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Highlights

Birth-Death Process

Homotopy Perturbation Method

Function of n-variable

Exponentially Stretching Cylinder

Discovering Thoughts, Inventing Future

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Creation of a Summation Formula Attached with Recurrence Relation and Hypergeometric Function

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Abstract- The main aim of present paper is the creation of a summation formula attached with recurrence relation and Hypergeometric function. We have used computational method using Mathematical.

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CREATION OFA SUMMATION FORMULA ATTACHED WITH RECURREN RELATION AND HYPERGEOMETRI UNCTION

Creation of a Summation Formula Attached with Recurrence Relation and Hypergeometric Function

Jai Bhagwan [α] & Salahuddin [σ]

Abstract- The main aim of present paper is the creation of a summation formula attached with recurrence relation and Hypergeometric function. We have used computational method using Mathematical.

I. Introduction

a) Generalized Hypergeometric Functions

A generalized hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\, z. \tag{1}$$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{2}$$

or

$${}_pF_q\left[\begin{array}{ccc} (a_p) & ; & \\ & & z \\ (b_q) & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_p))_k z^k}{((b_q))_k k!} \tag{3}$$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$[Luke p.156(3)].

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

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The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \tag{4}$$

The solution of this equation is

$$y = A_0\Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] \tag{5}$$

This is the so-called regular solution, denoted

$${}_2F_1(a,b;c;z) = \Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] = \sum_{k=0}^{\infty}\frac{(a)_k\ (b)_k z^k}{(c)_k k!} \tag{6}$$

which converges if c is not a negative integer for all $\mid z \mid< 1$ and on the unit circle $\mid z \mid= 1$ if $R(c-a-b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

Many of the common mathematical functions can be expressed in terms of the hypergeometric function. Some typical examples are

$$(1-z)^{-a} = z\ {}_2F_1(1,1;2;-z) \tag{7}$$

$$\sin^{-1} z = z\ {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2) \tag{8}$$

b) Gauss' Relations For Contiguous Functions

The six functions $F(a\pm 1,b;c;z), F(a,b\pm 1;c;z), F(a,b;c\pm 1;z)$ are called contiguous to $F(a,b;c;z)$. Relation between $F(a,b;c;z)$ and any two contiguous functions have been given by Gauss.

[Abramowitz p.558(15.2.19)]

$$(a-b)\ (1-z)\ {}_2F_1\left[\begin{matrix} a,\ b\ ; & \\ c\ \ ; & z\end{matrix}\right] = (c-b)\ {}_2F_1\left[\begin{matrix} a,\ b-1\ ; & \\ c\ \ ; & z\end{matrix}\right] + (a-c)\ {}_2F_1\left[\begin{matrix} a-1,\ b\ ; & \\ c\ ; & z\end{matrix}\right] \tag{9}$$

c) Recurrence Relation

In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms.The recurrence relation of Gamma function is defined by [Temme p.42(3.1.1)]

$$\Gamma(z+1) = z\ \Gamma(z) \tag{10}$$

d) Legendre's Duplication Formula

Following Bells and Wong [p.26(2.3.1)], we have

$$\sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{11}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{12}$$

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$$= \frac{2^{(a-1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{13}$$

Following Prudnikov, Brychkov & Marichev [p.491.(7.3.7.3)], we have

$${}_2F_1\left[\begin{matrix} a, & b & ; & \\ \frac{a+b-1}{2} & & ; & \end{matrix}\ \frac{1}{2}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\Gamma(b)}\right] \tag{14}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above formula can be written in the form

$${}_2F_1\left[\begin{matrix} a, & b & ; & \\ \frac{a+b-1}{2} & & ; & \end{matrix}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{15}$$

It is noted that the above formula [Prudnikov,491.(7.3.7.3)], i.e. equation(14) or (15) is not correct.The correct form of equation(14) or (15) is obtained by [Asish et. al(2008), p.337(10)]

$${}_2F_1\left[\begin{matrix} a, & b & ; & \\ \frac{a+b-1}{2} & & ; & \end{matrix}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})}\left\{\frac{(b+a-1)}{(a-1)}\right\} + \frac{2\,\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\right] \tag{16}$$

Involving the derived formula obtained by [Salahuddin,p .45(9)], we establish the main formula.

II. Main Summation Formula

For the main formula $a \neq b$

For $a < 1$ and $a > 30$

$${}_2F_1\left[\begin{matrix} a, & b & ; & \\ \frac{a+b-30}{2} & & ; & \end{matrix}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-30}{2})}{(a-b)\Gamma(b)}\left[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-29}{2})}\left\{\frac{(-42849873690624000a)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+\right.\right.$$

$$+\frac{(102174938785382400a^2 - 90168185255362560a^3 + 48238440075952128a^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-14222757092524032a^5 + 3434748504629248a^6 - 476013869035520a^7 + 63240546512640a^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-4603989262336a^9 + 362770106624a^{10} - 14085527040a^{11} + 665833376a^{12} - 12792832a^{13})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(343728a^{14} - 2480a^{15} + 31a^{16} + 42849873690624000b - 31082092167168000ab)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-73098173806018560a^2b + 108868047825272832a^3b - 57797146289504256a^4b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

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$$+\frac{(22066910445879296a^5b-4278218966429696a^6b+807376117319680a^7b)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-74955432767488a^8b+8169315368448a^9b-385635936768a^{10}b+25593453184a^{11}b)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-582589696a^{12}b+23404256a^{13}b-198896a^{14}b+4464a^{15}b-71092846618214400b^2)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(112755981161594880ab^2-36482063700197376a^2b^2-29343138531508224a^3b^2)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(30395328111255552a^4b^2-10197276680204288a^5b^2+2946095411127296a^6b^2)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-374548696047616a^7b^2+56962219478272a^8b^2-3393695599104a^9b^2+308716484928a^{10}b^2)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-8514867712a^{11}b^2+480008464a^{12}b^2-4847408a^{13}b^2+165416a^{14}b^2+50510377899786240b^3)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-99910969798164480ab^3+54967053635026944a^2b^3-10666441061793792a^3b^3)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-4166767625613312a^4b^3+3206675295074304a^5b^3-686051126108160a^6b^3)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(156167659677696a^7b^3-12540657489408a^8b^3+1572782347392a^9b^3-54323189760a^{10}b^3)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(4180393152a^{11}b^3-51048816a^{12}b^3+2459664a^{13}b^3-20713454402863104b^4)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(40852434546720768ab^4-32041240778039296a^2b^4+8698430393331712a^3b^4)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-1154676393076224a^4b^4-247159742668800a^5b^4+147864940085760a^6b^4)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

$$+\frac{(-19581576698880a^7b^4+3617566362720a^8b^4-166428633600a^9b^4+17522948400a^{10}b^4)}{\prod\limits_{\zeta=1}^{15}\left\{a-(2\zeta-1)\right\}}+$$

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$$+\frac{(-266463600a^{11}b^4+17530500a^{12}b^4+5543553731788800b^5-12155400984281088ab^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(8180404586819584a^2b^5-3671679306500096a^3b^5+552177907875840a^4b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-54103342402560a^5b^5-6502833792000a^6b^5+3126362400000a^7b^5-233673258240a^8b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(35839030560a^9b^5-719559600a^{10}b^5+64512240a^{11}b^5-1033904237649920b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(2099766917263360ab^6-1867829847333888a^2b^6+595797693046784a^3b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-178743343074816a^4b^6+15207496657920a^5b^6-1156337239680a^6b^6-73365304320a^7b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(29075170320a^8b^6-937908720a^9b^6+121580760a^{10}b^6+139675243868160b^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-315216253958144ab^7+210439757914112a^2b^7-108338835331072a^3b^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(17944242066432a^4b^7-3924412035840a^5b^7+177220085760a^6b^7-10838056320a^7b^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-282861360a^8b^7+94287120a^9b^7-13985569165568b^8+27853051633664ab^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-25906772757248a^2b^8+7697887351296a^3b^8-2669359676832a^4b^8+222457190400a^5b^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-37501461360a^6b^8+699709680a^7b^8-35357670a^8b^8+1050576384000b^9)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-2375213023744ab^9+1454416768512a^2b^9-790386876288a^3b^9+108636023040a^4b^9)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-27632383200a^5b^9+918058800a^6b^9-124062000a^7b^9-59398127360b^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

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$$+\frac{(111928327680ab^{10}-105528303296a^2b^{10}+25592473088a^3b^{10}-9433933392a^4b^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(490854000a^5b^{10}-96768360a^6b^{10}+2513871360b^{11}-5584355712ab^{11}+2860642304a^2b^{11})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-1587936064a^3b^{11}+134073264a^4b^{11}-36463440a^5b^{11}-78393952b^{12}+132474368ab^{12})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-123782256a^2b^{12}+18886192a^3b^{12}-7152444a^4b^{12}+1747200b^{13}-3719968ab^{13})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\left.\frac{-704816a^3b^{13}-26320b^{14}+34480ab^{14}-31000a^2b^{14}+240b^{15}-464ab^{15}-b^{16})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}\right\}+$$

$$+\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-30}{2})}\left\{\frac{(-42849873690624000a+71092846618214400a^2-50510377899786240a^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+\right.$$

$$+\frac{(20713454402863104a^4-5543553731788800a^5+1033904237649920a^6-139675243868160a^7)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(13985569165568a^8-1050576384000a^9+59398127360a^{10}-2513871360a^{11}+78393952a^{12})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-1747200a^{13}+26320a^{14}-240a^{15}+a^{16}+42849873690624000b+31082092167168000ab)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-112755981161594880a^2b+99910969798164480a^3b-40852434546720768a^4b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(12155400984281088a^5b-2099766917263360a^6b+315216253958144a^7b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-27853051633664a^8b+2375213023744a^9b-111928327680a^{10}b+5584355712a^{11}b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-132474368a^{12}b+3719968a^{13}b-34480a^{14}b+464a^{15}b-102174938785382400b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(73098173806018560ab^2+36482063700197376a^2b^2-54967053635026944a^3b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

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$$+\frac{(32041240778039296a^4b^2-8180404586819584a^5b^2+1867829847333888a^6b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-210439757914112a^7b^2+25906772757248a^8b^2-1454416768512a^9b^2+105528303296a^{10}b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-2860642304a^{11}b^2+123782256a^{12}b^2-1274224a^{13}b^2+31000a^{14}b^2+90168185255362560b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-108868047825272832ab^3+29343138531508224a^2b^3+10666441061793792a^3b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-8698430393331712a^4b^3+3671679306500096a^5b^3-595797693046784a^6b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(108338835331072a^7b^3-7697887351296a^8b^3+790386876288a^9b^3-25592473088a^{10}b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(1587936064a^{11}b^3-18886192a^{12}b^3+704816a^{13}b^3-48238440075952128b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(57797146289504256ab^4-30395328111255552a^2b^4+4166767625613312a^3b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(1154676393076224a^4b^4-552177907875840a^5b^4+178743343074816a^6b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-17944242066432a^7b^4+2669359676832a^8b^4-108636023040a^9b^4+9433933392a^{10}b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-134073264a^{11}b^4+7152444a^{12}b^4+14222757092524032b^5-22066910445879296ab^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(10197276680204288a^2b^5-3206675295074304a^3b^5+247159742668800a^4b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(54103342402560a^5b^5-15207496657920a^6b^5+3924412035840a^7b^5-222457190400a^8b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(27632383200a^9b^5-490854000a^{10}b^5+36463440a^{11}b^5-3434748504629248b^6)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(4278218966429696ab^{6}-2946095411127296a^{2}b^{6}+686051126108160a^{3}b^{6})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-147864940085760a^{4}b^{6}+6502833792000a^{5}b^{6}+1156337239680a^{6}b^{6}-177220085760a^{7}b^{6})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(37501461360a^{8}b^{6}-918058800a^{9}b^{6}+96768360a^{10}b^{6}+4760138690355520b^{7})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-807376117319680ab^{7}+374548696047616a^{2}b^{7}-156167659677696a^{3}b^{7})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(19581576698880a^{4}b^{7}-3126362400000a^{5}b^{7}+73365304320a^{6}b^{7}+10838056320a^{7}b^{7})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-699709680a^{8}b^{7}+124062000a^{9}b^{7}-632405465126400b^{8}+74955432767488ab^{8})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-56962219478272a^{2}b^{8}+12540657489408a^{3}b^{8}-3617566362720a^{4}b^{8}+233673258240a^{5}b^{8})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-29075170320a^{6}b^{8}+282861360a^{7}b^{8}+35357670a^{8}b^{8}+4603989262336b^{9})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-8169315368448ab^{9}+3393695599104a^{2}b^{9}-1572782347392a^{3}b^{9}+166428633600a^{4}b^{9})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-35839030560a^{5}b^{9}+937908720a^{6}b^{9}-94287120a^{7}b^{9}-362770106624b^{10})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(385635936768ab^{10}-308716484928a^{2}b^{10}+54323189760a^{3}b^{10}-17522948400a^{4}b^{10})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(719559600a^{5}b^{10}-121580760a^{6}b^{10}+14085527040b^{11}-25593453184ab^{11}+8514867712a^{2}b^{11})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-4180393152a^{3}b^{11}+266463600a^{4}b^{11}-64512240a^{5}b^{11}-665833376b^{12}+582589696ab^{12})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-480008464a^{2}b^{12}+51048816a^{3}b^{12}-17530500a^{4}b^{12}+12792832b^{13}-23404256ab^{13})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

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$$+\frac{(4847408a^2b^{13}-2459664a^3b^{13}-343728b^{14}+198896ab^{14}-165416a^2b^{14}+2480b^{15})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-4464ab^{15}-31b^{16})}{\prod_{\eta=1}^{15}\{a-2\eta\}}\Bigg\}\Bigg] \quad (17)$$

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III. Evaluation Of Main Summation Formula

Substituting $c=\frac{a+b-30}{2}$ and $z=\frac{1}{2}$ in equation (24), we get

$$(a-b)\,{}_2F_1\left[\begin{matrix}a,\ b & ; \\ \frac{a+b-30}{2} & ;\end{matrix}\ \frac{1}{2}\right]=(a-b-30)\,{}_2F_1\left[\begin{matrix}a,\ b-1 & ; \\ \frac{a+b-30}{2} & ;\end{matrix}\ \frac{1}{2}\right]+(a-b+30)\,{}_2F_1\left[\begin{matrix}a-1,\ b & ; \\ \frac{a+b-30}{2} & ;\end{matrix}\ \frac{1}{2}\right]$$

Now involving the the formula obtained by salahuddin[Salahuddin,p .45(9)], we get

$$L.H.S=\frac{2^{(b-1)}\,\Gamma(\frac{a+b-30}{2})}{\Gamma(b)}\Bigg[\frac{(a-b-30)}{(a-b+1)}\,\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-29}{2})}\Bigg\{\frac{(-42849873690624000)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(56468406848716800a+10090943081349120a^2-4564682122729881 6a^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(3011999453242982 4a^4-1065924237463552 0a^5+2384177957457920a^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-372420039991552a^7+41100528816128a^8-3372406380160a^9+199271883200a^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-8841030848a^{11}+271930672a^{12}-6056120a^{13}+80260a^{14}-659a^{15}+a^{16})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(113942720308838400b-192432640779878400ab+71179471376547840a^2b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(31565349178638336a^3b-35990351353896960a^4b+14307315699220480a^5b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-3391367994865920a^6b+532460361363712a^7b-59560797031680a^8b+4745684818240a^9b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-279195240480a^{10}b+11619237968a^{11}b-349449360a^{12}b+6800780a^{13}b-84165a^{14}b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

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$$+\frac{(434a^{15}b - 1216032245180006640b^2 + 2296633293147340880ab^2 - 1325623103491276880a^2b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(177163611736473600a^3b^2 + 128176061140377600a^4b^2 - 72685019219366400a^5b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(1903248210151680a^6b^2 - 309340014067200a^7b^2 + 34669345543680a^8b^2 - 2718132397440a^9b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(154142180400a^{10}b^2 - 6044516400a^{11}b^2 + 166890360a^{12}b^2 - 2737455a^{13}b^2 + 26970a^{14}b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(712238323026493444b^3 - 1424112696256757776ab^3 + 965027356200960000a^2b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-271585057215283200a^3b^3 + 10460526963187200a^4b^3 + 16454110056460800a^5b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-5401448380416000a^6b^3 + 974223475468800a^7b^3 - 106890252782400a^8b^3 + 8550463077600a^9b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-435878352000a^{10}b^3 + 16955499600a^{11}b^3 - 352363050a^{12}b^3 + 566370a^{13}b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-262570081346519040b^4 + 541705059819110400ab^4 - 394557256147968000a^2b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(136792040535974400a^3b^4 - 21208903346880000a^4b^4 - 190255269312000a^5b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(849202448160000a^6b^4 - 166770138528000a^7b^4 + 21165020352000a^8b^4 - 1454141490000a^9b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(8158694700a^10b^4 - 212995575a^{11}b^4 + 5259150a^{12}b^4 + 65774579694387200b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-138202864672768000ab^5 + 103936705650086400a^2b^5 - 39180813025804800a^3b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(7918457936256000a^4b^5 - 694927834272000a^5b^5 - 24839991388800a^6b^5 + 19106242747200a^7b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

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$$+\frac{(-212369331600a^8b^5+19049720100a^9b^5-641090385a^{10}b^5+24192090a^{11}b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-11735794815180080b^6+2469322022918400ab^6-1911816899339520a^2b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(732192289313280a^3b^6-163543093881600a^4b^6+20082084664320a^5b^6-941035082400a^6b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-48979677600a^7b^6+18237114000a^8b^6-901000275a^9b^6+56448210a^{10}b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(1536608130333728b^7-326583853541632ab^7+245643692083200a^2b^7-99106124778240a^3b^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(21459053764800a^4b^7-3012890332320a^5b^7+216761126400a^6b^7-4242920400a^7b^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-265182525a^8b^7+58929450a^9b^7-150361455549568b^8+31156475560320ab^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-24451722638400a^2b^8+8940052710720a^3b^8-2152280437200a^4b^8+260046358200a^5b^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-23317452900a^6b^8+795547575a^7b^8+1109974511360b^9-2340242526400ab^9)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(1645789231200a^2b^9-673510158960a^3b^9+125836626000a^4b^9-19235813100a^5b^9)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(1052976225a^6b^9-58929450a^7b^9-61911998720b^{10}+122169740160ab^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-95144334480a^2b^{10}+29274532560a^3b^{10}-7159456200a^4b^{10}+568514115a^5b^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-56448210a^6b^{10}+2592265312b^{11}-5337467408ab^{11}+3197563200a^2b^{11}-1300061880a^3b^{11})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(155144925a^4b^{11}-24192090a^5b^{11}-80141152b^{12}+142613640ab^{12})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-108689100a^2b^{12}+21562515a^3b^{12}-5259150a^4b^{12}+1773520b^{13}-3493700ab^{13})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(1415925a^2b^{13}-566370a^3b^{13}-26560b^{14}+36735ab^{14}-26970a^2b^{14}+241b^{15})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-434ab^{15}-b^{16})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}\Bigg\}+\frac{(a-b-30)(b-1)}{(a-b+1)}\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-30}{2})}\Bigg\{\frac{(42849873690624000)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-26814643804569600a-21476290178580480a^2+29081047157637120a^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-14200532432486400a^4+4036298413056000a^5-759685713346560a^6+100366817591040a^7)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-9589432195200a^8+671530516800a^9-34499905440a^{10}+1285480560a^{11}-33797400a^{12})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(594300a^{13}-6270a^{14}+30a^{15}-10074660966236160 0b+105369623089643520ab)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-18497359471509504a^2b-19697219949428736a^3b+13886382286397440a^4b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-4339068030189568a^5b+857574469125888a^6b-113217893178368a^7b+10859483768000a^8b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-731396205504a^9b+36817157936a^{10}b-1257372896a^{11}b+31361460a^{12}b-443548a^{13}b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(4030a^{14}b+92490335838535680b^2-113987664871096320ab^2+46383618741829632a^2b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-1926305147437056a^3b^2-4380147255427072a^4b^2+1878105598159616a^5b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-386934853266944a^6b^2+54537090886912a^7b^2-4938029127456a^8b^2+342285504528a^9b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-15032099888a^{10}b^2+523484104a^{11}b^2-9419722a^{12}b^2+138446a^{13}b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

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$$+\frac{(-466321658320650024b^3+62163571923615744ab^3-30822099684507648a^2b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(6525530932051968a^3b^3+131032468218624a^4b^3-348720370467840a^5b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(98495967926016a^6b^3-13207757720832a^7b^3+1340626529232a^8b^3-76951667040a^9b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(3775087992a^{10}b^3-84987864a^{11}b^3+1893294a^{12}b^3+148596630978232b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-20711447922302976ab^4+11169873454153728a^2b^4-2983163756323584a^3b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(363675061996800a^4b^4+20359470898560a^5b^4-113537339395 20a^6b^4+2209706471520a^7b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-178725335400a^8b^4+13011676500a^9b^4-369893550a^{10}b^4+12271350a^{11}b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-3271193580412928b^5+4569102979137536ab^5-2610314589022976a^2b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(753958524042240a^3b^5-120686089584000a^4b^5+8674613539200a^5b^5+705515781600a^6b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-150958641600a^7b^5+20749369500a^8b^5-787793700a^9b^5+40320150a^{10}b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(505025512816640b^6-733037562791680ab^6+405911801172480a^2b^6-127441219311360a^3b^6)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(21411848928960a^4b^6-2132804429280a^5b^6+86079178080a^6b^6+8858026800a^7b^6)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-664352010a^8b^6+65132550a^9b^6-59551759843072b^7+81170061386752ab^7)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-48474589768960a^2b^7+137332745141 76a^3b^7-2650892519520a^4b^7+253026930240a^5b^7)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-15527599920a^6b^7+282861360a^7b^7+35357670a^8b^7+4947215345536b^8)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-7286981375168ab^8+3740645877984a^2b^8-1232794859472a^3b^8+183751464600a^4b^8)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-22893160860a^5b^8+1003041270a^6b^8-35357670a^7b^8-339191763264b^9+424211354688ab^9)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-257569498128a^2b^9+60858560160a^3b^9-12407548500a^4b^9+800199900a^5b^9)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-65132550a^6b^9+15279519840b^10-22632383280ab^{10}+9573055440a^2b^{10})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-3218599800a^3b^{10}+303277650a^4b^{10}-40320150a^5b^{10}-621837424b^{11}+649171744ab^{11})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-397825480a^2b^{11}+58621992a^3b^{11}-12271350a^4b^{11}+139394332b^{12}-20735156ab^{12})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(5539638a^2b^{12}-1893294a^3b^{12}-323708b^{13}+223076ab^{13}-138446a^2b^{13}+2690b^{14})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\left.\left.\frac{(-4030ab^{14}-30b^{15})}{\prod_{\eta=1}^{15}\{a-2\eta\}}\right\}\right]+\frac{2^{(b-1)}\,\Gamma(\frac{a+b-30}{2})}{\Gamma(b)}\left[\frac{(a-b+30)}{(a-b-1)}\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-30}{2})}\times\right.$$

$$+\frac{(42849873690624000-113942720308838400a+121603224518000640a^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-71223832302649344a^3+26257008134651904a^4-6577457969438720a^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(1173579481518080a^6-153660813033728a^7+15036145549568a^8-1109974511360a^9)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(61911998720a^{10}-2592265312a^{11}+80141152a^{12}-1773520a^{13}+26560a^{14}-241a^{15}+a^{16})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-56468406848716800b+192432640779878400ab-229663329314734080a^2b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

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$$+\frac{(142411269625675776a^3b-541705059819110 40a^4b+138202864672768 00a^5b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-2469322022918400a^6b+326583853541632a^7b-31156475560320a^8b+2340242526400a^9b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-122169740160a^{10}b+5337467408a^{11}b-142613640a^{12}b+3493700a^{13}b-36735a^{14}b)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(434a^{15}b-100909430813491200b^2-71179471376547840ab^2+132562310349127680a^2b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-96502735620096000a^3b^2+39455725614796800a^4b^2-10393670565008640a^5b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(1911816899339520a^6b^2-245643692083200a^7b^2+24451722638400a^8b^2-1645789231200a^9b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(95144334480a^{10}b^2-3197563200a^{11}b^2+108689100a^{12}b^2-1415925a^{13}b^2+26970a^{14}b^2)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(45646821227298816b^3-31565349178638336ab^3-17716361173647360a^2b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(27158505721528320a^3b^3-13679204053597440a^4b^3+3918081302580480a^5b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-732192289313280a^6b^3+99106124778240a^7b^3-8940052710720a^8b^3+673510158960a^9b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-29274532560a^{10}b^3+1300061880a^{11}b^3-21562515a^{12}b^3+566370a^{13}b^3)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-30119994532429824b^4+35990351353896960ab^4-12817606114037760a^2b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-1046052696318720a^3b^4+2120890334688000a^4b^4-791845793625600a^5b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(163543093881600a^6b^4-21459053764800a^7b^4+2152280437200a^8b^4-125836626000a^9b^4)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(7159456200a^10b^4 - 155144925a^{11}b^4 + 5259150a^{12}b^4 + 10659242374635520b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-14307315699220480ab^5 + 7268501921936640a^2b^5 - 1645411005646080a^3b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(19025526931200a^4b^5 + 69492783427200a^5b^5 - 20082084664320a^6b^5 + 3012890332320a^7b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-260046358200a^8b^5 + 19235813100a^9b^5 - 568514115a^{10}b^5 + 24192090a^{11}b^5)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-2384177957457920b^6 + 3391367994865920ab^6 - 1903248210151680a^2b^6)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(540144838041600a^3b^6 - 84920244816000a^4b^6 + 2483999138880a^5b^6 + 941035082400a^6b^6)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-216761126400a^7b^6 + 23317452900a^8b^6 - 1052976225a^9b^6 + 56448210a^{10}b^6)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(372420039991552b^7 - 532460361363712ab^7 + 309340014067200a^2b^7 - 97422347546880a^3b^7)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(16677013852800a^4b^7 - 1910624274720a^5b^7 + 48979677600a^6b^7 + 4242920400a^7b^7)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-795547575a^8b^7 + 58929450a^9b^7 - 41100528816128b^8 + 59560797031680ab^8)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-34669345543680a^2b^8 + 10689025278240a^3b^8 - 2116502035200a^4b^8 + 212369331600a^5b^8)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-18237114000a^6b^8 + 265182525a^7b^8 + 3372406380160b^9 - 4745684818240ab^9)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(2718132397440a^2b^9 - 855046307760a^3b^9 + 145414149000a^4b^9 - 19049720100a^5b^9)}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(901000275a^6b^9 - 58929450a^7b^9 - 199271883200b^{10} + 279195240480ab^{10})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

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$$+\frac{(-154142180400a^2b^{10}+43587835200a^3b^{10}-8158694700a^4b^{10}+641090385a^5b^{10})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-56448210a^6b^{10}+88410308848b^{11}-11619237968ab^{11}+6044516400a^2b^{11}-1695549960a^3b^{11})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(212995575a^4b^{11}-24192090a^5b^{11}-271930672b^{12}+349449360ab^{12}-166890360a^2b^{12})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(35236305a^3b^{12}-5259150a^4b^{12}+6056120b^{13}-6800780ab^{13}+2737455a^2b^{13}-566370a^3b^{13})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+$$

$$+\frac{(-80260b^{14}+84165ab^{14}-26970a^2b^{14}+659b^{15}-434ab^{15}-b^{16})}{\prod_{\eta=1}^{15}\{a-2\eta\}}+\frac{(a-b+30)}{(a-b-1)}\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-29}{2})}\times$$

$$\times\Bigg\{\frac{(-42849873690624000+100746609662361600a-92490335838535680a^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(46632165832065024a^3-14859663097823232a^4+3271193580412928a^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-505025512816640a^6+59551759843072a^7-4947215345536a^8+339191763264a^9)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-15279519840a^{10}+621837424a^{11}-13939432a^{12}+323708a^{13}-2690a^{14}+30a^{15})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(26814643804569600b-105369623089643520ab+113987664871096320a^2b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-62163571923615744a^3b+20711447922302976a^4b-4569102979137536a^5b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(733037562791680a^6b-81170061386752a^7b+7286981375168a^8b-424211354688a^9b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(22632383280a^{10}b-649171744a^{11}b+20735156a^{12}b-223076a^{13}b+4030a^{14}b)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(21476290178580480b^2+18497359471509504ab^2-46383618741829632a^2b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(30822099684507648a^3b^2 - 11169873454153728a^4b^2 + 2610314589022976a^5b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-405911801172480a^6b^2 + 48474589768960a^7b^2 - 3740645877984a^8b^2 + 257569498128a^9b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-9573055440a^{10}b^2 + 397825480a^{11}b^2 - 5539638a^{12}b^2 + 138446a^{13}b^2)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-29081047157637120b^3 + 19697219949428736ab^3 + 1926305147437056a^2b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-6525530932051968a^3b^3 + 2983163756323584a^4b^3 - 753958524042240a^5b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(127441219311360a^6b^3 - 13733274514176a^7b^3 + 1232794859472a^8b^3 - 60858560160a^9b^3)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(3218599800a^10b^3 - 58621992a^{11}b^3 + 1893294a^{12}b^3 + 14200532432486400b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-13886382286397440ab^4 + 4380147255427072a^2b^4 - 131032468218624a^3b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-363675061996800a^4b^4 + 120686089584000a^5b^4 - 21411848928960a^6b^4 + 2650892519520a^7b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-183751464600a^8b^4 + 12407548500a^9b^4 - 303277650a^{10}b^4 + 12271350a^{11}b^4)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-4036298413056000b^5 + 4339068030189568ab^5 - 1878105598159616a^2b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(348720370467840a^3b^5 - 20359470898560a^4b^5 - 8674613539200a^5b^5 + 2132804429280a^6b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-253026930240a^7b^5 + 22893160860a^8b^5 - 800199900a^9b^5 + 40320150a^{10}b^5)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(759685713346560b^6 - 857574469125888ab^6 + 386934853266944a^2b^6 - 98495967926016a^3b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

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$$+\frac{(+113537339395 20a^4b^6-705515781600a^5b^6-86079178080a^6b^6+15527599920a^7b^6)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-1003041270a^8b^6+65132550a^9b^6-100366817591040b^7+113217893178368ab^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-54537090886912a^2b^7+13207757720832a^3b^7-2209706471520a^4b^7+150958641600a^5b^7)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-8858026800a^6b^7-282861360a^7b^7+35357670a^8b^7+9589432195200b^8-10859483768000ab^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(4938029127456a^2b^8-1340626529232a^3b^8+178725335400a^4b^8-20749369500a^5b^8)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(664352010a^6b^8-35357670a^7b^8-671530516800b^9+731396205504ab^9-342285504528a^2b^9)}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(76951667040a^3b^9-13011676500a^4b^9+787793700a^5b^9-65132550a^6b^9+34499905440b^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-36817157936ab^{10}+15032099888a^2b^{10}-3775087992a^3b^{10}+369893550a^4b^{10})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-40320150a^5b^{10}-1285480560b^11+1257372896ab^{11}-523484104a^2b^{11}+84987864a^3b^{11})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\frac{(-12271350a^4b^{11}+337974 00b^{12}-31361460ab^{12}+9419722a^2b^{12}-1893294a^3b^{12}-594300b^{13})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}+$$

$$+\left.\left.\frac{(443548ab^{13}-138446a^2b^{13}+6270b^{14}-4030ab^{14}-30b^{15})}{\prod_{\zeta=1}^{15}\{a-(2\zeta-1)\}}\right\}\right]$$

In this way, we get the main formula.

IV. Conclusion

In this paper we have derived a summation formula with the help of contiguous relation . However, the formula presented herein may be further developed to extend this result .Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions. Just as the mathematical properties of the Gauss hypergeometric function are already of immense and significant utility in mathematical sciences and numerous other areas of pure and applied mathematics, the elucidation and discovery of the formula of hyperge-

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ometric functions considered herein should certainly eventually prove useful to further developments in the broad areas alluded to above.

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On Markovian Queueing Model as Birth-Death Process

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Abstract- Markovian queueing model has so many application in real life situations. Places where Markovian queueing model can be applied include, Supermarket, Production system, Post office, data communication, parking place, assembly of printed circuit boards, call center of an insurance company, main frame computer, toll booths, traffic lights, e.t.c. Birth-death process has being markovian foundation on queueing models. This article is an eye opener to novice researchers, since it explore Markovian queueing model in real life situation. The fundamental of Markovian Queueing model as birth and death process is hereby reviewed in this article, with fundamental results applications in $M/M/1$, $M/M/S$, $M/M/1/K$, and $M/M/s/K$. Here we reexamined; Average Number of Customers and average number of time in the system, waiting in the queue, in service respectfully. These summaries of these results are also tabulated.

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GJSFR-F Classification : MSC 2010: 60K25, 65C10

ON MARKOVIAN QUEUEING MODEL AS BIRTH-DEATH PROCESS

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Abstract- Markovian queueing model has so many application in real life situations. Places where Markovian queueing model can be applied include, Supermarket, Production system, Post office, data communication, parking place, assembly of printed circuit boards, call center of an insurance company, main frame computer, toll booths, traffic lights, e.t.c. Birth-death process has being markovian foundation on queueing models. This article is an eye opener to novice researchers, since it explore Markovian queueing model in real life situation. The fundamental of Markovian Queueing model as birth and death process is hereby reviewed in this article, with fundamental results applications in $M / M / 1$, $M / M / S$, $M / M / 1/ K$, and $M / M / s / K$. Here we reexamined; Average Number of Customers and average number of time in the system, waiting in the queue, in service respectfully. These summaries of these results are also tabulated.

Keywords: markovian properties, random process, poisson and exponential probability functions, sum to infinity of a G.P.

I. Introduction

This article is a review on Markovian queuing model. The general expression for an explicit markovian queueing model by definition is given as $M/M/./.$ the first M is a Poisson rate of arrival with an exponential time distribution and the second M represent the exponential service time. The other dots represent other attributes similar to general queueing model.

The need for queueing models cannot be overemphasize because in any service station, the owner may be interested to know when to increase service points or number of queues, putting cost into consideration, In a bank, or a selling out feet, how long will one have to wait, and how can we decompose the waiting time during rush period. A production manager will want to know the lead time production for an order or for the production in mounting vertical components on printed circuit boards, how can this lead time be reduced and what will be the effect in the production system also when order are prioritized. The information or computer technologist will want to estimate the number of cell delay at the switches, the fraction of cell lost, and the size of the buffer that will be good enough to accommodate more cells. In air and sea port or any other out feet, it is important to maximize the available parking space. Managers of Call centers will want to minimize the waiting time of customers, by increasing call centers, operators, pooling teams for better efficiency and also traffic light regulators

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and tollbooth managers will need to give acceptable waiting time and acceptable amount to pay to motorist respectively. There may be need for Server managers to increase efficiency and capacity of their servers in order to handle more transactions. All these and more can be modeled using Markov queueing processing. Ivo Adan and Jacques in 2002 give some application of queue model.

Birth and death process has been regarded as an important subclass of Markov Chains and is frequently used to model growth of biological population, Zhong Li 2013. He also compute the expected extinction time of birth-death chain. In 2010 on the application queueing theory to epidemic model Carlos M. H. and others give an expression that relates basic reproductive number, R_0 and the server utilization, ρ. also they derive new approximations to quasistationary distribution (QSD) of SIS (Susceptible- Infected- Susceptible) and SEIS (Susceptible- Latent- Infected-Susceptible) stochastic epidemic models. In their work they considered all individual in a close population to be server of which this individual may either be busy (infected) or idle (susceptible). Research work on epidemiology continuous markov chain in queueing model is just too few. In 1971 n −phase generalization of the typical $M/M/1$ queueing model, were considered, where the queueing-type birth-and-death process is defined on a continuous-time n-state Markov chain. It was conclude that the n −phase generalization of the steady-state $M/M/1$ queue will not yield, in general closed-form solution. Hence there will be need to employ numerical method to solve any specific case. Some applications to classical birth-death Markov process are given by Carlos M. and Carlos C. 1999. John Willey in 2006 and son give a thorough treatment of queue system and queueing network; among other method used, continuous Markov chain was employed. Forrest and Marc, 2011 used the continuous-time Markov chain that counts particles in a system over a time as a birth and death proceces to obtain expressions for Laplace transforms of transition probabilities in a general birth-death process with arbitrary birth and death rates and make explicit important derivation connecting transition probabilities and continued fraction. Markovian model from Markov chain where used in application and examples to illustrate key points. Solution techniques of Markova regeneration processes where investigated.

It surprising to note that, no so much research has been done using $M/M/./.$ queue theory and model in *Epidemiology* analysis which has to do with study of disease origin and spread pattern of disease development. In this article our focus is on Markovian queueing model as a birth-death process with emphasis on epidemiological analysis.

II. Definitions from Queue Model

a) *Memoryless property of the exponential distribution*

Memoryless means that the probability of time of occurrence of the event no matter how long since the last event occur is the same. That is, $Pr(x \leq T + t \mid x > T) = Pr(x \leq t)$ in real world situation this not always true. In most cases it is applicable to

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phenomena that follow random variable and random processes. For example, the longer a real traffic light has been red, the greater the probability that it will turn green in the next, say, 10 seconds, this situation is not a random process. If the probability of a traffic light turning green in the next 10 seconds does not change independent of how long it has been red, then the distribution of the red light is memoryless. Only two distributions are memoryless - the exponential (continuous) and geometric (discrete). Here is the memoryless proof for the exponential distribution…

$$\Pr(x \le T+t|x>T) = \frac{Pr[(x \le T+t) \cap (x>T)]}{\Pr(x>T)} = \frac{Pr[(x \le T+t) - (x \le T)]}{\Pr(x>T)}$$

$$= \frac{\left(1-e^{-\lambda(T+t)}\right)-\left(1-e^{-\lambda T}\right)}{1-\left(1-e^{-\lambda T}\right)} = \frac{e^{-\lambda T}\left(1-e^{-\lambda t}\right)}{e^{-\lambda T}} = 1-e^{-\lambda t}$$

$P(x > T+\ t|x >\ T) =\ P(X\ >\ x) = 1 - Pr(x \le T+t|x>T) = 1-\ \left(1-e^{-\lambda t}\right) =$

$e^{-\lambda t}$. This memoryless property state that

b) *The Little's Formula*

Assume that entering customers are required to pay an entrance fee (according to some rule) to the system. Then we have Average rate at which the system earns $= \lambda_a \times$ average amount an entering customer pays where λ_a is the average arrival rate of entering customers

$\lambda_a = \lim_{t\to\infty} \frac{X(t)}{t}$ and $X(t)$ denotes the number of customer arrivals by time t.(see [1] and [2])

c) *Birth-Death Process of Markov Chain*

The birth-Death process is a case of Markov time continuous process. The current size of the population represent the state. For a birth-death Markov time continuous process the movement from one state to another; known as transition is limited to birth and death. Let i represent each state such that the state can move from i to $i+1$ by birth and $i-1$ by death, we assume that the movement from one state to another is independent from each other. Let λ_i and μ_i for $i = 1, 2, \ldots$ represent birth and death process respectively. We define pure death process as $\mu_i$such that $\lambda_i = 0$ and pure birth process as $\lambda_i$such that $\mu_i = 0$ for all i. the probability transition from state i to $i+1$ and to $i-1$, is $\frac{\lambda_i}{\lambda_i+\mu_i} = P[B(i) < D(i)]$ and $\frac{\mu_i}{\lambda_i+\mu_i} = P[D(i) < B(i)]$, respectively where $P[B(i) < D(i)]$ and $P[D(i) < B(i)]$ are the probability of the time until a birth $B(i)$ is less than the time until a death $D(i)$ and probability of the time until a birth $B(i)$ is less than the time until a death$D(i)$ respectively. The process remain in state i with exponential distribution $\lambda_i + \mu_i$. For a death to occur there must be a birth, for there to be any first noticeable change in the system, the process must move from state i to state $i+1$ which implies one birth and no death, of which its probability is given by

$$P_{i,i+1}(h) = P(X(t+h) - X(t) = 1|X(t) = i) = \frac{(\lambda_i h)^1 e^{-\lambda_i h}}{1!}\frac{(\mu_i h)e^{-\mu_i h}}{0!} + 0(h)$$

$$= (\lambda_i h)e^{-\lambda_i h}e^{-\mu_i h} + o(h) = (\lambda_i h)e^{-h(\lambda_i+\mu_i)} = (\lambda_i h)\sum_{n=0}^{\infty}\frac{(-h(\lambda_i+\mu_i))^n}{n!}$$

$$= (\lambda_i h)\left(1 - h(\lambda_i+\mu_i) - \frac{1}{2!}h^2(\lambda_i+\mu_i)^2 \ - \cdots\right) + 0(h)$$

$$= \lambda_i h + o(h)$$

The probability for moving from state i to $i-1$ is given by $P_{i,i-1}(h) = P(X(t+h) - X(t) = -1|X(t) = i) = \mu_i h + o(h)$. The probability of having any other moves other than this two is non-zero instead is given by $P(X(t+h) - X(t) > 1|X(t) = i) = o(h)$, for $\mu_0 = 0, \lambda_0 > 0;\ \mu_i, \lambda_i > 0; for\ i = 1, 2, 3, \ldots$. This also implies that $P(X(t+h) - X(t) = 0|X(t) = i) = 1 - h(\lambda_i+\mu_i) + o(h)$. Generally we can represent the birth and death process by

$$p_{i,j}(h) = \begin{cases} \lambda_i h + o(h) & if\ j = i+1 \\ \mu_i h + o(h) & if\ j = i-1 \\ \quad 1 - h(\lambda_i+\mu_i) + o(h) & if\ j = i \\ o(h) & otherwise \end{cases}$$

This can be $p_{i,j}(h) = \delta_{i,j} + r_{i,j}(h) + h(0)$

$\delta_{i,j} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$ the Kronecker's delta

$$g_{i,j}(h) = \begin{cases} \lambda_i & if\ j = i+1 \\ \mu_i & if\ j = i-1 \\ -(\lambda_i+\mu_i) & if\ j = i \\ o & otherwise \end{cases}$$

Then the matrix G is the infinitesimal generator of the process $X(t)$ define by $[g_{i,j}]$, where $g_{i,j}$ are called transition rate.

$$G = \left\|\begin{matrix} -\lambda_0 & \lambda_0 & 0 & 0 & . \ . \ . \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & . \ . \ . \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & . \ . \ . \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & . \ . \ . \\ . & . & . & . & . \ . \ . \\ . & . & . & . & . \ . \ . \\ . & . & . & . & . \ . \ . \end{matrix}\right\|$$

Note that $\delta_{i,j} = p_{i,j}(0)$ since the process remain in the same state in zero step with probability one and move to another state in zero step with probability one. We have

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$g_{i,j} = \frac{p_{i,j}(h)-\delta_{i,j}}{h} = \frac{p_{i,j}(h)- p_{i,j}(0)}{h} = p'_{i,j}(0)$ hence differentiating term by term and setting

$t = 0 \;\; \sum_j g_{i,j}(0) = 0$ this implies that $g_{i,j}(t) = p'_{i,j}(0) \geq 0 \qquad for\ i \neq j$

$g_{i,j}(t) = p'_{i,j}(0) \leq 0$ otherwise

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Also $\sum_j p_{i,j}(t) = 1$

d) Probability Transition of Birth-Death Process and Differential Equation from Kolmogorov

Kolmogorov backward differential equation describe the transition probabilities in their dependence on the initial point i

Basically

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(t)$$

$$= P_{i,i-1}(h)P_{i-1,j}(t) + P_{i,i+1}(h)P_{i+1,j}(t) + P_{i,i}(h)P_{i,j}(t) + \sum_{k}' P_{ik}(h)P_{kj}(t)$$

The last summation is for $k \neq i-1, i, i+1$.

$$P_{ij}(t+h) = (\mu_i h + o(h))P_{i-1,j}(t) + (\lambda_i h + o(h))P_{i+1,j}(t) + (1 - h(\lambda_i + \mu_i) + o(h))P_{i,j}(t) + \sum_k' P_{ik}(h)P_{kj}(t)$$

But $\sum_k' P_{ik}(h)P_{kj}(t) \leq \sum_k' P_{ik}(h) = 1 - \left(P_{i,i}(h) + P_{i,i-1}(h) + P_{i,i+1}(h)\right)$

$$= 1 - (1 - h(\lambda_i + \mu_i) + o(h) + (\mu_i h + o(h) + (\lambda_i h + o(h))$$

$$= o(h)$$

hence we have

$$P_{ij}(t+h) = \mu_i h P_{i-1,j}(t) + \lambda_i h P_{i+1,j}(t) + (1 - h(\lambda_i + \mu_i))P_{i,j}(t) + o(h)\left(P_{i-1,j}(t) + P_{i+1,j}(t) + P_{i,j}(t) + 1\right)$$

and so;

$$P_{ij}(t+h) = \mu_i h P_{i-1,j}(t) + \lambda_i h P_{i+1,j}(t) + (1 - h(\lambda_i + \mu_i)P_{i,j}(t) + o(h)$$

$$= \mu_i h P_{i-1,j}(t) + \lambda_i h P_{i+1,j}(t) + P_{i,j}(t) - P_{i,j}(t)h(\lambda_i + \mu_i) + o(h)$$

also we have

$$\frac{P_{ij}(t+h)-P_{i,j}(t)}{h}=\frac{\mu_i hP_{i-1,j}(t)+\lambda_i hP_{i+1,j}(t)-P_{i,j}(t)h(\lambda_i+\mu_i)+o(h)}{h}$$

$$=\mu_i P_{i-1,j}(t)+\lambda_i P_{i+1,j}(t)-P_{i,j}(t)(\lambda_i+\mu_i)+o(1)$$

$$P'_{ij}(t)=\mu_i P_{i-1,j}(t)+\lambda_i P_{i+1,j}(t)-P_{i,j}(t)(\lambda_i+\mu_i),$$

we then derive the differential equation knowing there is no birth without death that $\mu_0=0$

$$P'_{0j}(t)=\mu_0 P_{0-1,j}(t)+\lambda_0 P_{0+1,j}(t)-P_{0,j}(t)(\lambda_0+\mu_0)$$

$$P'_{0j}(t)=\lambda_0 P_{1,j}(t)-\lambda_0 P_{0,j}(t)$$

and again

$$P'_{ij}(t)=\mu_i P_{i-1,j}(t)+\lambda_i P_{i+1,j}(t)-P_{i,j}(t)(\lambda_i+\mu_i)$$

we know that

$$P_{ij}(s+t)=\sum_k P_{ik}(s)P_{kj}(t)$$

by Chapman-Kolmogorov

Differentiating with respect to s we have

$$P'_{ij}(s+t)=\sum_k P'_{ik}(s)P_{kj}(t)$$

Setting $s=0$ gives

$$P'_{ij}(s+t)=\sum_k P'_{ik}(0)P_{kj}(t)$$

$$=\sum_k g_{ik}(0)P_{kj}(t)$$

Therefore

$$\boldsymbol{P}'(\boldsymbol{t})=\boldsymbol{GP}(\boldsymbol{t})$$

While the forward Kolmogorov differential equation describes the probability distribution of a state in time t keeping the initial point fixed, decomposing the interval $(0,t+h)$ into $(0,t)$ and $(t,t+h)$

Notes

Notes

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(h)$$

$$= P_{i,j-1}(t)P_{j-1,j}(h) + P_{i,j+1}(t)P_{j+1,j}(h) + P_{i,j}(t)P_{j,j}(h) + \sum_{k}{}' P_{ik}(t)P_{kj}(h)$$

The last summation is for $k \neq j-1, j, j+1$.

$$P_{ij}(t+h) = P_{i,j-1}(t)\lambda_{j-1}h + P_{i,j+1}(t)\mu_{j+1}h + P_{i,j}(t)\left(1 - h(\lambda_j + \mu_j)\right) + o(h)$$

Similar apply here as in Kolmogorov back differential equation

$$P'_{ij}(t) = P_{i,j-1}(t)\lambda_{j-1} + P_{i,j+1}(t)\mu_{j+1} - P_{i,j}(t)(\lambda_j + \mu_j)$$

$$P'_{i0}(t) = P_{i,1}(t)\mu_1 - P_{i,0}(t)\lambda_0$$

With the same initial condition $P_{ij}(0) = \delta_{ij}$.

We know that

$$P_{ij}(s+t) = \sum_k P_{ik}(s)P_{kj}(t)$$

by Chapman-Kolmogorov

Differentiating with respect to t we have

$$P'_{ij}(s+t) = \sum_k P_{ik}(s)P'_{kj}(t)$$

Setting $t = 0$ gives

$$P'_{ij}(s) = \sum_k P_{ik}(s)P'_{kj}(0)$$

$$= \sum_k P_{ik}(t)g_{kj}$$

therefore

$$\boldsymbol{P}'(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{t})\boldsymbol{G}$$

For $\boldsymbol{p}(\boldsymbol{0}) = \boldsymbol{1}$

$$\boldsymbol{P}(\boldsymbol{t}) = \mathrm{e}^{\mathbf{Gt}}$$

Where $\mathrm{e}^{\mathbf{Gt}} = \sum_{n=0}^{\infty} \frac{\boldsymbol{G}^n \boldsymbol{t}^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\boldsymbol{G}^n \boldsymbol{t}^n}{n!}$

Also changes or number of event $j-i$ over a period of time of let T for a Poisson pure birth process with the infinitesimal transition rate

$$\begin{cases} p_{i,j} = \lambda_i & if\ j = i+1 \\ p_{i,j} = \lambda_i & if\ j = i \\ 0 & otherwise \end{cases}$$

follows a Bernoulli trial performed n times with the probability of success λ hence we have

$$p_{ij}^{(n)} = \begin{cases} \binom{n}{j-i} \lambda^{j-i}(-\lambda)^{n-(j-i)} & if\ 0 \le j-i \le n \\ 0 & otherwise \end{cases}$$

$$\binom{n}{j-i} \lambda^{j-i}(-\lambda)^{n-(j-i)} = \binom{n}{j-i} \lambda^{n}(-1)^{n-(j-i)}$$

From the formula

$$\boldsymbol{P}(t) = e^{Gt} = \sum_{n=0}^{\infty} \frac{\boldsymbol{G}^n \boldsymbol{t}^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\boldsymbol{G}^n \boldsymbol{t}^n}{n!}$$

$$p_{ij}(t) = \delta_{ij} + \frac{\boldsymbol{t}^1}{1!} g_{ij} + \frac{\boldsymbol{t}^2}{2!} g_{ij}^2 + \cdots$$

$$p_{ij}(t) = \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^n}{n!} g_{ij}^n = \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^n}{n!} \binom{n}{j-i} \lambda^{j-i}(-\lambda)^{n-(j-i)}$$

$$= \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^n}{n!} \frac{n!}{(n-(j-i))!\,(j-i)!} \lambda^{j-i}(-\lambda)^{n-(j-i)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(n-(j-i))!\,(j-i)!} \lambda^{j-i}(-\lambda)^{n-(j-i)}$$

$$= \sum_{n=0}^{\infty} \frac{\boldsymbol{t}^{(j-i)}}{(j-i)!} \lambda^{j-i} \frac{t^{(n-(j-i))}}{(n-(j-i))!} (-\lambda)^{n-(j-i)}$$

$$= \frac{(\lambda t)^{(j-i)}}{(j-i)!} \sum_{n=0}^{\infty} \frac{(-\lambda t)^{(n-(j-i))}}{(n-(j-i))!}$$

But $j-i \neq 0$ hence n cannot start from zero, but from $j-i$ and letting $k = n-(j-i)$ we have

$$p_{ij}(t) = \frac{(\lambda t)^{(j-i)}}{(j-i)!} \sum_{n=j-i}^{\infty} \frac{(-\lambda t)^{(n-(j-i))}}{(n-(j-i))!}$$

Notes

$$= \frac{(\lambda t)^{(j-i)}}{(j-i)!} \sum_{n=0}^{\infty} \frac{(-\lambda t)^k}{k!} = \frac{(\lambda t)^{(j-i)}}{(j-i)!} e^{-\lambda t}$$

Which is Poisson process

Notes

e) The law of Rare Events

This law holds when the probability of success p occurrence from large number N of independent Bernoulli trails is small and constant from one occurrence to another. Let $X_{N,p}$ follows the binomial distribution, such that $X_{N,p}$ is the total number of success in trails for $k = 0,1,2,\dots,N$.

$$P = \binom{N}{k} p^k (1-p)^{N-k}$$

$$P(X_{N,p} = k) = \binom{N}{k} p^k (1-p)^{N-k} = \frac{N(N-1)(N-2)\dots(1-k+1)}{k!} p^k (1-p)^{N-k} \quad k = 0,1,\dots,N$$

Multiplying and dividing the right-hand side by N^k, we have

$$p_X(k) = \binom{N}{k} p^k (1-p)^{N-k} = \frac{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\dots\left(1-\frac{k-1}{N}\right)}{k!} (Np)^k p^k \left(1-\frac{Np}{n}\right)^{N-k}$$

If we let $n \to \infty$ in such a way that $Np = \lambda$ remains constant, then

$$\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\dots\left(1-\frac{k-1}{N}\right) \xrightarrow[N\to\infty]{} 1$$

$$\left(1-\frac{Np}{N}\right)^{N-k} = \left(1-\frac{\mu}{N}\right)^{n}\left(1-\frac{\mu}{N}\right)^{-k} \xrightarrow[N\to\infty]{} e^{-\mu}(1) = e^{-\mu}$$

Where we need the fact that

$$\lim_{n\to\infty}\left(1-\frac{\mu}{N}\right)^{N} = e^{-\mu}$$

Hence, in the limit as $N \to \infty$ with $Np = \mu$ (and as $p = \frac{\mu}{N} \to 0$),

$$\binom{N}{k} p^k (1-p)^{N-k} \xrightarrow[n\to\infty]{} e^{-\mu}\frac{\mu^k}{k!} \qquad Np = \mu$$

Thus, in the case of large N and small p

$$\binom{N}{k} p^k (1-p)^{N-k} \approx e^{-\mu}\frac{\mu^k}{k!} \qquad Np = \mu$$

Which, indicate that the binomial distribution can be approximated by the Poisson under some circumstance in Stochastic modeling. The Poisson distribution has independent increment, that is for $t_0 \le t_1 < t_2 < \cdots < t_n$, and $t_0 = 0$ such that $\{X(t); t \ge 0\}$ which is the Stochastic process of event happening we have $X(t_1) - X(t_0), X(t_2) - X(t_1), X(t_3) - X(t_2) \ldots , X(t_n) - X(t_{n-1})$ this is the independent increment property Poisson process such that for each increment for s to $s + t$, the probability associated to the stochastic process $\{X(t); t \ge 0\}$ for $\lambda > 0$ as the intensity of the process is given by $P\big(X(s+t) - X(s) = k\big) = \frac{(\lambda t)e^{-\lambda t}}{k!}$. for exactly one event happening over a period h we have $P\big(X(s+t) - X(s) = 1\big) = \frac{(\lambda h)e^{-\lambda h}}{1!}$

$$= (\lambda h)\sum_{n=0}^{\infty}\frac{(-\lambda h)^n}{k!} = (\lambda h)\left(\frac{(-\lambda h)^0}{0!} + \frac{(-\lambda h)^1}{1!} + \frac{(-\lambda h)^2}{2!} + \cdots\right)$$

$$= (\lambda h)\left(1 - \lambda h - \tfrac{1}{2!}\lambda^2 h^2 - \cdots\right) = \lambda h + 0(h)$$

Note: Poisson process count the number of events entering into the system while the time in between these events, also known as sojourn time, follows an exponential probability distribution. Suppose at this time interval no arrival $P(X = 0) = e^{-\lambda t}$ this can interpreted that the time T of the first occurrence greater than t, hence we have $P(T > t) = P(X = 0|\lambda t) = e^{-\lambda t}$ hence this implies that $P(T \le t) = 1 - P(X = 0|\lambda t) = 1 - e^{-\lambda t}$, differentiating we have $f(t) = \lambda e^{-\lambda t}$ which is an exponential distribution. This give the Connection between Poisson And Exponential Distribution. The distribution of the Sojourn time of the random variable $X(t)$ in state i can adequately be describe by the random variable J_i. Hence the distribution of the time j_i the process $X(t)$ first leaves state i can be determine; that is $Z_i(t) = P(J_i \ge t)$, by the independency of Markovian property $h \to 0$ we have $Z_i(t + h) = Z_i(t) + Z_i(h) = Z_i(t)\big[P_{i,i}(h) + o(h)\big]$

$$= Z_i(t)[1 - h(\lambda_i + \mu_i)] + o(h)$$

$$= Z_i(t) - Z_i(t)h(\lambda_i + \mu_i) + o(h)$$

Subtracting $Z_i(t)$ from both sides and dividing by h gives

$$\frac{Z_i(t+h) - Z_i(t)}{h} = \frac{Z_i(t) - Z_i(t)h(\lambda_i + \mu_i) + o(h) - Z_i(t)}{h}$$

$$= \frac{Z_i(t+h) - Z_i(t)}{h} = -Z_i(t)(\lambda_i + \mu_i) + o(h)$$

$Z_i'(t) = -Z_i(t)(\lambda_i + \mu_i)$ hence we have $Z_i(t) = e^{-t(\lambda_i + \mu_i)}$

Notes

For the fact that $Z_i(t) = P(J_i \geq t) = 1 - P(J_i \leq t) = 1 - e^{-t(\lambda_i+\mu_i)}$, it follows that J_i follows exponential distribution with parameter $\frac{1}{\lambda_i+\mu_i}$ as mean.

Notes

f) Steady-State Probability of Birth-Death Process

Steady state is reach under the condition $\frac{\lambda_n}{\mu_n} < 1$, where λ_n and μ_n are birth and death rate respectively. So from

$$P'_{i0}(t) = P_{i,1}(t)\mu_1 - P_{i,0}(t)\lambda_0$$

$$P'_{ij}(t) = P_{i,j-1}(t)\lambda_{j-1} + P_{i,j+1}(t)\mu_{j+1} - P_{i,j}(t)(\lambda_j + \mu_j),$$

With the same initial condition $P_{ij}(0) = \delta_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$ we assume steady such that

$$0 = P_{i,1}(t)\mu_1 - P_{i,0}(t)\lambda_0$$

$$0 = P_{i,j-1}(t)\lambda_{j-1} + P_{i,j+1}(t)\mu_{j+1} - P_{i,j}(t)(\lambda_j + \mu_j),$$

$$P_{i,1}(t) = P_{i,0}(t)\frac{\lambda_0}{\mu_1}$$

$$P_{i,2}(t) = P_{i,1}(t)\frac{\lambda_1}{\mu_2} = P_{i,0}(t)\frac{\lambda_0\lambda_1}{\mu_1\mu_2} \qquad e.t.c$$

$$P_{i,n}(t) = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2 \dots \mu_n}P_{i,0}(t) \qquad e.t.c$$

We know that $\sum_{n=0}^{\infty} P_{i,n}(t) = 1$ for the ith system

$$P_{i,0}(t) = \frac{1}{\left\{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \cdots\right\}}$$

The necessary condition for the existence of a steady-state solution is that $\sum_{n=0}^{\infty} \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2 \dots \mu_n}$ must converge to ensure that $P_{i,0}(t) \neq 0$

III. The Epidemiology M/M/./. Queue Model

In a system, entry into the system, can be regarded as arrival, the Epidemiologist might interested to know the rate of entry into the system, the number in the system at time t how long will someone be in the system to be infected, when will infection begin, when will the signs or symptom start manifesting, at what time will infection elapse, at what stage or rate will the process of removal change from real death to recovering. Other case may arise where entry in the system may not mean infection, may be because of vaccination or high resistance of that individual, the researcher will want to know how effective is the vaccine, and how significant is the resistance. Sometime removal may not necessarily mean recovery or removal, but rendering the patient incapacitated, the epidemiologist will want to know the time, duration for such to happen. The $M/M/./.$ queue model can be used to give exact or approximated answers to the above questions. On this light we will discuss well known $M/M/./.$ queue model in Epidemiologist point of view.

Notes

a) $M/M/1$ Queueing Model

This can be divided into two. There might be a situation when entry into the system, sometime may elapse before one is infected, the other case is the situation one get infected immediately he/she enter the system, the former will be discuss later.

$\lambda_n = \lambda \qquad n \geq 0$ the rate of infection

$\mu_n = \mu \qquad n \geq 1$ the rate of removal (recovering or death)

$p_{i0}(t) = P_i\{N(t) = 0\} = 1 - \frac{\lambda}{\mu} = 1 - \rho$ The probability of infection, such that non is infected.

$$p_{in}(t) = P_i\{N(t) = n\} = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n$$

The probability of infection, such that n, has been infected.

where $\rho = \frac{\lambda}{\mu} < 1$, and $i = 1$ number of server which implies that the rate of infection which implies rate of removal, on the average, must process faster than their average entry rate into the system.

The average number in the system that are infected is given by

$$N = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

in this case there is no waiting time for infection to take place once you find yourself in the system you became infected.

The average waiting time for a removal to occur in the system is given by

$$T = \frac{1}{\mu - \lambda}$$

On the other hand there might be a situation where, entering the system does not q your infection, this case there is a waiting time. We give the following expression for the above as follow;

Average waiting time in the system before infection is given by

$$= W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

here it is assume they do not get infected once they enter the system, but have to wait for some time.

The expected number of person in system before infection

$$= L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

Notes

if there exist a great deviation from this, the epidemiologist will be interested to why it is so. This is for just one type of infection, however this is just the foundation for other model on Epidemiology and it can also be decompose.

b) The $M/M/S$ Queueing Model

Situation may arise, where there are more than one system infected with diseases, such that diseases are not the same from system to system. Ideally system real or abstract, depend on the purpose of research, in some cases this model have to be modify. However fundamentally, this may be model after the well-known $M/M/s$ queue model, where s is the number of systems, here i take the value from 1 to s or any value that may suit the analysis.

$\lambda_n = \begin{cases} \lambda, & 0 \le n < K \\ 0, & n \ge K \end{cases}$ here we assume that the rates infection are the same for each system of infection

$\mu_n = \begin{cases} n\mu, & 0 < n < s \\ s\mu, & n \ge s \end{cases}$ here we assume that the rate of removal depend on the number of infected system.

Note that the removal parameter μ_n is state dependent.

Balance equations

$$\lambda p_{i-1} = \begin{cases} i\mu p_i, & for\ \ i < s \\ s\mu p_i, & for\ \ i = s \end{cases}$$

$p_{i0}(t) = P_i\{N(t) = 0\} = \left[\sum_{n=0}^{s-1} \frac{(s\rho)}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1}$ for each i in the number of s system, here we make use of some past record for the computation of $p_{i0}(t)$

$$p_{in}(t) = P_i\{N(t) = n\} = \begin{cases} \frac{(s\rho)^n}{n!} p_{i0}, & n < s \\ \frac{\rho^n s^s}{s!} p_{i0}, & n \ge s \end{cases}$$

where $\rho = \frac{\lambda}{(s\mu)} < 1$. Note that the ratio $\rho = \frac{\lambda}{(s\mu)}$ is the infection intensity of the $M/M/s$ queueing model.

the expected number in the system before infection is given by;

$$E(N_q) = L_q = \lambda W_q = L - \frac{\lambda}{\mu} = \frac{\rho(s\rho)^s}{s!\,(1-\rho)^2} p_{i0}$$

The expected number in the system before and during infection,

$$E(N) = L = \lambda W = \frac{\lambda}{\mu} + \lambda W_q = \frac{\lambda}{\mu} + \frac{\rho(s\rho)^s}{s!(1-\rho)^2} p_{i0}$$

because after removal it is expected they will no longer be in the system, however some model will not allow this.

While the expected time spent before infection in the system

$$= W_q = \frac{L_q}{\lambda} = w - \frac{1}{\mu} = \frac{\rho(s\rho)^s}{\lambda s!\,(1-\rho)^2} p_{i0}$$

The expected time lapse spent in the system before and during infection

$$= W = \frac{L}{\lambda} = \frac{1}{\mu} + W_q = \frac{1}{\mu} + \frac{\rho(s\rho)^s}{\lambda s!(1-\rho)^2} p_{i0}$$

When analytical computation of μ is very difficult or almost impossible, a Monte Carlo simulation is applied in order to get estimations. A standard Monte Carlo simulation algorithm fix a regenerative state and generate a sample of regenerative cycles, and then use this sample to construct a likelihood estimator of state (Nasroallah, 2004). Monte Carlo simulation uses the mathematical models to generate random variables for the artificial events and collect observations. (Banks, 2001).

c) The $M/M/1/K$ Queueing Model

This model may be built for the purpose of research. Here the population or the capacity of the system to be considered is known which is assume to be K. The $M/M/1/K$ queueing system can be modeled as a birth-death process with the following assumed parameters:

$$\lambda_n = \begin{cases} \lambda, & 0 \le n < K \\ 0, & n \ge K \end{cases}$$

$$\mu_n = \mu \qquad n \ge 1$$

$$p_{i0} = \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1-\rho}{1-\rho^{k+1}} \qquad \rho \neq 1$$

Notes

$$p_{in} = \begin{cases} \rho^n p_{i0} = \left(\frac{\lambda}{\mu}\right)^n p_0 = \frac{(1-\rho)\rho^n}{1-\rho^{k+1}}, \; for\; 0 \le n \le K & n = 1, \dots, K \\ 0, \qquad\qquad\qquad\qquad\quad for\;\; n > K \end{cases}$$

where $\rho = \frac{\lambda}{\mu}$

Notes

It is important to note that it is no longer necessary for infection intensity $\rho = \frac{\lambda}{\mu}$ *to* be less than 1. Patient will not be infected when the system is in state K

the expected number in the system before infection is given by;

$$L_q = L - (1 - p_{i0})$$

The expected number in the system before and during infection is given by,

$$L = \rho \frac{1-(K+1)\rho^K + Kp^{K+1}}{(1-\rho)(1-\rho^{K+1})}, \qquad \rho = \frac{\lambda}{\mu}$$

While the expected time spent before infection in the system is given by;

$$W_q = \frac{1}{\mu} L$$

The expected time lapse spent in the system before and during infection is given by;

$$W = \frac{1}{\mu}(L + 1)$$

Since the fraction of infection that actually enter into the is $1 — p_{iK}$, the infective arrival rate is given by

$$\lambda_e = \lambda(1 - p_{ik})$$

the expected number in the system before infection is equivalent to;

$$L_q = \lambda_e W_q = \lambda(1 - p_{ik}) W_q$$

The expected time lapse spent in the system before and during infection is equivalent to;

$$W = \frac{L}{\lambda_e} = \frac{L}{\lambda(1-p_K)}$$

While the expected time spent before infection in the system is equivalent to;

$$W_q = W - \frac{1}{\mu}$$

d) The $M/M/s/K$ *Queueing Model*

This general model contains only limited number K in the system. However, if there are unlimited number then $K = \infty$, then our model will be labeled as $M/M/s$ (Hillier & Lieberman, 2001.). The $M/M/s/K$ queueing system can be modeled as a birth-death process with the following parameter:

$$\lambda_n = \begin{cases} \lambda, & 0 \le n < K \\ 0, & n \ge K \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 < n < s \\ s\mu, & n \ge s \end{cases}$$

$$p_{i0} = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right)\right]^{-1}$$

$$p_{in} = \begin{cases} \frac{(s\rho)^n}{n!} p_{i0}, & n < s \\ \frac{\rho^n s^s}{s!} p_{i0}, & s \le n \ge K \end{cases}$$

where $\rho = \frac{\lambda}{(s\mu)}$.

The expected number in the system before infection is given by;

$$L_q = \rho_0 \frac{\rho(s\rho)^s}{s!\,(1-\rho)^2}\{1 - [1 + (1-\rho)(K-s)]\rho^{K-s}\}$$

The expected number in the system before and during infection is given by,

$$L = L_q + \frac{\lambda_e}{\mu} = L_q + \frac{\lambda}{\mu}(1 - p_K)$$

The quantities W and W_q which is the expected time spent before infection in the system and The expected time lapse spent in the system before and during infection respectively are given by;

$$W = \frac{L}{\lambda_e} = L_q + \frac{1}{\mu}$$

$$W_q = \frac{L_q}{\lambda_e} = \frac{L_q}{\lambda(1 - p_K)}$$

e) Tabulated Result

However, for easy understanding these results has been tabulated as follow;

Notes

	$p_0\,(t)$	$p_n(t)$	L	L_q	W	W_q
$M/M/1$	$1-\frac{\lambda}{\mu}$, $1-\rho$	$\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$, $(1-\rho)\rho^n$ $\rho=\frac{\lambda}{\mu}<1$	λW $\frac{\lambda}{\mu-\lambda}$ $\frac{\rho}{1-\rho}$	$\frac{\lambda^2}{\mu(\mu-\lambda)}$ $\frac{\rho^2}{1-\rho}$	$\frac{1}{\mu-\lambda}$ $\frac{1}{\mu(1-\rho)}$	$\frac{\lambda}{\mu(\mu-\lambda)}$ $\frac{\rho}{\mu(1-\rho)}$
$M/M/s$	$\left[\sum_{n=0}^{s-1}\frac{(s\rho)}{n!}\frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1}$ $\rho=\frac{\lambda}{(s\mu)}$	$\begin{cases}\frac{(s\rho)^n}{n!}p_0, & n<s\\ \frac{\rho^n s^s}{s!}p_0, & n\geq s\end{cases}$ $\rho=\frac{\lambda}{(sp)}<1$	λW $\frac{\lambda}{\mu}+\lambda W_q$ $\frac{\lambda}{\mu}+\frac{\rho(s\rho)^s}{s!(1-\rho)^2}p_0$	λW_q $L-\frac{\lambda}{\mu}$ $\frac{\rho(s\rho)^s}{s!\,(1-\rho)^2}p_0$	$\frac{L}{\lambda}$ $\frac{1}{\mu}+W_q$ $\frac{1}{\mu}+\frac{\rho(s\rho)^s}{\lambda s!\,(1-\rho)^2}p_0$	$\frac{L_q}{\lambda}$ $w-\frac{1}{\mu}$ $\frac{\rho(s\rho)^s}{\lambda s!(1-\rho)^2}p_0$
$M/M/1/K$	$\frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$ $\frac{1-\rho}{1-\rho^{k+1}}$ $\rho\neq 1$	$\left(\frac{\lambda}{\mu}\right)^n p_0$ $\frac{(1-\rho)\rho^n}{1-\rho^{k+1}}$ $n=1,\dots,K$	$\rho\frac{1-(K+1)\rho^K+Kp^{K+1}}{(1-\rho)(1-\rho^{K+1})}$ $\rho=\frac{\lambda}{\mu}$	$L-(1-p_0)$ Or $\lambda_e W_q$ $\lambda(1-p_k)W_q$ $\lambda_e=\lambda(1-p_k)$	$\frac{1}{\mu}(L+1)$ Or $\frac{L}{\lambda_e}$ $\frac{L}{\lambda(1-p_K)}$	$\frac{1}{\mu}L$ $W-\frac{1}{\mu}$
$(M/M)/s/K$	$\left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!}+\frac{(s\rho)^s}{s!}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right)\right]^{-1}$	$\begin{cases}\frac{(s\rho)^n}{n!}p_0, & n<s\\ \frac{\rho^n s^s}{s!}p_0, & s\leq n\geq K\end{cases}$ $\rho=\frac{\lambda}{(s\mu)}$	$\rho_0\frac{\rho(s\rho)^s}{s!(1-\rho)^2}\{1-[1+(1-\rho)(K-s)]\rho^{K-s}\}$	$L_q+\frac{\lambda_e}{\mu}=L_q+\frac{\lambda}{\mu}(1-p_K)$	$\frac{L}{\lambda_e}$ $L_q+\frac{1}{\mu}$	$\frac{L_q}{\lambda_e}$ $\frac{L_q}{\lambda(1-p_K)}$

IV. The Epidemiology $M/M/./././$ Queue Model

Applications abound in $M/M/./.$ these rages from theoretical and analytical applications and numerical, (Ivo Adan and Jacques Resing, (2002))with the advent of computer programming it is much easier to simulate using these model. In Epidemiology, no much work can be seen,

3.1 Let W_E denote the amount of time lapse before infection in the $M/M/1$ queueing system. the distribution of W_E is given by

$$F_{W_E}=P\{W_a\leq a\}=\sum_{n=0}^{\infty}\left[\int_0^a \mu e^{-\mu t}\frac{(\mu t)^n}{n!}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n dt\right]$$

$$=\int_0^a(\mu-\lambda)e^{-\mu t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}dt$$

$$=\int_0^a(\mu-\lambda)e^{-(\mu-\lambda)t}dt=1-e^{-(\mu-\lambda)a}$$

Note; $(W_E)=\frac{1}{\mu-\lambda}$, since $W=E(W_E)$. (Hwei Hsu, (2011))

3.2. The rate λ of entry into infected system follow a Poisson process and the service time is an exponential r.v. with mean μ. the expected number in the system is L, the w, and the waiting time before infection W_q. When the rate λ, of entry into infected system increases by any percent. It increases the expected number of infection in the system. The average time spent in the system waiting to be infected also increased.

3.3. For an $M/M/1$ queueing model/system rate of infection most be at least $\sqrt[k]{\frac{100\lambda^k}{1-n}}$ with entry rate λ, when there are k number yet to be infected in the system up n percent of the capacity of the system accommodating number yet to be infected but in the system.

3.4. The probability that an arrival may enter the infected system s is given by

$$P(that\ an\ arrival\ may\ enter\ the\ infected\ system\ s) = \sum_{n=s}^{\infty} p_n = p_0 \frac{s^s}{s!} \sum_{n=s}^{\infty} \rho^n$$

$$= p_0 \frac{(s\rho)^s}{s!\,(1-\rho)} = \frac{\frac{(s\rho)^s}{s!\,(1-\rho)}}{\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!\,(1-\rho)}}$$

This is the modify Erlang's delay $(or\ C)$ formula and denoted by $C(s, \frac{\lambda}{\mu})$. (Hwei Hsu, (2011).

3.5 In an $M/M/1$ queue model one can compute the expected number of a arrival during an infected period and the probability that no arrival during that period, the expected number of arrival= $E[E\{number\ of\ arrival | infected\ period\ T\}] = E[\lambda T] = \frac{\lambda}{\mu}$. while 0 arrival probability $P\{zero\ arrival\} = E[P\{zero\ arrival | infected\ period\ T\}] = E[P\{N(T) = 0\}] = E\left[e^{-\lambda T}\right] = \int_0^x e^{-\lambda T} \mu e^{-\lambda s} ds = \frac{\mu}{\lambda+\mu}$

IV. Conclusion

We have considered result on $M/M/./.$/ Queue model. In some of the section model was form without showing its derivation, since these derivation can be found in any queue model text book [e.g., Hwei Hsu, 2011] and in order for the article not be too lighten. Markovian queueing model as a birth-death process is very vital in epidemiology study. For a more detail work, one may need to discuss on the following the rate of entry, infection and recovery, waiting time of entry, infection, and recovery. Also one may also discuss on number in the system, infected, and recovered. Heuristics data may be needed to validate this claim, simulation can also be employ in complex cases. All these result are observable at steady-state. It should be noted that result obtain here can actually be different from simulated result, why?. Queueing model has been so important in performance analysis in manufacturing system, which is another open area of research. We are next

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concerned about how to obtain solution for a queuing model with a network of queues? Such questions require running Queuing Simulation. Simulation can be used for more refined analysis to represent complex systems. The queuing system is when classified as M/M/c with multiple queues where number in the system and in a queue is infinite, the solution for such models are difficult to compute. When analytical computation of T is very difficult or almost impossible, a Monte Carlo simulation may be applied in order to get estimations.

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Study of Viscus Incompressible Fluid Past a Hot Vertical Porous Wall in the Presence of Transverse Magnetic Field with Periodic Temperature using the Homotopy Perturbation Method

By Vivek Kumar Sharma & Divya Saxena
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Abstract- Analytical solution of flow of viscous incompressible fluid past a hot vertical porous wall in the presence of transverse magnetic field with periodic temperature is discussed by using regular perturbation and Homotopy Perturbation Method. The effect of various physical parameters on velocity and temperature of fluid are calculated numerically and are shown through the graphs. The numerical values of the skin friction and Nusselt number are calculated for various physical parameters.

Keywords: *free convective flow; fluid dynamics; porous medium; homotopy perturbation technique; coupled non linear partial differential equations.*

GJSFR-F Classification : *MSC 2010: 65H20, 34D10*

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Strictly as per the compliance and regulations of:

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I. Introduction

The phenomenon of free convection arises in the fluid when temperature varies, this cause density variations leading to buoyancy forces acting on the fluid elements. This process of heat transfer is encountered in aeronautics, chemical engineering and fluid fuel nuclear reactor. But in case of fluid fuel nuclear reactors, the problems of heat transfer become complicated due to variation in wall temperature. Kafoussias. et.al. (1992) investigate the problem of MHD thermal-diffusion effects on free convective and mass transfer flow over an infinite moving plate. Three-dimensional free convective flow and heat transfer through a porous medium was discussed by Ahmed and Sharma (1997). Unsteady free convective MHD flow of a viscous incompressible fluid in porous medium between two long vertical walls discussed by Sarangi and Jose (1998).

Singh and Chand were consider the unsteady free convective MHD flow past a vertical porous plate with variable temperature (2000). Flow of an electrically conducting viscous incompressible fluid past a hot vertical porous wall in the presence of transverse magnetic field with periodic temperature was studied by Sharma (2002). Jain, Khendelwal and Goyal (2002) discussed MHD Three dimensional flow past a Vertical Porous Plate with Periodic Temperature in slip flow Regime. Unsteady free convective MHD flow past an infinite porous vertical plate with variable suction and heat absorbing sink discussed by Sharma (2007).

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Homotopy Perturbation Method was discussed as 'Applications of Homotopy Perturbation Method to Nonlinear wave equations' by J. H. He(2005). A. A. Hemeda (2012) considered the Homotopy Perturbation Method for solving System of Nonlinear Coupled Equations.

II. Homotopy Perturbation Method

Notes

The Homotopy Perturbation Method is a combination of classical Perturbation Technique and Homotopy Theory, which has eliminated the limitations of the traditional perturbation methods. A brief introduction of Homotopy Perturbation Method is given below:

$$L(u) + N(u) - f(r) = 0, r \in \Omega \tag{1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \tag{2}$$

here L is the linear operator, N is Nonlinear operator, B is boundary operator and $f(r)$ is known analytic function and Γ is the boundary of the domain Ω .

A Homotopy $v(r,p): \Omega \times [0,1] \to R$ for the problem mentioned in equation (1) is

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{3}$$

Or

$$H(v,p) = L(v) - L(v_0) + p[L(v_0) + N(v) - f(r)] = 0 \tag{4}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an initial approximation of equation (1) which satisfies boundary conditions. It follows from equation (3) and equation (4) that

$$H(v,0) = L(v) - L(v_0) \; and \; H(v,1) = L(v) + N(v) - f(r) \tag{5}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $v_0(r)$ to $v(r)$. In topology, this is called deformation and $L(v) - L(v_0)$ and $L(v) + N(v) - f(r)$ are called homotopic in topology.

Let

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{6}$$

And setting $p = 1$ result in an approximate solution of equation (1)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{7}$$

The series of equation (7) is convergent for most of the cases. However, the convergent rate is depends upon the nonlinear operator $N(v)$, the following options are already suggested by He (1999):

1. The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large i.e. $p \to 1$.
2. The norm of $L^{-1}\left(\frac{\partial N}{\partial u}\right)$ must be smaller than one so that the series is convergent.

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III. Formulation of Problem

Let the wall be along the x^*z^* -plane and y^* axis to be taken normal to it. The magnetic field B_0 is applied normal to the wall in the presence of constant suction velocity v_0. Let the span-wise co-sinusoidal temperature be $\theta_w^* = \theta_0\left(1+\epsilon cos\frac{\pi z}{L}\right)$, is taken at the wall.

Where ϵ (<<1) is a small positive value, L is the wave length and θ_0 is a constant and using the Bousinesque approximation, the governing equations of the fluid flow are:

a) *Equation of Momentum*

$$v_0\frac{\partial u^*}{\partial y^*} = \upsilon\left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right) + g\beta\theta^* - \frac{\sigma B_0^2(u^*-U)}{\rho} \tag{8}$$

b) *Equation of Energy*

$$\rho C_p v_0\frac{\partial \theta^*}{\partial y^*} = \kappa\left(\frac{\partial^2 \theta^*}{\partial y^{*2}} + \frac{\partial^2 \theta^*}{\partial z^{*2}}\right) + \mu\left[\left(\frac{\partial u^*}{\partial y^*}\right)^2 + \left(\frac{\partial u^*}{\partial y^*}\right)^2 + \sigma B_0^2(u^*-U)^2\right] \tag{9}$$

Where ρ is the density, μ is the coefficient of viscosity, υ is the kinematic viscosity, g is the acceleration due to gravity, β is the coefficient of volumetric expansion, σ is the coefficient of electrical conductivity, B_0 is the coefficient of electromagnetic induction, C_P the coefficient of specific heat, κ is the thermal conductivity, U is the free stream velocity in x^*- direction, θ^* the temperature at any point and v_0 is the suction velocity.

The corresponding boundary conditions are

$$y^* = 0: u^* = 0, \theta^* = \theta_0\left(1+\epsilon\cos\left(\pi z/_L\right)\right);$$

$$y^* \to \infty: u^* \to U, \theta^* \to 0 \tag{10}$$

IV. Method of Solution

Introducing the following dimensionless quantities:

$$u = \frac{u^*}{U}, y = \frac{y^*}{L}, z = \frac{z^*}{L}, \theta = \frac{\theta^*}{\theta_0}, Re = -\frac{v_0 L}{\upsilon}, Pr = \frac{UC_p}{\kappa}, Gr = \frac{g\beta\theta_0\upsilon}{Uv_0^2}, Ec = \frac{U^2}{C_p\theta_0}, M^2 = \frac{\sigma B_0^2 L^2}{\mu}$$

Substituting the dimensionless quantities into equations (8) and (9) and corresponding boundary conditions, we get

$$\frac{\partial u}{\partial y} = -\frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - GrRe\theta + \frac{M^2(u-1)}{Re}, \tag{11}$$

$$\frac{\partial \theta}{\partial y} = -\frac{1}{PrRe}\left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right) - \frac{Ec}{Re}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] - \frac{EcM^2}{Re}(u-1)^2 \tag{12}$$

where Pr is the Prandtl number, M the Hartmann number, Ec the Eckert number and Re the Reynolds number.
The corresponding boundary conditions are

$$y = 0: u = 0, \theta = 1 + \epsilon\cos(\pi z),$$

$$y \to \infty: u \to 1, \theta \to 0 \tag{13}$$

Assuming that,

$$u(y,z) = u_0(y) + \in u_1(y,z) + O(\in^2)$$

$$\theta(y,z) = \varphi(y) + \in \theta_1(y,z) + O(\in^2) \tag{14}$$

Using these assumptions into equations (11) and (12) and equating the coefficients of like powers of $\in$, we get

a) Zeroth-Order Equations

$$\frac{d^2 u_0}{dy^2} + Re\frac{du_0}{dy} - M^2 u_0 = -GrRe^2\varphi - M^2 \tag{15}$$

$$\frac{d^2\varphi}{dy^2} + PrRe\frac{d\varphi}{dy} = -EcPr\left[\left(\frac{du_0}{dy}\right)^2 + M^2(u_0-1)^2\right] \tag{16}$$

The corresponding boundary conditions are:

$$y = 0: u_0 = 0, \varphi = 1;$$

and

$$y \to \infty: u_0 \to 1, \varphi \to 0 \tag{17}$$

b) First-Order Equations

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} + Re\frac{\partial u_1}{\partial y} - M^2 u_1 = -GrRe^2\theta_1 \tag{18}$$

$$\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} + PrRe\frac{\partial \theta_1}{\partial y} = -2EcPr\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} - 2EcM^2Pru_1(u_0-1) \tag{19}$$

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The corresponding boundary conditions are:

$$y = 0 : u_1 = 0, \theta_1 = \cos \pi z ;$$

$$y \to \infty : u_1 = 0, \theta_1 = 0 \tag{20}$$

The Homotopy for zeroth order equations are following:

$$H(u_0, p) = (1-p)\left[\frac{d^2u_0}{dy^2} + Re\frac{du_0}{dy} - M^2u_0 + (1 - Re - M^2)e^{-y} + M^2\right] + p\left[\frac{d^2u_0}{dy^2} + Re\frac{du_0}{dy} - M^2u_0 + GrRe^2\varphi + M^2\right] = 0 \ , \tag{21}$$

and

$$H(\varphi, p) = (1-p)\left[\frac{d^2\varphi}{dy^2} + PrRe\frac{d\varphi}{dy} - (1 - PrRe)e^{-y}\right] + p\left[\frac{d^2\varphi}{dy^2} + PrRe\frac{d\varphi}{dy} + EcPr\left(\left(\frac{du_0}{dy}\right)^2 + M^2(u_0 - 1)^2\right)\right] = 0 \tag{22}$$

Let

$$u_0 = u_{00} + pu_{01} + p^2u_{02} + \cdots \quad ,$$

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + \cdots \tag{23}$$

Substituting the assumptions from equations (23) into the equations (21) and (22) and comparing the coefficients of like powers of p, we get

$$p^0 : \frac{d^2u_{00}}{dy^2} + Re\frac{du_{00}}{dy} - M^2u_{00} + (1 - Re - M^2)e^{-y} + M^2 = 0 \ , \tag{24}$$

$$p^1 : \frac{d^2u_{01}}{dy^2} + Re\frac{du_{01}}{dy} - M^2u_{01} - (1 - Re - M^2)e^{-y} + GrRe^2\varphi_0 = 0 \ , \tag{25}$$

$$p^0 : \frac{d^2\varphi_0}{dy^2} + PrRe\frac{d\varphi_0}{dy} - (1 - PrRe)e^{-y} = 0 \tag{26}$$

$$p^1 : \frac{d^2\varphi_1}{dy^2} + PrRe\frac{d\varphi_1}{dy} + (1 - PrRe)e^{-y} + EcPr\left(\frac{du_{00}}{dy}\right)^2 + EcPrM^2(u_{00} - 1)^2 = 0 \tag{27}$$

Now, the corresponding boundary conditions are:

$$at\ y = 0 : u_{00} = 0, u_{01} = 0, \ldots \ and \ \ \varphi_0 = 1, \varphi_1 = 0, \ldots$$

$$at\ y \to \infty : u_{00} = 1, u_{01} = 0, \ldots and \ \ \varphi_0 = 0, \varphi_1 = 0, \ldots \tag{28}$$

The solutions of equation (24) to equation (27) under the corresponding boundary conditions are

$$u_{00} = 1 - e^{-y} \tag{29}$$

$$u_{01} = b_1(e^{-y} - e^{-a_1 y}) \tag{30}$$

$$\varphi_0 = e^{-y} \tag{31}$$

$$\varphi_1 = -e^{-y} + \alpha_1 e^{-2y} + (1-\alpha_1)e^{-PrRey} \tag{32}$$

Notes

When embedding parameter $p \to 0$, we get

$$u_0 = 1 + (b_1 - 1)e^{-y} - b_1 e^{-a_1 y} \tag{33}$$

$$\varphi = (1-\alpha_1)e^{-PrRey} + \alpha_1 e^{-2y} + (1-\alpha_1)e^{-PrRey} \tag{34}$$

Here a_1, b_1 and α_1 are constants and are not mentioned here due to shake of brevity.

To find the solution of first order equations, introducing

$$u_1(y,z) = V(y)\cos \pi z\ ,$$

$$\theta_1(y,z) = \psi(y)\cos \pi z \tag{35}$$

Substitute these values in equations (18) & (19), we get

$$\frac{d^2V}{dy^2} + Re\frac{dV}{dy} - (M^2 + \pi^2)V = -GrRe^2\psi\ , \tag{36}$$

$$\frac{d^2\psi}{dy^2} + PrRe\frac{d\psi}{dy} - \pi^2\psi = -2EcPr[(1-b_1)e^{-y} + a_1 b_1 e^{-a_1 y}]\frac{dV}{dy} - 2EcPrM^2[(b_1 - 1)e^{-y} - b_1 e^{-a_1 y}]V \tag{37}$$

Now, the corresponding boundary conditions are:

$$y = 0: V = 0, \psi = 1;$$

$$y \to \infty: V = 0, \psi = 0 \tag{38}$$

Following are the Homotopy for first order equation

$$H(V,p) = (1-p)\left[\frac{d^2v}{dy^2} + Re\frac{dv}{dy} - (M^2+\pi^2)v - (1 + Re + M^2 + \pi^2)e^{-y} + (4 + 2Re + M^2 + \pi^2)e^{-2y}\right] + p\left[\frac{d^2v}{dy^2} + Re\frac{dv}{dy} - (M^2+\pi^2)v + GrRe^2\psi\right] = 0 \tag{39}$$

$$H(\psi,p) = (1-p)\left[\frac{d^2\psi}{dy^2} + PrRe\frac{d\psi}{dy} - \pi^2\psi - (1 - PrRe - \pi^2)e^{-y}\right] + p\left[\frac{d^2\psi}{dy^2} + PrRe\frac{d\psi}{dy} - \pi^2\psi + 2EcPr[(1-b_1)e^{-y} + a_1 b_1 e^{-a_1 y}]\frac{dV}{dy} + 2EcPrM^2[(b_1-1)e^{-y} - b_1 e^{-a_1 y}]V\right] = 0 \tag{40}$$

Let

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \quad ,$$

$$\psi = \psi_0 + p\psi_1 + p^2\psi_2 + \cdots \tag{41}$$

Substituting the assumptions made in equation (41) into the equations (39) & (40) and comparing the coefficients of like powers of p, we get

Notes

$$p^0: \quad \frac{d^2 v_0}{dy^2} + Re\frac{dv_0}{dy} - (M^2+\pi^2)v_0 - (1 - Re - M^2 - \pi^2)e^{-y} + (4 - 2Re - M^2 - \pi^2)e^{-2y} = 0 \tag{42}$$

$$p^1: \quad \frac{d^2 v_1}{dy^2} + Re\frac{dv_1}{dy} - (M^2+\pi^2)v_1 + (1 - Re - M^2 - \pi^2)e^{-y} - (4 - 2Re - M^2 - \pi^2)e^{-2y} + GrRe^2\psi_0 = 0 \tag{43}$$

$$p^0: \quad \frac{d^2 \psi_0}{dy^2} + PrRe\frac{d\psi_0}{dy} - \pi^2\psi_0 - (1 - PrRe - \pi^2)e^{-y} = 0 \tag{44}$$

$$p^1: \quad \frac{d^2 \psi_1}{dy^2} + PrRe\frac{d\psi_1}{dy} - \pi^2\psi_1 + (1 - PrRe - \pi^2)e^{-y} +$$

$$2EcPr[(1-b_1)e^{-y} + a_1 b_1 e^{-a_1 y}]\frac{dv_0}{dy} + 2EcPrM^2[(b_1 - 1)e^{-y} - b_1 e^{-a_1 y}]v_0 = 0 \tag{45}$$

Now, the corresponding boundary conditions are:

$$at\ y = 0: v_0 = 0, v_1 = 0, \ldots\ and\ \ \psi_0 = 1, \psi_1 = 0, \ldots ,$$

$$at\ y \to \infty: v_0 = 0, v_1 = 0, \ldots\ and\ \ \psi_0 = 0, \psi_1 = 0, \ldots \tag{46}$$

Solutions of equation (42) to equation (45) under the corresponding boundary conditions

$$v_0 = e^{-y} - e^{-2y} \tag{47}$$

$$v_1 = \beta_1 e^{-c_1 y} - (1+\beta_1)e^{-y} + e^{-2y} \tag{48}$$

$$\psi_0 = e^{-y} \tag{49}$$

$$\psi_1 = (1 - \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4)e^{-c_2 y} - e^{-y} + \gamma_1 e^{-2y} + \gamma_2 e^{-3y} + \gamma_3 e^{-(a_1+1)y} + \gamma_4 e^{-(a_1+2)y} \tag{50}$$

Taking limit on embedding parameter as $p \to 0$, we get

$$v = \beta_1 e^{-c_1 y} - \beta_1 e^{-y}, \tag{51}$$

$$\psi = (1 - \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4)e^{-c_2 y} + \gamma_1 e^{-2y} + \gamma_2 e^{-3y} + \gamma_3 e^{-(a_1+1)y} + \gamma_4 e^{-(a_1+2)y} \tag{52}$$

Here $a_1, c_1, c_2, \beta_1, \gamma_1, \gamma_2, \gamma_3$ and γ_4 are constants but are not mentioned due to shake of brevity.
Now, using equation (51) and equation (52) into equation (35), we have

$$u_1(y,z) = (\beta_1 e^{-c_1 y} - \beta_1 e^{-y}) \cos \pi z \,, \tag{53}$$

$$\theta_1(y,z) = \left[(1-\gamma_1-\gamma_2-\gamma_3-\gamma_4)e^{-c_2 y} + \gamma_1 e^{-2y} + \gamma_2 e^{-3y} + \gamma_3 e^{-(a_1+1)y} + \gamma_4 e^{-(a_1+2)y}\right] \cos \pi z \tag{54}$$

Finally, we have

$$u(y,z) = 1 + (b_1 - 1)e^{-y} - b_1 e^{-a_1 y} + \epsilon(\beta_1 e^{-c_1 y} - \beta_1 e^{-y}) \cos \pi z \tag{55}$$

$$\theta(y,z) = (1-\alpha_1)e^{-PrRey} + \alpha_1 e^{-2y} + \epsilon\left[(1-\gamma_1-\gamma_2-\gamma_3-\gamma_4)e^{-c_2 y} + \gamma_1 e^{-2y} + \gamma_2 e^{-3y} + \gamma_3 e^{-(a_1+1)y} + \gamma_4 e^{-(a_1+2)y}\right] \cos \pi z \tag{56}$$

c) Skin Friction Coefficient

The coefficient of skin friction at the wall is given by

$$\tau_w = \left(\frac{\partial u}{\partial y}\right)_{y=0} = -b_1 + 1 + a_1 b_1 + \epsilon \beta_1 (1 - c_1) \cos \pi z \tag{57}$$

d) Heat Transfer Coefficient (Nusselt Number)

The rate of heat transfer in terms of Nusselt Number at the wall is given by

$$N_u = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = PrRe(1-\alpha_1) + 2\alpha_1 - \epsilon[-c_2(1-\gamma_1-\gamma_2-\gamma_3-\gamma_4) - 2\gamma_1 - 3\gamma_2 - (a_1+1)\gamma_3 - (a_1+2)\gamma_4] \cos \pi z \tag{58}$$

Table 1 : The coefficient of skin-friction and Nusselt number

Re	**M**	**Gr**	**Ec**	**Pr**	**τ_w**	**τ_w**	**Nu**	**Nu**
					z=1/4	z=1/3	z=1/4	z=1/3
1	0.1	5	0.001	0.7	6.05688	6.02876	0.88645	0.83201
2	0.1	5	0.001	0.7	22.3344	22.2089	1.67060	1.59138
1	0.2	5	0.001	0.7	5.94895	5.92087	0.93244	0.86454
1	0.1	6	0.001	0.7	7.06627	7.03253	0.87398	0.82320
1	0.1	5	0.002	0.7	6.05688	6.02876	0.82463	0.78848
1	0.1	5	0.001	0.75	6.05688	6.02876	0.93423	0.88044

Notes

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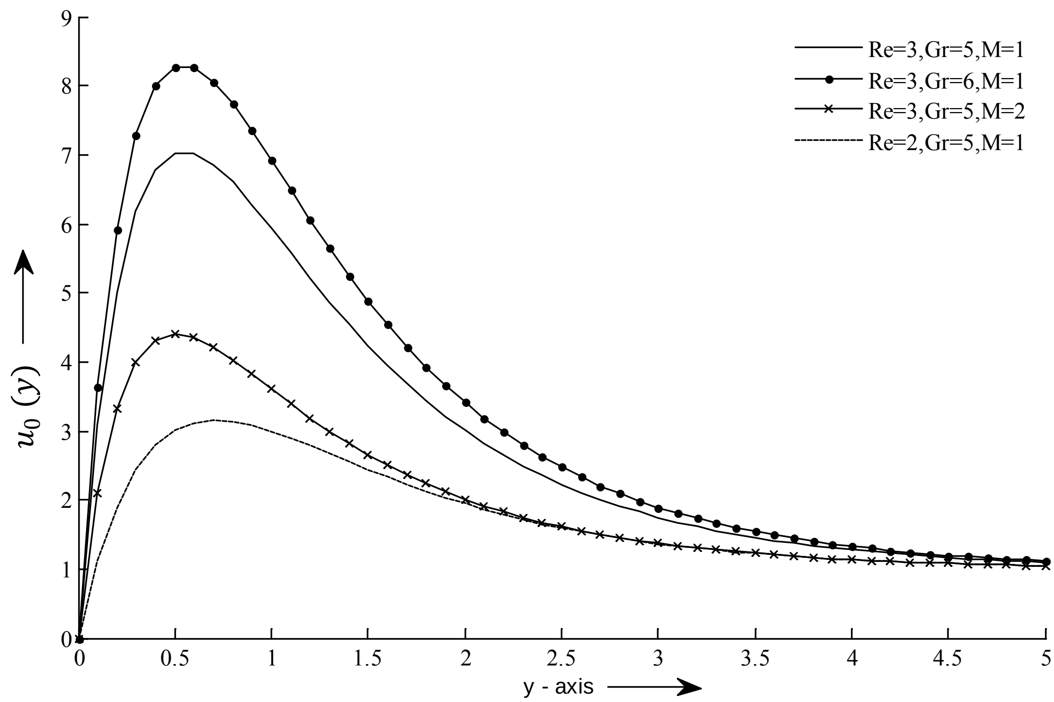

Figure 1.1 : Zeroth Order Velocity distribution versus y

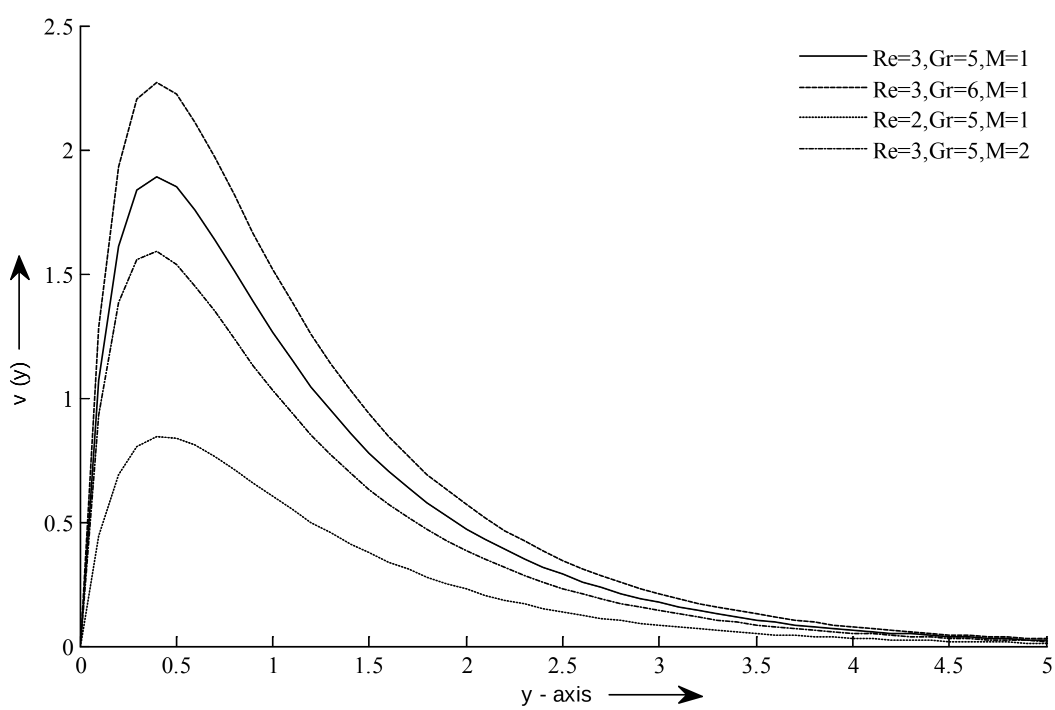

Figure 1.2 : First Order Velocity distribution versus y

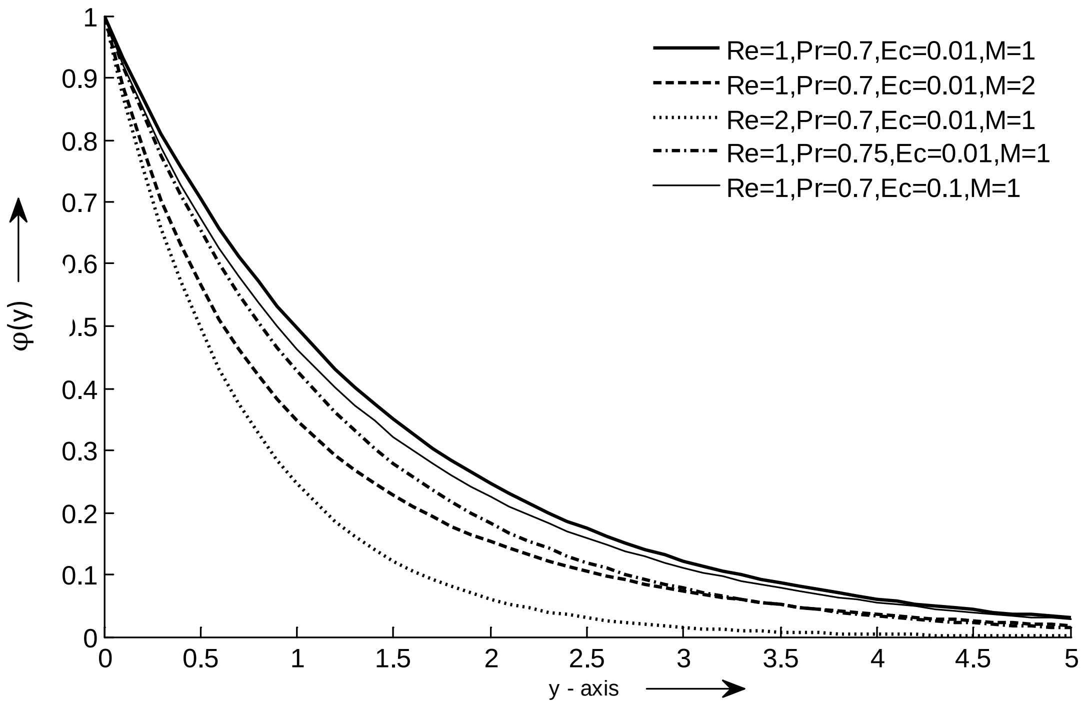

Figure: 1.3 : Zeroth Order Temperature distribution versus y

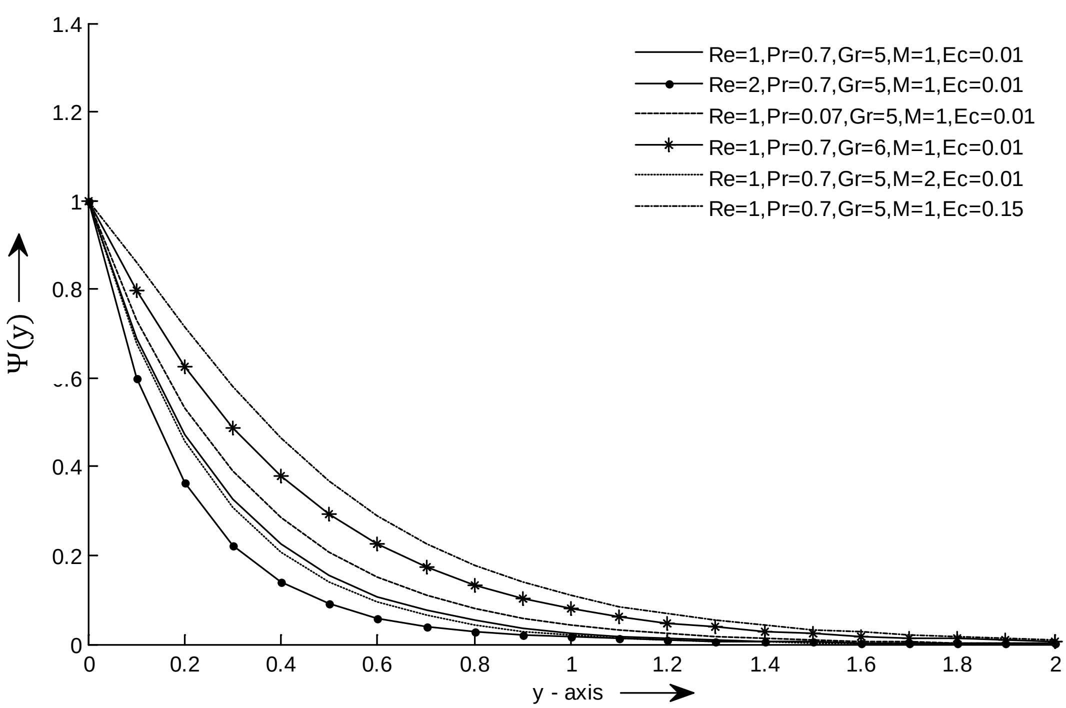

Figure 1.4 : First Order Temperature distribution versus y

Notes

Notes

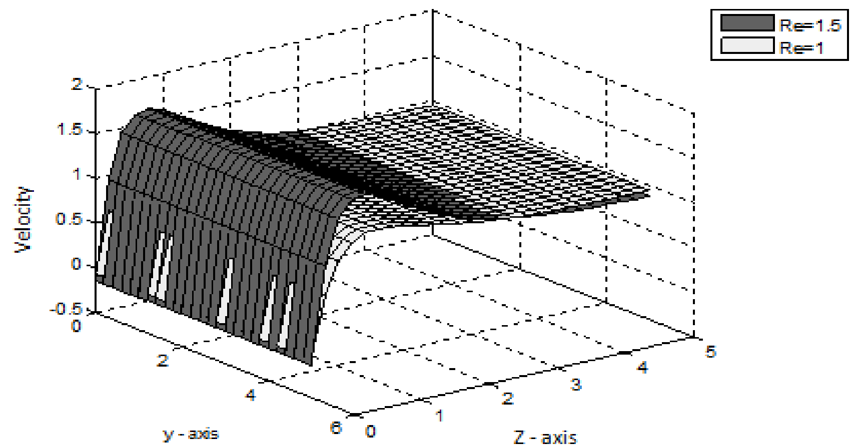

Figure 1.5 : Velocity Distribution for various values of Reynold's Number

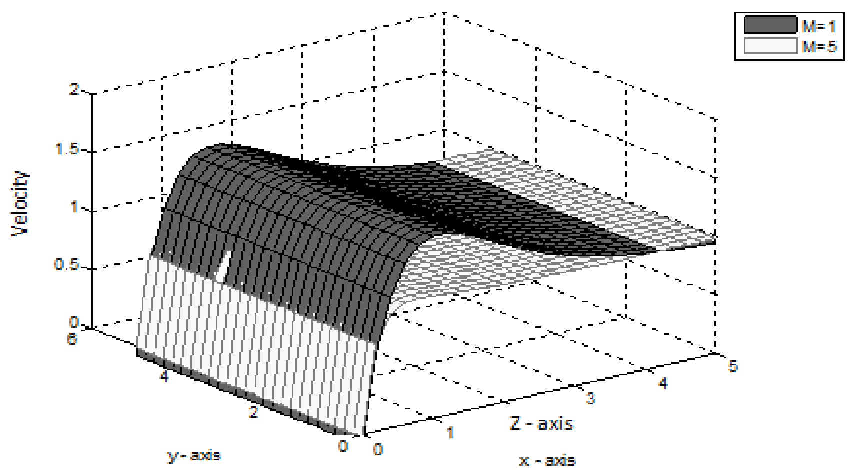

Figure 1.6 : Velocity Distribution for various values of Hartman number

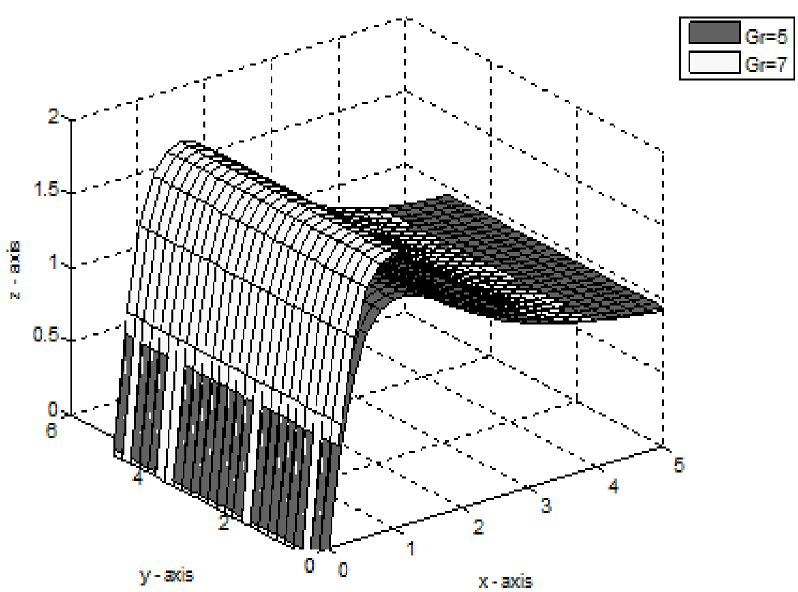

Figure 1.7 : Velocity Distribution for various values of Grasoff Number

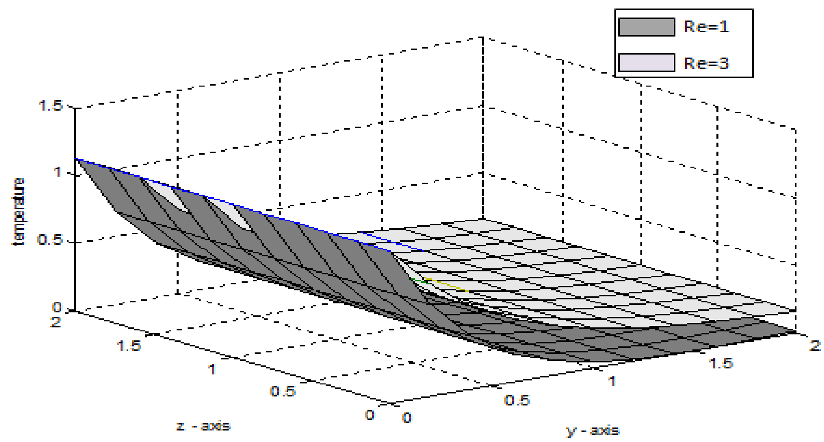

Figure 1.8 : Temperature Distribution for various values of Reynold number

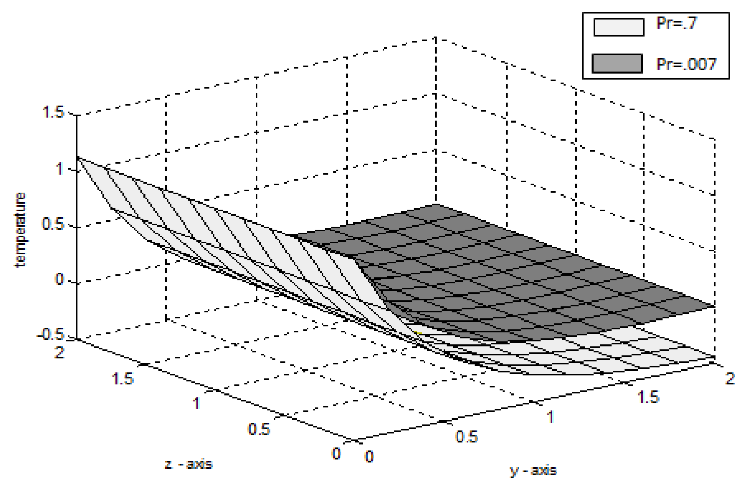

Figure 1.9 : Temperature Distribution for various values of Prandtl number

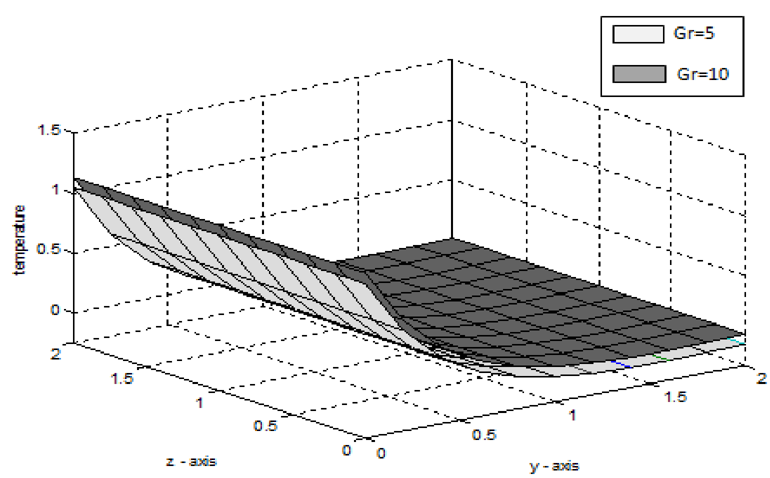

Figure 1.10 : Temperature Distribution for various values of Grasoff Number

Notes

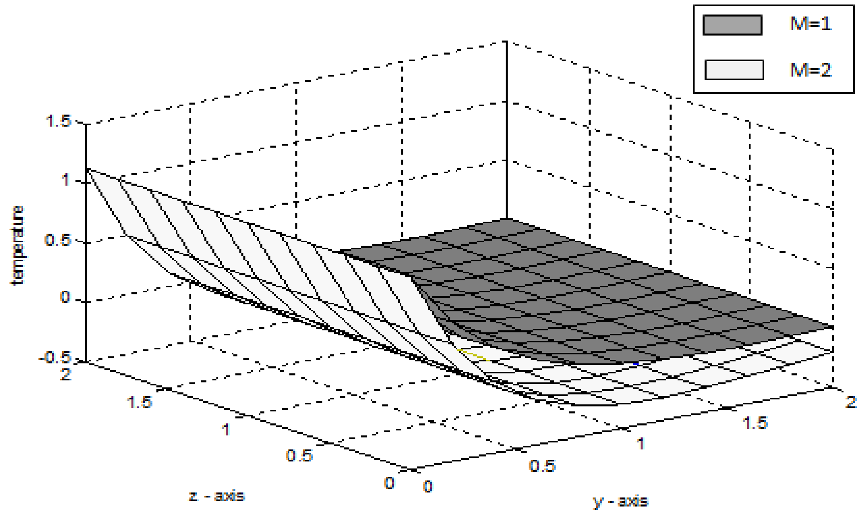

Figure 1.11 : Temperature Distribution for various values of Hartman Number

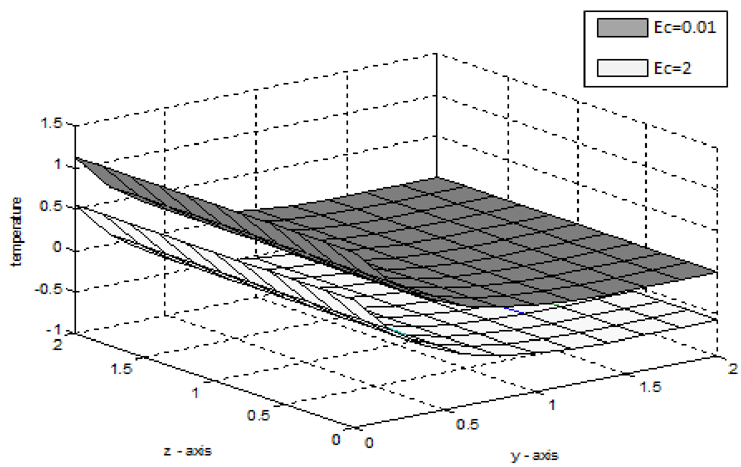

Figure 1.12 : Temperature Distribution for various values of Eckert Number

V. Results and Discussions

It has been observed from the **Table 1** that the coefficient of skin-friction increases due to increase in Reynold Number and Grasoff Number and decreases due to increase in Hartman number.

Again, it is observed that Nusselt number increases due to increase in Reynold Number, Hartman number and Prandtl Number and decreases due to increase in Eckert Number and Grasoff Number.

It has been observed from figure (1.1) and figure (1.2) that Zeroth Order Velocity and First Order Velocity both increases due to increase in Reynold Number and Grasoff Number and decreases due to increase in Hartman number. It has been observed from figure (1.3) that Zeroth Order Temperature decreases due to increase in Reynold Number, Eckert Number, Hartman number and Prandtl Number. Again, it is observed from figure (1.4) that First Order Temperature increase due to increase in Grasoff Number and Eckert Number and it decreases due to increase in Reynold Number, Hartman number and Prandtl Number.

It has been observed from figure (1.5), (1.6) and (1.7) that the velocity increases due to increase in Reynold Number and Grasoff Number and decreases due to increase in Hartman number. It has been observed from figure (1.8), (1.9), (1.10), (1.11) and (1.12) that the temperature increases due to increase in Reynold Number and Grasoff Number and it decreases due to increase in Hartman number, Prandtl Number and Eckert Number.

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Non Split Geodetic Number of a Line Graph

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Abstract- A set $S \subseteq V[L(G)]$ is a non split geodetic set of *L(G),* if *S* is a geodetic set and $\langle V-S \rangle$ is connected. The non split geodetic number of a line graph *L(G),* is denoted by $g_{ns}[L(G)]$, is the minimum cardinality of a non split geodetic set of *L(G)*. In this paper we obtain the non split geodetic number of line graph of any graph. Also obtain many bounds on non split geodetic number in terms of elements of *G* and covering number of *G*. We investigate the relationship between non split geodetic number and geodetic number.

Keywords: *cartesian product, distance, edge covering number, line graph, non split geodetic number, vertex covering number.*

GJSFR-F Classification : *MSC 2010: 05C05, 05C12*

NON SPLIT GEODETIC NUMBER OF A LINE GRAPH

Strictly as per the compliance and regulations of:

Ref

[3] F.Harary, Graph Theory, Addison-Wesely, Reading, MA,(1969)

Non Split Geodetic Number of a Line Graph

Ashalatha K.S [α], Venkanagouda M Goudar [σ] & Venkatesha [ρ]

Abstract- A set $S \subseteq V[L(G)]$ is a non split geodetic set of $L(G)$, if S is a geodetic set and $\langle V-S \rangle$ is connected. The non split geodetic number of a line graph $L(G)$, is denoted by $g_{ns}[L(G)]$, is the minimum cardinality of a non split geodetic set of $L(G)$. In this paper we obtain the non split geodetic number of line graph of any graph. Also obtain many bounds on non split geodetic number in terms of elements of G and covering number of G. We investigate the relationship between non split geodetic number and geodetic number.

Keywords: cartesian product, distance, edge covering number, line graph, non split geodetic number, vertex covering number.

I. Introduction

In this paper we follow the notations of [3]. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively.

The graphs considered here have at least one component which is not complete or at least two non trivial components.

For any graph $G(V, E)$, the line graph $L(G)$ whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A $u-v$ path of length d(u, v) is called a $u-v$ geodesic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u-v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by $g(G)$.

Non split geodetic number of a graph was studied by in [5]. A geodetic set S of a graph $G = (V, E)$ is a non split geodetic set if the induced subgraph

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$< V - S >$ is connected. The non split geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a non split geodetic set. Geodetic number of a line graph was studied by in [4]. Geodetic number of a line graph $L(G)$ of G is a set S' of vertices of $L(G) = H$ is called the geodetic set in H if $I(S') = V(H)$ and a geodetic set of minimum cardinality is the geodetic number of $L(G)$ and is denoted by $g[L(G)]$. Now we define non split geodetic number of a line graph. A set S' of vertices of $L(G) = H$ is called the non split geodetic set in H if the induced subgraph $V(H) - S'$ is connected and a non split geodetic set of minimum cardinality is the non split geodetic number of $L(G)$ and is denoted by $g_{ns}[L(G)]$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbors is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G.

For any undefined term in this paper, see [2] and [3].

II. Preliminary Notes

We need the following results to prove further results.

Theorem 2.1 (1) *Every geodetic set of a graph contains its extreme vertices.*

Proposition 2.2 *For any graph G, $g(G) \leq g_{ns}(G)$.*

Proposition 2.3 *For any tree T of order n and number of cut vertices c_i then the number of end edges is $n - c_i$.*

III. Main Results

Theorem 3.1 *For any tree T with k end edges and c_i be the number of cut vertices, then $g_{ns}[L(T)] = n - c_i$.*

Proof. Let S be the set of all extreme vertices of a line graph $L(T)$ of a tree T. By Theorem 2.1 $g_{ns}[L(T)] \geq |S|$. On the other hand, for an internal vertex v of $L(T)$, there exists x, y of $L(T)$ such that v lies on the unique $x - y$ geodesic in $L(T)$. The end edges of T are the extreme vertices of $L(T)$ and the induced subgraph $V - S$ is connected. Thus $g_{ns}[L(T)] \leq |S|$. Also every split geodetic set S_1 of $L(T)$ must contain S which is the unique minimum split geodetic set. Thus $|S| = |S_1| = k$, by proposition 2.3 $|S_1| = n - c_i$. Hence $g_{ns}[L(T)] = n - c_i$.

Corollary 3.2 *For any path P_n, $n \geq 6$, $g_{ns}[L(P_n)] = 2$.*

Proof. Clearly the set of two end vertices of a path P_n is its unique geodetic set. From Theorem 3.1 the results follows.

Ref

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Theorem 3.3 *For the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,

$$g_{ns}[L(W_n)] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Notes

Proof. Let $W_n = K_1 + C_{n-1}(n \geq 6)$ and let $V(W_n) = \{x, v_1, v_2, ..., v_{n-1}\}$, where $deg(x) = n - 1 > 3$ and $deg(v_i) = 3$ for each $i \in \{1, 2, ..., n-1\}$. Now $U = \{u_1, u_2, ..., u_j\}$ are the vertices of $L(W_n)$ formed from edges of C_{n-1}, i.e $U \subseteq V[L(W_n)]$ and $Y = \{y_1, y_2, ..., y_j\}$ are the vertices of $L(W_n)$ formed from internal edges of W_n, i.e $Y \subseteq V[L(W_n)]$.
We have the following cases
Case 1. For n is even.
Let $H \subseteq U$, now $S = H \cup \{y_j\}$ forms a minimum geodetic set of $L(W_n)$ and $V - S$ is connected. Thus S itself is the minimum non split geodetic set of $L(W_n)$. Clearly $|H \cup \{y_j\}| = \frac{n}{2}$. Therefore $g_{ns}[L(W_n)] = \frac{n}{2}$.
Case 2. For n is odd.
Let $H \subseteq U$, now $S = H \cup \{y_j, y_{j-1}\}$ forms a minimum geodetic set of $L(W_n)$ and $V - S$ is connected. Thus S itself is the minimum non split geodetic set of $L(W_n)$. Clearly $|H \cup \{y_j, y_{j-1}\}| = \frac{n+1}{2}$. Therefore $g_{ns}[L(W_n)] = \frac{n+1}{2}$.

As an immediate consequence of the above theorem we have the following.

Corollary 3.4 *For the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,

$$g_{ns}[L(W_n)] = \begin{cases} \frac{\Delta}{2} & \text{if } n \text{ is even} \\ \frac{\Delta+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Maximum degree(Δ) of $L(W_n)$ is equal to n. i,e number of vertices in W_n.
Case 1. For n is even.
We have from case 1. of Theorem 3.3 $g_{ns}[L(W_n)] = \frac{n}{2}$.
$g_{ns}[L(W_n)] = \frac{\Delta}{2}$.
Case 2. For n is odd.
We have from case 2. of Theorem 3.3 $g_{ns}[L(W_n)] = \frac{n+1}{2}$.
$g_{ns}[L(W_n)] = \frac{\Delta+1}{2}$.

Theorem 3.5 *For any tree* T, *with* m *edges,* $g_{ns}[L(T)] \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$. *Where* α_1 *is the edge covering number.*

Proof. Suppose $S = \{e_1, e_2, ..., e_k\}$ be the set of all end edges in T. Then $S \cup J$ where $J \subseteq E(T) - S$, be the minimal set of edges which covers all the vertices of T and is not covered by S, such that $|S \cup J| = \alpha_1(T)$. Now without loss of generality in $L(T)$, let $S' = \{u_1, u_2, ..., u_n\} \subseteq V[L(T)]$ be the set of vertices in $L(T)$ formed by the end edges in T and $V - S'$ is connected which is the minimal non split geodetic set of $L(T)$. Clearly it follows that $g_{ns}[L(T)] \leq |E(T)| - |\lceil \frac{S \cup J}{2} \rceil| + 2 \Rightarrow g_{ns}[L(T)] \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$.

Theorem 3.6 *For any connected graph G of order n, $g_{ns}(G) + g_{ns}[L(G)] \leq 2n$.*

Proof. Let $S = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ be the minimum non split geodetic set of G. Now without loss of generality in $L(G)$, if $F = \{u_1, u_2, ..., u_k\}$ be the set of all end vertices in $L(G)$. Then $F \cup H$ where $H \subseteq V[L(G)] - F$ forms a minimum non split geodetic set of $L(G)$. Since each vertex in $L(G)$ corresponds to two adjacent vertices of G, it follows that $|S| \cup |F \cup H| \leq 2n$. Therefore $g_{ns}(G) + g_{ns}[L(G)] \leq 2n$.

Theorem 3.7 *Let G be a connected graph of order n and diameter d. Then $g_{ns}[L(G)] \leq n - d + 1$.*

Proof. Let u and v be vertices of $L(G)$ for which $d(u, v) = d$ and let $u = v_0, v_1, ..., v_d = v$ be the $u - v$ path of length d. Now let $S = V[L(G)] - \{v_1, v_2, ..., v_{d-1}\}$. Then $I(S) = V[L(G)]$, $V[L(G)] - S$ is connected and $g_{ns}[L(G)] \leq |S| = n - d + 1$.

Observation 3.8 *For cycle C_n of order n $g_{ns}[L(C_n)] = n - d + 1$.*

Theorem 3.9 *For cycle C_n of order $n > 3$*

$$g_{ns}[L(C_n)] = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Line graph of a cycle is again a cycle of same order and d be the diameter. We have the following cases.

Case 1. If n is even. Let n be the number of vertices of $L(C_n)$ and diameter of $L(C_{2n})$ is $\frac{n}{2}$. Hence by Theorem 3.7 and Observation 3.8, $g_{ns}[L(C_n)] = n - d + 1$.

Now we have

$g_{ns}[L(C_n)] = n - \frac{n}{2} + 1$.

$\Rightarrow g_{ns}[L(C_n)] = \frac{n}{2} + 1$.

$\Rightarrow g_{ns}[L(C_n)] = \frac{n+2}{2}$.

Case 2. If n is odd. Let n be the number of vertices of $L(C_n)$ and diameter of $L(C_{2n+1})$ is $\frac{n-1}{2}$. Hence by Theorem 3.7 and Observation 3.8, $g_{ns}[L(C_n)] = n - d + 1$. We have

$g_{ns}[L(C_n)] = n - \frac{n-1}{2} + 1$.

Notes

Notes

$\Rightarrow g_{ns}[L(C_n)] = \frac{n}{2} + \frac{1}{2} + 1.$

$\Rightarrow g_{ns}[L(C_n)] = \frac{n+3}{2}.$

Theorem 3.10 *For any integers $r, s > 2$, $g_{ns}[L(K_{r,s})] \leq rs - 1$.*

Proof. Let $r + s$ and rs be the number of vertices and edges of the given graph $K_{r,s}$ and d be the diameter. Since diameter of $L(K_{r,s}) = 2$, the number of vertices in $L(K_{r,s})$ is rs. Hence by Theorem 3.7 $g_{ns}[L(G)] \leq n - d + 1$. Now we have $g_{ns}[L(K_{r,s})] \leq rs - 2 + 1$. $\Rightarrow g_{ns}[L(K_{r,s})] \leq rs - 1$.

Theorem 3.11 *For any integer $n \geq 4$, $g_{ns}[L(K_n)] \leq \frac{(n+1)(n-2)}{2}$.*

Proof. Let $n \geq 4$ be the vertices of the given graph K_n and d be the diameter. Since diameter of $L(K_n)$ is 2 and the number of vertices in $L(K_n)$ is $\frac{n(n-1)}{2}$, hence by Theorem 3.7 $g_{ns}[L(G)] \leq n - d + 1$. We have

$g_{ns}[L(K_n)] \leq \frac{n(n-1)}{2} - 2 + 1.$

$\Rightarrow g_{ns}[L(K_n)] \leq \frac{n(n-1)}{2} - 1.$

$\Rightarrow g_{ns}[L(K_n)] \leq \frac{n^2-n-2}{2}.$

$\Rightarrow g_{ns}[L(K_n)] \leq \frac{(n+1)(n-2)}{2}.$

IV. Adding an End Edge

For an edge $e = (u, v)$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$, we call e an end-edge and u an end-vertex.

Theorem 4.1 *G' be the graph obtained by adding k end edges $\{(u, v_1), (u, v_2), ..., (u, v_k)\}$ to a cycle $C_n = G$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, ..., v_k\} \notin G$. Then*

$$g_{ns}[L(G')] = \begin{cases} k + \frac{n}{2} + 1 & \text{if } n \text{ is even} \\ k + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\{e_1, e_2, ..., e_n, e_1\}$ be a cycle with n vertices and let G' be the graph obtained from $G = C_n$ by adding end edges (u, v_i), $i = 1, 2, ..., k$. Such that $u \in G$ and $v_i \notin G$.

Case 1. If n is even.

By the definition of line graph, $L(G')$ has $\langle K_{k+2} \rangle$ as an induced subgraph. Also the edges (u, v_i), $i = 1, 2, ..., k$ becomes vertices of $L(G')$ and it belongs to some geodetic set of $L(G')$. Hence $\{e_1, e_2, ..., e_k, e_l, e_m\}$ are the vertices of $L(G')$ where e_l, e_m are the edges incident on the antipodal vertex of u in G' and these vertices belongs to some geodetic set of $L(G')$. $L(G') = C_n \cup K_{k+2}$.

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Let$S = \{e_1, e_2, ..., e_k, e_l, e_m\}$ be the minimum geodetic set. Consider $H \subseteq V - S$, now $S' = S \cup H_1$ where $H_1 \subseteq H$ forms minimum non split geodetic number of $L(G')$. Therefore $g_{ns}[L(G')] = k + \frac{n}{2} + 1$.

Case 2. If n is odd.

By the definition of line graph, $L(G')$ has $\langle K_{k+2} \rangle$ as an induced subgraph, also the edges $(u, v_i) = \{e_1, e_2, ..., e_k\}$ becomes vertices of $L(G')$. Let $e_l = (a, b) \in G$ such that $d(u, a) = d(u, b)$ in the graph $L(G')$. Let $S = \{e_1, e_2, ..., e_k, e_l\}$ be the minimum geodetic set. Since $V - S$ is connected S is the minimum non split geodetic set. Therefore $g_{ns}[L(G')] = k + 1$.

Theorem 4.2 *Let G' be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, ..., n$, $j = 1, 2, ..., n$ to each vertex of $G = C_n$ of order $n > 3$ such that $u_i \in G$, $v_j \notin G$. Then $g_{ns}[L(G')] = n$.*

Proof. Let $\{e_1, e_2, ..., e_n, e_1\}$ be a cycle with n vertices and $G = C_n$. Let G' be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, ..., n$, $j = 1, 2, ..., n$ to each vertex of G, such that $u_i \in G$, $v_j \notin G$. Clearly n be the number of end vertices of G'. By the definition of line graph $L(G')$ have n copies of K_3 as an induced subgraph. The edges $(u_i, v_j) = e_j$ for all j, becomes n vertices of $L(G')$ and those lies on geodetic set of $L(G')$. Since they forms the extreme vertices of $L(G')$. By Theorem 2.1 $S = \{e_1, e_2, ..., e_n\}$ forms minimum geodetic set. Since $V - S$ is connected S is the minimum non split geodetic set. Therefore $g_{ns}[L(G')] = n$.

V. Cartesian Product

The cartesian product of the graphs H_1 and H_2, written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 5.1 *For any path P_n of order n,*

$$g_{ns}[L(K_2 \times P_n)] = \begin{cases} 3 & \text{for } n = 2 \\ 4 & \text{for } n = 3 \\ 5 & \text{for } n > 3. \end{cases}$$

Proof. Let $K_2 \times P_n$ be formed from two copies of G_1 and G_2 of P_n. Now $L(K_2 \times P_n)$ formed from two copies of G'_1, G'_2 of $L(P_n)$. And let $U = \{u_1, u_2, ..., u_n\} \in V(G'_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G'_2)$. We have the following cases.

Case 1. If $n = 2$.

Notes

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Let $S = \{u_1, w_2\}$ forms minimum geodetic set of $L(K_2 \times P_2)$. Consider $H = \{u_2, w_1\} \subseteq V - S$. Now $S' = S \cup \{w_1\}$ or $\{u_2\}$, where w_1, u_2 are isolated vertices in $V - S$ forms minimum non split geodetic set of $L(K_2 \times P_2)$. Therefore $g_{ns}[L(K_2 \times P_2)] = 3$.
Case 2. If $n = 3$.

Let $S = \{u_2, w_1, w_3\}$ forms minimum geodetic set of $L(K_2 \times P_3)$. Consider $H = \{u_1, u_3, u_4, w_2\} \subseteq V - S$. Now $S' = S \cup \{u_1\}$, where u_1 is a isolated vertex in $V - S$ forms minimum non split geodetic set of $L(K_2 \times P_3)$. Therefore $g_{ns}[L(K_2 \times P_2)] = 4$.
Case 3. Suppose $n > 3$. Let S be the non split geodetic set of $L(K_2 \times P_n)$.

We claim that $S = \{u_1, u_n, w_1, w_{n-1}, w_n\}$ and $V - S$ is connected. Since $I(S) = V[L(K_2 \times P_n)]$, it follows that $g_{ns}[L(K_2 \times P_n)] = 5$. If S' is a four element subset of $V[L(K_2 \times P_n)]$ then $V - S$ is disconnected. It remains to show that if S' is a three element subset of $V[L(K_2 \times P_n)]$ then $I(S') \neq V[L(K_2 \times P_n)]$. First assume that S' is a subset U or W, say the farmer. Then $I(S') = S' \cup W \neq V$. Therefore, we may take that $S' \cap U = \{u_i, u_j\}$ and $S' \cap W = \{w_k\}$. Then $I(S') = \{u_i, u_j\} \cup W \neq V[L(K_2 \times P_n)]$.

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Traveling Wave Solutions of Nonlinear Evolution Equations via Exp($-\Phi(\eta)$)-Expansion Method

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Abstract- In this article, we implement the exp($-\Phi(\eta)$)-expansion method for seeking the exact solutions of NLEEs via the Benjamin-Ono equation and achieve exact solutions involving parameters. Abundant traveling wave solutions with arbitrary parameters are successfully obtained by this method and the wave solutions are expressed in terms of the hyperbolic, trigonometric, and rational functions. It is established that the exp($-\Phi(\eta)$)-expansion method offers a further influential mathematical tool for constructing the exact solutions of NLEEs in mathematical physics. The obtained results show that exp($-\Phi(\eta)$)-expansion method is very powerful and concise mathematical tool for nonlinear evolution equations in science and engineering.

Keywords: *the exp($-\Phi(\eta)$)-expansion method; complixiton soliton solutions; the benjamin-ono equation; traveling wave solutions; nonlinear evolution equation.*

GJSFR-F Classification : *MSC 2010: 35C07, 35C08, 35P99.*

TRAVELING WAVE SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS VIA EXPEXPANSION METHOD

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Traveling Wave Solutions of Nonlinear Evolution Equations via Exp$(-\Phi(\eta))$-Expansion Method

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Abstract- In this article, we implement the $\exp(-\Phi(\eta))$-expansion method for seeking the exact solutions of NLEEs via the Benjamin-Ono equation and achieve exact solutions involving parameters. Abundant traveling wave solutions with arbitrary parameters are successfully obtained by this method and the wave solutions are expressed in terms of the hyperbolic, trigonometric, and rational functions. It is established that the $\exp(-\Phi(\eta))$-expansion method offers a further influential mathematical tool for constructing the exact solutions of NLEEs in mathematical physics. The obtained results show that $\exp(-\Phi(\eta))$-expansion method is very powerful and concise mathematical tool for nonlinear evolution equations in science and engineering.

Keywords: the $\exp(-\Phi(\eta))$ *-expansion method; complixiton soliton solutions; the benjamin-ono equation; traveling wave solutions; nonlinear evolution equation.*

I. Introduction

Nonlinear wave phenomena appears in various scientific and engineering fields such as fluid mechanics, plasma physics, optical fibers, biophysics, geochemistry, electricity, propagation of shallow water waves, high-energy physics, condensed matter physics, quantum mechanics, elastic media, biology, solid state physics, chemical kinematics, chemical physics and so on. This is also noticed to arise in engineering, chemical and biological applications. The application of nonlinear traveling waves has been brought prosperity in the field of applied science. In order to understand better the nonlinear phenomena as well as further application in the practical life, it is important to seek their more exact travelling wave solutions. Essentially all the fundamental equations in physical sciences are nonlinear and, in general, such NLEEs are often very complicated to solve explicitly. The exact solutions of NLEEs play an important role in the study of nonlinear physical phenomena. Therefore, the powerful and efficient methods to find exact solutions of nonlinear equations still have drawn a lot of interest by diverse group of scientists. In the past three decades, there has been significant progress in the development of finding effective methods for obtaining exact solutions of NLEEs. These methods are the homogeneous balance method [1], the tanh-function method [2], the extended tanh-function method [3, 4], the Exp-function method [5, 6], the sine-cosine method [7], the modified Exp-function method [8], the generalized Riccati equation [9],

Ref

2. A.M. Wazwaz, The tanh method for generalized forms of nonlinear heat conduction and Burgers–Fisher equations, Appl. Math.Comput. 169 (2005) 321–338.

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the Jacobi elliptic function expansion method [10, 11], the Hirota's bilinear method [12], the Miura transformation [13], the (G'/G)-expansion method [14-18], the novel (G'/G)-expansion method [19, 20], the modified simple equation method [21, 22], the improved (G'/G)-expansion method [23], the inverse scattering transform [24], the Jacobi elliptic function expansion method [25, 26], the new generalized (G'/G)-expansion method [27-31], the $\exp(-\Phi(\eta))$-expansion method [32, 33] and so on.

The objective of this article is to apply the $\exp(-\Phi(\eta))$-expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the Benjamin-Ono equation.

The outline of this paper is organized as follows: In Section 2, we give the description of the $\exp(-\Phi(\eta))$-expansion method. In Section 3, we apply this method to the Benjamin-Ono equation, graphical representation of solutions. Conclusions are given in the last section.

II. Description of the $\exp(-\Phi(\eta))$-Expansion Method

Let us consider a general nonlinear PDE in the form

$$F(v, v_t, v_x, v_{xx}, v_{tt}, v_{tx}, \ldots), \tag{1}$$

where $v = v(x,t)$ is an unknown function, F is a polynomial in $v(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. In the following, we give the main steps of this method:

Step 1: We combine the real variables x and t by a complex variable η

$$v(x,t) = v(\eta), \; \eta = x \pm V t, \tag{2}$$

where V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation (ODE) for $v = v(\eta)$:

$$\Re(v, v', v'', v''', \cdots), \tag{3}$$

where $\Re$ is a polynomial of v and its derivatives and the superscripts indicate the ordinary derivatives with respect to η.

Step 2. Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$v(\eta) = \sum_{i=0}^{N} A_i (\exp(-\Phi(\eta)))^i, \tag{4}$$

where A_i $(0 \le i \le N)$ are constants to be determined, such that $A_N \ne 0$ and $\Phi = \Phi(\eta)$ satisfies the following ordinary differential equation:

$$\Phi'(\eta) = \exp(-\Phi(\eta)) + \mu \exp(\Phi(\eta)) + \lambda, \tag{5}$$

Eq. (5) gives the following solutions:

Ref

13. T.L. Bock and M.D. Kruskal, A two-parameter Miura transformation of the Benjamin-One equation, Phys. Lett. A 74 (1979) 173-176.

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Family 1: When $\mu \neq 0,\ \lambda^2 - 4\mu > 0,$

$$\Phi(\eta) = \text{In}\left(\frac{-\sqrt{(\lambda^2-4\mu)}\ \tan\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)-\lambda}{2\mu}\right) \tag{6}$$

Family 2: When $\mu \neq 0,\ \lambda^2 - 4\mu < 0,$

$$\Phi(\eta) = \text{In}\left(\frac{\sqrt{(4\mu-\lambda^2)}\ \tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda}{2\mu}\right) \tag{7}$$

Family 3: When $\mu = 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu > 0,$

$$\Phi(\eta) = -\text{In}\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right) \tag{8}$$

Family 4: When $\mu \neq 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Phi(\eta) = \text{In}\left(-\frac{2(\lambda(\eta+E)+2)}{\lambda^2(\eta+E)}\right) \tag{9}$$

Family 5: When $\mu = 0,\ \lambda = 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Phi(\eta) = \text{In}(\eta+E) \tag{10}$$

$A_N, \cdots\cdots, V, \lambda, \mu$ are constants to be determined latter, $A_N \neq 0$, the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: We substitute Eq. (4) into Eq. (3) and then we account the function $\exp(-\Phi(\eta))$. As a result of this substitution, we get a polynomial of $\exp(-\Phi(\eta))$. We equate all the coefficients of same power of $\exp(-\Phi(\eta))$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find $A_N, \cdots\cdots, V, \lambda, \mu$. Substituting the values of $A_N, \cdots\cdots, V, \lambda, \mu$ into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

III. Application of the Method

In this section, we will present the $\exp(-\Phi(\eta))$-expansion method to construct the exact solutions and then the solitary wave solutions of the Benjamin-Ono equation [34]. Let us consider the Benjamin-Ono equation,

$$v_t + hv_{xx} + vv_x = 0\,. \tag{11}$$

Ref

34. A. Neyrame, A. Roozi, S. S .Hosseini, S. M. Shafiof, Exact travelling wave solutions for some nonlinear partial differential equations, journal of King Saud University (Science), (2010), 22, 275-278.

We utilize the traveling wave variable $v(\eta)=v(x,t)$, $\eta=x-\omega t$, Eq. (11) is carried to an ODE

$$-Vv'+hv''+\frac{1}{2}(v^2)'=0. \tag{12}$$

Eq. (12) is integrable, therefore, integrating with respect to η once yields:

$$P-\omega v+hv'+\frac{1}{2}v^2=0, \tag{13}$$

where P is an integration constant which is to be determined.

Taking the homogeneous balance between highest order nonlinear term w^2 and linear term of the highest order w' in Eq. (13), we obtain $N=1$. Therefore, the solution of Eq. (13) is of the form:

$$v(\eta)=A_0+A_1(\exp(-\Phi(\eta))), \tag{14}$$

where A_0, A_1 are constants to be determined such that $A_N\neq 0$, while λ,μ are arbitrary constants. It is easy to see that

$$v'(\eta)=-A_1(\exp(-2\Phi(\eta))+\mu+\lambda\exp(-\Phi(\eta))). \tag{15}$$

$$v^2(\eta)=A_0^{\ 2}+2A_0A_1\exp(-\Phi(\eta))+A_1^{\ 2}\exp(-2\Phi(\eta))). \tag{16}$$

Substituting v, v', v^2 into Eq. (13) and then equating the coefficients of $\exp(-\Phi(\eta))$ to zero, we get

$$\frac{1}{2}A_1^{\ 2}-hA_1=0, \tag{17}$$

$$A_0A_1-hA_1\lambda-\omega A_1=0, \tag{18}$$

$$P-\omega A_0+\frac{1}{2}A_0^{\ 2}-hA_1\mu=0, \tag{19}$$

Solving the Eq. (17)-Eq. (19) yields

$$P=\frac{1}{2}A_0^{\ 2}-A_0h\lambda+2h^2\mu,\ \ \omega=A_0-h\lambda,\ A_0=A_0,\ \ A_1=2h.$$

where λ,μ are arbitrary constants.

Now substituting the values of ω, A_o, A_1 into Eq. (14) yields

$$v(\eta)=A_0+2h(\exp(-\Phi(\eta))), \tag{20}$$

where $\eta=x-(A_0-h\lambda)t$.

Notes

Now substituting Eq. (6)- Eq. (10) into Eq. (20) respectively, we get the following five traveling wave solutions of the Benjamin-Ono equation.

When $\mu \neq 0,\ \lambda^2 - 4\mu > 0,$

Notes

$$v_1(\eta) = A_0 - \frac{4h\mu}{\sqrt{\lambda^2 - 4\mu}\ \tan(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\eta + E)) + \lambda}.$$

where $\eta = x - (A_0 - h\lambda)t$. E is an arbitrary constant.

When $\mu \neq 0,\ \lambda^2 - 4\mu < 0,$

$$v_2(\eta) = A_0 + \frac{4h\mu}{\sqrt{4\mu - \lambda^2}\ \tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\eta + E)) - \lambda}.$$

where $\eta = x - (A_0 - h\lambda)t$. E is an arbitrary constant.

When $\mu = 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu > 0,$

$$v_3(\eta) = A_0 + \frac{2h\lambda}{\exp(\lambda(\eta + E)) - 1}.$$

where $\eta = x - (A_0 - h\lambda)t$. E is an arbitrary constant.

When $\mu \neq 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu = 0,$

$$v_4(\eta) = A_0 - \frac{h\lambda^2(\eta + E)}{(\lambda(\eta + E)) + 2)}.$$

where $\eta = x - (A_0 - h\lambda)t$. E is an arbitrary constant.

When $\mu = 0,\ \lambda = 0,$ and $\lambda^2 - 4\mu = 0,$

$$v_5(\eta) = A_0 - \frac{2h}{(\eta + E)}.$$

where $\eta = x - (A_0 - h\lambda)t$. E is an arbitrary constant.

a) Graphical representation of the solutions

The graphical illustrations of the solutions are given below in the figures with the aid of Maple.

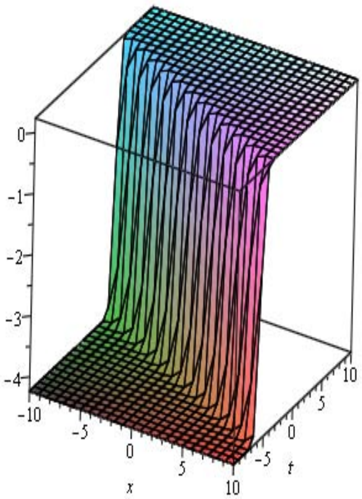

Fig. 1 : The modulus of Kink wave solution $v_1(\eta)$.

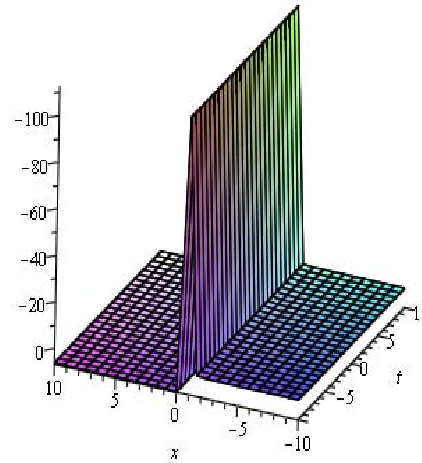

Fig. 2 : The modulus of solitary wave solution $v_2(\eta)$.

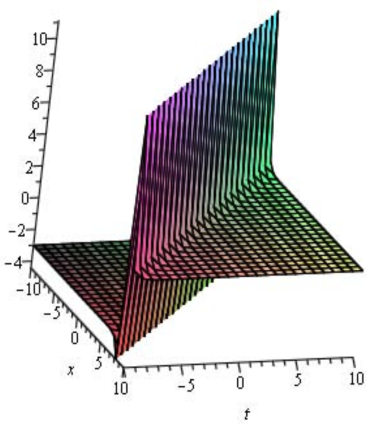

Fig. 3 : The modulus of solitary wave solution $v_3(\eta)$

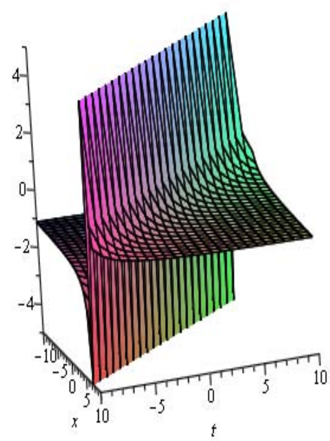

Fig. 4 : The modulus of solitary wave solution $v_4(\eta)$

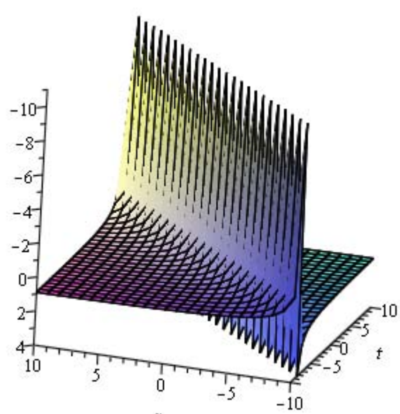

Fig. 5 : The modulus of solitary wave solution $v_5(\eta)$.

Notes

IV. Conclusion

In this study, we considered the Benjamin-Ono equation. We apply the $\exp(-\Phi(\eta))$-expansion method for the exact solution of this equation and constructed some new solutions which are not found in the previous literature. The method offers solutions with free parameters that might be imperative to explain some intricate physical phenomena. This study shows that the $\exp(-\Phi(\eta))$-expansion method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. Also, we observe that the $\exp(-\Phi(\eta))$-expansion method is straightforward and can be applied to many other nonlinear evolution equations.

Notes

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Heat Transfer Analysis of the Boundary Layer Flow over a Vertical Exponentially Stretching Cylinder

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Abstract- In this article an analysis is presented to obtain the similarity solution of the steady boundary layer flow and heat transfer of a viscous fluid flowing through a vertical cylinder that is stretching exponentially along its surface. The governing partial differential equations along with the boundary conditions are reduced to into system of nonlinear ordinary differential equations by using the boundary layer approach and a suitable similarity transformation. The resulting coupled system of equations subject to the appropriate boundary conditions is solved with the help of powerful numerical technique, the Keller-box method. The effects of the involved parameters such as Reynolds numbers, Prandtl numbers and the natural convection parameter are presented through sketches. The associated physical properties on the flow and heat transfer characteristics that is the skinfriction coefficient and Nusselt numbers are presented for different parameters.

Keywords: *boundary layer flow; vertical cylinder; viscous fluid; natural convection heat transfer; keller-box technique.*

GJSFR-F Classification : *MSC 2010: 53A17, 33C10*

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Heat Transfer Analysis of the Boundary Layer Flow over a Vertical Exponentially Stretching Cylinder

Abdul Rehman [α] & S. Nadeem [σ]

Abstract- In this article an analysis is presented to obtain the similarity solution of the steady boundary layer flow and heat transfer of a viscous fluid flowing through a vertical cylinder that is stretching exponentially along its surface. The governing partial differential equations along with the boundary conditions are reduced to into system of nonlinear ordinary differential equations by using the boundary layer approach and a suitable similarity transformation. The resulting coupled system of equations subject to the appropriate boundary conditions is solved with the help of powerful numerical technique, the Keller-box method. The effects of the involved parameters such as Reynolds numbers, Prandtl numbers and the natural convection parameter are presented through sketches. The associated physical properties on the flow and heat transfer characteristics that is the skinfriction coefficient and Nusselt numbers are presented for different parameters.

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I. Introduction

Fluid flow over a cylindrical shaped geometry has been an area of rigorous investment for the researchers in recent past. The notion is deliberated for Newtonian as well as non-Newtonian fluids with both steady and unsteady flows. The concept is debated with reference to boundary layer theory and stagnation points. Also, in several situations the researchers have accomplished the effects of free or natural convection heat transfer. Ahmad et al [1] have determined the impact of free convection over the problem of boundary layer flow over a horizontal cylinder having constant heat flux at the exterior. The analysis was carried for both slender and blunt orientations. In another work, Buchlin [2] has examined the experimental effects of natural and free convection heat transfer over a cylinder with slender configuration. The natural convection boundary layer flow of steady viscous fluid curving from an isothermal horizontal circular cylinder in presence of heat generation was replicated by Molla et al [3]. Further, Chang [4] has provided a difference scheme concerning the natural convection heat transfer flow of a micropolar fluid along a circular cylinder taking care of wall conduction effects. Moreover, Anwar et al [5] have analyzed the problem of boundary layer flow of a viscoelastic fluid over a horizontal circular cylinder under the influence of natural convection heat transfer. The problem of axisymmetric stagnation flow of non-Newtonian micropolar fluids, containing nanoparticles, flowing through the annular region

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between two concentric cylinders was studied by Nadeem et al [6]. The article comprised the impact of thermophoresis and Brownian motion parameters over the flow, heat and concentration profiles. Recently, Nadeem et al [7] have examined the topic of boundary layer flow of a second grade fluid in a cylinder with heat transfer analysis. Few other interesting works regarding fluid flow through cylindrical geometry are referred in [8-18].

Many authors have made successful attempts to solve the problem of flow and heat transfer over a cylinder that is stretched linearly along its axial direction. Ishak et al [19] have uncovered the effects of uniform suction for the problem of heat transfer and fluid flow due to a stretching cylinder. More recently, Wang [20] has contributed an interesting work about the natural convection heat transfer effects over a vertical stretching cylinder. To the best of author's knowledge, no successful attempt is marketed for the flow and heat transfer over a stretching cylinder that is stretched exponentially along its axial axis. Useful similarities for exponential stretching in cylindrical coordinates have been found and simplified the partial differential equations and its related boundary conditions. The solution of the problem is obtained through the powerful numeric technique the Keller-box method. The behavior of velocity and temperature profiles presented for a large range of Reynolds and Prandtl numbers. At the end the important physical quantities associated with the problem such as the skinfriction coefficient and the local Nusselt numbers are presented.

II. Formulation

Consider the problem of natural convection boundary layer flow of a viscous fluid flowing over a vertical circular cylinder of radius a. The cylinder is assumed to be stretched exponentially along the axial direction with velocity U_w. The temperature at the surface of the cylinder is assumed to be T_w and the uniform ambient temperature is taken as T_∞ such that the quantity $T_w - T_\infty > 0$ in case of the assisting flow, while $T_w - T_\infty < 0$ in case of the opposing flow, respectively. Under these assumptions the boundary layer equations of motion and heat transfer are

$$u_r + \frac{u}{r} + w_z = 0, \tag{1}$$

$$uw_r + ww_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(w_{rr} + \frac{1}{r}w_r) + g\beta(T - T_\infty), \tag{2}$$

$$uT_r + wT_x = \alpha(T_{rr} + \frac{1}{r}T_r), \tag{3}$$

where the velocity components along the (r,z) axes are (u,w), ρ is density, ν is the kinematic viscosity, p is pressure, g is the gravitational acceleration along the $z-$ direction, β is the coefficient of thermal expansion, T is the temperature, α is the thermal diffusivity. The corresponding boundary conditions for the problem are

$$u(a,z) = 0, \quad w(a,z) = U_w \quad w(r,z) \to 0 \text{ as } r \to \infty, \tag{4}$$

$$T(a,z) = T_w(z), \quad T(r,z) \to T_\infty \text{ as } r \to \infty, \tag{5}$$

where $U_w = 2ake^{z/a}$ is the fluid velocity at the surface of the cylinder.

Ref

6. S. Nadeem, Abdul Rehman, K. Vajravelu, J. Lee, C. Lee, Axisymmetric stagnation flow of a micropolar nanofluid in a moving cylinder, Math. Prob. Eng. Volume 2012 (2012), Article ID 378259, 17 pages, doi:10.1155/2012/378259

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III. Solution of the Problem

Introduce the following similarity transformations:

$$u = -\frac{1}{2}U_w \frac{f(\eta)}{\sqrt{\eta}}, \qquad w = U_w f'(\eta), \tag{6}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \eta = \frac{r^2}{a^2}, \tag{7}$$

where the characteristic temperature difference is calculated from the relations $T_w - T_\infty = ce^{z/a}$. With the help of transformations (6) and (7), *Eqs*. (1) to (3) take the form

$$\eta f''' + f'' + \mathrm{Re}(ff'' - f'^2) + \mathrm{Re}\,\lambda\theta = 0, \tag{8}$$

$$\eta\theta'' + \theta' + \frac{1}{2}\mathrm{Re}\,\mathrm{Pr}(f\theta' - f'\theta) = 0, \tag{9}$$

in which $\lambda = g\beta a(T_w - T_\infty)/U_w^2$ is the natural convection parameter, $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number and $\mathrm{Re} = aU_w/4\nu$ is the Reynolds number. The boundary conditions in nondimensional form become

$$f(1) = 0, \quad f'(1) = 1, \quad f' \to 0, \text{ as } \eta \to \infty, \tag{10}$$

$$\theta(1) = 1, \quad \theta \to 0, \text{ as } \eta \to \infty. \tag{11}$$

The important physical quantities such as the shear stress at the surface τ_w, the skinfriction coefficient c_f, the heat flux at the surface of the cylinder q_w and the local Nusselt number Nu are

$$c_f\,\mathrm{Re}^{1/2} = f''(1), \qquad Nu/\mathrm{Re}^{1/2} = -\theta'(1), \tag{12}$$

The solution of the present problem is obtained by using the powerful numerical technique the Keller-box method. To develop the technique the system of differential equations $(8-9)$ along with the boundary conditions $(10-11)$ is converted into a first order differential system by taking

$$R = f', \quad S = R', \quad G = \theta'. \tag{13}$$

With the help of $Eq.(13)$, $Eqs.(8)$ and (9) take the form

$$\eta S' + S + \mathrm{Re}(fS - R^2) + \mathrm{Re}\,\lambda\theta = 0, \tag{14}$$

$$\eta G' + G + \frac{1}{2}\mathrm{Re}\,\mathrm{Pr}(fG - R\theta) = 0. \tag{15}$$

The corresponding boundary conditions are

$$f(1)=0,\ \ u(1)=1,\quad u\to 0,\ \text{as}\ \eta\to\infty, \tag{16}$$

$$\theta(1)=1,\quad \theta\to 0,\ \text{as}\ \eta\to\infty. \tag{17}$$

Further details of the numerical solution can be found in references [21-24]. The detailed discussion about the obtained numerical solutions is presented in the next section.

IV. Results and Discussion

The problem of natural convection boundary layer flow of a viscous fluid over an exponentially stretched cylinder is studied in this article. The cylinder is assumed to be stretching exponentially along its radial direction. The exponential stretching velocity at the surface of the cylinder is assumed to be $U_w = 2ake^{z/a}$. The solution of the problem is obtained with the assistance of powerful second order finite difference scheme known as the Keller-box technique. The impact of the involved parameters such as the Reynolds number Re, the Prandtl number Pr and the natural convection parameter λ over the nondimensional velocity and temperature profiles is presented graphically and in the form of tables. $Fig.1$ shows the influence of Reynolds number Re and the natural convection parameter λ over the velocity function f' for waters Prandtl number (that is $\mathrm{Pr}=7$). From $Fig.1$ it is observed that with increase in both Reynolds number and the natural convection parameter λ the velocity profile increases. This observation is consistent with the fact that enhanced natural convection parameter λ acquires greater density difference in fluid that asks increase in fluid velocity. The Reynolds numbers range selected in $Fig.1$ is up to $\mathrm{Re}=50$. These values of Re are associated with the laminar type flow and the increasing effect of velocity profile due to increase in Re reveals the fact that high Reynolds number corresponds to enhance the rate of momentum transfer which in return demands increase in velocity profile f'. $Fig.2$ compiles the influence of Prandtl number Pr and Reynolds number Re over the velocity profile f' when natural convection parameter $\lambda=0.5$, the curves are provided for $\mathrm{Pr}=0.72$ (air) and $\mathrm{Pr}=7$ (water) for high Reynolds numbers. From $Fig.2$ it is conveyed that for both laminar ($\mathrm{Re}=5,50,500$) and turbulent ($\mathrm{Re}=5000$) velocities, inflation in velocity losses dependence over increase in Pr with increasing Re. Moreover, with the increase in Re, the boundary layer thickness increases. The natural convection parameter λ impact over the velocity profile is sketched in $Fig.3$ for different values of Prandtl number Pr when $\mathrm{Re}=10$. $Fig.3$ inculcates that with increase in λ the velocity profile increases. The rate of increase restricts with increase in Pr. $Figs.4$ and 5 telecast the influence of Prandtl numbers and Reynolds numbers over the temperature profile θ. From $Fig.4$ it is noted that with increase in Prandtl number Pr the temperature profile decreases while the thermal boundary layer is achieved much rapidly. The curves are plotted for Prandtl numbers up to $\mathrm{Pr}=20000$ (such high Prandtl numbers are usually for hydrocarbon based engine oils). This decrease in temperature profile is consistent with the observation that with increase in Prandtl numbers, thermal diffusion rate decreases that also deduces the temperature function θ. $Fig.5$ indicates that with increase in Reynolds numbers, the temperature profile decreases. This signal to the fact that high Reynolds numbers corresponds to more dense fluids which corresponds to reduced temperature flow rate in the fluid and hence have low fluid temperature. $Figs.6$ and 7 gathers the behaviors of Prandtl

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21. S. Nadeem, Abdul Rehman, M. Y. Malik, Boundary layer stagnation-point flow of third grade fluid over an exponentially stretching sheet, Braz. Soci. Che. Eng, in press

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numbers and Reynolds numbers over the temperature profile θ for different λ. From these two graphs it is obvious that with increase in λ temperature profile decreases for the presented ranges of Prandtl and Reynolds numbers. *Figs*.8 and 9 signifies the influence of small Reynolds numbers over velocity and temperature profiles f' and θ for different pairs of Prandtl numbers and the natural convection parameter. From *Fig*.8 it is observed that for small λ, increase in Reynolds numbers causes to decrease the velocity profile but for larger λ, increase in Reynolds numbers results to increases the velocity profile. From *Fig*.9 it is clear that with increase in Reynolds numbers, a decrease in the temperature profile is observed. The behavior of skinfriction coefficient for different combinations of Prandtl numbers and λ is graphed in *Fig*.10 against Reynolds numbers. From this plot it is shown that with increase in Prandtl number and λ, the skinfriction coefficient decreases, while increase in Reynolds numbers enhances the skinfriction coefficient. *Fig*.11 contains the curves predicting the behavior of local Nusselt numbers Nu for different Reynolds numbers and λ plotted against Prandtl numbers Pr. It is predicted from *Fig*.11 that with increase in the Pr, Re and λ, increases the local Nusselt numbers. The stream lines pattern associated with the fluid flow are presented in *Figs*.12 and 13 for $\lambda = 0.25$, and 1.2 respectively, graphed in the (r, z) plane. From these plots the decaying pattern observed is much rapid for high λ. The behavior of shear stress at the surface of the cylinder τ_w and the surface heat flux q_w is tabulated in *Tables*.1 and 2 for different values of the involved parameters. From these tables it is observed that Prandtl numbers and λ enhances the shear stress while Reynolds numbers reduces the shear stress at the surface of the cylinder, whereas local Nusselt number increases with increase in these parameters.

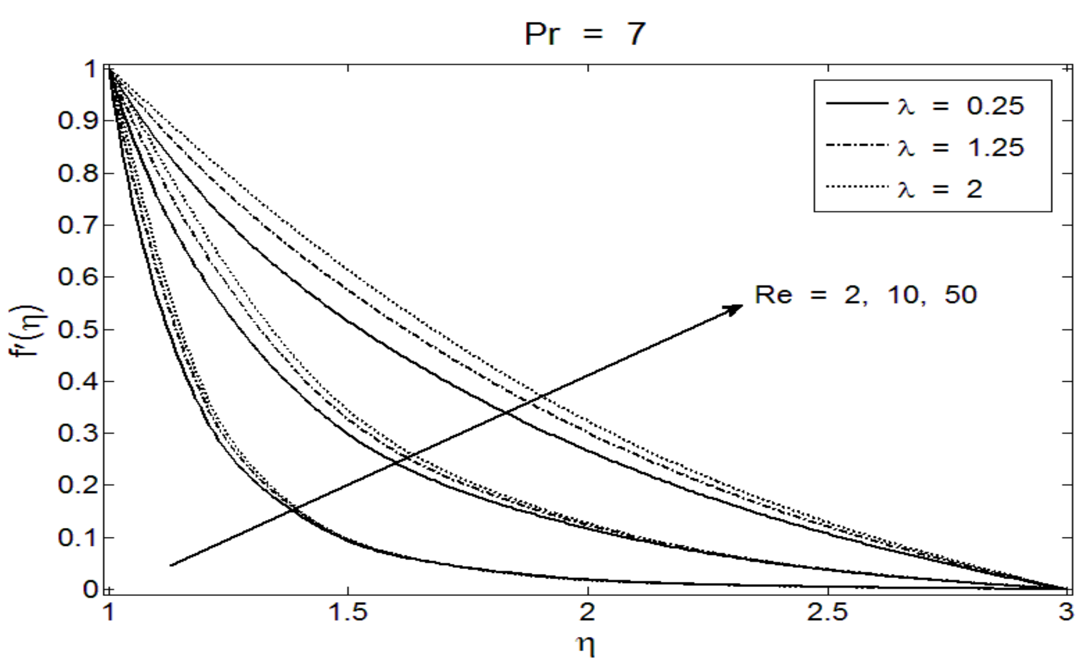

Fig.1. Influence of Reynolds numbers Re over velocity profile f' for different λ

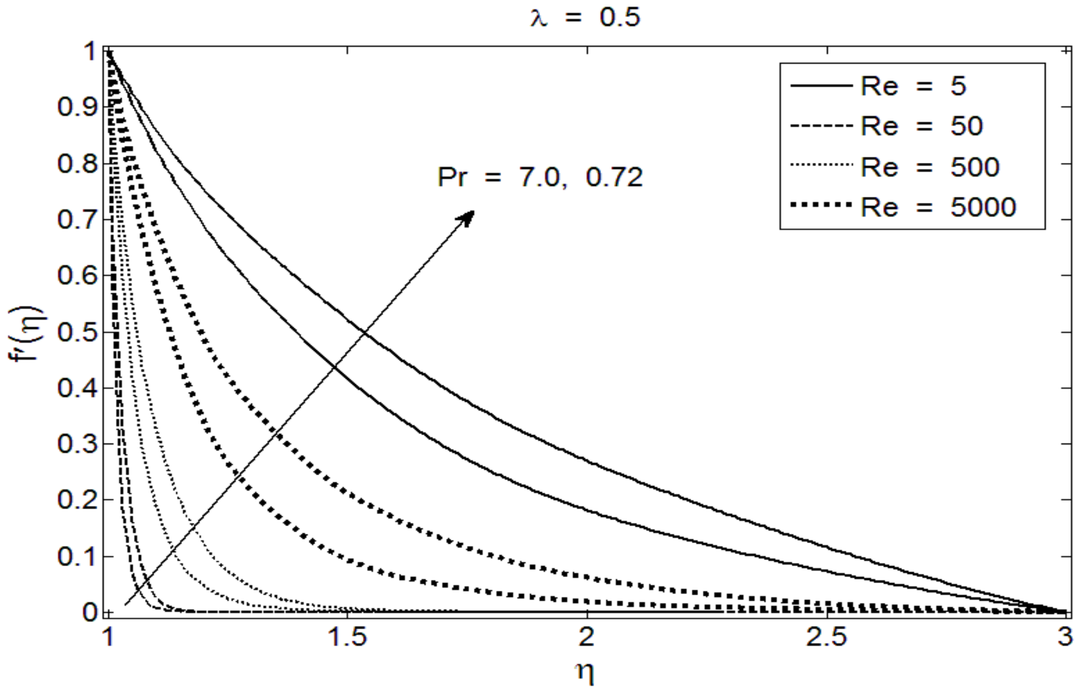

Fig.2. Influence of $\Pr$ over velocity profile f' for different Re

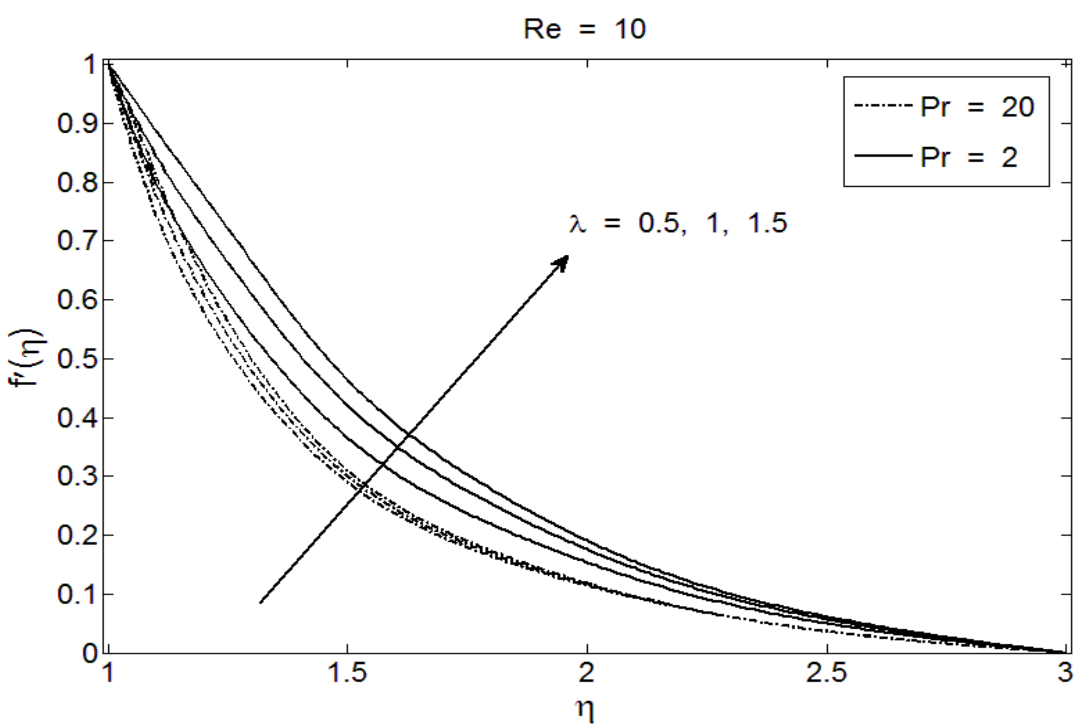

Fig.3. Influence of λ over velocity profile f' for different $\Pr$

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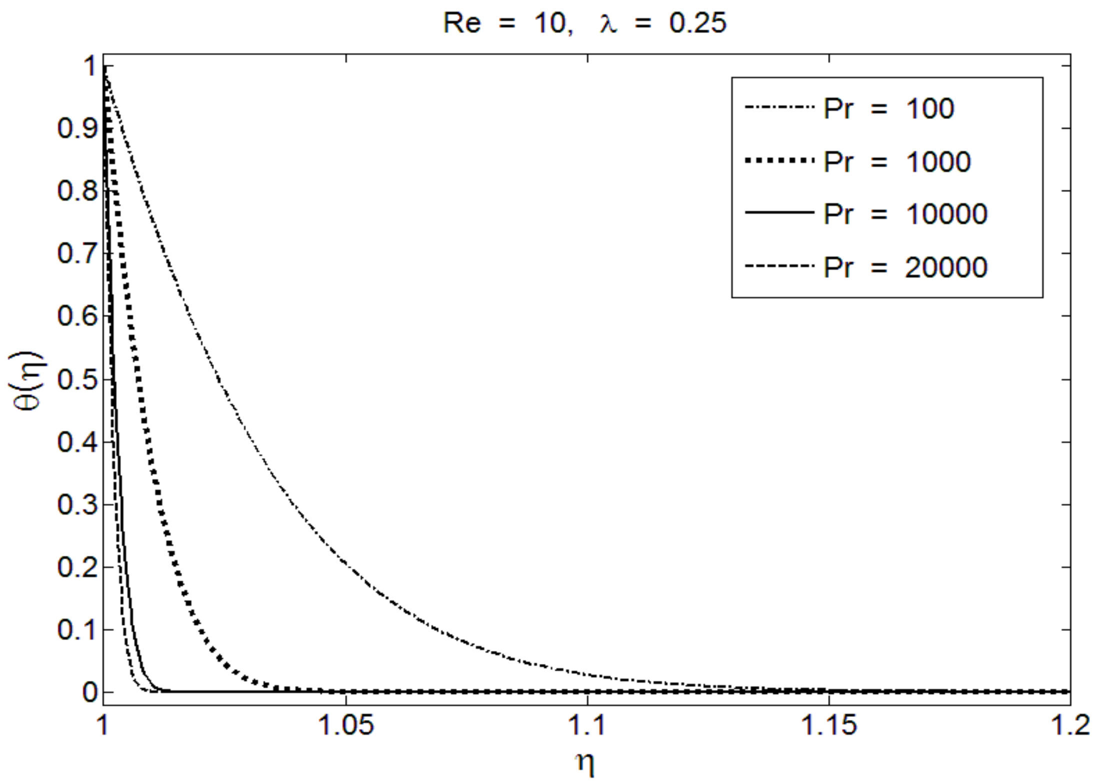

Fig.4. Influence of Pr over temperature profile θ

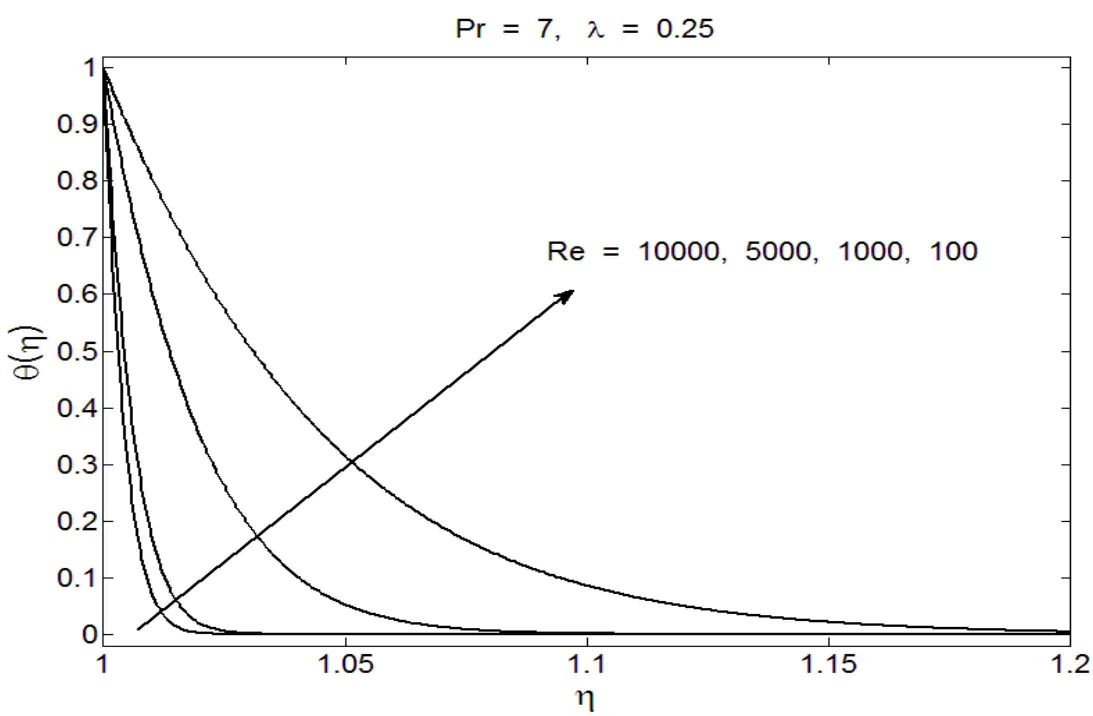

Fig.5. Influence of Re over temperature profile θ

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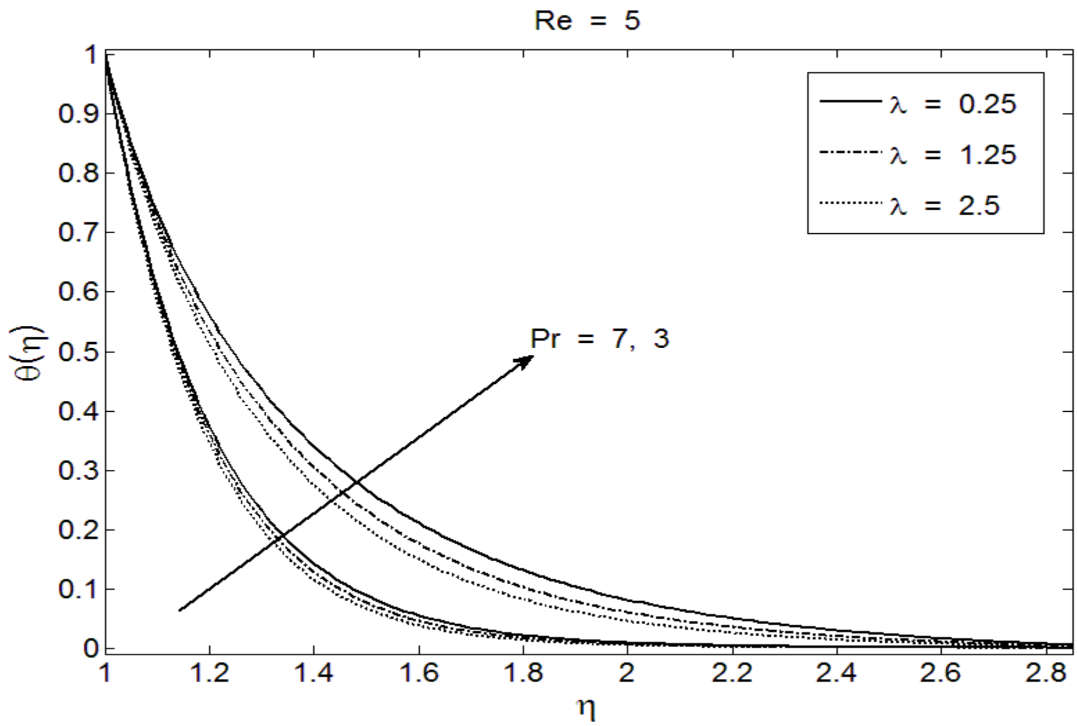

Fig.6. Influence of λ over temperature profile θ for different Pr

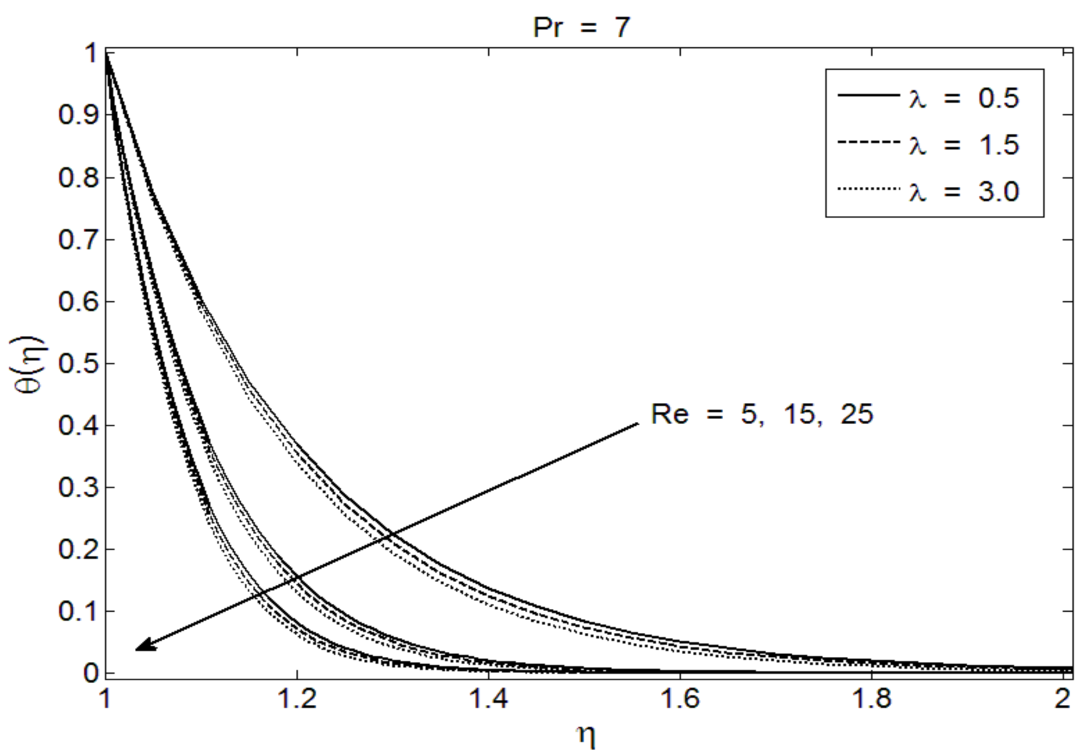

Fig.7. Influence of λ over temperature profile θ for different Re

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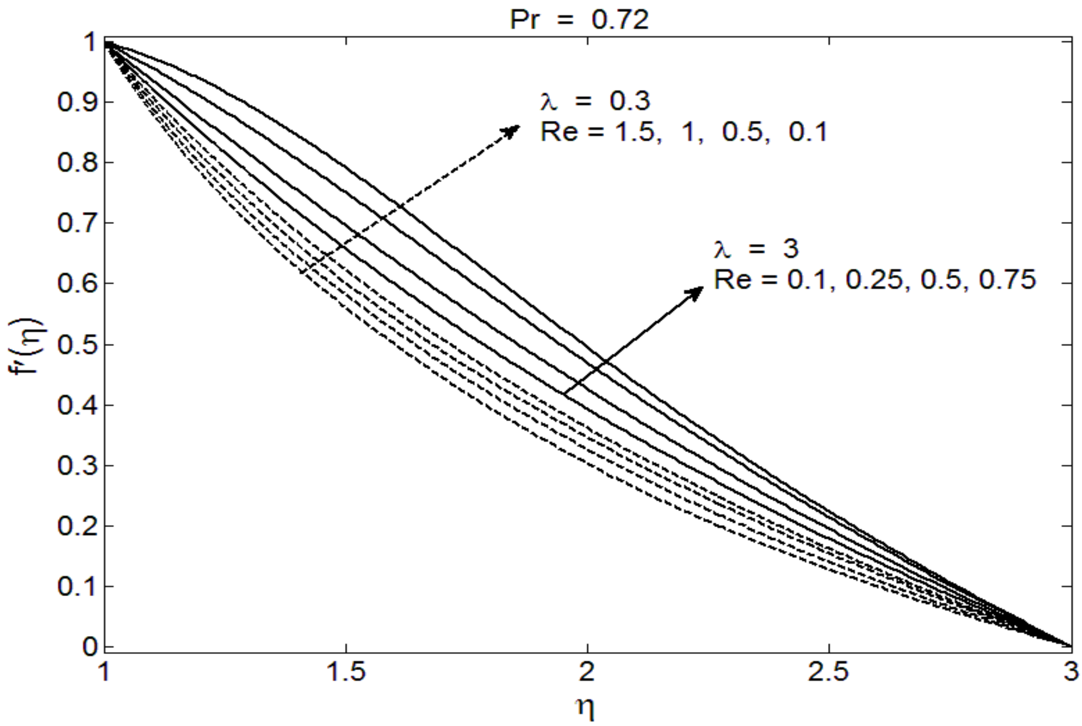

Fig.8. Behavior of small Re over the velocity profile f'

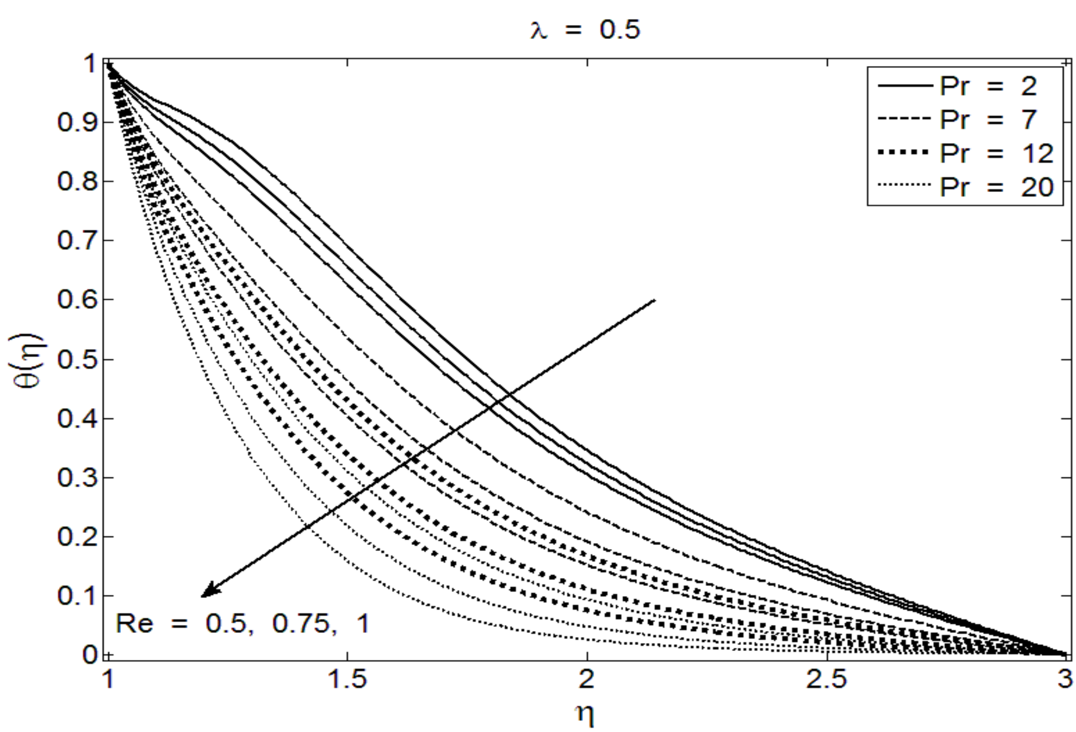

Fig.9. Behavior of small Re over the temperature profile θ

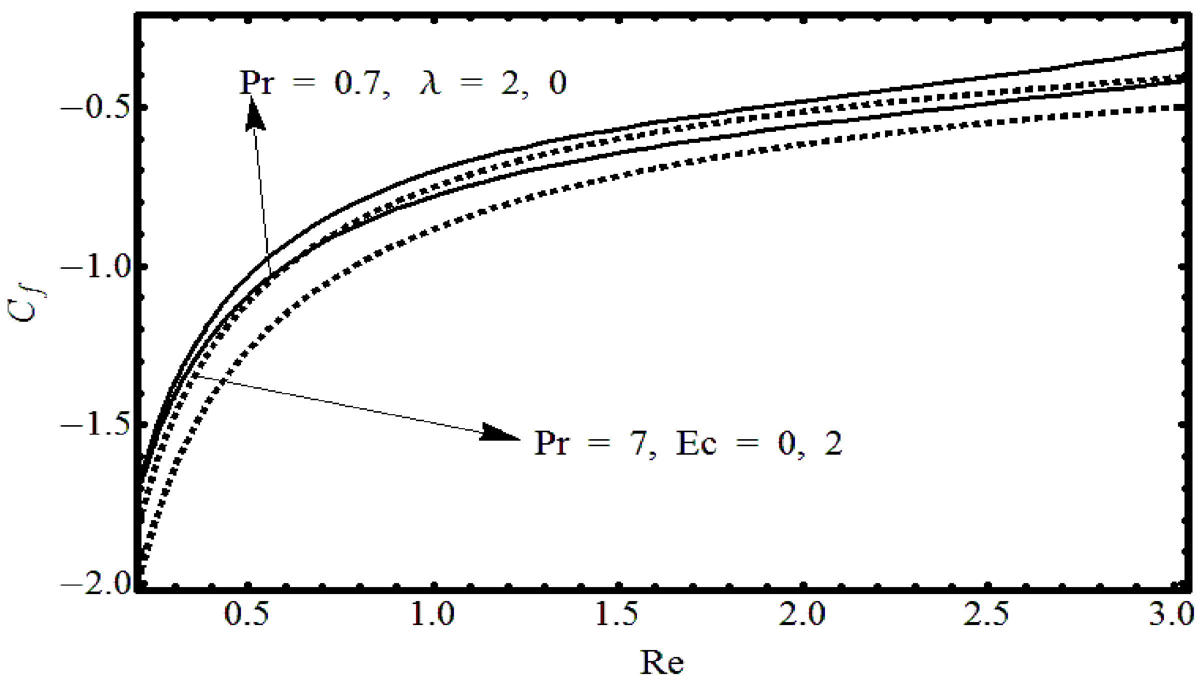

Fig.10. Influence of Pr and λ over C_f against Re

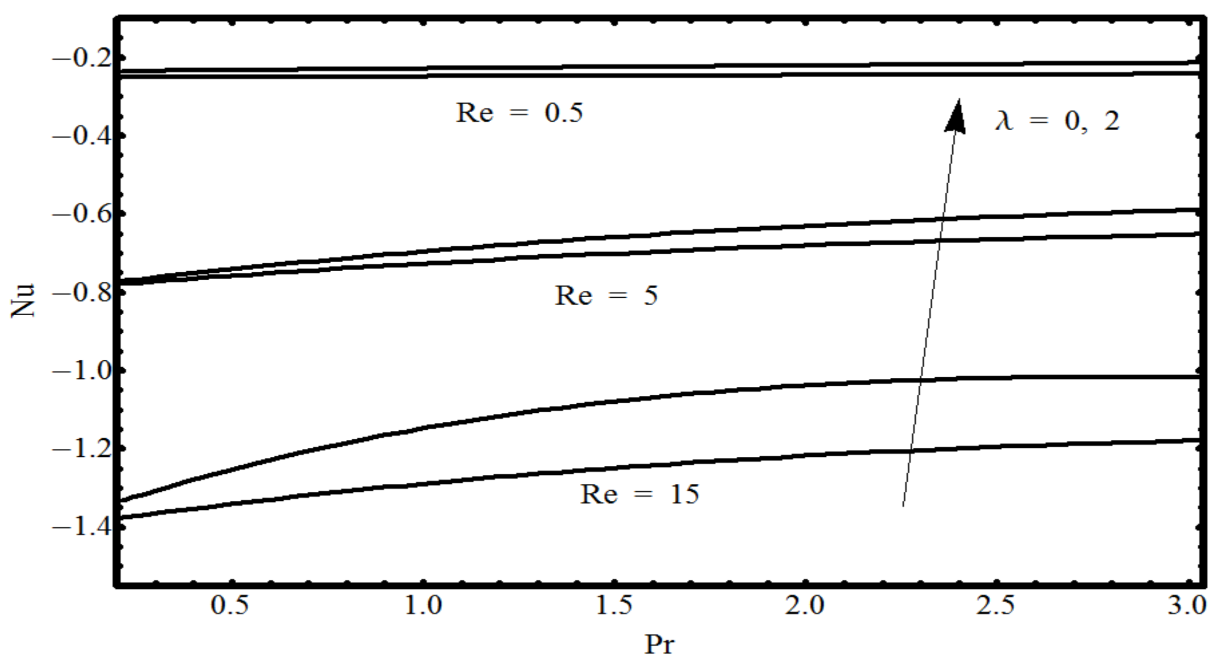

Fig.11. Influence of Re and λ over *Nu* against Pr

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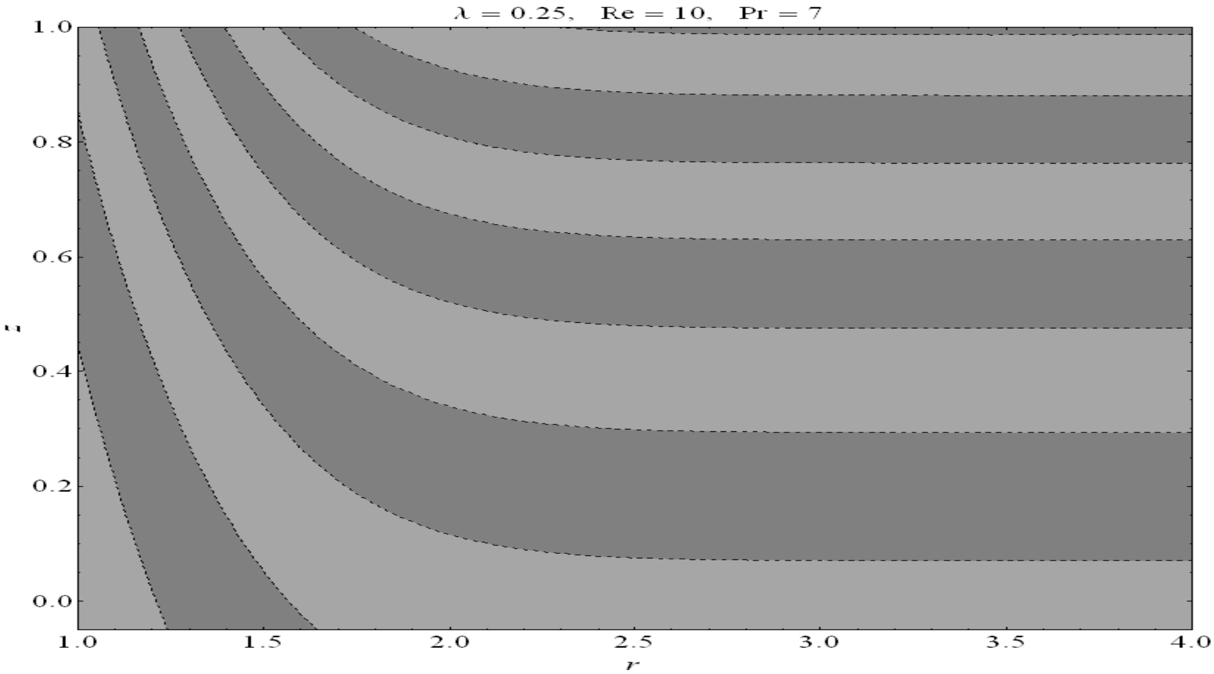
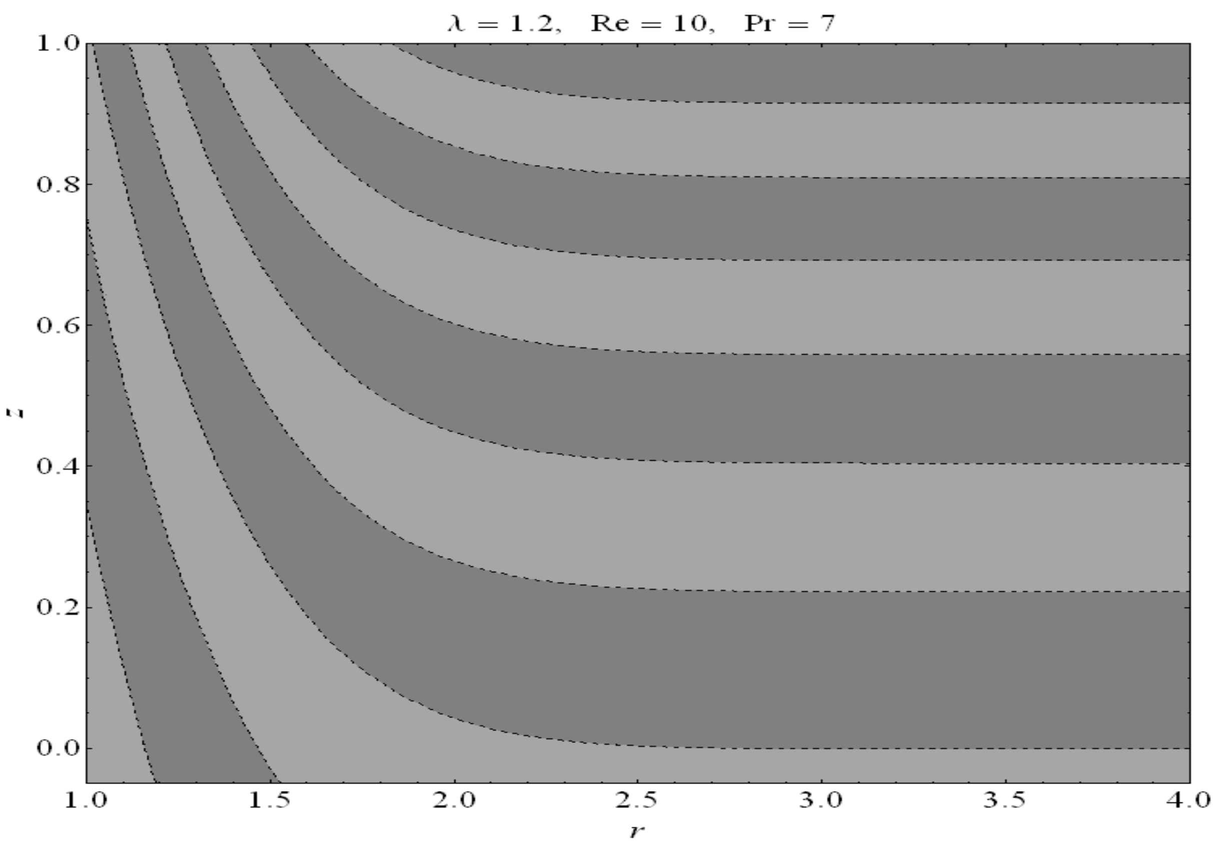


$Fig.12.$ Stream lines pattern for $\lambda = 0.25$



$Fig.13.$ Stream lines pattern for $\lambda = 1.2$

		f″(1)						
	λ\Re	0.5	1	5	10	100	1000	10000
Pr = 0.7	0.25	1.0691	1.1943	1.9559	2.6148	7.8386	24.6839	81.5422
	0.75	0.9384	0.9608	1.2858	1.6661	4.9948	15.8302	53.8606
	1.50	0.7454	0.6202	0.3844	0.4155	1.2255	4.0216	16.2816
	2.50	0.4905	0.1821	0.7015	1.0833	3.3322	10.3141	30.0166
Pr = 7	0.25	1.0838	1.2348	2.1561	2.9292	8.8292	27.6638	90.0127
	0.75	0.9836	1.0817	1.8294	2.4794	7.4529	23.3204	75.4808
	1.50	0.8364	0.8600	1.3592	1.8300	5.4571	17.0109	54.2655
	2.50	0.6451	0.5768	0.7610	1.0012	2.8987	8.9081	30.9185

*Table.*1. Behavior of shear stress at the surface of the cylinder for different values of the involved parameters

		-Θ′(1)							
	Re\Pr	0.2	0.7	7	10	70	150	1500	15000
λ = 0	1	1.2281	1.3355	2.4696	2.8960	7.4317	10.8491	34.4118	113.4154
	5	1.3633	1.7833	5.0510	6.0576	16.3167	24.0243	78.0879	297.9081
	10	1.4905	2.1932	7.0066	8.4368	22.9968	33.9505	112.7487	491.7423
	20	1.6789	2.7991	9.7683	11.8010	32.4741	48.0992	166.1533	861.8583
	50	2.0891	4.0877	15.2755	18.5037	51.4576	76.7794	293.7000	1941.5102
λ = 0.5	1	1.2324	1.3530	2.5058	2.9234	7.4943	10.8660	34.4135	113.4162
	5	1.4124	18865	5.1156	6.1103	16.3405	24.0832	78.0887	297.9041
	10	1.6013	2.3794	7.0846	8.5018	23.0068	33.9665	112.7556	491.7535
	20	1.9185	3.1213	9.8782	11.8927	32.5090	48.2209	166.1675	961.9010
	50	2.6281	4.6405	15.4471	18.6462	51.5160	76.8235	293.7440	1941.5985

*Table.*2. Behavior of heat flux at the surface of the cylinder for different values of the involved parameters

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A New Approach of Iteration Method for Solving Some Nonlinear Jerk Equations

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Abstract- A new approach of the Mickens' iteration method has been presented to obtain approximate analytic solutions of some nonlinear jerk equations. It has been shown that the partial derivatives of integral functions are valid for iteration method in each step of iteration. Also the solutions give more accurate result than other existing methods.

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A NEWAPPROACHOFITERATIONMETHODFORSOLVINGSOMENONLINEARJERKEQUATIONS

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Abstract- A new approach of the Mickens' iteration method has been presented to obtain approximate analytic solutions of some nonlinear jerk equations. It has been shown that the partial derivatives of integral functions are valid for iteration method in each step of iteration. Also the solutions give more accurate result than other existing methods.

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I. Introduction

The subject of differential equation not only is one of the most beautiful parts of mathematics, but it is also an essential tool for modeling many physical situations such as mechanical vibration, nonlinear circuits, chemical oscillation, space dynamics and so forth. These equations have also demonstrated their usefulness in ecology, business cycle and biology. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. The differential equations are linear or nonlinear, autonomous or non-autonomous. Practically, numerous differential equations involving physical phenomena are nonlinear. Methods of solutions of linear differential equations are comparatively easy and highly developed. Whereas, very little of a general character is known about nonlinear equations.

Generally, the nonlinear problems are solved by converting into linear equations imposing some conditions; but such linearization is always not possible. In this situation there are several analytical approaches to find approximate solutions to nonlinear problems, such as: Perturbation [11,19,20], Standard and modified Linstedt-Poincaré [20-22], Harmonic Balance [1-4,6-9,13,14,16,24-29], Homotopy [5,15], Iterative [10, 12,18] methods, etc.

Among them the most widely used method is the perturbation method where the nonlinear term is small. Another recent technique is developed by Mickens [17] and farther work has

[11] Hu, H.: *Perturbation method for periodic solutions of nonlinear jerk equations*, Phys. Lett. A, vol. 372, pp. 4205-4209, 2008.

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been done by Wu [29], Gottlieb [7,8], Alam [1] and so forth for handling the strong nonlinear problems named HB method. Recently, some authors used an iteration procedure which is valid for small as well as large amplitude of oscillation to obtain the approximate frequency and the corresponding periodic solution of such nonlinear problems.

The term "jerk" i.e. the third-order derivative of the displacement (which might also be termed "triceleration") was first introduced by Schot in 1978 [27]. We know most of the efforts on dynamical systems are related to second-order differential equations, but some dynamical systems can be described by nonlinear jerk (third-order) differential equations. As for example oscillations in a nonlinear vacuum tube circuit [24] and third order mechanical oscillators [6]. Originally jerk equations

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) \,. \tag{1}$$

being of some interest in mechanics [4,26], these equations are finding increasing importance in the study of chaos [4]. Many third-order nonlinear systems (three simultaneous first order differential equations), both mathematically and physically motivated, such as the now–classical Rössler system [25], may be recast into a single nonlinear third-order differential (jerk) equation involving only one of the dependent variables [7,14]. Some early investigations into nonlinear jerk equations (although not termed as such) include oscillations in a nonlinear vacuum tube circuit [24], and third order mechanical oscillators [6]. Other physical situations in which nonlinear jerk-type equations have been investigated with more emphasis in chaotic solutions (called a periodic in earlier works), include a thermo–mechanical oscillator model with thermal dissipation [16], fluid dynamical convection [2], and stellar ionization zone oscillations [3]. Jerk equations, even if not nearly as common as acceleration (or force) equations $\ddot{x} = f(x, \dot{x})$, are therefore of direct physical interest. Moreover, simple forms of the jerk function J which lead to maybe the simplest manifestation if chaoses have been found by Sprott [28].

In this article we have investigated not chaotic solutions to jerk equations (as many of the above references do), but the analytical approximate periodic solutions and corresponding frequencies of some nonlinear jerk equations using the iteration method.

Ref

[7] Gottlieb, H.P.W.: *Question #38. What is the simplest jerk function that gives chaos?* American Journal of Physics, vol.64, pp. 525, 1996.

Gottlieb [8] used the lowest order harmonic balance method to calculate approximations to the periodic solution and the angular frequencies obtained by Gottlieb were not accurate enough. But it is very difficult to construct the higher order approximation by harmonic balance method for the reason that the method requires analytical solutions of a set of complicated nonlinear algebraic equations. Wu et al. [29] and Leung et al. [13] applied, respectively, an improved harmonic balance approach and a residue harmonic balance approach to solve nonlinear jerk equations, and their higher order approximations give accurate results to a large range of initial velocity amplitudes. Ma et al. [15] and Hu [11] used, respectively, Homotopy perturbation method and parameter perturbation method to determine the high order approximate solutions of nonlinear jerk equations, and their results obtained are more accurate than those obtained by the low order harmonic balance method. Recently, Ramos [21-23] presented an order reduction method, two-level iterative procedure and a volterra integral formulation, respectively, to solve nonlinear jerk equations. He found that the second reduction method provides accurate solutions only for initial velocities close to unity, the third reduction method produces very accurate for the first and second differential equation in [21], the fourth reduction method provides as accurate results as or more accurate results obtained by parameter perturbation method. Hu et al. [12] presented the modified Mickens iteration procedure for a nonlinear jerk equation, and the second order approximate angular frequency was obtained by Newton's method. Leung and Guo [13] has established a residue harmonic balance approach for determining limit cycles to parity- and time-reversal invariant general non-linear jerk equations with cubic nonlinearities.

To obtain periodic solutions for jerk-oscillators certain restrictions must be placed on the mathematical structure of the oscillator. We only consider the following cubic nonlinear functions investigated by Gottlieb [8]:

$$x\,\dot{x}\,\ddot{x}\,. \tag{2}$$

The most general jerk function with invariance of time- and space-reversal and which has only cubic nonlinearities may be written in the form [8]

$$\dddot{x} = \alpha\, x\,\dot{x}\,\ddot{x} - \beta\,\dot{x}\,\ddot{x}^2 - \gamma\,\dot{x} - \delta\, x^2\,\dot{x} - \varepsilon\,\dot{x}^3\,. \tag{3}$$

Ref

[8] Gottlieb, H.P.W.: *Harmonic balance approach to periodic solutions of nonlinear jerk equation*, J. Sound Vib., vol. 271, pp. 671-683, 2004.

where an over-dot denotes the time derivative and the parameters α, β, γ, δ and ε are given real constants. The corresponding initial conditions are

$$x(0) = 0,\ \dot{x}(0) = A \ \text{ and } \ \ddot{x}(0) = 0. \qquad (4)$$

These three initial conditions in Eq. (4) are to satisfy the periodic requirement. Here, at least one of α, β, δ and ε should be non-zero.

For $\alpha = 1$ with cubic nonlinearity includes $x\dot{x}\ddot{x}$ only. The resulting standard jerk equation, after rescaling of both x and t, is taken to be

$$\dddot{x} + \dot{x} - x\dot{x}\ddot{x} = 0, x(0) = 0,\ \dot{x}(0) = A,\ \ddot{x}(0) = 0. \qquad (5)$$

In this article, we present a new approach of the Mickens' iteration method [16] for the determination of approximate solutions of some nonlinear jerk equations. Here we have defined the function which contains dependent variable, its derivatives of first and second order also the integrals of the dependent variable. Previously the function contains only dependent variable and its derivatives of first and second order. It is mentioned that our method is valid for second and higher order period of oscillations and show a good agreement compared to other existing solutions.

II. The Method

Let us consider a nonlinear oscillator modeled by

$$\ddot{x} + f(\ddot{x}, \dot{x}, x', x) = 0, \qquad x(0) = A, \quad \dot{x}(0) = 0, \qquad (6)$$

where over dots denote differentiation with respect to time, t over dash denotes integration with respect to time, t.

We choose the frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (6), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x, x', \dot{x}, \ddot{x}) \equiv G(x, x', \dot{x}, \ddot{x}). \qquad (7)$$

Following [18], we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, x_k', \dot{x}_k, \ddot{x}_k); \quad k = 0, 1, 2, \cdots, \qquad (8)$$

Ref

[16] Moore, D.W. and Spigel, E.A.: *A thermally excited nonlinear oscillator*, *The Astrophysical Journal*, vol. 143, pp. 871-887, 1966.

together with

$$x_0(t) = A\cos(\Omega_0 t)\ . \tag{9}$$

Herein x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0\,. \tag{10}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms (see [19] for details) should not occur in the solution. This procedure gives the sequence of solutions: $x_0(t), x_1(t), \cdots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [18].

III. Solution Procedure

Here we have considered the particular Jerk function $x\,\dot{x}\,\ddot{x} - \dot{x}$

i.e. Jerk function containing displacement times velocity time's acceleration, and velocity:

Let us consider the nonlinear jerk oscillator [8]

$$\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}\ . \tag{11}$$

Introducing the phase space variable (y,t) by the relation $\dot{x} = y$ then Eq. (11) becomes

$$\ddot{y} + y = y\,\dot{y}\,y'\,. \tag{12}$$

Obviously, Eq. (12) can be written as

$$\ddot{y} + \Omega^2\, y = (\Omega^2 - 1 + \dot{y}\,y')\,y\,. \tag{13}$$

Now the iteration scheme is (according to Eq. (8))

$$\ddot{y}_{k+1} + \Omega_k^2\, y_{k+1} = (\Omega_k^2 \ -1 - \dot{y}_k\, y'_k)\, y_k\,. \tag{14}$$

Equation (9) is rewritten as

$$y_0 = y_0(t) = A\cos\theta\ \ , \tag{15}$$

Ref

[19] Nayfeh, A.H.: *Perturbation Method*, John Wiley & Sons, New York, 1973.

where $\theta = \Omega t$. For $k = 0$, the Eq. (14) becomes

$$\ddot{y}_1 + \Omega_0^{\ 2}\, y_1 = (\Omega_0^{\ 2} - 1 - \dot{y}_0\, y_0')\, y_0 \,. \tag{16}$$

Substituting the initial function Eq. (15) into the right hand side of Eq. (16) and expanding in a Cosine series, we obtain

$$\ddot{y}_1 + \Omega_0^{\ 2}\, y_1 = a_1 \cos\theta + a_3 \cos 3\theta \,. \tag{17}$$

where,

$$a_1 = \frac{A}{4}(-4 - A^2 + 4\,\Omega_0^{\ 2}), \quad a_3 = \frac{1}{4}A^3 \quad . \tag{18}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (17). Thus we have

$$\Omega_0^{\ 2} = 1 + \frac{A^2}{4} \quad . \tag{19}$$

Then solving Eq. (17) and satisfying the initial condition $y_1(0) = A$, we obtain

$$y_1(t) = \left(A + \frac{a_3}{8\,\Omega_0^{\ 2}} \right) \cos\theta - \frac{a_3}{8\,\Omega_0^{\ 2}} \cos 3\theta \quad . \tag{20}$$

This is the first approximate solution of Eq. (12) and the related Ω_1 is to be determined. The value of Ω_1 will be obtained from the solution of

$$\ddot{y}_2 + \Omega_1^{\ 2}\, y_2 = (\Omega_1^{\ 2} - 1 - \dot{y}_1\, y_1')\, y_1 \,. \tag{21}$$

Substituting $y_1(t)$ from Eq. (20) into the right hand side of Eq. (21) and then expanding in a truncated Cosine series, we obtain

$$\ddot{y}_2 + \Omega_1^2\, y_2 = \sum_{r=1}^{4} b_{2r-1} \cos(2r-1)\theta \,. \tag{22}$$

where,

$$b_1 = -\frac{A(32 + 9A^2)}{3072\,(4 + A^2)^3} \{6144 + 4608A^2 + 1136A^4 + 93A^6 - 384(4 + A^2)^2\, \Omega_1^2\}$$

Notes

Notes

$$b_3 = \frac{A^3(32+9A^2)}{512(4+A^2)^3}\{9216+7936A^2+2296A^4+223A^6-64(4+A^2)^2\,\Omega_1^2\}$$

$$b_5 = -\frac{A^5}{1536(4+A^2)^3}\{3328+1928A^2+279A^4\}$$

$$b_7 = \frac{13A^7}{6144(4+A^2)^3}(32+9A^2) \tag{23}$$

Again avoiding secular terms in the solution of Eq. (22), we obtain

$$\Omega_1^2 = \frac{(6144+4608A^2+1136A^4+93A^6)}{384(4+A^2)^2}. \tag{24}$$

Then solving Eq. (22) and satisfying initial condition, we obtain the second approximate solution,

$$y_2(t) = \left(A + \frac{1}{\Omega_1^{\,2}}\left(\frac{b_3}{8}+\frac{b_5}{24}+\frac{b_7}{48}\right)\right)\cos\theta - \frac{1}{\Omega_1^{\,2}}\left(\frac{b_3}{8}\cos 3\theta + \frac{b_5}{24}\cos 5\theta + \frac{b_7}{48}\cos 7\theta\right). \tag{25}$$

This is the second approximate solution of Eq. (12).

The third approximation y_3 and the value of Ω_2 will be obtained from the solution of

$$\ddot{y}_3 + \Omega_2^{\,2}\, y_3 = (\Omega_2^{\,2} - 1 - \dot{y}_2\, y_2')\, y_2 . \tag{26}$$

After substituting the second approximate solution $y_2(t)$ of Eq. (12) from Eq. (25) into the right hand side of Eq. (26) and avoiding secular terms in the solution, we obtain

$$\begin{aligned}
\Omega_2^2 = \{&768(14843406974976+48704929136640A^2+73635603677184A^4 \\
&+67914279419904A^6+213141799305216A^8/5+2884018883133344A^{10}/15 \\
&+6409791774720A^{12}+100727452289312A^{14}/63+40272237000484A^{16}/135 \\
&+20716379322491A^{18}/504+2621464123311569A^{20}/645120 \\
&+5631142163784439A^{22}/20643840+817336388391143A^{24}/73400320 \\
&+489874066020999A^{26}/2348810240)\}/\{(24576+24576A^2+9152A^4 \\
&+1508A^6+93A^8)^2(18874368+19464192A^2+7518208A^4+1294816A^6 \\
&+84249A^8)\}
\end{aligned} \tag{27}$$

In a similar way the method can be proceeded higher order approximations and Ω_0, Ω_1, Ω_2, … respectively obtained by Eqs. (19), (24), (27), … represent the approximation of frequencies of oscillator (11). However due to growing algebraic complexity, most of the approximate techniques are applied to second or third approximations.

IV. Results and Discussions

An iteration method is developed based on Mickens [16] iteration method to solve a class of nonlinear jerk equations. In this section, we express the accuracy of the modified approach of iteration method by comparing with the existing results from different methods and with the exact results of the nonlinear jerk equations. To show the accuracy, we have calculated the percentage errors (denoted by Er(%)) by the definitions $\left|100(T_e - T_i)/T_e\right|$, where $T_i = 2\pi/\Omega_i\,; i = 0, 1, 2, \cdots$ represents the approximate periods obtained by the present method and T_e represents the corresponding exact period of the oscillator.

Now we demonstrate the comparison of results for the oscillator.

i.e., For the Jerk function containing displacement time's velocity time's acceleration, and velocity $x\,\dot{x}\,\ddot{x} - \dot{x}$.

Recently, Gottlieb [8], Ma *et al.* [15], Ramos [21] and Leung & Guo [13] has found approximate solutions, frequencies and as well as approximate periods of nonlinear jerk oscillator (given by Eq. (11)) by different methods other than Iteration method.

We have used a modified iteration procedure to obtaining approximate solutions of the oscillator. The procedure is very simple. It has been shown that, in most of the cases our solution gives significantly better result than other existing results and sometimes it is almost similar to other existing results.

Herein we have calculated the first second and third approximate frequencies which are denoted by Ω_1, Ω_2 and Ω_3 respectively and corresponding periods are T_1, T_2 and T_3. All the results are given in Table 5.a and Table 5.b, to compare the approximate frequencies we have also given the existing results determined separately by Gottlieb [8], Ma *et al.* [15], Ramos [21] and Leung & Guo [13] respectively.

Ref

[15] Ma, X., Wei, L. and Guo, Z.: *He's homotopy perturbation method to periodic solutions of nonlinear jerk equations*, J. Sound Vib., vol. 314, pp. 217-227, 2008.

Notes

Table 5.a

Comparison of the approximate periods with exact periods T_e [8] of $\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}$:

A	T_e	Modified T_0 Er(%)	Modified T_1 Er(%)	Modified T_2 Er(%)
0.1	6.275347	6.275346 1.56 e^{-5}	6.275347 2.65 e^{-6}	6.275347 2.60 e^{-6}
0.2	6.252016	6.252003 2.07 e^{-4}	6.252016 3.35 e^{-6}	6.252016 1.42 e^{-7}
0.5	6.096061	6.095585 7.8 e^{-3}	6.096018 7.0 e^{-4}	6.096060 2.1 e^{-5}
1	5.626007	5.619852 1.09 e^{-1}	5.624306 3.0 e^{-2}	5.625880 2.3 e^{-3}
2	4.491214	4.442883	4.463270	4.484913
		1.08	6.2 e^{-1}	1.4 e^{-1}

T_0 , T_1 and T_2 respectively denote initial, first and second modified approximate periods. Er(%) denotes percentage error.

Table 5.b

Comparison of the approximate periods obtained by our method and other existing result with exact periods T_e [8] of $\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}$:

A	T_e	Modified T_2 Er(%)	Gottlieb T_{G2} (2004) [8] Er(%)	Ma et al. T_{M2} (2008) [15] Er(%)	Ramos T_{R2} (2010) [21] Er(%)	Leung-Guo T_{LG2} (2011) [13] Er(%)
0.1	6.275347	6.275347 2.60 e^{-6}	6.275346 1.3 e^{-5}	6.275347 2.5 e^{-6}	6.275347 9.0 e^{-8}	6.275347 2.6 e^{-6}
0.2	6.252016	6.252016 1.42 e^{-7}	6.252003 2.11 e^{-4}	6.252016 1.6 e^{-7}	6.252016 6.4 e^{-6}	6.252016 1.6 e^{-7}
0.5	6.096061	6.096060 2.1 e^{-5}	6.095585 7.8 e^{-3}	6.096059 3.21 e^{-5}	6.096025 6.06 e^{-4}	6.096061 8.2 e^{-6}
1	5.626007	5.625880 2.3 e^{-3}	5.619852 1.09 e^{-1}	5.625795 3.8 e^{-3}	5.624539 2.6 e^{-2}	5.625993 2.5 e^{-4}
2	4.491214	4.484913 1.4 e^{-1}	4.442883 1.08	4.482081 2.03	4.466144 5.6 e^{-1}	4.490125 2.4 e^{-2}

T_2 denotes second modified approximate periods; T_{G2} , T_{M2} , T_{R2} and T_{LG2} respectively denote second approximate periods obtained by Gottlieb, Ma et al., Ramos and Leung-Guo. Er(%) denotes percentage error.

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CESÁRO Mean of Product Summability of Partial Differential Equations of Sequences

By Suyash Narayan Mishra

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Abstract- In [4], the definition of product summability method $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{l})$ for functions was given and some of its properties were investigated. In [2], $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})$ ($k > 0$, $\boldsymbol{\alpha} > 0$, $\boldsymbol{\beta} > -1$) summability for functions are defined and some of its properties were investigated. In [1], the Cesàro means and Cesàro summability were discussed for sequences. In this paper, we study some results of Cesàro mean of product summability $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})$ ($k > 0$, $\boldsymbol{\alpha} > 0$, $\boldsymbol{\beta} > -1$) of partial differential equations of sequences.

GJSFR-F Classification : MSC 2010: 32W50, 42B08

Ref

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CESȦRO Mean of Product Summability of Partial Differential Equations of Sequences

Suyash Narayan Mishra

Abstract- In [4], the definition of product summability method $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{l})$ for functions was given and some of its properties were investigated. In [2], $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})$ $(k > 0,\ \alpha > 0,\ \beta > -1)$ summability for functions are defined and some of its properties were investigated. In [1], the Cesȧro means and Cesȧro summability were discussed for sequences. In this paper, we study some results of Cesȧro mean of product summability $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})$ $(k > 0,\ \alpha > 0,\ \beta > -1)$ of partial differential equations of sequences.

I. Introduction

Kuttner [1], introduced the summability method for functions and investigated some of its properties. Pathak [4], defined the product summability method for functions and investigated some of its properties. Mishra and Srivastava [3], introduced the summability method for functions by generalizing summability method. Mishra and Mishra [2], introduced the summability method for functions and investigated some of its properties. In this paper, we study some results of Cesȧro mean of product summability $(D,k)(C,\alpha,\beta)$ $(k>0, \alpha>0, \beta>-1)$ of partial differential equations of sequences.

II. Some relations and Definitions

Let $f(x)$ be any function which is Lebesgue-measurable, and that $f:[0,+\infty)\to R$, and integrable in $(0,\ x)$, for any finite x and which is bounded in some right hand neighbourhood of origin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim_{x\to\infty}\int_0^x$, $\int_0^x$ being a Lebesgue integral. For any n>o, we write $a_n(x)$ for the n^{th} integral,

$$a_n(x) = \frac{1}{\Gamma(n)}\int_0^x (x-y)^{n-1}\, a(y)dy\,,$$

$$\boldsymbol{a\_(0\,)\,(x) = a(x)}$$

The (C,α,β) transform of $a\,(t)$, which we denote by $\partial_{\alpha,\beta}(t)$ is given by

$$a(t) \qquad (\alpha=0)$$

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$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{1}{t^{\alpha+\beta}}\int_0^x (t-u)^{\alpha-1}u^{\beta}a(y)dy\,,\ \ (\alpha>0,\beta>-1)\,, \qquad (2.1)$$

If, for t > 0, the integral defining $\partial_{\alpha,\beta}(t)$ exists and if $\partial_{\alpha,\beta}(t)\rightarrow s$ as $t\rightarrow\infty$, we say that a (x) is summable (C,α,β) to s, and we write $\boldsymbol{a(x)}\rightarrow$ s (C, α, β) . We write

$g(t)=g^{(k)}(t)=kt\int_0^{\infty}\frac{x^{k-1}}{(x+t)^{k+1}}a(x)dx$, (k > 0) (2.2) if this exists, We also write

$U_{k,\alpha,\beta}(t)=kt\int_0^{\infty}\frac{x^{k-1}}{(x+t)^{k+1}}\partial_{\alpha,\beta}(x)dx$, (2.3) if this exists.

With the usual terminology, we say that the sequence a_n is summable,

(I) (D, k) to the sum s, if g (t) tends to a limit s as t→ ∞ ,

(II) (D, k) (C,α,β) to s, if $U_{k,\alpha,\beta}(t)$ tends to s as t→ ∞. When$\beta=0$,$(D,k)(C,\alpha,\beta)$ and $(D,k)(C,\alpha)$ denote the same method. The case $\beta=0$ is due to Pathak[5] . We know that for any fixed t > 0, k > 0, it is necessary and sufficient for the convergence of (2.3) that $\int_1^{\infty}\frac{\partial_{\alpha,\beta}(x)}{x^2}\,dx$ should converge. (2.4)
If (2.4) converges, write for x > 0, $F_{\alpha,\beta}(x)=\int_x^{\infty}\frac{\partial_{\alpha,\beta}(t)}{t^2}\,dt$.

We note that $F_{\alpha,\beta}(x)=o(1)$ as x→ ∞ . Further, (since f(x) is bounded in some right hand neighbourhood of the origin) we have,

$F_{\alpha,\beta}(x)=o\left(\frac{1}{x}\right)$ as x→ 0 + .

III. Main Results

In this section, we have following theorems for sequences analogous to [2].

Theorem 3.1 : If $\alpha>\gamma\geq 1,\ k>0$ then $a\,(x)\rightarrow s\,(D,k)(C,\alpha-1,\beta)$,whenever $a(x)\rightarrow s(D,k)(C,\gamma-1,\beta)$.

Theorem 3.2 : Let $\alpha>\gamma\geq 0,\ \beta>-1$, and suppose that a(x) is summable (C,γ,β) to s and that $\int_1^{\infty}\frac{\partial_{\gamma,\beta}(x)}{x^2}\,dx$ converges . Then a(x) is summable $(D,k)(C,\alpha,\beta)$ to s.

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Notes

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The Errors in the Fields Medals, 1982 to S. T. Yau and 1990 to E. Witten

By C. Y. Lo

Abstract- Due to inadequate understanding in physics, mathematicians have made misleading erroneous claims in general relativity that result in awarding three times the Fields Medal in 1982 to S. T. Yau, and 1990 to E. Witten, and the 2011 Shaw Prize to D. Christodoulou. It is pointed out that they failed to discover that there are no bounded dynamic solutions for the Einstein equation as Gullstrand suspected. Thus, the existence of bounded dynamic solutions was implicitly, but incorrectly assumed. This error currently prevents the necessary rectification of general relativity that leads to the unification of gravitation and electromagnetism. Also, similar errors have been made by D. Hilbert and M. Atiyah.

Keywords: *stable solution; dynamic solution.*

GJSFR-F Classification : *MSC 2010: 00A69*

Ref

1. R. Schoen and S.-T. Yau, "Proof of the Positive Mass Theorem. II," Commun. Math. Phys. **79**, 231-260 (1981).

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Abstract- Due to inadequate understanding in physics, mathematicians have made misleading erroneous claims in general relativity that result in awarding three times the Fields Medal in 1982 to S. T. Yau, and 1990 to E. Witten, and the 2011 Shaw Prize to D. Christodoulou. It is pointed out that they failed to discover that there are no bounded dynamic solutions for the Einstein equation as Gullstrand suspected. Thus, the existence of bounded dynamic solutions was implicitly, but incorrectly assumed. This error currently prevents the necessary rectification of general relativity that leads to the unification of gravitation and electromagnetism. Also, similar errors have been made by D. Hilbert and M. Atiyah.

Keywords: stable solution; dynamic solution.

I. Introduction

In mathematics, it is commonly known that an assertion can be either right or wrong. However, in logic, there is actually a third case that the conditions in a theorem are valid in mathematics but some implicit assumption is not generally valid. Thus, the theorem is not simply right or wrong, but misleading. In fact, such an error can be made by top mathematicians such as M. Atiyah [1)] and consequently such misleading errors in mathematics were cited as a main reason to award the 1982 and the 1990 Fields Medal to Yau and Witten [2)] and to award the 2011 Shaw Prize in mathematics to Christodoulou. [3)] To this end, the Positive Energy Theorem of Yau and Schoen [1, 2] for general relativity is an example. Briefly, the positive mass conjecture says that if a three-dimensional manifold has positive scalar curvature and is asymptotically flat, then the mass in the asymptotic expansion of the metric is positive (Wikipedia). As in the space-time singularity theorems, the unique coupling signs are also implicitly used in the positive energy theorem of Schoen and Yau [1, 2]. A crucial assumption in the theorem of Schoen and Yau is that the solution is asymptotically flat. To be more specific, they [1] requires the metric,

$$g_{ij} = \delta_{ij} + O(r^{-1}). \tag{1}$$

The motivation of (1) is clearly the linearized equation of Einstein (see eq. (C3) in Appendix C). Moreover, such an assumption can be considered as common in physics since this condition is satisfied in stable solutions such as the Schwarzschild solution, the harmonic solution, the Kerr solution, etc. [4)]

Thus, it could be "natural" to assert (as in Wikipedia) that their proof of the positive energy theorem in general relativity demonstrated—sixty years after its

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discovery—that Einstein's theory is consistent and stable. However, if one understands the physics in general relativity as well as Gullstrand, the Chairman (1922-1929) of the Nobel Prize Committee for Physics does, the above statement is clearly incorrect [3-5]. Note that the condition of asymptotically flat does not necessarily imply the inclusion of a dynamic solution. Apparently, Schoen and Yau assumed it did because they failed to see that, for a dynamic case the linearized equation and the non-linear Einstein equation are not compatible [6].

However, it has been proven that the Einstein equation has no dynamic solution, which is bounded [3-5]. Thus, the assumption of asymptotically flat implies the exclusion of the most important class of solutions, the dynamic solutions. However, the notion of a dynamic solution was not critical to mathematicians Schoen and Yau. So, they have not considered such a problem in their theorem. Therefore, they actually prove a trivial result that the total mass of a static (or stable) solution is positive. In other words, the conclusions drawn from the positive theorem are grossly misleading. This illustrates that an inadequate understanding in physics can lead to beautiful, but actually completely invalid statements in physics.

The problem rises from the Einstein equation that does not have a bounded dynamic solution as Gullstrand suspected [7]. Thus, Yau and Schoen used an implicit assumption, the existence of bounded dynamic solutions, which is actually false but was not stated in their theorem. Similarly, Witten is also essentially a mathematician because his major concern is self-consistency instead of agreement with observation. Thus he also overlooked the problem of the dynamic solutions. Moreover, Atiyah, being a pure mathematician, was not aware of the problem of non-existence of bounded dynamic solutions. Thus, one should not be surprised that such an error was made twice over eight years (1982-1990) by the Fields medal. Note that the proof for the nonexistence of a bounded dynamic solution was published in 2000 [4].[5)]

It should be noted that D. Hilbert also made a mistake on approving Einstein's calculation of the perihelion of Mercury because he was not aware that this calculation requires a bounded solution of the many-body problem [7]. However, Hilbert was lucky because he understood that the related Einstein's calculation is not valid, but Atiyah was not as lucky. Nevertheless, because of Atiyah's reputation as an outstanding mathematician, some journals such as Nature would not dare to criticize him.

In fact theorists such as Yau [1], Christodoulou [8], Wald, and Penrose & Hawking [9] make essentially the same error of defining a set of solutions that actually includes no dynamic solutions [10-13]. The fatal error is that they neglected to find explicit examples to support their claims. Had they tried, they should have discovered their errors. Moreover, the same error [5] was cited in awarding to Christodoulou the 2011 Shaw Prize.[3), 6)] Subsequently, Christodoulou was elected to the Member of U.S. National Academy of Sciences (2012). It would be interesting to see how this special case would end up. The problem of Christodoulou represents an accumulation of long standing errors committed by the top mathematicians and physicists.

The non-existence of a bounded dynamic solution for the Einstein equation was not recognized because they did not try to obtain such a solution. Thus the need of modifying the Einstein equation with an additional gravitational energy-stress tensor with the anti-gravitational coupling as the source was overlooked [3]. Then, the energy-mass formula $E = mc^2$ was still incorrectly considered as unconditionally valid [12]. Consequently, the charge-mass interaction was not only overlooked, but also explicitly denied by Einstein and his colleagues. Hence, the need of unification between gravitation and

Ref

3. C. Y. Lo, Astrophys. J. 455, 421-428 (1995); Editor S. Chandrasekhar, a Nobel Laureate, suggests and approves the Appendix therein.

electromagnetism is missed [14]. Thus, the positive mass theorem is actually an obstacle for the progress in physics.

An urgent problem is that Misner, Thorne. & Wheeler [15] used the errors of Yau [1] and Witten [2] to strengthen their incorrect claim on the existence of bounded dynamic solutions. For instance, they incorrectly claim that for their eq. (35. 31), $L'' + (\beta')^2 L = 0$, there are dynamic solutions without a proof (see Appendix A). If such an error was overlooked, one could easily fall into agreeing with the other errors [16].

Ref

14. C. Y. Lo, Phys. Essays, **23** (2), 258-267 (2010).

After P. Morrison passed away, general relativity at MIT is dominated by the Wheeler School whose errors are in the open courses Phys 8.033 and Phys 8.962. Although E. Bertschinger and Scott A. Hughes studied the linearized equation of the Einstein equation, they failed to understand that for the dynamic case, the non-linear Einstein equation and its linearized equation do not have any compatible solutions [3-6]. Apparently, they failed to see that this process of linearization is not valid in mathematics [6]. Moreover, Max Tegmark even failed to tell the different between mathematics and physics [16]. Thus, in the Physics Department of MIT, currently nobody understands the basic essence of general relativity, and has up-to-dated knowledge.

Moreover, MIT is not the only victim among universities because of the influences of the Wheeler School [5, 14]. Thus, it is necessary to point out their errors with a paper, [7] such that it is clearly understood that Fields Medals, 1982 to S. T. Yau and 1990 to E. Witten [8] were misleading. Different from a mathematician, a physicist usually understands the problem of dynamics and the principle of causality [3, 4].

Currently, mathematicians are often being considered in terms of a hierarchy system.[9] However, such a practice would result in errors in mathematics not being corrected. This article attempts to break such a practice by showing that the current top mathematicians can also make an elementary mistake just as Hilbert did because of inadequate consideration in physics (see also Appendices A, B, C).

II. Acknowledgments

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Appendix A: Invalidity of Linearization for the Dynamic case & the Principle of Causality

The earliest reference of the definite non-existence of dynamics solution for the Einstein equation is probably the 1953 thesis of J. E. Hogarth [13], who conjectured that, for an exact solution of the two-particle problem, the energy-momentum tensor did not vanish in the surrounding space and would represent the energy of gravitational radiation. In 1995 and subsequently, it is proven that this is indeed the case [3].

Historically, Einstein and Rosen were the first that questioned the existence of a wave solution [17] because they found a singularity in such a solution. However, the Physical Review shows that such a singularity is removable, and thus claimed a wave solution does exist because they failed to see that a wave solution (or a dynamic solution) must be bounded in amplitude according to the principle of causality [9]. Thus, it is clear that this boundedness is needed for a dynamic solution.

A1. Errors of Misner, Thorne, & Wheeler

An example is that Misner et al. [15] claimed that there is a bounded wave solution of the form,

$$ds^2 = c^2dt^2 - dx^2 - L^2\left(e^{2\beta}dy^2 + e^{-2\beta}dz^2\right) \quad \text{(A1)}$$

where $L = L(u)$, $\beta = \beta\ (u)$, $u = ct - x$, and c is the light speed. Then, the Einstein equation $G_{\mu\nu} = 0$ becomes

$$\frac{d^2L}{du^2} + L\left(\frac{d\beta}{du}\right)^2 = 0 \quad \text{(A2)}$$

They claimed that Eq. (A2) has a bounded solution, compatible with a linearization of metric (A1). It has been shown with mathematics at the undergraduate level that Misner et al. are incorrect [12, 16] and Eq. (A2) does not have a physical solution that satisfies Einstein's requirement on weak gravity.

Misner et al. [15] claimed that Eq. (A2) has a bounded solution, compatible with a linearization of metric (A1). Such a claim is in conflict with the non-existence of dynamic solutions [3, 4]. They further claimed,

"The linearized version of L″ = 0 since $(\beta')^2$ is a second–order quantity.

Therefore the solution corresponding to linearized theory is

$$L = 1, \quad \beta(u) \text{ arbitrary but small.} \quad \text{(A3)}$$

The corresponding metric is

$$ds^2 = (1 + 2\beta)dx^2 + (1 - 2\beta)dy^2 + dz^2 - dt^2, \qquad \beta = \beta(t\text{-}z)." \quad \text{(A4)}$$

However, these claims are actually incorrect. In fact, *L(u)* is unbounded even for a very small $\beta\ (u)$. *It should be noted that their book* [15] *includes also factual errors, in addition to a misrepresentation of Einstein* [16].

Linearization of (A2) yields $L'' = 0$, and in turn this leads to $\beta'\ (u) = 0$. Thus, this leads to a solution $L = C_1u + C_2$ where C_1 & C_2 are constants. Therefore, the requirement $L \approx 1$ implies $C_1 = 0$. *However,* $\beta'\ (u) = 0$ *implies* $\beta\ (u)$*=constant, i.e. no waves. Thus, metric (6) is not derived, but only claimed.*

To prove Eq. (A2) having no wave solution, it is sufficient to consider the case of weak gravity. According to Einstein, for weak gravity of metric (A1), one would have

$$L^2e^{2\beta} \cong 1 \quad \text{and} \quad L^2e^{-2\beta} \cong 1 \quad \text{(A5a)}$$

It follows that

$$L^4 \cong 1, \quad e^{\pm 2\beta} \cong 1 \quad \text{and} \quad L(u) >> |\beta(u)| \quad \text{(A5b)}$$

Since *L(u)* is bounded, *L′(u)* cannot be a monotonic function of u, unless L′–¿ 0. Thus, there is an interval of *u* such that the average,

$$\langle L'' \rangle = 0 \quad \text{(A6)}$$

Ref

15. C. W. Misner, K. S. Thorne, and J. A. Wheeler, ***Gravitation*** (W. H. Freeman, San Francisco, 1973).

On the other hand, the average of the second term of equation (A2) is always larger than zero unless $\beta'(u) = 0$ in the whole interval of u.

Also, from eq. (A2), one would obtain L ($\cong 1$) > 0, and one has $0 > L''(u)$ *if* $\beta'(u) \neq 0$. Thus, $- L'(u)$ is a monotonic increasing function in any finite interval of u since $\beta'(u) = 0$ means $L'' = 0$, i.e., no wave. In turn, *since* $\beta'(u)$ *is a "wave factor",* this implies that $L(u)$ is an unbounded function of u. Therefore, this would contradict the requirement that L is bounded. In other words, eq. (A2) does not have a bounded wave solution. Moreover, the second order term L'' would give a very large term to L, after integration.

Notes

Now, let us investigate the errors of Misner et al. [15; p. 958]. Their assumption is that the signal $\beta(u)$ has duration of $2T$. For simplicity, it is assumed that definitely $|\beta'(u)| = \delta$ in the period $2T$. Before the arrival of the signal at $u = x$, one has

$$L(u) = 1, \quad \text{and} \quad \beta(u) = 0 \tag{A7}$$

If the assumption of weak gravity is compatible with Eq. (A2), one would have L(u) ≅ 1. It thus follows one has

$$L'(u) = 0 - \int_x^u \beta'^2 dy \approx -\int_x^u \delta^2 dy = \delta^2 (u - x) \quad \text{for} \quad x + 2T > u > x,$$

$$\text{or} \quad \approx -\delta^2 2T \qquad \text{for} \quad u > x + 2T \tag{A8}$$

Hence

$$\left. \begin{aligned} L(u) &= 1 + \int_x^u L' dy \\ &\approx 1 - \int_x^u \delta^2 (y - x) dy = 1 - \frac{\delta^2 (u - x)^2}{2} \end{aligned} \right) \text{for} \quad x + 2T > u > x$$

$$\text{or} \quad \left. \begin{aligned} &\approx 1 - \int_x^{x+2T} \delta^2 (y - x) dy - \delta^2 2T \int_{x+2T}^u dy \\ &= 1 - \delta^2 2T(u - T - x) \end{aligned} \right) \quad \text{for} \quad u > x + 2T \tag{A9}$$

Thus, independent of the smallness of $2\delta^2 T$ (or details of $|\beta'(u)|^2$), L could be approximately zero and violates the condition for weak gravity. Thus, eq. (A2) has no weak wave solution. Moreover, —$L(u)$— is not bounded since it would become very large as u increases. Thus, restriction of $2\delta^2 T$ being small [15] does not help.

Thus, one can get a no wave solution through linearization of Eq. (A2), which has no bounded solution. The assumption of metric form (A1) is bounded [15], and has a weak form (A4), is not valid. Thus, there is no bounded wave solution for the non-linear Einstein equation, which violates the principle of causality.

The root of their errors was that they incorrectly assumed that a linearization of the Einstein non-linear equation would produce a valid approximation. Thus, they implicitly and incorrectly assume the existence of a bounded wave solution without the necessary verification, and thus obtain incorrect conclusions.

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On the other hand, from the linearization of the Einstein equation (Maxwell-Newton approximation) in vacuum, Einstein [18] obtained a solution independently as follows:

$$ds^2 = c^2dt^2 - dx^2 - (1+2\phi)dy^2 - (1-2\phi)dz^2 \qquad \text{(A10)}$$

where ϕ is a bounded function of u $(= ct - x)$. Note that metric (A10) is the linearization of metric (A1) if $\phi = \beta\ (u)$. Thus, the problem of waves illustrates that the linearization may not be valid for the dynamic case when gravitational waves are involved since eq. (A2) does not have a weak wave solution.

The error of Misner et al. is clearly due to the combination of a blind faith on the Einstein equation and inadequacy in mathematics at the undergraduate level. Such a blind faith is often shown in the literature.

A2. Errors of Wald

According to Einstein [19], in general relativity weak sources would produce a weak field, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \text{ where } |\gamma_{\mu\nu}| << 1 \qquad \text{(A11)}$$

and $\eta_{\mu\nu}$ is the flat metric when there is no source. However, this is true only if the equation is valid in physics. Many theorists failed to see this because they failed to see the difference between physics and mathematics clearly [14]. When the Einstein equation has a weak solution, an approximate weak solution can be derived through the approach of the field equation being linearized. However, the non-linear equation may not have a bounded solution. The linearized Einstein equation with the linearized harmonic gauge

$$\partial^\mu \bar{\gamma}_{\mu\nu} = 0 \text{ is } \quad \frac{1}{2}\partial^\alpha\partial_\alpha \bar{\gamma}_{\mu\nu} = \kappa T_{\mu\nu} \quad \text{where } \bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma \quad \text{and } \gamma = \eta^{\alpha\beta}\gamma_{\alpha\beta} \qquad \text{(A12)}$$

Note that we have

$$G_{\mu\nu} = G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(2)} \text{ and } G^{(1)}_{\mu\nu} = \frac{1}{2}\partial^\alpha\partial_\alpha \bar{\gamma}_{\mu\nu} - \partial^\alpha\partial_\mu \bar{\gamma}_{\nu\alpha} - \partial^\alpha\partial_\nu \bar{\gamma}_{\mu\alpha} + \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{\gamma}_{\alpha\beta} \qquad \text{(A13)}$$

The linearized vacuum Einstein equation means

$$G^{(1)}_{\mu\nu}[\gamma^{(1)}_{\alpha\beta}] = 0 \qquad \text{(A14)}$$

Thus, as pointed out by Wald [9], in order to maintain a solution of the vacuum Einstein equation to second order we must correct $\gamma^{(1)}{}_{\mu\nu}$ by adding to it the term $\gamma^{(2)}{}_{\mu\nu}$, where $\gamma^{(2)}{}_{\mu\nu}$ satisfies

$$G^{(1)}_{\mu\nu}[\gamma^{(2)}{}_{\alpha\beta}] + G^{(2)}_{\mu\nu}\left[\gamma_{\alpha\beta}\right] = 0\,, \qquad \text{where } \gamma_{\mu\nu} = \gamma^{(1)}{}_{\mu\nu} + \gamma^{(2)}{}_{\mu\nu} \qquad \text{(A15)}$$

which is the correct form of eq. (4.4.52) in Wald's book. (In Wald's book, he did not distinguish $\gamma_{\mu\nu}$ from $\gamma^{(1)}{}_{\mu\nu}$) This equation does have a solution for the static case. However, detailed calculation shows that this equation does not have a solution for the dynamic case [3, 14]. The fact that there is no bounded solution for eq. (A15) a dynamic case means also that the Einstein equation does not have a dynamic solution.

Ref

18. A. Einstein, Sitzungsberi, Preuss, Acad. Wis. 1918, 1: 154 (1918).

For instance, a well-known example is the metric of Bondi, Pirani, & Robinson [20] as follows:

$$ds^2 = e^{2\varphi}\left(d\tau^2 - d\xi^2\right) - u^2\begin{bmatrix}\cosh 2\beta\left(d\eta^2 + d\varsigma^2\right) \\ +\sinh 2\beta\cos 2\theta\left(d\eta^2 - d\varsigma^2\right) \\ -2\sinh 2\beta\sin 2\theta\, d\eta\, d\varsigma\end{bmatrix} \qquad \text{(A16a)}$$

Ref

where ϕ, β and θ are functions of u $(= \tau - \xi)$. It satisfies the equation (i.e., their Eq. [2.8]),

$$2\phi' = u\left(\beta'^2 + \theta'^2 \sinh^2 2\beta\right). \qquad \text{(A16b)}$$

Eq. (A16b) implies ϕ *cannot be a periodic function.* The metric is irreducibly unbounded because of the factor u^2. Both eq. (A2) and eq. (A16b) are special cases of $G_{\mu\nu} = 0$. However, linearization of (A16b) does not make sense since variable u is not bounded. Thus, they incorrectly claim Einstein's notion of weak gravity invalid because they do not understand the principle of causality adequately.
Moreover, when gravity is absent, it is necessary to reduce (A16a) to

$$ds^2 = \left(d\tau^2 - d\xi^2\right) - u^2\left(d\eta^2 - d\zeta^2\right) \qquad \text{(A16c)}$$

because $\phi = \sinh 2\beta = \sin 2\theta = 0$. However, this metric is not equivalent to the flat metric, and thus violates the principle of causality. Also it is impossible to adjust metric (A16a) to become equivalent to the flat metric.

This challenges the view that both Einstein's notion of weak gravity and his covariance principle are valid. These conflicting views are supported respectively by the editorials of the "Royal Society Proceedings A" and the "Physical Review D"; thus there is no general consensus. As the Royal Society correctly pointed out [21], Einstein's notion of weak gravity is inconsistent with his covariance principle. In fact, Einstein's covariance principle has been proven invalid by counter examples [22, 23].

Due to confusion between mathematics and physics, Wald [9] made errors in mathematics at the undergraduate level. Wald did not see that the Einstein equation can fail the principle of causality. The principle of causality requires the existence of a dynamic solution, but Wald did not see that the Einstein equation can fail this requirement. Thus, his theory does not include the dynamic solutions [3-5].

A3. The Principle of Causality

There are other theorists who also ignore the principle of causality. For example, another "plane wave", which is intrinsically non-physical, is the metric accepted by Penrose [24] as follows:

$$ds^2 = du\, dv + H du^2 - dx_i dx_i\,, \quad \text{where } H = h_{ij}(u) x_i x_j \text{ and } u = ct - z\,, v = ct + z\,. \qquad \text{(A17)}$$

However, there are arbitrary non-physical parameters (the choice of origin) that are unrelated to any physical causes. Being essentially only a mathematician, Penrose [24] naturally over-looked the principle of causality.

20. H. Bondi, F. A. E. Pirani, & I. Robinson, Proc. R. Soc. London A **251**, 519-533 (1959).

Also, the plane wave solution of Liu & Zhou [25], which satisfies the harmonic gauge, is as follows:

$$ds^2 = dt^2 - dx^2 + 2\,F(dt - dx)^2 - \cosh 2\psi(e^{2\phi}\,dy^2 + e^{-2\phi}dz^2) - 2\sinh 2\psi\,dy\,dz. \qquad (A18)$$

where $\phi = \phi(u)$ and $\psi = \psi(u)$. Moreover, $F = F_P + H$, where

$$F_P = \frac{1}{2}(\dot{\psi}^2 + \dot{\phi}^2\cosh^2 2\psi)\,[\cosh 2\psi\,(e^{2\phi}\,y^2 + e^{-2\phi}\,z^2) + 2\sinh 2\phi\,yz], \qquad (A19)$$

and H satisfies the equation,

$$\cosh 2\psi\,(e^{-2\phi}H_{,22} + e^{2\phi}\,H_{,33}) - 2\sinh 2\psi\,H_{,23} = 0. \qquad (A20)$$

For the weak fields one has $1 >> |\phi|$, $1 >> |\psi|$, but there is no weak approximation as claimed to be

$$ds^2 = dt^2 - dx^2 - (1 + 2\phi)\,dy^2 - (1 - 2\phi)dz^2 - 4\psi dydz \qquad (A21)$$

because F_p is not bounded unless $\dot{\phi}$ and $\dot{\psi}$ are zero (i.e., no wave).

A4. Other Supporting Evidence and Conclusions

Moreover, there is no bounded wave solution in the literature. The reason is later identified as the missing of a gravitational energy-momentum tensor with a coupling constant of different sign [3, 11]. An independent convincing evidence for the absence of a bounded dynamic solution is, as shown by Hu, Zhang & Ting [26], that gravitational radiation calculated would depend on the approach used. This is also a manifestation that there is no bounded solution. A similar problem in approximation schemes such as post-Newtonian approximation [8, 14, 27] is that their validity is also only assumed.

The linearized equation for a dynamic case has been illustrated as incompatible with the non-linear Einstein equation. Thus, Eq. (A2), Eq. (A16b), and Eq. (A19) serve as good simple examples that can be shown through explicit calculation that linearization of the Einstein equation is not valid. Also, metric (A17) suggests that the cause of having no physical solution would be due to inadequate source terms [3, 26, 28].

Appendix B: The So-called Space-time Singularity Theorems and the Speculation of $E = mc^2$

A surprising conclusion, from the investigation of the Einstein equation, is that the space-time singularity theorems of Penrose and Hawking are actually irrelevant to physics. This is so because their theorems have a common implicit assumption that all the couplings have the same sign. However, from the investigation of dynamic solutions, such an assumption is necessarily invalid in physics [3, 29] because it implies no dynamic solution. These theorems were accepted because Penrose won the arguments against a Russian scientist E. M. Lifshitz who claimed, with the same set of assumptions, that there is no space-time singularity [30]. However, the problem is not the mathematics in the theorems, but the earlier historical errors in mathematics and physics.

As Pauli [31] pointed out, in principle general relativity can have different signs for their coupling constants. The fact that nobody questioned the assumption of unique sign for all coupling, is probably due to the unverified speculation of formula $E = mc^2$ being generally true. This formula comes from special relativity, and the conversion of

Ref

25. Liu H. Y., & Zhou, P.-Y.: Scientia Sincia (Series A) 1985, XXVIII (6) 628-637.

some mass to various combinations of energy is verified by the fission and fusion in nuclear physics. However, the conversion of a single type of energy to mass actually has never been verified [19], but this is currently proven as the invalid main speculation.

Einstein and theorists have shown that the photons can be converted into mass thorough absorption [32]. This conversion is supported by the fact that the π_0 meson can be decayed into two photons. Thus, it was claimed that the electromagnetic energy can be converted into mass because they failed to see that the photons must have non-electromagnetic energy. When Einstein proposed the notion of photons, he had not conceived general relativity yet. Thus, understandably he neglected the gravitational component of light. However, after general relativity, a light ray consists of a gravitational component is natural because the electron has a mass. Besides, the electromagnetic energy-momentum tensor has a zero trace. In fact, Einstein failed to show the general validity of $E = mc^2$ in spite of several years effort [33]. Experimentally, in contrast of Einstein's claim, $E = mc^2$ is not always valid because a piece of heated up metal has reduced weight [34].

Physically the dynamic solution must exist for a rectified equation. A problem of the Einstein equation is that it does not include the gravitational energy-stress tensor of its gravitational waves in the source and thus the principle of causality is violated. Since a gravitational wave carried energy-momentum and the source of gravity is the energy-stress tensors, as Hogarth [13] pointed out, the presence of a non-zero energy-momentum in the source is necessary for a gravitational wave. Thus, to fit the Hulse-Taylor data of the binary pulsar, it is necessary [3] to modify the Einstein equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T(m)_{\mu\nu} \qquad \text{(B1a)}$$

to

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa \left[T(m)_{\mu\nu} - t(g)_{\mu\nu} \right] \qquad \text{(B1b)}$$

where $t(g)_{\mu\nu}$ is the energy-stress tensor for gravity. For radiation, the tensor $t(g)_{\mu\nu}$ is equivalent to Einstein's notion of the gravitational energy-stress. However, his notion is a pseudo-tensor and can become zero by choosing a suitable coordinate system, but the energy-momentum of a radiation cannot be zero [3].

Moreover, the geodesic equation is not the exact equation of motion for a particle because the radiation reaction force is not included. Moreover, the mass-charge interaction is only partially involved. Thus, general relativity is clearly not yet a complete theory [35].

It is crucial to note that for the existence of a dynamic solution, the new tensor necessarily has a different sign for its coupling [3]. Thus, the implicit assumption of Penrose and Hawking is proven necessarily invalid. Note that the absence of a dynamic solution and the presence of space-time singularities are related to the same invalid assumption. It is the long standing bias and errors in mathematics that some theorists accepted one but rejected the other. Other victims are the positive mass theorem of Yau [1] and Witten [2] because they used the same invalid implicit assumption as Hawking and Penrose.

Ref

35. C. Y. Lo, On the Nobel Prize in Physics, Controversies and Influences, GJSFR Vol. 13-A Issue 3 Version 1.0, 59-73 (June 2013).

Appendix C: The Necessity of the Maxwell-Newton Approximation

A problem in general relativity [3] is that, for a dynamic case, there is no bounded solution,

$$|g_{ab}(x, y, z, t)| < \text{constant}, \quad \text{(C1)}$$

for the Einstein equation, where g_{ab} is the space-time metric. In fact, eq. (C1) is also a necessary implicit assumption in Einstein's radiation formula [27] and the light bending [28]. One might argue that requirement (C1) violates the covariance principle. However, the covariant principle is proven invalid in physics [36]. Moreover, Einstein's notion of weak gravity [19] is also in agreement with the principle of causality. It will be shown that weak gravity is also compatible with Einstein's equivalence principle.

The question of dynamic solutions was raised by Gullstrand [37]. He challenged Einstein and also D. Hilbert who approved Einstein's calculations [7]. However, Hilbert did not participate in the subsequent defense and he would probably have seen the deficiency. Nevertheless, theorists such as Christodoulou & Klainerman [8], Misner et al. [15] and Wald [9] etc. failed to see this, and tried very hard to prove otherwise.

The failure of producing a dynamic solution would cast a strong doubt to the validity of the linearized equation that produces many effects including the gravitational waves. In fact, for the case that the source is an electromagnetic plane wave, the linearized equation actually does not have a bounded solution [38].

Nevertheless, when the sources are massive, some of such results from the linearized equation have been verified by observation. Thus, there must be a way to justify the linearized equation, independently. To this end, Einstein's equivalence principle [29] is needed, although rejected by the 1993 Nobel Prize Committee for Physics implicitly [39]. As a result, it becomes even clearer that the non-existence of a bounded dynamic solution for massive sources is due to a violation of the principle of causality [12].

C1. Gravitational Waves and the Einstein Equation of 1915

Relativity requires the existence of gravitational waves because physical influence must be propagated with a finite speed [40]. To this end, let us consider the Einstein equation of 1915 [19], which is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T(m)_{\mu\nu} . \quad \text{(C2)}$$

Einstein believed that his equation satisfied this requirement since its linearized "approximation" gives a wave.

The linearized equation with massive sources [19] is the Maxwell-Newton Approximation [3],

$$\frac{1}{2} \partial_c \partial^c \bar{\gamma}_{ab} = -\kappa T(m)_{ab} \quad \text{(C3a)}$$

where $\bar{\gamma}_{ab} = \gamma_{ab} - (1/2)\eta_{ab}$, $\gamma_{ab} = g_{ab} - \eta_{ab}$, $\gamma = \eta^{cd} \gamma_{cd}$, and η_{ab} is the flat metric. Eq. (C3a) has a mathematical structure similar to that of Maxwell's equations. A solution of eq. (C3a) is

$$\bar{\gamma}_{ab}(x_i,t) = -\frac{\kappa}{2\pi} \int \frac{1}{R} T_{ab}[y^i, (t-R)]\, d^3y, \quad \text{where } R^2 = \sum_{i=1}^{3} \left(x^i - y^i\right)^2 \quad \text{(C3b)}$$

40. H. A. Lorentz, Proc. K. Ak. Amsterdam 8, 603 (1900).

Note that the Schwarzschild solution, after a gauge transformation, can also be approximated by (C3b). Solution (C3b) would represent a wave if T_{ab} has a dynamical dependency on time t' (= t − R). Thus, the theoretical existence of gravitational waves seems to be assured as a certainty as believed [27, 31, 41].

However, for non-linear equations, the physical second order terms can be crucial for the mathematical existence of bounded solutions. For the Einstein equation, the Cauchy initial condition is restricted by four constraints since there is no second order time derivatives in G_{at} (a = x, y, z, t) [27]. This suggests that the Einstein equation (C2) and (C3) may not be compatible for a dynamic problem. Einstein discovered that his equation does not admit a propagating wave solution [42, 43]. Recently, it has been shown that the linearization procedure is not generally valid [3, 44]. Thus, it is necessary to justify wave solution (A18) independently since it is the basis of Einstein's radiation formula.

Ref

27. S. Weinberg, "Gravitation and Cosmology" (John Wiley, New York, 1972), pp. 69, 251-273.

C2. The Weak Gravity of Massive Matter and Einstein Equation of the 1995 Update

For a massive source, the linear equation (C3), as a first order approximation, is supported by experiments [3, 27]. However, for the dynamic case, the Einstein equation is clearly invalid.

It will be shown that eq. (C3a) can be derived from Einstein's equivalence principle. Based on this, the equation of motion for a neutral particle is the geodesic equation. In comparison with Newton's second law, one obtains that the Newtonian potential of gravity is approximately $c^2g_{tt}/2$. Then, in accord with the Poisson equation and special relativity, the most general equation for the first order approximation of g_{ab} is,

$$\frac{1}{2}\partial_c\partial^c\gamma_{ab} = -\frac{\kappa}{2}[\alpha T(m)_{ab} + \beta\hat{T}(m)\eta_{ab}], \qquad \text{(C4a)}$$

where

$$\hat{T}(m) = \eta^{cd}T(m)_{cd}, \quad \kappa = 8\pi Kc^{-2}, \quad \text{and} \quad \alpha + \beta = 1, \qquad \text{(C4b)}$$

where α and β are constants since Newton's theory is not gauge invariant.

Then, according to Riemannian geometry [27], the exact equation would be

$$R_{ab} + X^{(2)}{}_{ab} = -\frac{\kappa}{2}[\alpha T(m)_{ab} + \beta T(m)g_{ab}], \quad \text{where } T(m) = g^{cd}T(m)_{cd} \qquad \text{(C5a)}$$

and $X^{(2)}{}_{ab}$ is an unknown tensor of second order in K, if R_{ab} consists of no net sum of first order other than the term $(1/2)\,\partial_c\partial^c\gamma_{ab}$. This requires that the sum

$$-\frac{1}{2}\partial^c[\partial_b\gamma_{ac} + \partial_a\gamma_{bc}] + \frac{1}{2}\partial_a\partial_b\gamma, \qquad \text{(C5b)}$$

must be of second order. To this end, let us consider eq. (C4a), and obtain

$$\frac{1}{2}\partial_c\partial^c(\partial^a\gamma_{ab}) = -\frac{\kappa}{2}[\alpha\partial^a T(m)_{ab} + \beta\partial_b\hat{T}(m)]. \qquad \text{(C6a)}$$

From $\nabla^c T(m)_{cb} = 0$, it is clear that K $\partial^c T(m)_{cb}$ is of second order but $K\partial_b \hat{T}(m)$ is not. However, one may obtain a second order term by a suitable linear combination of $\nabla^c \gamma_{cb}$ and $\partial_b \gamma$. From (A6a), one has

$$\frac{1}{2}\partial_c \partial^c (\partial^a \gamma_{ab} + C\,\partial_b \gamma) = -\frac{\kappa}{2}[\alpha \partial^a T(m)_{ab} + (\beta + 4C\beta + C\alpha)\partial_b \hat{T}(m)]. \quad \text{(C6b)}$$

Thus, the harmonic coordinates (i.e., $\partial^a \gamma_{ab} - \partial_b \gamma/2 \approx 0$), can lead to inconsistency. It follows eqs. (C5b) and (C6b) that, for the other terms to be of second order, one must have C = -1/2, α = 2, and β = -1.
Hence, eq. (C4a) becomes,

$$\frac{1}{2}\partial_c \partial^c \gamma_{ab} = -\kappa[T(m)_{ab} - \frac{1}{2}\hat{T}(m)\,\eta_{ab}]. \quad \text{(C7)}$$

which is equivalent to eq. (C3a), has been determined to be the field equation of massive matter. This derivation is independent of the exact form of equation (C5a). The implicit gauge condition is that the flat metric η_{ab} is the asymptotic limit. Eq. (C7) is compatible with the equivalence principle as demonstrated by Einstein in his calculation of the bending of light. Thus, the derivation is self-consistent.

Einstein obtained the same values for α and β by considering eq. (C5a) after assuming $X^{(2)}_{ab} = 0$. An advantage of the approach of considering eq. (C4) and eq. (C5b) is that the over simplification $X^{(2)}_{ab} = 0$ is not needed. Then, it is possible to obtain from eq. (C5a) an equation different from (C2),

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R = -\kappa\,[T(m)_{ab} - Y^{(1)}{}_{ab}], \quad \text{(C8)}$$

where

$$-\kappa Y^{(1)}{}_{ab} = X^{(2)}{}_{ab} - \frac{1}{2}g_{ab}\{X^{(2)}{}_{cd}\,g^{cd}\}.$$

The conservation law $\nabla^c T(m)_{cb} = 0$ and $\nabla^c G_{cb} \equiv 0$ implies also $\nabla^a Y^{(1)}{}_{ab} = 0$. If $Y^{(1)}{}_{ab}$ is identified as the gravitational energy tensor of $t(g)_{ab}$, Einstein equation of the 1995 update [3] is reaffirmed. Note that eq. (C3a) is the first order approximation of eq. (C8) but may not be of (C2). Note, however, that in Einstein's initial consideration, $t(g)_{ab}$ is a pseudo-tensor. It has been shown that it must be a tensor [3].

Endnotes:

1) Michael Francis Atiyah has been president of the Royal Society (1990-1995), master of *Trinity College, Cambridge* (1990-1997), chancellor of the *University of Leicester* (1995-2005), and President of the *Royal Society of Edinburgh* (2005-2008). Since 1997, he has been an honorary professor at the *University of Edinburgh* (Wikipedia).
2) Ludwig D. Faddeev, the Chairman of the Fields Medal Committee, wrote ("On the work of Edward Witten"):

"Now I turn to another beautiful result of Witten – proof of positivity of energy in Einstein's theory of gravitation.

Hamiltonian approach to this theory proposed by Dirac in the beginning of the fifties and developed further by many people has led to the natural definition of energy. In

Ref

8. D. Christodoulou & S. Klainerman, ***The Global Nonlinear Stability of the Minkowski Space*** (Princeton. Univ. Press, 1993); No. 42 of the Princeton Mathematical Series.

this approach a metric γ and external curvature h on a space-like initial surface $S^{(3)}$ embedded in space-time $M^{(4)}$ are used as parameters in the corresponding phase space. These data are not independent. They satisfy Gauss-Codazzi constraints – highly non-linear PDE, The energy H in the asymptotically flat case is given as an integral of indefinite quadratic form of $\nabla\gamma$ and h. Thus, it is not manifestly positive. The important statement that it is nevertheless positive may be proved only by taking into the account the constraints – a formidable problem solved by Yau and Schoen in the late seventy as Atiyah mentions, 'leading in part to Yau's Fields Medal at the Warsaw Congress'.

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Witten proposed an alternative expression for energy in terms of solutions of a linear PDE with the coefficients expressed through γ and h ....."

3) The 2011 Shaw Prize also made a mistake by awarding a half prize to Christodoulou for his work, based on obscure errors, against the honorable Gullstrand [37] of the Nobel Committee. Although Christodoulou has misled many including the 1993 Nobel Committee [39], his errors are now well-established and they have been illustrated with mathematics at the undergraduate level [5]. Christodoulou claimed in his autobiography that his work is essentially based on two sources: 1) The claims of Christodoulou and Klainerman on general relativity as shown in their book ***The Global Nonlinear Stability of the Minkowski Space*** [8]; 2) Roger Penrose had introduced, in 1965, the concept of a trapped surface and had proved that a space-time containing such a surface cannot be complete [9]. However, this work of Penrose, which uses an implicit assumption of unique sign for all coupling constants, actually depends on the errors of Christodoulou and Klainerman [8]. However, such a relation was not clear until 1995 [3] (see Appendix B).
4) These solutions have no gravitational radiation.
5) At MIT, only P. Morrison surely read the proof for the non-existence of a dynamic solution. Apparently, Yau probably did not read such a proof since his interest is no longer in general relativity since 1993 [8].
6) M. Atiyah was in the 2011 Selection Committee for the Shaw Prize in Mathematics Sciences.
7) MIT President Reif would be able to do little without our help to counter his incompetent subordinates [45], who disobey his directive of communication because of their out-dated knowledge [45, 46].
8) E. Witten is a leader of string theorists. Thus, his error in general relativity represents a common deficiency.
9) Thus, many journals just decline to consider a critical article as this since Atiyah is a well-known mathematician and was the President of the Royal Society (1990-1995). The intention is to avoid his mistake in physics becoming an embarrassment to the scientific community. Moreover, the schools also have an informal hierarchy system. For instance, MIT would decline to think Harvard University could be wrong. In spite of an eloquent speech of the MIT President Reif on basic research, so far no MIT professor has made a single move to correct the errors of Harvard professor S. T. Yau [1, 2].

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Con-s-k-EP Generalized Inverses

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Abstract- In this paper, equivalent conditions for various generalized inverses of a con-s-k-EPr matrix to be con-s-k-EPr are determined. Generalized inverses belonging to the sets A{1,2}, A{1,2,3} and A{1,2,4} of a con-s-k-EPr matrix A are characterized.

Keywords: *Con-s-k-EP matrix, generalized inverse.*

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Con-s-k-EP Generalized Inverses

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Abstract- In this paper, equivalent conditions for various generalized inverses of a con-s-k-EPr matrix to be con-s-k-EPr are determined. Generalized inverses belonging to the sets A{1,2}, A{1,2,3} and A{1,2,4} of a con-s-k-EPr matrix *A* are characterized.

Keywords: Con-s-k-EP matrix, generalized inverse.

I. Introduction

Let c_{nxn} be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For Aϵc_{nxn} . Let $\bar{A}, A^T, A^*, A^S, \bar{A}^S, A^\dagger$, R(A), N(A) and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose , conjugate secondary transpose, Moore Penrose inverse range space, null space and rank of A respectively. A solution X of the equation AXA = A is called generalized inverse of A and is denoted by A^- . If A ϵ c_{nxn} then the unique solution of the equations A XA =A , XAX = X, $[AX]^*$ = AX , $(XA)^*$ = XA **[2]** is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$. A matrix A is called con-s-k-EP$_r$ if (A) = r and N(A) = N(A^TVK) (or) R(A)=R(KVA^T). Throughout this paper let "$\mathscr{k}$" be the fixed product of disjoint transposition in S_n = { 1,2,….n} and k be the associated permutation matrix .

Let us define the function $\mathscr{k}(x) = \left(x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)}\right)$. A matrix A = (a_{ij}) ϵ c_{nxn} is s-k-symmetric if $a_{ij} = a_{n-k(j)+1, n-k(i)+1}$ for i, j = 1,2,….n . A matrix A ϵ c_{nxn}

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is said to be Con-s-k-EP if it satisfies the condition $A_x = 0 <=> A^s k(x) = 0$ or equivalently N(A) =N(A^TVK). In addition to that A is con-s-k-EP $<=> KVA$ is con-EP or AVK is con-EP and A is con-s-k-EP$<=> A^T$ is con-s-k-EP$_r$ moreover A is said to be Con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer **[1]**.

In **Theorem (2.11) [1]**, it is shown that $\boldsymbol{A}$ is con-s-k-EP$_r$, if and only if $\boldsymbol{A}^{\dagger}$ is con-s-k-EP$_r$. Thus the con-s-k-EP$_r$ property of complex matrices is preserved for its Moore-Penrose inverses. However, all other generalized inverses of a con-s-k-EP$_r$ matrix need not be con-s-k-EP$_r$. For instance, let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ for k=(1,2)(3), the associated permutation matrix be $\boldsymbol{K} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{V} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Here, $\boldsymbol{A}$ is con-s-k-EP$_1$. But $\boldsymbol{A}^{-} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a inverse of $\boldsymbol{A}$ which is not con-s-k-EP$_1$.

A generalized inverse $\boldsymbol{A}^{=} \in \boldsymbol{A}\{1,2\}$ is shown to be con-s-k-EP$_r$ whenever $\boldsymbol{A}$ is con-s-k-EP$_r$ under certain conditions in the following way.

Theorem 2.1

Let $\boldsymbol{A} \in \boldsymbol{C}_{n \times n}, \boldsymbol{X} \in \boldsymbol{A}\{1,2\}$ and $\boldsymbol{AX}, \boldsymbol{XA}$ are con-s-k-EP$_r$ matrices. Then $\boldsymbol{A}$ is con-s-k-EP$_r$ $\Leftrightarrow$ $\boldsymbol{X}$ is con-s-k-EP$_r$.

Proof

Since $\boldsymbol{AX}$ and $\boldsymbol{XA}$ are con-s-k-EP$_r$, by **Theorem (2.11) [1]**, we have $\boldsymbol{R}(\boldsymbol{AX}) = \boldsymbol{R}(\boldsymbol{KV}(\boldsymbol{AX})^{\boldsymbol{T}})$ and $\boldsymbol{R}(\boldsymbol{KV}(\boldsymbol{XA})^{\boldsymbol{T}}) = \boldsymbol{R}(\boldsymbol{XA})$. Since $\boldsymbol{X} \in \boldsymbol{A}\{1,2\}$ we have $\boldsymbol{AXA} = \boldsymbol{A}, \boldsymbol{XAX} = \boldsymbol{X}$.

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Notes

Now,

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{A}) &= \boldsymbol{R}(\boldsymbol{AX}) \\ &= \boldsymbol{R}(\boldsymbol{KV}(\boldsymbol{AX})^{\boldsymbol{T}}) \\ &= \boldsymbol{R}(\boldsymbol{KVX}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}) \\ &= \boldsymbol{R}(\boldsymbol{KVX}^{\boldsymbol{T}})\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{KVA}^{\boldsymbol{T}}) &= \boldsymbol{R}(\boldsymbol{KVA}^{\boldsymbol{T}}\boldsymbol{X}^{\boldsymbol{T}}) \\ &= \boldsymbol{R}(\boldsymbol{KV}(\boldsymbol{XA})^{\boldsymbol{T}}) \\ &= \boldsymbol{R}(\boldsymbol{XA}) \\ &= \boldsymbol{R}(\boldsymbol{X})\end{aligned}$$

Now, $\boldsymbol{A}$ is con-s-k-EP$_r$ $\Leftrightarrow \boldsymbol{R}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{KVA}^{\boldsymbol{T}})$ and $(\boldsymbol{A}) = \boldsymbol{r}$ $\Leftrightarrow \boldsymbol{R}(\boldsymbol{KVX}^{\boldsymbol{T}}) = \boldsymbol{R}(\boldsymbol{X})$ and $(\boldsymbol{A}) = \ (\boldsymbol{X}) = \boldsymbol{r} \Leftrightarrow \boldsymbol{X}$ is con-s-k-EP$_r$.

Remark 2.2

In the above Theorem, the conditions that both $\boldsymbol{AX}$ and $\boldsymbol{XA}$ to be con-s-k-EP$_r$ are essential. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ for k=(1,2)(3), the associated permutation matrix $\boldsymbol{K} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{V} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. $\boldsymbol{A}$ is con-s-k-EP$_1$.

$\boldsymbol{X} = \boldsymbol{A}^{-} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \boldsymbol{A}\{1,2\}$ $\boldsymbol{AX} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; $\boldsymbol{XA} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $\boldsymbol{AX}$ and $\boldsymbol{XA}$ are not con-s-k-EP$_1$. Also $\boldsymbol{X}$ is not con-s-k-EP$_1$.

Now, we show that generalized inverses belonging to the sets A{1,2,3} and A{1,2,4} of a con-s-k-EP$_r$ matrix $\boldsymbol{A}$ is also con-s-k-EP$_r$, under certain conditions in the following Theorems.

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Theorem 2.3

Let $\boldsymbol{A} \in \boldsymbol{C}_{n\times n}, \boldsymbol{X} \in \boldsymbol{A}\{1,2,3\}, \boldsymbol{R}(\boldsymbol{X}) = \boldsymbol{R}(\boldsymbol{A}^T)$. Then $\boldsymbol{A}$ is con-s-k-EP$_r$ $\Leftrightarrow \boldsymbol{X}$ is con-s-k-EP$_r$.

Proof

Since $\boldsymbol{X} \in \boldsymbol{A}\{1,2,3\}$, we have $\boldsymbol{AXA} = \boldsymbol{A}, \boldsymbol{XAX} = \boldsymbol{X}, (\boldsymbol{AX})^T = \boldsymbol{AX}$. Therefore, $\boldsymbol{R}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{AX}) = \boldsymbol{R}((\boldsymbol{AX})^T) = \boldsymbol{R}(\boldsymbol{X}^T)$

$$
\begin{aligned}
\boldsymbol{R}(\boldsymbol{X}) = \boldsymbol{R}(\boldsymbol{A}^T) &\Rightarrow \boldsymbol{XX}^\dagger = \boldsymbol{A}^T(\boldsymbol{A}^T)^\dagger \\
&\Rightarrow \boldsymbol{XX}^\dagger = \boldsymbol{A}^T(\boldsymbol{A}^\dagger)^T \\
&\Rightarrow \boldsymbol{XX}^\dagger = (\boldsymbol{A}^\dagger \boldsymbol{A})^T \\
&\Rightarrow \boldsymbol{XX}^\dagger = \boldsymbol{A}^\dagger \boldsymbol{A} \\
&\Rightarrow \boldsymbol{KVXX}^\dagger \boldsymbol{VK} = \boldsymbol{KVA}^\dagger \boldsymbol{AVK} \\
&\Rightarrow (\boldsymbol{KVX})(\boldsymbol{KVX})^\dagger = (\boldsymbol{AVK})^\dagger \boldsymbol{AVK} \\
&\Rightarrow \boldsymbol{R}(\boldsymbol{KVX}) = \boldsymbol{R}((\boldsymbol{AVK})^T) \\
&\Rightarrow \boldsymbol{R}(\boldsymbol{KVX}) = \boldsymbol{R}(\boldsymbol{KVA}^T)
\end{aligned}
$$

A is con-s-k-EP$_r$ $\Leftrightarrow \boldsymbol{R}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{KVA}^T)$ and $(\boldsymbol{A}) = \boldsymbol{r}$ $\Leftrightarrow \boldsymbol{R}(\boldsymbol{X}^T) = \boldsymbol{R}(\boldsymbol{KVX})$ and $(\boldsymbol{A}) = (\boldsymbol{X}) = \boldsymbol{r} \Leftrightarrow \boldsymbol{X}$ is con-s-k-EP$_r$.

Theorem 2.4

Let $\boldsymbol{A} \in \boldsymbol{C}_{n\times n}, \boldsymbol{X} \in \boldsymbol{A}\{1,2,4\}, \boldsymbol{R}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{X}^T)$. Then $\boldsymbol{A}$ is con-s-k-EP$_r$ $\Leftrightarrow \boldsymbol{X}$ is con-s-k-EP$_r$.

Proof

Since $\boldsymbol{X} \in \boldsymbol{A}\{1,2,4\}$, we have $\boldsymbol{AXA} = \boldsymbol{A}, \boldsymbol{XAX} = \boldsymbol{X}, (\boldsymbol{XA})^T = \boldsymbol{XA}$ and $\boldsymbol{R}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{X}^T)$.

$$
\begin{aligned}
\text{Now, } \boldsymbol{R}(\boldsymbol{KVA}^T) &= \boldsymbol{R}(\boldsymbol{KVA}^T \boldsymbol{X}^T) \\
&= \boldsymbol{R}(\boldsymbol{KV}(\boldsymbol{XA})^T) \\
&= \boldsymbol{R}(\boldsymbol{KVXA}) \\
&= \boldsymbol{R}(\boldsymbol{KVX})
\end{aligned}
$$

A is con-s-k-EP$_r$ $\Leftrightarrow R(A)=R(KVA^T)$ and $(A)=r \Leftrightarrow R(X^T)=R(KVX)$

and $(A)=\ (X)=r \Leftrightarrow X$ is con-s-k-EP$_r$.

Remark 2.5

Notes

In particular, if $X=A^\dagger$ then $R(A^\dagger)=R(A^T)$ holds. Hence A is con-s-k-EP$_r$ is equivalent to $A^\dagger$ is con-s-k-EP$_r$.

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A Note on Basic Hypergeometric Function of N-Variable

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Abstract- In this paper an attempt has been made to establish new transformation formula for the basic hypergeometric function of n variable.

Keywords: *q-series, basic hypergeometric function, transformation.*

GJSFR-F Classification : *MSC 2010: 33C05, 33D70*

A NOTE ON BASIC HYPERGEOMETRIC FUNCTION OF N-VARIABLE

A Note on Basic Hypergeometric Function of N-Variable

Pankaj Srivastava [α] & Mohan Rudravarapu [σ]

Abstract- In this paper an attempt has been made to establish new transformation formula for the basic hypergeometric function of n variable.

Keywords: q-series, basic hypergeometric function, transformation.

I. Introduction

Basic hypergeometric functions are among the most important functions with very diverse applications to Engineering,Physics and Mathematical Analysis.Nowadays, the importance of basic hypergeometric function and its unique role as the strategic resource in the Ramanujan's Mathematics became more important than the past time. The importance given to the Ramanujan's Mathematics has increased considerably in recent years and great deal of attention in basic hypergeometric function's literature is being given for the numerous topics that have been addressed by mathematicians working in the field of Basic hypergeometric functions, notably R.P Agarwal [1], G.E. Andrews and B.C. Berndt [2], G.E. Andrews [3, 4], R.Askey [5, 6],W.N. Bailey [7], S. Bhargava and Chandrashekar Adiga [8], R.Y.Denis [9], R.Y. Denis *et al.* [10], G.Gasper [11], V.K.jain [12, 13], M.S. Mahadeva Naika and B.N. Dharmendra [14], TH.M.Rassias and S.N. Singh [15], L.J.Slater [16], Pankaj Srivastava [17, 18], Pankaj Srivastava and Mohan Rudravarapu [19], A. Verma [20], G.N.Watson [21] and many others published large number of studies.In this paper, we are interested to develop certain new transformation formula for the basic hypergeometric function of n variable with the help of technique developed by Andrews [4], special cases also developed.

II. Notations and Definitions

A basic hypergeometric series of n-variables is defined as

$$\phi\left[\begin{array}{c} (a_p):(b^1_{M_1});(b^2_{M_2});\ldots;(b^n_{Mn}) \\ \\ (c_t):(d^1_{N_1});(d^2_{N_2});\ldots;(d^n_{N_n}) \end{array};x_1,x_2,\ldots;x_n\right]$$

Ref

[1] R.P.Agarwal, Certain basic hypergeometric identities associated with Mock theta functions, *Quarterly Journal Of Mathematics*,**20(1)** (1969), 121-128.

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$$= \sum_{m_1, m_2, \ldots, m_n \geq 0}^{\infty} \frac{[(a_p)]_{m_1+m_2+\ldots+m_n}[(b^1_{M_1})]_{m_1} \ldots [(b^n_{M_n})]_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{[q]_{m_1} \ldots [q]_{m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}[(d^1_{N_1})]_{m_1} \ldots [(d^n_{N_n})]_{m_n}}, \quad (2.1)$$

Where (a_p) stands for p-parameters $a_1, a_2, \ldots, a_p$. For the convergence of this series we require $\max(|q|, |x_1|, \ldots, |x_n|) < 1$. The q-shifted factorial is defined as

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n = 1, 2, 3 \ldots . \end{cases} \quad (2.2)$$

We also define

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \text{ for } |q| < 1. \quad (2.3)$$

The infinite product diverges when $a \neq 0$.
Also

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_r;q)_n. \quad (2.4)$$

And

$${}_1\phi_0\left[a; -; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, |z| < 1, |q| < 1. \quad (2.5)$$

The product formula

$${}_1\phi_0\left(a; -; q, z\right) {}_1\phi_0\left(b; -; q, az\right) = {}_1\phi_0\left(ab; -; q, z\right). \quad (2.6)$$

$${}_1\phi_1\left[a; c; q, c/a\right] = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \quad (2.7)$$

III. Main Results

In this section we shall establish the following main result .

$$\phi\left[\begin{array}{l} a, (a_p) : (b^1_{M_1}); (b^2_{M_2}); \ldots; (b^n_{Mn}) \\ c, (c_t) : (d^1_{N_1}); (d^2_{N_2}); \ldots; (d^n_{N_n}) \end{array} ; x_1, x_2, \ldots; x_n\right]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \phi\left[\begin{array}{l} (a_p) : (b^1_{M_1}); \ldots; (b^n_{Mn}) \\ (c_t) : (d^1_{N_1}); \ldots; (d^n_{N_n}) \end{array} ; x_1 q^r, x_2 q^r, \ldots; x_n q^r\right]. \quad (3.1)$$

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IV. Proof of (3.1)

By using (2.1) the left hand side of (3.1) can be put in the following form

$$\sum_{m_1,m_2,\ldots,m_n\geq 0}^{\infty} \frac{[a]_{m_1+m_2+\ldots+m_n}[(a_p)]_{m_1+m_2+\ldots+m_n}}{[q]_{m_1}\ldots[q]_{m_n}[c]_{m_1+m_2+\ldots+m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}} \times \frac{[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n} x_1^{m_1}\ldots x_n^{m_n}}{[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}}$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{m_1,m_2,\ldots,m_n\geq 0}^{\infty} \frac{[(a_p)]_{m_1+m_2+\ldots+m_n}[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n}}{[q]_{m_1}\ldots[q]_{m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}} \times \frac{(cq^{m_1+\ldots+m_n})_\infty x_1^{m_1}\ldots x_n^{m_n}}{[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}(aq^{m_1+m_2+\ldots+m_n})_\infty}$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{m_1,\ldots,m_n\geq 0}^{\infty} \frac{[(a_p)]_{m_1+\ldots+m_n}[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n} x_1^{m_1}\ldots x_n^{m_n}}{[q]_{m_1}\ldots[q]_{m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}} \times \sum_{r=0}^{\infty} \frac{(c/a)_r a^r q^{r(m_1+m_2+\ldots+m_n)}}{[q]_r}.$$

By changing the order of summation in the above equation, the right hand side of (3.1) can be obtained.

V. Particular Cases

Putting $p = t = M_1 = \ldots = M_n = N_1 = \ldots = N_n = 0$ in (3.1), we get

$$ {}_1\phi_1 \left[\begin{array}{l} a : -; \ldots; -; \\ \\ c : -; \ldots; -; \end{array} ; x_1, x_2, \ldots; x_n \right] $$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \times {}_0\phi_0 [-; -; x_1 q^r] \ldots {}_0\phi_0 [-; -; x_n q^r]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r [x_1 q^r]_\infty \ldots [x_n q^r]_\infty}$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r [x_1]_r \ldots [x_n]_r}{[q]_r [x_1]_\infty \ldots [x_n]_\infty}$$

$$= \frac{[a]_\infty}{[c]_\infty [x_1]_\infty \ldots [x_n]_\infty} {}_{n+1}\phi_0 [x_1, \ldots x_n, c/a; -; a],$$

which is valid if $|a| < 1$. (5.1)

Putting $p = t = N_1 = \ldots = N_n = 0, M_1 = \ldots = M_n = 1, b_1^1 = b_1, b_1^2 = b_2, \ldots, b_1^n = b_n$ in (3.1), we get:

$$\phi \begin{bmatrix} a : b_1; \ldots ; b_n \\ & ; x_1, x_2, \ldots ; x_n \\ c : -; \ldots ; - \end{bmatrix}$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \times {}_1\phi_0 \left[b_1; -; x_1 q^r\right] \ldots {}_1\phi_0 \left[b_n; -; x_n q^r\right]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \frac{[b_1 x_1 q^r]_\infty}{[x_1 q^r]_\infty} \ldots \frac{[b_n x_n q^r]_\infty}{[x_n q^r]_\infty}$$

$$= \frac{[a]_\infty [b_1 x_1]_\infty \ldots [b_n x_n]_\infty}{[c]_\infty [x_1]_\infty \ldots [x_n]_\infty} \times \sum_{r=0}^{\infty} \frac{[c/a]_r a^r [x_1]_r \ldots [x_n]_r}{[q]_r [b_1 x_1]_r \ldots [b_n x_n]_r}$$

$$= \frac{[a]_\infty [b_1 x_1]_\infty \ldots [b_n x_n]_\infty}{[c]_\infty [x_1]_\infty \ldots [x_n]_\infty} \times {}_{n+1}\phi_n \left[x_1, \ldots, x_n, c/a; b_1 x_1, \ldots, b_n x_n; a\right], \quad (5.2)$$

Putting $x_3 = x_4 = \ldots = x_n = 0$ in (5.2), we get

$$\phi \begin{bmatrix} a : b_1; b_2; \ldots ; \\ & ; x_1, x_2 \\ c : -; \ldots ; - \end{bmatrix} = \phi^{(1)} \left[a; b_1, b_2; c; x_1, x_2\right]$$

$$= \frac{[a]_\infty [b_1 x_1]_\infty [b_2 x_2]_\infty}{[c]_\infty [x_1]_\infty [x_2]_\infty} \times {}_3\phi_2 \begin{bmatrix} x_1, x_2, c/a; \\ & ; a \\ b_1 x_1, b_2 x_2 \end{bmatrix}, \quad (5.3)$$

This is the result due to Denis ([9],5.5).
Now, putting $b_2 = x_1/x_2, b_1 x_1 = c x_2$ in (5.3) and evaluating ${}_2\phi_1$ series on the right hand side with the help of Slater ([16];Appendix.IV.2), we get

$$\phi^{(1)} \left[a; c x_2/x_1, x_1/x_2; c; x_1, x_2\right] = \frac{[x_2 a]_\infty}{[x_2]_\infty}. \quad (5.4)$$

Now, substituting $p = t = 0,\ N_1 = \ldots = N_n = 1, M_1 = \ldots = M_n = 1, b_1^1 = b_1, b_1^2 = b_2, \ldots, b_1^n = b_n, d_1^1 = d_1, d_1^2 = d_2, \ldots, d_1^n = d_n$ in (3.1), we get:

$$\phi \begin{bmatrix} a : b_1; \ldots ; b_n \\ & ; x_1, x_2, \ldots ; x_n \\ c : d_1; \ldots ; d_n \end{bmatrix}$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \times {}_1\phi_1 \left[b_1; d_1; x_1 q^r\right] \ldots {}_1\phi_1 \left[b_n; d_n; x_n q^r\right], \quad (5.5)$$

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Taking $d_1 = b_1x_1q^r, d_2 = b_2x_2q^r, \ldots, d_n = b_nx_nq^r$ in (5.5), we get the following result after simplification

$$\phi\left[\begin{array}{ll} a : b_1; \ldots; b_n & \\ & ; x_1, x_2, \ldots; x_n \\ c : b_1x_1q^r; \ldots; b_nx_nq^r & \end{array}\right]$$

$$= \frac{[a]_\infty[x_1]_\infty \ldots [x_n]_\infty}{[c]_\infty[b_1x_1]_\infty \ldots [b_nx_n]_\infty} \times {}_{n+1}\phi_n \left[b_1x_1, \ldots, b_nx_n, c/a; x_1, \ldots, x_n; a\right], \qquad (5.6)$$

Now, putting $x_3 = x_4 = \ldots = x_n = 0$ in (5.6), we get

$$\phi\left[\begin{array}{ll} a : b_1; b_2 & \\ & ; x_1, x_2 \\ c : b_1x_1q^r; b_2x_2q^r & \end{array}\right]$$

$$= \frac{[a]_\infty[x_1]_\infty[x_2]_\infty}{[c]_\infty[b_1x_1]_\infty[b_2x_2]_\infty} \times {}_3\phi_2 \left[\begin{array}{ll} b_1x_1, b_2x_2, c/a; & \\ & ; a \\ x_1, x_2 & \end{array}\right], \qquad (5.7)$$

Now, substituting $b_1x_1 = x_2, b_2x_2 = x_1/c$ in (5.7) and evaluating ${}_2\phi_1$ series in the right hand side with the help of Slater ([16]:Appendix IV.2), we get

$$\phi\left[\begin{array}{ll} a : x_2/x_1; x_1/cx_2 & \\ & ; x_1, x_2 \\ c : x_2q^r; x_1q^r/c & \end{array}\right] = \frac{(ax_1/c; q)_\infty}{(x_1/c; q)_\infty}. \qquad (5.8)$$

A variety of similar interesting results can be scored.

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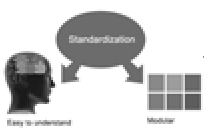

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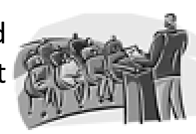
The IFOARS institution is entitled to form a Board comprised of one Chairperson and three to five board members preferably from different streams. The Board will be recognized as "Institutional Board of Open Association of Research Society"-(IBOARS).

The Institute will be entitled to following benefits:

The IBOARS can initially review research papers of their institute and recommend them to publish with respective journal of Global Journals. It can also review the papers of other institutions after obtaining our consent. The second review will be done by peer reviewer of Global Journals Incorporation (USA)
The Board is at liberty to appoint a peer reviewer with the approval of chairperson after consulting us.
The author fees of such paper may be waived off up to 40%.



The Global Journals Incorporation (USA) at its discretion can also refer double blind peer reviewed paper at their end to the board for the verification and to get recommendation for final stage of acceptance of publication.

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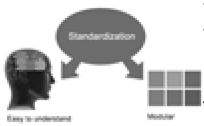

After nomination of your institution as "Institutional Fellow" and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.

The board can also take up the additional allied activities for betterment after our consultation.

The following entitlements are applicable to individual Fellows:

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Note :

"

- In future, if the board feels the necessity to change any board member, the same can be done with the consent of the chairperson along with anyone board member without our approval.
- In case, the chairperson needs to be replaced then consent of 2/3rd board members are required and they are also required to jointly pass the resolution copy of which should be sent to us. In such case, it will be compulsory to obtain our approval before replacement.
- In case of "Difference of Opinion [if any]" among the Board members, our decision will be final and binding to everyone.

"

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The Author can submit the paper either online or offline. The authors should prefer online submission.Online Submission: There are three ways to submit your paper:

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1. General,

2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript’s Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

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Research articles: These are handled with small investigation and applications

Research letters: The letters are small and concise comments on previously published matters.

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The recommended size of original research paper is less than seven thousand words, review papers fewer than seven thousands words also.Preparation of research paper or how to write research paper, are major hurdle, while writing manuscript. The research articles and research letters should be fewer than three thousand words, the structure original research paper; sometime review paper should be as follows:

Papers: These are reports of significant research (typically less than 7000 words equivalent, including tables, figures, references), and comprise:

(a)Title should be relevant and commensurate with the theme of the paper.

(b) A brief Summary, "Abstract" (less than 150 words) containing the major results and conclusions.

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(e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition; sources of information must be given and numerical methods must be specified by reference, unless non-standard.

(f) Results should be presented concisely, by well-designed tables and/or figures; the same data may not be used in both; suitable statistical data should be given. All data must be obtained with attention to numerical detail in the planning stage. As reproduced design has been recognized to be important to experiments for a considerable time, the Editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned un-refereed;

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Authors should very cautiously consider the preparation of papers to ensure that they communicate efficiently. Papers are much more likely to be accepted, if they are cautiously designed and laid out, contain few or no errors, are summarizing, and be conventional to the approach and instructions. They will in addition, be published with much less delays than those that require much technical and editorial correction.

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Abbreviations supposed to be used carefully. The abbreviated name or expression is supposed to be cited in full at first usage, followed by the conventional abbreviation in parentheses.

Metric SI units are supposed to generally be used excluding where they conflict with current practice or are confusing. For illustration, 1.4 l rather than 1.4 × 10-3 m3, or 4 mm somewhat than 4 × 10-3 m. Chemical formula and solutions must identify the form used, e.g. anhydrous or hydrated, and the concentration must be in clearly defined units. Common species names should be followed by underlines at the first mention. For following use the generic name should be constricted to a single letter, if it is clear.

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Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

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References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

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5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

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7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

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- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.

- Do not give permission to anyone else to "PROOFREAD" your manuscript.

- Methods to avoid Plagiarism is applied by us on every paper, if found guilty, you will be blacklisted by all of our collaborated research groups, your institution will be informed for this and strict legal actions will be taken immediately.)
- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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