



GLOBAL JOURNAL OF SCIENCE FRONTIER RESEARCH  
PHYSICS AND SPACE SCIENCE  
Volume 13 Issue 1 Version 1.0 Year 2013  
Type : Double Blind Peer Reviewed International Research Journal  
Publisher: Global Journals Inc. (USA)  
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

# Harmonic and Wave Maps Coupled with Einstein's Gravitation

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*GJSFR-A Classification* : *FOR Code: 020105*



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# Harmonic and Wave Maps Coupled with Einstein's Gravitation

Rainer Schimming<sup>α</sup> & Ragab M. Gad<sup>σ</sup>

**Abstract** - In this paper we discuss the coupled dynamics, following from a suitable Lagrangian, of a harmonic or wave map  $\phi$  and Einstein's gravitation described by a metric  $g$ . The main results concern energy conditions for wave maps, harmonic maps from warped product manifolds, and wave maps from wave-like Lorentzian manifolds.

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## 1. INTRODUCTION

Scalar fields on a space or space-time manifold  $(X, g)$ , which satisfy a linear or nonlinear field equation, attract enduring attention; cf, e.g., [1, 2, 3]. The *Dirichlet Lagrangian* or *energy density*

$$e = \frac{1}{2}|d\phi|^2 = \frac{1}{2}g^{ab}(\partial_a\phi)(\partial_b\phi), \quad (1.1)$$

of a one-component scalar field  $\phi = \phi(x)$  leads to a linear field equation of Laplace or D'Alembert type. Here we denote

$$g = g_{ab}dx^a dx^b, \quad (g^{ab}) := (g_{ab})^{-1}, \quad \partial_a := \frac{\partial}{\partial x^a}.$$

The Dirichlet Lagrangian  $e$  admits a natural generalization to a multi-component scalar field  $\phi = \phi(x)$  if the range of  $\phi$  is suitably geometrized, namely if  $im\phi$  lies in a Riemannian manifold  $(Y, h)$ . That means,  $\phi$  becomes a map  $X \rightarrow Y$ ,  $x \mapsto y$  between a source manifold  $(X, g)$  and a target manifold  $(Y, h)$ . The choice of the Lagrangian

$$e = e[\phi] = e[\phi, g] = \frac{1}{2}|d\phi|^2, \quad (1.2)$$

where now

$$|d\phi|^2 := tr(\phi^*h) \equiv g^{ab}(\partial_a\phi^i)(\partial_b\phi^j)h_{ij}(\phi^k),$$

leads to the generally semi-linear field equation with Laplace-like or D'Alembert-like principal part

$$tr(\nabla d\phi)^i \equiv g^{ab}\nabla_a\partial_b\phi^i = 0. \quad (1.3)$$

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Here we denote  $h = h_{ij}dy^i dy^j$  and the special covariant derivative  $\nabla$  is built from  $g, h, \phi$  as will be explained in section 2.

A map  $\phi : X \rightarrow Y$  between properly Riemannian manifolds  $(X, g), (Y, h)$  which satisfies (1.3) is called a *harmonic map* [4, 5]. A map  $\phi : X \rightarrow Y$  from a Lorentzian manifold  $(X, g)$  to a properly Riemannian manifold  $(Y, h)$  which satisfies (1.3) is called a *wave map* [6, 7, 8, 9, 3].

In this paper, we take  $(Y, h)$  as a fixed background and consider  $g$  and  $\phi$  as dynamical objects. The dynamics shall follow from the Lagrangian

$$L = \kappa R - e, \tag{1.4}$$

where  $R$  denotes the scalar curvature of  $g$  and  $\kappa \neq 0$  is a coupling constant. Variation with respect to  $g$  yields, for  $\dim X \geq 3$ , the *Einstein equation* in the form

$$\kappa Ric = \phi^* h, \tag{1.5}$$

in components

$$\kappa R_{ab} = (\partial_a \phi^i)(\partial_b \phi^j) h_{ij}. \tag{1.6}$$

Variation with respect to  $\phi$  yields (1.3).

A Lorentzian metric  $g$  can be interpreted as *gravitation*. According to the Kaluza-Klein principle, the space-time manifold  $(X, g)$  may have a higher dimension.

A positive definite metric  $g$  on  $X$  can be given a physical interpretation through a Lorentzian metric constructed from it as follows. Consider the product manifold  $\tilde{X} := \mathfrak{R} \times X$  with points  $(t, x)$  and equip it with the Lorentzian metric  $\tilde{g} := dt^2 - g$ . Extend  $\phi : X \rightarrow Y$  to  $\tilde{\phi} : \tilde{X} \rightarrow Y$  by setting  $\tilde{\phi}(t, x) := \phi(x)$ , that means  $\tilde{\phi} := \phi \circ pr_2$ , where  $pr_2$  is the projection  $\tilde{X} \rightarrow X$  to the second factor. The field equations (1.5), (1.3) for  $(\tilde{X}, \tilde{g}), \tilde{\phi}$  reduce to (1.5), (1.3) for  $(X, g), \phi$ .

Let us sketch our main results.

- The energy-momentum tensor  $T$  of a wave map obeys several energy conditions. In particular, if  $v$  is a causal (i.e. non-spacelike) vector field then the momentum one-form  $I := T(\cdot, v)$  is causal again.
- The Einstein equation (1.5) implies that the conditions  $Ric(v, v) = 0, Ric(\cdot, v) = 0, \phi_* v = 0$  for a vector field  $v$  are equivalent to each other. Moreover, then  $v$  is a Ricci collineation, i.e.  $\mathcal{L}_v Ric = 0$ . Some conclusions are drawn from this latter fact.
- A submersive map  $\phi : X \rightarrow Y$  between a pure manifold  $X$  and a Riemannian manifold  $(Y, h)$  can locally be made to a harmonic or wave map by a suitable choice of a metric  $g$  on  $X$ .
- We study the case of warped product  $X = 'X \times ''X, g = 'g \oplus w^2 ''g$ . Several propositions are proved by means of the argument that the integral of a Laplace expression over a closed manifold  $'X$  vanishes.
- We study radiation conditions for a Lorentzian manifold  $(X, g)$ . The Einstein equation (1.5) leads from one condition to a stronger condition.



## II. PRELIMINARIES

We consider dimensions

$$m := \dim X \geq 3, \quad n := \dim Y \geq 1$$

and adopt the following index convention: indices  $a, b, c, \dots$  label the components of geometric objects on  $X$ ; indices  $i, j, k, \dots$  label the components of geometric objects on  $Y$ .

Tensor fields on  $Y$  become multi-component scalar fields on  $X$  by insertion of  $y = \phi(x)$ , where  $x \in X, y \in Y$ . For covariant tensor fields on  $Y$  there is also the conventional pull-back map  $\phi^*$ . For instance, the metric  $h = h_{ij}dy^i dy^j$  yields  $h \circ \phi$  with components  $h_{ij}(\phi^k) = h_{ij}(\phi^k(x^a))$  on  $X$  and also the pull-back  $\phi^*h$  on  $X$  with components  $(\partial_a \phi^i)(\partial_b \phi^j)(h_{ij}(\phi^k))$ . The object  $\phi^*h$  is called *first fundamental form* of  $\phi : X \rightarrow Y$ .

We will occasionally write

$$\phi^*h = h(d\phi, d\phi)$$

and also use the bilinear symmetric form  $h(\cdot, \cdot)$  with tensorial entries.

Some natural covariant derivative  $\nabla$  with components  $\nabla_a$  is built from  $g, h, \phi$  according to the following rules.

1-  $\nabla$  applied to tensor fields on  $X$  equals the Levi-Civita derivative  ${}^g\nabla$  to  $g$ . For instance,

$$\nabla_a v^c := \partial_a v^c + {}^g\Gamma_{ab}^c v^b,$$

where  $v = v^a \partial_a$  is a vector field on  $X$  and  ${}^g\Gamma_{ab}^c$  are the Christoffel symbols to  $(g_{ab})$ .

2-  $\nabla$  applied to tensor fields on  $Y$  equals some pull-back of the Levi-Civita derivative  ${}^h\nabla$  to  $h$ . For instance,

$$\nabla_a w^k := (\partial_a \phi^i)(\partial_i w^k + {}^h\Gamma_{ij}^k w^j),$$

where  $w = w^i \partial_i$  is a vector field on  $Y$  and  ${}^h\Gamma_{ij}^k$  are the Christoffel symbols to  $(h_{ij})$ . Here  $\nabla_a w^k$  is understood to depend on  $y^l = \phi^l(x^c)$ .

3- There are natural product rules for mixed quantities, the components of which carry both indices  $a, b, \dots$  and  $i, j, \dots$ . For instance, the *differential*  $d\phi$  of  $\phi : X \rightarrow Y$  with components  $\partial_a \phi^i$  is a mixed tensor field. The covariant derivative  $\nabla d\phi$  of  $d\phi$  is called *second fundamental form* of  $\phi$ . It has the components

$$\nabla_a \partial_b \phi^k = \partial_a \partial_b \phi^k - {}^g\Gamma_{ab}^c \partial_c \phi^k + {}^h\Gamma_{ij}^k (\partial_a \phi^i)(\partial_b \phi^j)$$

and the symmetry property

$$\nabla_a \partial_b \phi^k = \nabla_b \partial_a \phi^k.$$

More on calculus for maps between (pseudo-) Riemannian manifolds can be found in the literature, e.g. [4, 5].

### III. THE FIELD EQUATIONS

The field theory for  $g$  and  $\phi$  considered here is based on the Lagrangian

$$L = \kappa R - e, \tag{3.1}$$

which is the sum of the *gravitational Lagrangian*  $\kappa R$  and the *matter Lagrangian*  $e$ . Here  $R = R[g]$  denotes the scalar curvature of  $g$ ,  $\kappa \neq 0$  is a coupling constant and  $e = e[\phi] = e[g, \phi]$  is given by (1.2).

The idea to couple a harmonic map, formerly also called *sigma model*, with Einstein's gravitation appeared in [10, 11, 12] and other early papers.

The following is well-known [10, 11, 12, 13, 14, 15, 16, 17, 18, 7, 8, 9]. We abbreviate  $\det g := \det(g_{ab})$ .

**Proposition 3.1** *Variation of  $L|\det g|^{\frac{1}{2}}$  with respect to  $g$  yields the Einstein equation in the form*

$$\kappa(\text{Ric} - \frac{1}{2}Rg) = \phi^*h - eg. \tag{3.2}$$

If  $m = \dim X \geq 3$  then this is equivalent to

$$\kappa \text{Ric} = \phi^*h. \tag{3.3}$$

Variation of  $L|\det g|^{\frac{1}{2}}$  with respect to  $\phi$  yields

$$\text{tr}(\nabla d\phi) = 0, \tag{3.4}$$

where  $\nabla d\phi$  is the second fundamental form of  $\phi$  and the trace  $\text{tr}$  refers to the metric  $g$ .

The right-hand side of (3.2)

$$T := \phi^*h - eg \tag{3.5}$$

is called *energy-momentum tensor of  $\phi$* . There holds  $e = \frac{1}{2}\text{tr}(\phi^*h)$  and (3.5) is equivalent to

$$T - (m - 2)^{-1}(\text{tr}T) = \phi^*h.$$

**Proposition 3.2** *From the field equation (3.4) for  $\phi$  there follows that  $T$  is divergence-free:*

$$\nabla^b T_{ab} = 0.$$

The *proof* follows from the identity

$$\nabla^b T_{ab} = h_{ij}(\partial_a \phi^i)(\text{tr} \nabla d\phi)^j.$$

### IV. ENERGY CONDITIONS FOR A WAVE MAP

Let now the metric  $g$  have Lorentzian signature  $(+ - \dots -)$ .

**Definition 4.1** Let a symmetric two-form  $T = T_{ab}dx^a dx^b$  on  $X$  be interpreted as an energy-momentum tensor field and let  $v = v^a \partial_a$  be a unit timelike vector field on  $X$ , i.e.,  $v_a v^a \equiv g_{ab}v^a v^b = 1$ . Then  $T(v, v) \equiv$

$T_{ab}v^av^b$  is called *energy density*,  $I := T(\cdot, v)$  with components  $I_a := T_{ab}v^b$  is called *momentum one-form*, and  $J := I - T(v, v)v$  with components  $J_a := I_a - T(v, v)v_a$  is called *proper momentum one-form*.

Physically,  $v$  is interpreted as the unit tangent vector to the world line of an observer. This observer measures the quantities  $T(v, v)$ ,  $I$ ,  $J$ .

Every unit timelike vector field  $v = v^a\partial_a$  on  $X$  gives rise to a positive definite metric  $g^+ = g_{ab}^+dx^adx^b$  on  $X$  with components  $g_{ab}^+ = 2v_av_b - g_{ab}$ . The inverse  $(g_+^{ab}) := (g_{ab}^+)^{-1}$  has the representation  $g_+^{ab} = 2v^av^b - g^{ab}$ .

**Theorem 4.1** *Consider the energy-momentum tensor*

$$T = \phi^*h - eg \tag{4.1}$$

of a map  $\phi : X \rightarrow Y$  between  $(X, g)$  and  $(Y, h)$ . The energy density equals

$$T(v, v) = e_+ := e[\phi, g_+] \equiv \frac{1}{2}g_+^{ab}(\partial_a\phi^i)(\partial_b\phi^j)h_{ij}. \tag{4.2}$$

It is a positive definite quadratic form in  $d\phi$ . The momentum one-form  $I$  obeys the estimate

$$e^2 \leq |I|^2 \leq e_+^2, \tag{4.3}$$

where

$$|I|^2 := I_aI^a \equiv g^{ab}I_aI_b. \tag{4.4}$$

**Proof:**

Let us abbreviate  $f := \phi^*h$  with components  $f_{ab} := (\phi^*h)_{ab} = (\partial_a\phi^i)(\partial_b\phi^j)h_{ij}$ . We calculate

$$\begin{aligned} T(v, v) &= T_{av}v^av^b = (f_{ab} - eg_{ab})v^av^b \\ &= f_{ab}v^av^b - e = \frac{1}{2}(2v^av^b - g^{ab})f_{ab} = \frac{1}{2}g_+^{ab}f_{ab} = e_+. \end{aligned}$$

The proper momentum one-form  $J$  is orthogonal to  $v$ , that means  $J_av^a = 0$ . Considering that, we find that

$$\begin{aligned} 0 &\leq g_+^{ab}J_aJ_b = -g^{ab}J_aJ_b \\ &= -g^{ab}J_a(I_b - e_+v_b) = -g^{ab}J_aI_b \\ &= -g^{ab}(I_a - e_+v_a)I_b = -I_aI^a + e_+^2. \end{aligned}$$

Thus, the right-hand side inequality of (4.3) is proved. In order to prove the left-hand side of (4.3), we start with the remark that the matrix  $(f_{ab})$  is positive semi-definite. Let us consider a fixed point  $x_0 \in X$  and use coordinates  $x^a$  such that

$$v^a = \delta_0^a, \quad g_+^{ab} = \delta^{ab} \tag{4.5}$$

in that very point  $x_0$ . Actually, such coordinates exist. The following  $2 \times 2$  subdeterminants of  $(f_{ab})$  are non-negative:

$$\begin{aligned} f_{00}f_{11} - f_{10}f_{10} &\geq 0, \\ f_{00}f_{22} - f_{20}f_{20} &\geq 0, \end{aligned}$$

$$\dots$$

$$f_{00}f_{mm} - f_{m0}f_{m0} \geq 0.$$

Let us sum up:

$$f_{00}f_{aa} - f_{a0}f_{a0} \geq 0. \tag{4.6}$$

Here a summation convention applies to the index  $a$  and the coordinate conditions (4.5) are assumed. The inequality (4.6) can be brought into a coordinate-invariant form

$$g_+^{ab}(f_{ab}v^c v^d f_{cd} - v^c f_{ac} v^d f_{bd}) \geq 0.$$

Here we insert

$$g_+^{ab} f_{ab} = 2e_+, \quad v^c v^d f_{cd} = e + e_+,$$

$$v^c f_{ac} = v^c(T_{ac} + eg_{ac}) = I_a + ev_a,$$

$$g_+^{ab}(I_a + ev_a)(I_b + ev_b) = 2e_+(e + e_+) + e^2 - |I|^2.$$

Taking all this together, the left-hand side inequality of (4.3) follows. The conditions

$$T(v, v) \geq 0, \quad |I|^2 \geq 0$$

together form the *dominant energy condition*, which expresses that the energy density is non-negative and that the momentum  $I$  is causal. The latter condition physically means that the momentum  $I$  propagates with a velocity which is not greater than the velocity of light.

The so-called *strongy energy condition* also holds in the present situation, namely in the form

$$(m - 2)T(v, v) \geq trT.$$

**Theorem 4.2** Consider the energy-momentum tensor  $T = T_{ab}dx^a dx^b$  to  $\phi$  as above and lightlike vector fields  $l = l^a \partial_a$ ,  $n = n^a \partial_a$  such that  $l^a n^a \equiv g_{ab} l^a n^b = 1$ . Then

$$T(l, l) \equiv T_{ab} l^a l^b = h(\phi_* l, \phi_* l) \geq 0, \tag{4.7}$$

and the one-form  $I := T(\cdot, l)$  with components  $I_a := T_{ab} l^b$  obeys

$$0 \leq |I|^2 \leq 2T(l, l)T(l, n). \tag{4.8}$$

**Proof:**

Assertion (4.7) follows from

$$T(l, l) = (f_{ab} - eg_{ab})l^a l^b = f_{ab} l^a l^b = h_{ij}(l^a \partial_a \phi^i)(l^b \partial_b \phi^j).$$

The projection tensor  $p$  with components

$$p_{ab} := l_a n_b + n_a l_b - g_{ab}$$

is a useful tool. It is orthogonal to  $l$  and  $n$ , i.e.

$$p_{ab} l^b = p_{ab} n^b = 0,$$

and it is positive semi-definite. Hence

$$0 \leq p^{ab}I_aI_b = (2l^an^b - g^{ab})I_aI_b = 2T(l, l)T(l, n) - |I|^2,$$

which proves the right-hand side inequality of (4.8)

Below we will also use

$$p^{ab}T_{ab} = (2l^an^b - g^{ab})T_{ab} = 2T(l, n) - trT = 2T(l, n) + (m - 2)e.$$

Let us, in order to complete the proof, consider a fixed point  $x_0 \in X$  and use coordinates  $x^a$  such that

$$l^a = l_0^a, \quad n^a = \delta_1^a, \quad (p_{ab}) = diag(0, 0, 1, \dots, 1)$$

in that very point  $x_0$ , where *diag* indicates a diagonal matrix. Formally the same  $2 \times 2$  subdeterminants of  $(f_{ab})$  as in the preceding proof are non-negative. Their sum is now in another way transformed into a coordinate-invariant form

$$p^{ab}[f_{ab}(l^cl^df_{cd}) - (l^cf_{ac})(l^df_{bd})] \geq 0.$$

Here we insert

$$p^{ab}f_{ab} = p^{ab}(T_{ab} + eg_{ab}) = p^{ab}T_{ab} - (m - 2)e = 2T(l, n),$$

$$l^cl^df_{cd} = T(l, l), \quad p^{ab}(l^cf_{ac})(l^df_{bd}) = p^{ab}I_aI_b = 2T(l, l)T(l, n) - |I|^2.$$

The result  $|I|^2 = g^{ab}I_aI_b \geq 0$  follows.

Physically, theorem (4.2) can be interpreted in terms of a fictional observer which moves faster and faster. In the limit, he reaches the velocity of light and  $v$  turns to  $l$ . The energy density  $T(l, l)$  then remains non-negative and the momentum  $I$  remains causal.

**Corollary 4.3** *There holds  $T(l, n) \geq 0$ . Especially,  $T(l, n) = 0$  iff  $I_a = T(l, l)n_a$ .*

**Proof:**

The formulas (4.7), (4.8) imply  $T(l, n) \geq 0$ . If  $T(l, n) = 0$  then  $|I|^2 = 0$  and

$$0 = p^{ab}(l^cf_{bc}) = p^{ab}I_b = (l^an^b + n^al^b - g^{ab})I_b = T(l, l)n^a - I^a$$

## V. IMPLICATIONS OF THE EINSTEIN EQUATION

Let us study

$$\kappa Ric = \phi^*h, \tag{5.1}$$

for given background  $(Y, h)$  as a relation between  $g$  and  $\phi$ . Contraction with  $g^{-1}$  yields

$$\kappa R = 2e. \tag{5.2}$$

Double Contraction with a vector field  $v = v^a\partial_a$  on  $X$  yields

$$\kappa Ric(v, v) = h(\phi_*v, \phi_*v), \tag{5.3}$$

with the interpretation that  $y = \phi(x)$  is to be inserted into the right-hand side of (5.3);  $\phi_*v$  denotes the push-forward of  $v$  with respect to  $\phi$ . As a conclusion,  $\kappa Ric(v, v)$  is positive definite in  $\phi_*v$  and is positive semi-definite in  $v$ .

**Proposition 5.1** *From (5.1) there follows that Ric and  $d\phi$ , interpreted as linear map, have the same rank:*

$$r := \text{rank}(Ric) = \text{rank}(d\phi). \tag{5.4}$$

*In particular:*

$r = 0$  iff  $\phi$  is constant.

$r = m \equiv \dim X$  iff  $\phi$  is an immersion.

$r = n \equiv \dim Y$  iff  $\phi$  is a submersion.

$r = m = n$  iff  $\phi$  is a local diffeomorphism.

The proof of (5.4) is an exercise in linear algebra.

Note that  $r = 0$  means that  $(X, g)$  is Ricci flat, i.e.,  $Ric = 0$ .

**Proposition 5.2** *If (5.1) holds then the conditions*

$$Ric(v, v) = 0, \tag{5.5}$$

$$Ric(\cdot, v) = 0, \tag{5.6}$$

$$\phi_*v = 0 \tag{5.7}$$

for a vector field  $v = v^a \partial_a$  on  $X$  are equivalent to each other. Moreover, they imply

$$\mathcal{L}_v Ric = 0, \tag{5.8}$$

where  $\mathcal{L}_v$  denotes the Lie derivative with respect to  $v$ .

**Proof:**

Equation (5.3) in components reads

$$\kappa R_{ab} v^a v^b = h_{ij} (\phi_*v)^i (\phi_*v)^j,$$

where  $(\phi_*v)^i = v^a \partial_a \phi^i$ . Moreover, (5.1) implies

$$\kappa R_{ab} v^b = h_{ij} (\partial_a \phi^i) (\phi_*v)^j.$$

These formulas and the definiteness of  $h$  give the first assertion. Next, we use comoving coordinates  $x^a$ , which are adapted to  $v$ , that means  $v^a = \delta_0^a$ , and we get

$$(\phi_*v)^i = v^a \partial_a \phi^i = \partial_0 \phi^i,$$

$$\kappa \mathcal{L}_v R_{ab} = \kappa \partial_0 R_{ab} = \partial_0 (\partial_a \phi^i \partial_b \phi^j h_{ij}(\phi^k)).$$

If  $\partial_0 \phi^i = 0$  then also  $\partial_0 R_{ab} = 0$ . This fact can be translated into the second assertion.

**Proposition 5.3** *If the Ricci tensor vanishes on the vectors of some integrable distribution in the tangent bundle  $TX$  and (5.1) holds then  $\phi$  is constant on each leaf of the foliation to the distribution.*



**Proof:**

A distribution of rank  $s$  in  $TX$  is integrable iff it admits adapted coordinates  $x^a$ , which means that the distribution is locally spanned by the coordinate vector fields  $\partial_1, \partial_2, \dots, \partial_s$ . The assumption becomes

$$Ric(\partial_a, \partial_b) = 0 \quad \text{for } a, b = 1, 2, \dots, s.$$

Proposition (5.2) then implies

$$\partial_1 \phi^i = 0, \partial_2 \phi^i = 0, \dots, \partial_s \phi^i = 0.$$

Hence  $\phi^i$  does not depend on  $x^1, x^2, \dots, x^s$  and is constant if the point  $x$  varies in a leaf of the foliation, i.e., if  $x^{s+1} = \text{const.}, \dots, x^m = \text{const.}$

**Proposition 5.4** *The Einstein equation (5.1) implies*

$$\kappa(\nabla_a R_{bc} + \nabla_b R_{ca} - \nabla_c R_{ab}) = 2h_{ij}(\nabla_a \partial_b \phi^i)(\partial_c \phi^j). \tag{5.9}$$

**Proof:**

Covariant differentiation of (5.1) gives

$$\kappa \nabla_c R_{ab} = h_{ij}[(\nabla_c \partial_a \phi^i)(\partial_b \phi^j) + (\partial_a \phi^i)(\nabla_c \partial_b \phi^j)].$$

Some rearrangement yields (5.9).

**Proposition 5.5** *The Einstein equation (5.1) implies*

$$h(\text{tr}(\nabla d\phi), d\phi) = 0. \tag{5.10}$$

*This equation for a submersion  $\phi$  implies the harmonic or wave map equation (3.4).*

**Proof:**

The Einstein tensor  $Ric - \frac{1}{2}Rg$  is divergence-free. This fact and (5.9) imply  $2h(\text{tr}(\nabla d\phi), d\phi)_c \equiv 2h_{ij} \text{tr}(\nabla d\phi)^i(\partial_c \phi^j) = \kappa g^{ab}(\nabla_a R_{bc} + \nabla_b R_{ca} - \nabla_c R_{ab}) = 0$ .

If  $\phi$  is a submersion, then the matrix with elements  $h(\cdot, d\phi)_{ic} = h_{ij} \partial_c \phi^j$  has maximal rank and therefore (5.10) implies (3.4).

**Proposition 5.6** *If the Ricci tensor is covariantly constant, i.e.,  $\nabla Ric = 0$ , and the Einstein equation (5.1) holds for a submersion  $\phi$ , then  $\phi$  is totally geodesic, that means*

$$\nabla d\phi = 0. \tag{5.11}$$

**Proof:**

If  $\nabla Ric = 0$  then (5.9) reduces to  $h_{ij}(\nabla_a \partial_b \phi^i)(\partial_c \phi^j) = 0$ . If, additionally,  $\phi$  is a submersion, then (5.11) follows by means of the rank argument already used in the preceding proof.

The next proposition needs some preparation. A diffeomorphism  $f : X \rightarrow X$  is called a *Ricci symmetry* iff

$$f^* Ric = Ric. \quad (5.12)$$

A vector field  $v = v^a \partial_a$  on  $X$  is called an *infinitesimal Ricci symmetry* or a *Ricci collineation* iff

$$\mathcal{L}_v Ric = 0. \quad (5.13)$$

It is well known that the flow  $f_t$  for  $t \in I$  of a Ricci collineation  $v$  is a one-parameter family of Ricci symmetries, i.e.  $f_t^* Ric = Ric$ . Here  $x = f_t(x_0)$  by definition represents the solution of the initial-value problem

$$\frac{dx}{dt} = v(x), \quad x|_{t=0} = x_0$$

and  $I$  denotes an open interval which contains 0.

**Proposition 5.7** *Let  $v = v^a \partial_a$  be a Ricci collineation of  $(X, g)$ . Then the Einstein equation (5.1) implies that  $\phi : X \rightarrow Y$  is invariant under the flow  $f_t$  for  $t \in I$  of  $v$ , that means*

$$\phi \circ f_t = \phi. \quad (5.14)$$

**Proof:**

Let us again use comoving coordinates such that  $v^a = \delta_0^a$ . In these coordinates,  $f_t$  is expressed by a translation  $x^0 \mapsto x^0 + t$ ,  $x^1 \mapsto x^1$ , ...,  $x^{m-1} \mapsto x^{m-1}$ . We know already from the proof of proposition (5.2)  $\partial_0 \phi^i = 0$ , i.e. each  $\phi^i$  is independent of  $x^0$ . Hence  $\phi^i = \phi^i(x^a)$  does not change under  $x^0 \mapsto x^0 + t$ , which is just expressed by (5.14).

The following definition is useful.

**Definition 5.1** A property of subsets of a manifold  $X$  holds *globally* if it is valid for  $X$ . It holds *locally* if every point  $x_0 \in X$  has a neighborhood  $U$  such that the property is valid for  $U$ .

**Theorem 5.8** *Let  $\phi : X \rightarrow Y$  be a submersion between smooth manifolds  $X, Y$  and let  $Y$  be equipped with a positive definite metric  $h$ .*

1- *Locally there is a positive definite metric  $g$  on  $X$  such that  $\phi$  becomes a harmonic map  $(X, g)$  and  $(Y, h)$ .*

2- *Locally there is a Lorentzian metric  $g$  on  $X$  such that  $\phi$  becomes a wave map.*

**Proof:**

We consider (5.1) and use the fact that the *problem of prescribed Ricci curvature* is locally solvable in the two cases [19, 20]. More precisely: cf., eg.

1- Einstein's equation (5.1) locally has a positive definite solution  $g$ . It can be constructed, e.g., through some boundary value problem [19]. By assumption,  $\phi$  is a submersion; proposition 5.5 implies  $tr(\nabla d\phi) = 0$ .

2- Einstein's equation (5.1) locally has a Lorentzian solution  $g$ . It can be constructed, e.g., through some Cauchy initial value problem cf., e.g. [20]. An argument like in item 1 completes the proof.

VI. PRODUCT AND WARPED PRODUCT SOURCE MANIFOLDS

**Definition 6.1:** The *product*  $(X, g)$  of two (pseudo-) Riemannian manifolds  $({}'X, {}'g)$ ,  $({}''X, {}''g)$  is given by  $X = {}'X \times {}''X$  as a product of manifolds and by

$$g(u, v) = {}'g({}'u, {}'v) + {}''g({}''u, {}''v)$$

where  $u, v$  are vector fields on  $X$ ,  $'u, 'v$  are the push-forwards of  $u, v$  with respect to the projection  $X \rightarrow {}'X$ , and  $''u, ''v$  are the push-forwards of  $u, v$  with respect to the projection  $X \rightarrow {}''X$ .

We write then

$$g = {}'g \oplus {}''g,$$

$$\dim {}'X = {}'m, \quad \dim {}''X = {}''m, \quad m = {}'m + {}''m.$$

We apply the index convention

$$'a, 'b, 'c, \dots = 1, 2, \dots, {}'m; \quad ''a, ''b, ''c, \dots = {}'m + 1, {}'m + 2, \dots, m.$$

**Definition 6.2:** The *warped product*  $(X, g)$  is given by  $X = {}'X \times {}''X$  like above and by

$$g(u, v) = {}'g({}'u, {}'v) + w^2 {}''g({}''u, {}''v),$$

where the *warping function*  $w$  is a map  $'X \rightarrow \mathfrak{R}$  with positive values  $w > 0$ . We write then

$$g = {}'g \oplus w^2 {}''g,$$

The following is known.

**Proposition 6.1** *If  $(X, g)$  is the product of  $({}'X, {}'g)$ ,  $({}''X, {}''g)$  then the Einstein equation  $\kappa Ric = \phi^*h \equiv h(d\phi, d\phi)$  decomposes into the two Einstein equations*

$$\kappa {}'Ric = h({}'d\phi, {}'d\phi), \quad \kappa {}''Ric = h({}''d\phi, {}''d\phi) \tag{6.1}$$

and the orthogonality condition with respect to  $h$ :

$$h({}'d\phi, {}''d\phi) = 0, \tag{6.2}$$

in components

$$\kappa {}'R_{a'b} = h_{ij}(\partial_a \phi^i)(\partial_b \phi^j), \quad \kappa {}''R_{a''b''} = h_{ij}(\partial_{a''} \phi^i)(\partial_{b''} \phi^j), \tag{6.3}$$

$$h_{ij}(\partial_a \phi^i)(\partial_{b''} \phi^j) = 0 \tag{6.4}$$

**Proposition 6.2** *If  $(X, g)$  is the warped product of  $({}'X, {}'g)$ ,  $({}''X, {}''g)$  with warping function  $w$  then the Einstein equation  $\kappa Ric = \phi^*h \equiv h(d\phi, d\phi)$  decomposes into*

$$\kappa({}'Ric - {}''mw^{-1}{}'\nabla'dw) = h({}'d\phi, {}'d\phi), \tag{6.5}$$

$$\kappa({}''Ric - \frac{1}{{}''m}w^{2-{}''m}{}''\Delta w{}''^m g) = h({}'d\phi, {}''d\phi), \tag{6.6}$$

$$h({}'d\phi, {}''d\phi) = 0. \tag{6.7}$$

The trace parts of (6.5), (6.6) read

$$\kappa('R - {}''mw^{-1}'\Delta w) = 2'e, \tag{6.8}$$

$$\kappa(''R - w^{2-''m}'\Delta w''^m) = 2''e. \tag{6.9}$$

The proof is an exercise in higher differential geometry

We say that  $\phi : 'X \times ''X \rightarrow Y$  does not depend on  $'x$  iff the restriction  $\phi(\cdot, ''x) : 'X \rightarrow Y$  is a constant map for every  $''x \in ''X$ . We say that  $\phi$  does not depend on  $''x \in ''X$  in the analogous situation.

**Theorem 6.3** *Let in the situation of proposition 6.2 the restriction  $\phi('x, \cdot) : ''X \rightarrow X$  be a submersion for every  $'x \in 'X$ . Then  $\phi$  does not depend on  $'x$ .*

**Proof:**

If every  $\phi('x, \cdot)$  is a submersion then the quantities  $h_{ij}\partial_{''b}\phi^j$  in (6.4) form a matrix of maximal rank, and (6.4) implies  $\partial_a\phi^i = 0$ , which gives the assertion.

**Theorem 6.4** *Let in the situation of proposition 6.2 the first factor  $('X, 'g)$  be properly Riemannian and closed. Then the following holds.*

- (i) *The symmetric two-form  $\kappa''Ric$  is positive semi-definite.*
- (ii) *If  $\kappa''Ric$  is not everywhere positive then  $w = \text{const}$ .*
- (iii)  *$\kappa \int_X w'Rd'vol \geq 0$ .*
- (iv) *If  $\int_X w'Rd'vol = 0$  then  $w = \text{const}$ . and  $\phi$  does not depend on  $'x$ .*

**Proof:**

(i) Multiply (6.6) by  $w''^{m-2}$  and evaluate the two-forms at a vector field  $''v \neq 0$  on  $''X$ :

$$\kappa[w''^{m-2}''Ric(''v, ''v) - \frac{1}{''m}(''\Delta w''^m)|''v|^2] = w''^{m-2}h(''d\phi(''v), ''d\phi(''v)). \tag{6.10}$$

Integrate this equation over  $'X$ ; the Laplacian term does not contribute, hence

$$\kappa''Ric(''v, ''v) \int_X w''^{m-2}d'vol = \int_X w''^{m-2}h(''d\phi(''v), ''d\phi(''v))d'vol.$$

Both the integrals are non-negative, hence  $\kappa''Ric(''v, ''v) \geq 0$  for every  $''v$ .

(ii) If  $''Ric(''v, ''v)(''x_0) = 0$  for some point  $''x_0 \in ''X$  then

$$\int_X w''^{m-2}h(''d\phi(''v(''x_0)), ''d\phi(''v(''x_0)))d'vol = 0,$$

which implies  $''d\phi(''v(''x_0)) = 0$ . Evaluation of (6.10) at the point  $''x_0$  reduces (6.10) to

$$(''\Delta w''^m)|''v(''x_0)|^2 = 0.$$

We can assume  $|''v(''x_0)|^2 \neq 0$ ; hence  $w''^m$  is a harmonic function. A harmonic function on a closed manifold is constant.

(iii) Multiply (6.8) by  $w$  and integrate then over  $'X$ . The Laplacian term does not contribute; hence

$$\kappa \int_X w'Rd'vol = \int_X v'ed'vol \geq 0.$$

(iv) The last integral vanishes only if  $\nabla e = 0$ , which implies  $\nabla \phi = 0$ , hence  $\phi(\cdot, x) = \text{const}$  for fixed  $x \in X$ . But then  $e(\cdot, x) = \text{const}$  and the standard separation argument can be applied to

$$\kappa w^{2-m} \Delta w^m = \kappa R - 2e. \tag{6.11}$$

Thus both sides of (6.11) equal a constant  $c$ . Integration of

$$\kappa \Delta w^m = c w^{m-2}$$

yields  $c = 0$ . Hence  $w^m$  is a harmonic function on the closed manifold  $X$ . We find  $w^m = \text{const}$ ,  $w = \text{const}$ .

**Proposition 6.5** *Let in the situation of the preceding theorem the second factor  $(X, g)$  be properly Riemannian with vanishing scalar curvature:  $R = 0$ . Then  $w = \text{const}$  and  $\phi$  does not depend on  $x \in X$ .*

**Proof:**

Now equation (6.9) reduces to

$$-\kappa \Delta w^m = 2w^{m-2} e$$

and we have  $e \geq 0$ . Integration over the closed manifold  $X$  yields

$$0 = \int_X w^{m-2} e \, d\text{vol},$$

which implies

$$e \equiv \text{tr } h(\nabla \phi, \nabla \phi) = 0.$$

Hence  $\nabla \phi = 0$ , i.e.  $\phi$  does not depend on  $x$ , and  $w^m$  becomes a harmonic function. We arrive at  $w^m = \text{const}$ ,  $w = \text{const}$ .

**Example 6.1.** Let in the situation of proposition 6.2 the first factor  $(X, g)$  be the unit circle  $S^1$ . It is a flat properly Riemannian closed manifold. Theorem 6.4 implies that  $w = \text{const}$  and  $\phi$  does not depend on  $x$ .

**Example 6.2.** A static metric  $g = w^2 dt^2 - g$  can be interpreted as a warped product metric on  $X = X \times X$  where there the second factor  $X$  is one-dimensional. The proof of proposition 6.5 works, with a slight modification, for this case. Hence Einstein's equation  $\kappa \text{Ric} = \phi^* h$  implies that  $w = \text{const}$  and  $\phi$  does not depend on  $t \in X$ .

## VII. SOURCE MANIFOLDS WITH RADIATION CONDITIONS

Lichnerowicz [21] proposed *conditions of pure radiation* for a Lorentzian manifold  $(X, g)$ :

$$R_{abcd} l^d = 0, \quad R_{ab[cd} l_{e]} = 0,$$

where  $R_{abcd}$  are the components of the Riemann curvature tensor and  $l = l^a \partial_a$  is a lightlike vector field. Bel [22] proposed weaker conditions:

$$R_{abcd} l^b l^d = 0, \quad l^b R_{ab[cd} l_{e]} = 0, \quad l_{[f} R_{ab][cd} l_{e]} = 0.$$



There are also radiation conditions for the Ricci tensor  $Ric$  or for the *Einstein tensor*  $G := Ric - \frac{1}{2}Rg$  with components

$$G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab},$$

namely

$$R_{ab}l^b = 0, \quad R_{a[b]c} = 0,$$

$$G_{ab}l^b = 0, \quad G_{a[b]c} = 0.$$

One of the present authors studied radiation conditions in [23, 24].

**Theorem 7.1** *If the Einstein equation  $\kappa Ric = \phi^*h$  holds then the conditions*

$$G_{ab}l^b = 0, \tag{7.1}$$

$$R_{a[b]c} = 0 \tag{7.2}$$

for a lightlike vector field  $l = l^a\partial_a$  are equivalent to each other.

**Proof:**

There is another lightlike vector field  $n = n^a\partial_a$  such that  $l^an_a = 1$ . Then the sum  $v := l + n$  is timelike with  $v^av_a = 2$  and  $g_{ab}^+ = v_av_b - g_{ab}$  are the components of a positive definite metric  $g_+$ .

Let us consider

$$\psi_a^i := 2v^bl_{[a}\partial_b]\phi^i \equiv l_a(\phi_*v)^i - \partial_a\phi^i$$

and calculate

$$g_+^{ab}h_{ij}\psi_a^i\psi_b^j = -g^{ab}h_{ij}\psi_a^i\psi_b^j = h_{ij}(\phi_*v)^i(\phi_*l)^j - e = \kappa G_{ab}v^al^b.$$

Here we made use of

$$g_+^{ab} = v^av^b - g^{ab}, \quad v^a\psi_a^i = 0$$

and of  $\kappa Ric = \phi_*h$ . If  $G_{ab}l^b = 0$  then the above positive definite expression vanishes and we get

$$\psi_a^i = 0, \quad \partial_a\phi^i = l_a(\phi_*v)^i,$$

$$\kappa R_{ab} = h_{ij}(\partial_a\phi^i)(\partial_b\phi^j) = l_al_bh(\phi_*v, \phi_*v).$$

Conversely, from  $R_{ab} = rl_al_b$  with some scalar  $r$  there follow  $R = 0$  and (7.1).

We here dwell again on the fact that to every smooth vector field  $v = v^a\partial_a$  on  $X$  there are comoving coordinates  $x^a = x^0, x^1, \dots, x^{m-1}$ , which means  $v^a = \delta_0^a$ . The adaptedness of coordinates is preserved by coordinate transformations of the form

$$\bar{x}^0 = x^0 + f(x^i), \quad \bar{x}^i = \bar{x}^i(x^j). \tag{7.3}$$

We apply in this section the index convention

$$a, b, c, \dots = 0, 1, 2, \dots, m - 1;$$

$$i, j, k, \dots = 1, 2, \dots, m - 1;$$

$$I, J, K, \dots = 2, 3, \dots, m - 1.$$

**Proposition 7.2** *A Lorentzian manifold  $(X, g)$  admits a lightlike ... hypersurface-orthogonal Killing vector field  $l = l^a \partial_a$ , that means*

$$l_{[c} \nabla_a l_{b]} = 0, \quad \nabla_{(a} l_{b)} = 0,$$

*iff in coordinates adapted to  $l$  the metric assumes the form*

$$g = 2g_{01} dx^0 dx^1 + g_{ij} dx^i dx^j, \tag{7.4}$$

*where  $g_{01}, g_{ij}$  do not depend on  $x^0$ . The component  $g_{01} = g_{01}(x^k)$  is invariant under gauge transformations (7.3) and the part  $g_{IJ} dx^I dx^J$  of (7.4) shows tensorial behavior under the part  $\bar{x}^i = \bar{x}^i(x^j)$  of (7.3). Moreover, the matrix  $(g_{IJ}) = (g_{IJ}(x^k))$  is negative definite.*

*... covariantly constant vector field  $l = l^a \partial_a$ , that means  $\nabla_a l_b = 0$ , iff in coordinates adapted to  $l$  the metric assumes the form*

$$g = 2dx^0 dx^1 + g_{ij} dx^i dx^j, \tag{7.5}$$

*where the components  $g_{ij}$  do not depend on  $x^0$ .*

*... covariantly constant vector field  $l = l^a \partial_a$  such that the Bel condition*

$$l_{[e} R_{ab][cd} l_{f]} = 0 \tag{7.6}$$

*holds iff there are coordinates adapted to  $l$  such that*

$$g = 2g_{01} dx^0 dx^1 + g_{11} (dx^1)^2 + 2g_{1I} dx^1 dx^I - dx^I dx^I, \tag{7.7}$$

*where  $g_{11}, g_{1I}$  do not depend on  $x^0$  and summation over  $I$  is applied.*

*... covariantly constant vector field  $l = l^a \partial_a$  such that the Lichnerowicz condition*

$$l_{[e} R_{ab]cd} = 0 \tag{7.8}$$

*holds iff there are coordinates adapted to  $l$  such that*

$$g = 2g_{01} dx^0 dx^1 + g_{11} (dx^1)^2 - dx^I dx^I, \tag{7.9}$$

*where  $g_{11}$  does not depend on  $x^0$  and summation over  $I$  is applied.*

All these facts together with *proof* and additional information are given in the papers [24, 23].

A Lorentzian manifold  $(X, g)$  which admits a covariantly constant lightlike vector  $l = l^a \partial_a$  is called a *plane-fronted gravitational wave with parallel rays*, abbreviated *pp-wave*. Note that from  $\nabla_a l_b = 0$  and the Ricci identity there follows the Lichnerowicz condition

$$R_{abcd} l^d = 0.$$

**Theorem 7.3** *A metric (7.4) satisfies an Einstein equation  $\kappa Ric = \phi^* h$  iff  $g_{01} = g_{01}(x^1, x^K)$  is a harmonic function of  $x^K = x^2, x^3, \dots, x^{m-1}$  with respect to the positive definite metric (which depends on  $x^1$  as a parameter)  $-g_{IJ} dx^I dx^J$ .*



**Proof:**

Some calculation gives the components

$$R_{00} = 0, \quad R_{01} = \frac{1}{2} \Delta g_{01}$$

of the Ricci tensor  $Ric = R_{ab} dx^a dx^b$ , where  $\Delta$  denotes the Laplace operator with respect to  $-g_{IJ} dx^I dx^J$ . By proposition 5.2, from  $R_{00} \equiv R_{abl^a l^b} = 0$  and  $\kappa Ric = \phi^* h$  there follows  $R_{01} \equiv R_{1b} l^b = 0$ . The assertion follows.

**Theorem 7.4** *Let  $m = \dim X = 4$ . A metric of the form (7.7) satisfies an Einstein equation  $\kappa Ric = \phi^* h$  iff it satisfies (7.8), that means iff it can be brought into the form (7.9).*

**Proof:**

Calculation of Ricci components gives  $R_{IJ} = 0$ ; in particular  $R_{II} = 0$  (without summation). By Proposition 5.2, from this and  $\kappa Ric = \phi^* h$  there follows  $R_{1I} = 0$ . For  $m = 4$ , there are only two independent curvature components of type  $R_{iJKL}$ , namely

$$R_{1223} = -R_{13}, \quad R_{1323} = R_{12}.$$

Thus we get  $R_{iJK2} = 0$  which is expressed by (7.9) in a coordinate invariant way.

## VIII. DISCUSSION

The literature on harmonic or wave maps is very extensive. There are good surveys on harmonic maps [4, 5]. Work on such maps in the role of matter fields coupled with gravitation began about 1980 [10, 11]. One of the authors of this paper worked, with coauthors, already on this subject; we refer to the paper [17] and the unpublished preprint [25].

The Einstein equation  $\kappa Ric = \phi^* h$  or  $\kappa G = T$ , where  $G = Ric - \frac{1}{2} Rg$  denotes the Einstein tensor and  $T = \phi^* h - eg$  the energy-momentum tensor of  $\phi$ , exhibits some remarkable properties:

- The rank of  $Ric$ , taken as a linear map, equals the rank of the differential  $d\phi$  (Proposition 5.1).
- The symmetries of the Ricci tensor  $Ric$  and of the map  $\phi$  are closely related to each other (Propositions 5.2, 5.3, 5.7).
- If  $\phi$  is submersive then the Einstein equation implies the harmonic or wave map equation (Proposition 5.5; cf. also Proposition 5.6).
- In the Lorentzian case there are identities and estimates for the energy-momentum tensor  $T$  which indicate a physically good behavior of  $T$  (Proposition 3.2, Theorems 4.1, 4.2).
- In the Lorentzian case there is a tendency to enhance radiation conditions. That means, the Einstein equation leads from one condition to a stronger condition (Section 7).

There is also a situation where the Einstein equation serves as an auxiliary construction: for a given submersion  $\phi$  there locally exists a metric  $g$  on  $X$  which makes  $\phi$  to a harmonic or wave map (Theorem 5.8).



One paper cannot touch all aspects of a subject. We did not discuss here:

- Bochner-Weitzenböck technique [5, 13, 14, 15],
- consequences of second variation formulas,
- factorizations of the map  $\phi$  [18],
- exact solutions [10, 7, 8, 9, 12, 13, 14, 15, 16, 11],
- coupling of  $\phi$  to a gravitational theory other than Einstein's theory.

These topics are by far not exhausted and could be subjects of further research.

## IX. ACKNOWLEDGMENTS

Former discussions and cooperations with T. Hirschmann and T. Deck are gratefully acknowledged. We thank the professors G. Huisken and A. D. Rendall for inspiring discussions.

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