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On Certain Indefinite Elliptic Integrals

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Abstract - In this paper we have developed some formulae related to indefinite integrals in association with Hypergeometric functions.

Keywords : pochhammer symbol; gaussian hypergeometric function; complete elliptic integrals; kampé de fériet double hypergeometric function and srivastava's triple hypergeometric function.

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On Certain Indefinite Elliptic Integrals

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Abstract - In this paper we have developed some formulae related to indefinite integrals in association with Hypergeometric functions.

Keywords and Phrases : pochhammer symbol; gaussian hypergeometric function; complete elliptic integrals; kampé de fériet double hypergeometric function and sri- vastava's triple hypergeometric function.

I. Introduction and Preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k)=(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k=1,2,3,\cdots \\ 1 \quad ; & \text{if } k=0 \\ k! \quad ; & \text{if } b=1,\ k=1,2,3,\cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

a) Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{ccc} a_1,a_2,\cdots,a_A & ; & \\ & & z \\ b_1,b_2,\cdots,b_B & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{ccc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right]\equiv {}_AF_B\left[\begin{array}{ccc} (a_j)_{j=1}^A & ; & \\ & & z \\ (b_j)_{j=1}^B & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.1}$$

where denominator parameters $b_1,b_2,\cdots,b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

b) Kampje de Fjeriet's General Double Hypergeometric Function

In 1921, Appell's four double hypergeometric functions F_1,F_2,F_3,F_4 and their confluent forms $\Phi_1,\Phi_2,\Phi_3,\Psi_1,\Psi_2,\Xi_1,\Xi_2$ were unified and generalized by Kampé de Fériet.

We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

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$$\mathrm{F}^{A:B;D}_{E:G;H}\left[\begin{array}{ll}(a_A):(b_B);(d_D) & ;\\ & x,\,y\\ (e_E):(g_G);(h_H) & ;\end{array}\right] = \sum_{m,n=0}^{\infty}\frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\;x^m\,y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\,m!\,n!} \quad (1.2)$$

where for convergence

(i) $A+B<E+G+1,\ A+D<E+H+1\quad ;|x|<\infty,\quad |y|<\infty,$ or

(ii) $A+B=E+G+1,\ A+D=E+H+1,$ and

$$\begin{cases}|x|^{\frac{1}{(A-E)}}+|y|^{\frac{1}{(A-E)}}<1 & ,\text{if } E<A\\ \max\ \{|x|,|y|\}<1 & ,\text{if } E\geq A\end{cases}$$

c) *Srivastava's General Triple Hypergeometric Function*

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$F^{(3)}\left[\begin{array}{ll}(a_A)::(b_B);\,(d_D);\,(e_E):(g_G);\,(h_H);\,(l_L); & \\ & x,\ y,\ z\\ (m_M)::(n_N);\,(p_P);\,(q_Q):(r_R);\,(s_S);\,(t_T); & \end{array}\right]$$

$$=\sum_{i,j,k=0}^{\infty}\frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\ y^j\ z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\ j!\ k!} \quad (1.3)$$

d) *Wright's Generalized Hypergeometric Function*

$${}_p\Psi_q\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ;\\ & x\\ (\lambda_1,B_1),\cdots,(\lambda_q,B_q) & ;\end{array}\right]=\sum_{m=0}^{\infty}\frac{\Gamma(\alpha_1+mA_1)\Gamma(\alpha_2+mA_2)\cdots\Gamma(\alpha_p+mA_p)x^m}{\Gamma(\lambda_1+mB_1)\Gamma(\lambda_2+mB_2)\cdots\Gamma(\lambda_q+mA_q)m!} \quad (1.4)$$

$${}_p\Psi^*_q\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ;\\ & x\\ (\lambda_1,B_1),\cdots,(\lambda_q,B_q) & ;\end{array}\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1)_{mA_1}(\alpha_2)_{mA_2}\cdots(\alpha_p)_{mA_p}x^m}{(\lambda_1)_{mB_1}(\lambda_2)_{mB_2}\cdots(\lambda_q)_{mB_q}m!} \quad (1.5)$$

II. Main Integrals

$$\int\frac{\mathrm{d}x}{\sqrt{(1+x\sinh x)}}=$$

$$=-\cosh x\sinh^{m+1}x(-\sinh^2 x)^{\frac{-m-1}{2}}\ F^{1;2}_{0;1}\left[\begin{array}{ll}\frac{1}{2}\ \ ;\frac{1}{2},\ \frac{1-m}{2} & ;\\ & -x,\ \cosh^2 x\\ -;\ \frac{3}{2} & ;\end{array}\right]+Constant \quad (2.1)$$

$$\int\frac{\mathrm{d}x}{\sqrt{(1+x\cosh x)}}=-\frac{\sinh x\cosh^{m+1}x}{(m+1)\sqrt{-\sinh^2 x}}\ F^{1;2}_{0;1}\left[\begin{array}{ll}\frac{1}{2}\ \ ;\frac{1}{2},\ \frac{m+1}{2} & ;\\ & -x,\ \cosh^2 x\\ -;\ \frac{m+3}{2} & ;\end{array}\right]+Constant \quad (2.2)$$

Notes

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$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\tanh x)}} = \frac{\tanh^{m+1} x}{(m+1)}\, F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;1, \frac{m+1}{2} & ; \\ & & -x,\ \tanh^2 x \\ \text{—} & ; \frac{m+3}{2} & ; \end{array}\right] + Constant \quad (2.3)$$

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\coth x)}} = \frac{\coth^{m+1} x}{(m+1)}\, F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;1, \frac{m+1}{2} & ; \\ & & -x,\ \coth^2 x \\ \text{—} & ; \frac{m+3}{2} & ; \end{array}\right] + Constant \quad (2.4)$$

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\ \mathrm{sech}\ x)}} =$$

$$= \sinh x \cosh^2(x)^{\frac{m+1}{2}} \mathrm{sech}^{m+1} x\ F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;\frac{1}{2}, \frac{1+m}{2} & ; \\ & & -x,\ -\sinh^2 x \\ \text{—} & ; \frac{3}{2} & ; \end{array}\right] + Constant \quad (2.5)$$

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\ \mathrm{cosech}\ x)}} =$$

$$= \cosh x(-\sinh^2(x))^{\frac{m+1}{2}} \mathrm{cosech}^{m+1} x\ F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;\frac{1}{2}, \frac{1+m}{2} & ; \\ & & -x,\ \cosh^2 x \\ \text{—} & ; \frac{3}{2} & ; \end{array}\right] + Constant \quad (2.6)$$

III. Derivation of Integrals

Derivation of integral (2.1)

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\sinh x)}} = \int (1+x\sinh x)^{-\frac{1}{2}}\mathrm{d}x = \int \{1-(-x\sinh x)\}^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \sinh^m x\ \mathrm{d}x = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(-\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \sinh^{\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!}(-\cosh x)\ \sinh^{m+1} x(-\sinh^2 x)^{\frac{-m-1}{2}} {}_2F_1\left[\begin{array}{cc} \frac{1}{2}, \frac{1-m}{2} & ; \\ \frac{3}{2} & ; \end{array} \cosh^2 x\right] + Constant$$

$$= -\cosh x \sinh^{m+1} x(-\sinh^2 x)^{\frac{-m-1}{2}}\ F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;\frac{1}{2}, \frac{1-m}{2} & ; \\ & & -x,\ \cosh^2 x \\ \text{—} & ; \frac{3}{2} & ; \end{array}\right] + Constant$$

Derivation of integral (2.2)

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\cosh x)}} = \int (1+x\cosh x)^{-\frac{1}{2}}\mathrm{d}x = \int \{1-(-x\cosh x)\}^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \cosh^m x\ \mathrm{d}x = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(-\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \cosh^{\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \frac{(-\sinh x)\cosh^{m+1} x}{(m+1)\sqrt{-\sinh^2 x}} {}_2F_1\left[\begin{array}{cc} \frac{1}{2}, \frac{m+1}{2} & ; \\ \frac{m+3}{2} & ; \end{array} \cosh^2 x\right] + Constant$$

$$= -\frac{\sinh x \cosh^{m+1} x}{(m+1)\sqrt{-\sinh^2 x}}\, F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;\frac{1}{2},\ \frac{m+1}{2} & ; \\ & & -x,\ \cosh^2 x \\ \text{—} & ;\ \frac{m+3}{2} & ; \end{array}\right] + Constant$$

Derivation of integral (2.3)

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\tanh x)}} = \int (1+x\tanh x)^{-\frac{1}{2}}\mathrm{d}x = \int \{1-(-x\tanh x)\}^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \tanh^m x \ \mathrm{d}x = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(-\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \tanh^{\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \frac{\tanh^{m+1} x}{(m+1)}\, {}_2F_1\left[\begin{array}{cc} 1,\ \frac{m+1}{2} & ; \\ \frac{m+3}{2} & ; \end{array}\ \tanh^2 x\right] + Constant$$

$$= \frac{\tanh^{m+1} x}{(m+1)}\, F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;1,\ \frac{m+1}{2} & ; \\ & & -x,\ \tanh^2 x \\ \text{—} & ;\ \frac{m+3}{2} & ; \end{array}\right] + Constant$$

Derivation of integral (2.4)

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\coth x)}} = \int (1+x\coth x)^{-\frac{1}{2}}\mathrm{d}x = \int \{1-(-x\coth x)\}^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \coth^m x \ \mathrm{d}x = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(-\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \coth^{\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \frac{\coth^{m+1} x}{(m+1)}\, {}_2F_1\left[\begin{array}{cc} 1,\ \frac{m+1}{2} & ; \\ \frac{m+3}{2} & ; \end{array}\ \coth^2 x\right] + Constant$$

$$= \frac{\coth^{m+1} x}{(m+1)}\, F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;1,\ \frac{m+1}{2} & ; \\ & & -x,\ \coth^2 x \\ \text{—} & ;\ \frac{m+3}{2} & ; \end{array}\right] + Constant$$

Derivation of integral (2.5)

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x\ \mathrm{sech}\ x)}} = \int (1+x\ \mathrm{sech}\ x)^{-\frac{1}{2}}\mathrm{d}x = \int \{1-(-x\ \mathrm{sech}\ x)\}^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \mathrm{sech}^m x \ \mathrm{d}x = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(-\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \mathrm{sech}^{\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!}\ \sinh x\ \cosh^2(x)^{\frac{m+1}{2}} \mathrm{sech}^{m+1}\ x\ {}_2F_1\left[\begin{array}{cc} \frac{1}{2},\ \frac{m+1}{2} & ; \\ \frac{3}{2} & ; \end{array}\ -\sinh^2 x\right] + Constant$$

$$= \sinh x \cosh^2(x)^{\frac{m+1}{2}} \mathrm{sech}^{m+1} x\ F^{1;2}_{0;1}\left[\begin{array}{ccc} \frac{1}{2} & ;\frac{1}{2},\ \frac{1+m}{2} & ; \\ & & -x,\ -\sinh^2 x \\ \text{—} & ;\ \frac{3}{2} & ; \end{array}\right] + Constant$$

Notes

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Derivation of integral (2.6)

$$\int \frac{\mathrm{d}x}{\sqrt{(1+x \text{ cosech } x)}} = \int (1+x \text{ cosech } x)^{-\frac{1}{2}} \mathrm{d}x = \int \{1-(-x \text{ cosech } x)\}^{-\frac{1}{2}} \mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \text{cosech}^m x \text{ dx} = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}}(-\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \text{cosech}^{\mathrm{m}} \mathrm{x} \text{ dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \cosh x \,(-\sinh^2(x))^{\frac{m+1}{2}} \text{cosech}^{m+1} x \; {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{m+1}{2} & ; \\ \frac{3}{2} & ; \end{matrix} \; \cosh^2 x \right] + Constant$$

$$= \cosh x(-\sinh^2(x))^{\frac{m+1}{2}} \text{cosech}^{m+1} x \; F^{1;2}_{0;1} \left[\begin{matrix} \frac{1}{2} \;;\frac{1}{2}, \frac{1+m}{2} & ; \\ & & -x, \cosh^2 x \\ \text{—}; \; \frac{3}{2} & ; \end{matrix} \right] + Constant$$

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