [R^+, R^+]: w(t) \in K \text{ and } 0 < w(s) < s,\ \ s > 0\}$$

$$PC_1(\rho) = \{\varphi \in PC([-\tau, 0], R^n): |\varphi|_1 < \rho\}$$

$$PC_2(\theta) = \{\varphi \in PC([-\tau, 0], R^n): |\varphi|_2 > \theta\}$$

Notes

We have the following definitions:
Definition: The trivial solution of (2.1) is said to be

(A_1) Strict practical stable, if for any $\sigma \geq t_0$ There exist (λ_1, A_1), $\lambda_1 \leq A_1$ such that $\varphi \in PC_1(\lambda_1)$ implies $\|x(t: \sigma, \varphi)\| < A_1$, $t \geq \sigma$, and for every $0 < \lambda_2 \leq \lambda_1$, there exist $0 < A_2 \leq \lambda_2$ such that $\varphi \in PC_2(\lambda_2)$ implies $\|x(t: \sigma, \varphi)\| > A_2$, $t \geq \sigma$.

(A_2) Strict Practically Uniformly Stable, if (A_1) holds for all $t \in R^+$.

Definition: The function V: $[t_0, \infty] \times s(\rho) \to R^+$ belongs to class v_0 if

I. The function V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times S(\rho)$ and for all, $t \geq t_0, V(t, 0) = 0$.

II. $V(t, x)$ is locally lipschitzian in $x \in S(\rho)$.

III. For each k = 1, 2,..... there exist finite limits.

$$\lim_{(t,y)\to(\tau_k^-, x)} V(t,y) = V(\tau_k^-, x)$$

$$\lim_{(t,y)\to(\tau_k^+, x)} V(t,y) = V(\tau_k^+, x)$$

With $V(\tau_k^+, x) = V(\tau_k, x)$ satisfied.
Definition: Let $V \in v_0$,for $(t, x) \in [\tau_{k-1}, \tau_k) \times S(\rho)$, D^+V is defined as:

$$D^+V(t,x(t)) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta} \{V(t+\delta, x(t+\delta)) - V(t, x(t))\}.$$

III. Main Result

Now we consider the strict practical stability of the Impulsive functional differential equation (2.1) with following results:
Theorem: Assume that

(i) There exist (λ_1, A_1), $0 < \lambda_1 \leq A_1$ and $V_1 \in v_0$, such that $b_1(\|x(t)\|) \leq V_1(t,x(t)) \leq a_1(\|x(t)\|)$, $a_1, b_1 \in K$

(ii) For any solution x(t) of (2.1), $V_1(t+s, x(t+s)) \leq V_1(t,x(t))$ for $s \in [-\tau,0]$, implies that

$$D^+V_1(t, x(t)) \leq 0.$$

Also for all $K \in Z^+$ and $x \in S(\rho)$

$$V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1+d_k) V_1(\tau_k, x(\tau_k^-)), \text{ where } d_k \geq 0 \text{ and } \sum_{k=1}^{\infty} d_k < \infty$$

(iii) For any $0 < \lambda_2 \leq \lambda_1$ and $V_2 \in v_0$

$$b_2(\|x(t)\|) \leq V_2(t,x(t)) \leq a_2(\|x(t)\|), \ \ a_2, b_2 \in K$$

(iv) For any solution x(t) of (2.1), V_2(t + s), x(t+ s)) ≥ V_2(t,(x(t)) for s∈ [-τ,0], implies

$$D^+V_2(t, x(t)) \geq 0.$$

Also for all K∈ Z^+ and x∈ S(ρ)

$$V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1 - c_k)\, V_2(\tau_k, x(\tau_k^-)),$$

Where $0 \leq c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$

Then the trivial solution of (2.1) is strict practical uniformly stable.

Proof: Since $\sum_{k=1}^{\infty} d_k < \infty$ and $\sum_{k=1}^{\infty} c_k < \infty$.

It follows that, $\prod_{k=1}^{\infty}(1+d_k)$ =Mand $\prod_{k=1}^{\infty}(1-c_k)$ =N, Obviously $1 \leq M < \infty$, $0 < N \leq 1$

Let $0 < A_1 < \rho$and $\sigma \geq t_0$ be given and σ∈ $[\tau_k, \tau_{k+1}]$ for some k ∈Z, Such that $Ma_1(\lambda_1) < b_1 A_1$

Then we claim that $\varphi \in PC_1(\lambda_1)$ implies $\|x(t)\| < A_1$, $t \geq \sigma$

Obviously for any t∈ [τ,σ], there exist θ∈ [τ,0], such that

$$V_1(t, x(t)) = V_1(\sigma + \theta, x(\sigma + \theta)) \leq a_1(\| x(\sigma + \theta) \|) =$$

$$a_1\| x_\sigma(\theta) | = a_1\| \varphi(\theta) \| \leq a_1(\lambda_1)$$

Then, we claim that

$$V_1(t, x(t)) \leq a_1, \quad \sigma \leq t < \tau_k \tag{3.1}$$

If the inequality (3.1), does not hold, then there exist a $\hat{t} \in (\sigma, \tau_k)$ such that

$$V_1(\hat{t}, x(\hat{t})) > a_1(\lambda_1) \geq V_1(\sigma, x(\sigma))$$

which implies that there exist a $\check{t} \in (\sigma, \hat{t}\,]$, such that

$$D^+V_1(\check{t}, x(\check{t})) > 0 \tag{3.2}$$

and

$$V_1(\check{t} + s, x(\check{t} + s)) \leq V_1(\hat{t}, (x(\hat{t})), \text{where } s \in [-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\check{t}, x(\check{t})) \leq 0$. This contradicts inequality (3.2)

So inequality (3.1) holds.

From condition (ii), we have

$$V_1(\tau_k, x(\tau_k)) = V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1+d_k)\, V_1(\tau_k, x(\tau_k^-)) \leq (1+ dk)\, a_1(\lambda 1)$$

Next, we claim that

$$V_1(t, x(t)) \leq (1 + dk) a_1(\lambda 1), \; \tau_k \leq t \leq \tau_{k+1} \tag{3.3}$$

Notes

Notes

If inequality (3.3) does not hold then, there exist $\hat{s} \in (\tau_k, \tau_{k+1})$, such that

$$V_1(\hat{s}, x(\hat{s})) \geq (1 + d_k)\, a_1(\lambda_1) \geq V_1(\tau_k, x(\tau_k))$$

which implies that there exist an $\check{s} \in (\tau_k, \tau_{k+1})$, such that

$$D^+V_1(\check{s}, x(\check{s})) > 0 \tag{3.4}$$

And

$$V_1(\check{s} + s), x(\check{s} + s)) \leq V_1(\hat{s}, (x(\hat{s})), \text{where } s \in [-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\hat{s}, x(\hat{s})) \leq 0$, This Contradicts inequality (3.4) so inequality (3.3) holds.
And from condition (ii), we have

$$V_1(\tau_{k+1}, x(\tau_{k+1}) = V_1(\tau_{k+1}, x(\tau_{k+1}{}^-) + I_k(x(\tau_{k+1}{}^-))) \leq (1 +$$

$$d_k)\, V_1(\tau_{k+1}, x(\tau_{k+1}{}^-)) \leq (1 + d_k)\,(1 + d_k)\, a_1(\lambda 1).$$

By a simple induction, we can easily prove that in general form for, m = 0,1,2,3 ……

$$V_1(t, x(t)) \leq (1 + d_{k+m}) \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. \leq (1 + d_k a_1(\lambda 1), \text{Where } \tau_{k+m} \leq t \leq \tau_{k+m+1}$$

Which together with inequality (3.1) and condition (i), we have

$$b_1(\|x(t)\|) \leq V_1(t, x(t)) \leq M\, a_1((\lambda_1) < b_1(A_1),\ t \geq \sigma$$

Thus we have

$$\|x(t)\| < A_1,\ t \geq \sigma$$

Now, let $0 < \lambda_2 \leq \lambda_1$ and choose $0 < A_2 < \lambda_2$, such that $a_2(A_2) < N b_2(\lambda_2)$.
Next, we claim that $\varphi \in PC_2(\lambda_2)$ implies $\| x(t) \| > A_2$, $t \geq \sigma$
If it holds,then $\varphi \in PC_1(\lambda_1) \cap PC_2(\lambda_2)$ implies

$$A_2 < \|x(t)\| < A_1,\ t \geq \sigma$$

Obviously for any, $t \in [\sigma - \tau, \sigma]$, there exist a $\theta \in [-\tau, 0]$, such that

$$V_2(t, x(t)) = V_2(\sigma + \theta, x(\sigma + \theta)) \geq b_2(\|x(\sigma + \theta)\|) = b_2\|x_\sigma(\theta)\|$$

$$= b_2\|\varphi(\theta)\| \geq b_2(\lambda_2)$$

Then, we claim that

$$V_2(t, x(t)) \geq b_2(\lambda_2),\ \sigma \leq t < \tau_k \tag{3.5}$$

If inequality (3.5) does not hold, then there exist $\bar{t}\in(\sigma, \tau_k)$, such that

$$V_2(\bar{t}, x(\bar{t})) < b_2(\lambda_2) \leq V_2(\sigma, x(\sigma))$$

Which implies that there exist a $t_1 \in(\sigma, \bar{t})$, such that

$$D^+V_2(t_1, x(t_1)) < 0 \tag{3.6}$$

And

$$V_2(t_1 + s, x(t_1 + s) \geq V_2(t_1, x(t_1)), \quad s\in [-\tau, 0]$$

By condition (iv), we have, $D^+V_2(t_1, x(t_1)) \geq 0$, This contradicts inequality (3.6) So, inequality (3.5) holds.
From condition (iv), we have

$$V_2(\tau_k, x(\tau_k)= V_2(\tau_k, x(\tau_k^-)+I_k(x(\tau_k^-)))\geq(1-c_k) V_2(\tau_k, x(\tau_k^-)) \geq (1-c_k)b_2(\lambda_2)$$

Next, we claim that

$$V_2(t, x(t))\geq(1-c_k)b_2(\lambda_2), \ \tau_k\leq t<\tau_{k+1} \tag{3.7}$$

If the inequality (3.7) does not hold then there exist an $\bar{r}\in(\tau_k, \tau_{k+1})$, such that

$$V_2(\bar{r}, x(\bar{r})) < (1- c_k) b_2(\lambda_2) \leq V_2(\tau_k, x(\tau_k))$$

Which implies that there exist an $\check{r}\in (\tau_k, \bar{r})$, such that

$$D^+V_2(\check{r}, x(\check{r}))<0 \tag{3.8}$$

And

$$V_2(\check{r}+ s, x(\check{r}+ s) \geq V_2(\check{r}, x(\check{r})), s\in [-\tau,0]$$

By condition (iv), we have, $D^+V_2(\check{r}, x(\check{r})) \geq 0$. Which contradicts inequality (3.8) So, inequality (3.7) holds
And from condition (iv), we have

$$V_2(\tau_{kk+1}, x(\tau_{kk+1})) = V_2(\tau_{k+1}, x(\tau_{k+1}^-) +I_k(x(\tau_{k+1}^-))) \geq (1- c_k)V_2(\tau_{k+1}, x(\tau_{k+1}^-))$$
$$\geq(1-c_{k+1})(1-c_k) b_2(\lambda_2).$$

And by a simple induction we can prove that, in general, for,m $= 0,1,2,3\ldots.$

$$V_2(t,x(t))\geq(1-c_{k+m})\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots.\leq(1-c_k)b_2(\lambda_2), \ \tau_{k+m}\leq t \leq \tau_{k+m+1}$$

Which together with inequality (3.5) and condition (iii), we have

$$a_2(\|x(t)\|) \leq V_2(t, x(t)) \leq Nb_2(\lambda_2) > a_2(A_2), \ \ t \geq \sigma$$

Notes

Thus we have, $\|x(t)\| > A_2$, $t \geq \sigma$
Thus the zero solution of (2.1) is strict practical uniformly stable.
The proof of theorem is complete.

IV. Conclusion

Notes

In this paper, we investigated the strict practical stability criteria in the form of theorem for impulsive functional differential equations, which is more useful as compared to practical stability. It gives rate of decay of the solution, so it is finer concept which can give us more precise information. In future we can modify this theorem to get less restricted conditions to verify strict practical stability.

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