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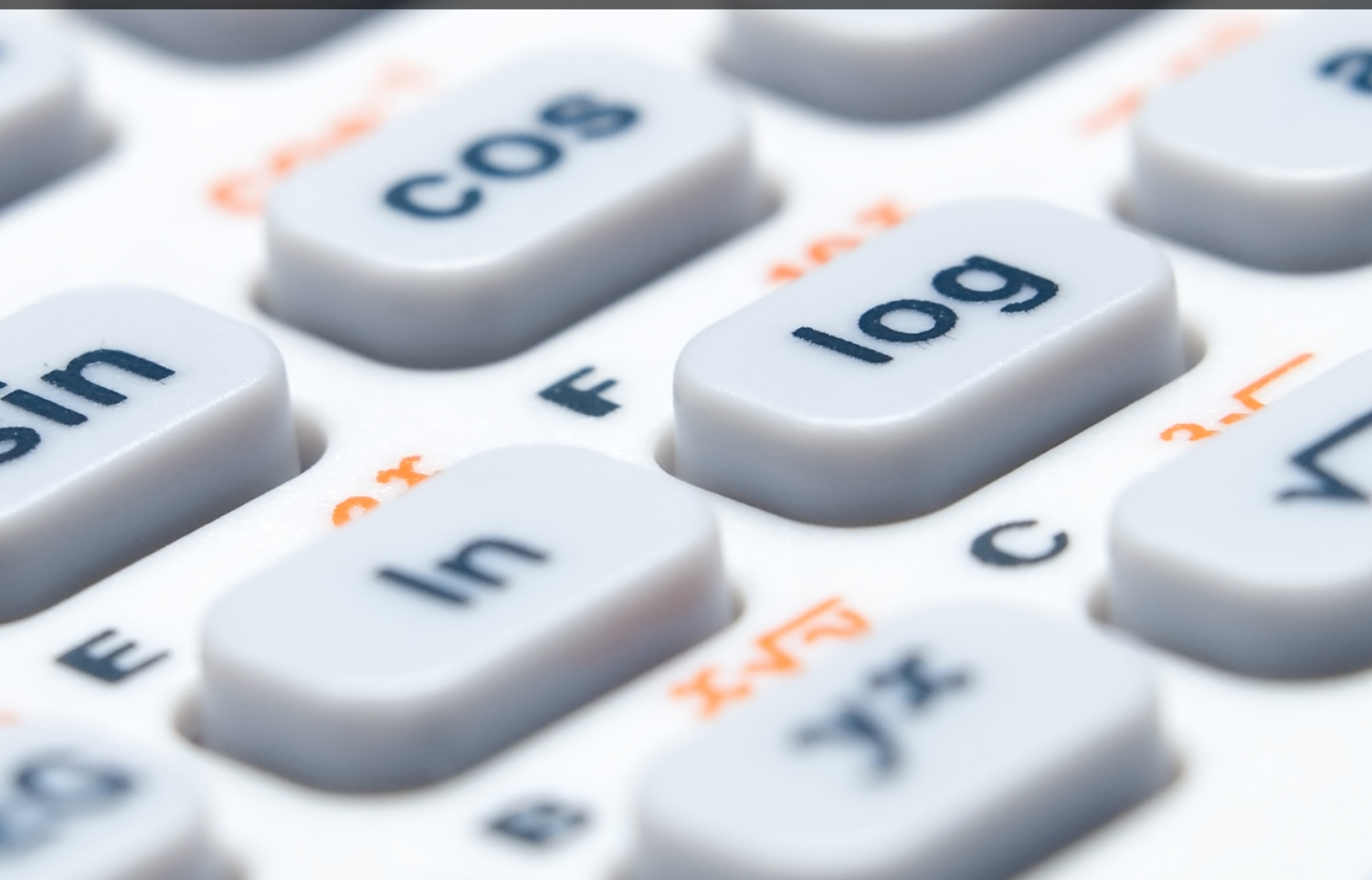

Highlights

Integral Operator Associated

Multipliers of Distributions Spaces

Analytic and Numeric Solution

Development of Discrete Version

Discovering Thoughts, Inventing Future

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Pathway Integral Operator Associated with Aleph-Function and General Polynomials

By Dr. Rinku Jain & Dr. Kirti Arekar

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Abstract - This paper is devoted to the study of a pathway fractional integral operator associated with the pathway model and pathway probability density for the ℵ- function and a generalized polynomial in the kernel. By specializing the coefficients and various parameters in the generalized polynomials and ℵ- function, our main theorem would readily yield several interesting results.

Keywords : pathway fractional integral operator, aleph function (ℵ- function), generalized polynomial.

GJSFR-F Classification : MSC 2010 : 11S23, 33C45

PATHWAY INTEGRAL OPERATOR ASSOCIATED WITH ALEPH-FUNCTION AND GENERAL POLYNOMIALS

Strictly as per the compliance and regulations of :

Ref.

7. S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, New York (1993).

Pathway Integral Operator Associated with Aleph-Function and General Polynomials

Dr. Rinku Jain α & Dr. Kirti Arekar σ

Abstract - This paper is devoted to the study of a pathway fractional integral operator associated with the pathway model and pathway probability density for the ℵ- function and a generalized polynomial in the kernel. By specializing the coefficients and various parameters in the generalized polynomials and ℵ- function, our main theorem would readily yield several interesting results.

Keywords : *pathway fractional integral operator, aleph function (ℵ-function), generalized polynomial.*

I. Introduction

In the last three decades several authors have made significant contribution in the field of fractional calculus. Fractional calculus has been applied to almost every field of science, engineering, and Mathematics. The most popular one, we are based on here, is the Riemann-Liouville fractional integral operator [7]. The Pathway fractional integral operator, as an extension of Riemann-Liouville fractional integral operator, introduced by Nair [8] is defined in the following manner

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt \qquad (1.1)$$

Where f(x) ∈ L (a,b), $\eta \in \mathbb{C}$, $Re(\eta) > 0$, $a > 0$ and 'pathway parameter' $\alpha < 1$.

The Pathway model is introduced by Mathai [1] and studied further by Mathai and Haubold[2], [3]. For real scalar α, the Pathway model for scalar random variables is represented by the following probability density function

$$f(x) = c\,|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}}, \qquad (1.2)$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \ge 0$, $\left[1-a(1-\alpha)|x|^{\delta}\right] > 0$, $\gamma > 0$, where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

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$$c = \frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)}, \quad for\ \alpha<1, \tag{1.3}$$

$$= \frac{1}{2}\frac{\delta[a(\alpha-1)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{1-\alpha}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta}\right)}, \quad for\ \frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0,\ \alpha>1, \tag{1.4}$$

$$= \frac{1}{2}\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \quad for\ \alpha\to 1. \tag{1.5}$$

For $\alpha<1$, it is a finite range density with $\left[1-a(1-\alpha)|x|^{\delta}\right]>0$ and (1.2) remains in the extended generalized type-1 beta family. The Pathway density in (1.2), for $\alpha<1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha>1$, we have

$$f(x)=c\ |x|^{\gamma-1}\left[1+a(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}}, \tag{1.6}$$

$-\infty<x<\infty$, $\delta>0$, $\beta\geq 0$, $\left[1-a(\alpha-1)|x|^{\delta}\right]>0, \gamma>0$, which is extended generalized type-2 beta model for real x. t includes the type-2 beta density, the F density, the Student-t density, and Cauchy density and many more.

Here, we consider only the case of Pathway parameter $\alpha<1$. For $\alpha\to 1$ both (1.2) and (1.6) take the exponential form, since

$$\begin{aligned}\lim_{\alpha\to 1} c\ |x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} &= c\ |x|^{\gamma-1}\left[1+a(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}}\\ &= c\ |x|^{\gamma-1}e^{-\alpha\eta|x|^{\delta}}\end{aligned} \tag{1.7}$$

This includes the generalized Gamma-, the Weibull-, the Chi-Square the Laplace and other related densities. For more details on the Pathway model, the reader is referred to the recent papers of [2], [3].

II. Pathway Integral Operator of An ℵ - Function

The Aleph ℵ-function introduced by Sudland et al. [6] which is defined as a contour integral of Mellin Barnes Type:

Ref.

2. A.M. Mathai and H.J. Haubold, On generalized distributions and pathways, Physics Letters 372 (2008), 2109-2113.

$$\aleph[Z]=\aleph^{m,n}_{p_i,q_i,\tau_i;r}[Z]=\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\left(\frac{x}{y}\right)^q \middle| \begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,pi}\\ (b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,qi}\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)z^{-s}ds \qquad (2.1)$$

for all z≠o, $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i-1}^{r}\tau_i\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)} \qquad (2.2)$$

The integration path $\ell=\ell_{i\gamma\infty},\gamma\in R$ extends from γ-i∞ to γ+i∞, and is such that the poles, assumed to be simple, of $\Gamma(1-a_j-A_js), j=\overline{1,n}$ do not coincide with the pole $\Gamma(b_j+B_js), j=\overline{1,m}$. The parameters pi, qi are non-negative integers satisfying 0 ≤ n ≤ pi, 1≤ m ≤ qi , $\tau_i>0$ for $i=\overline{1,r}$. The parameters Aj, B_j, A_{ji}, B_{ji} > 0 and aj, bj a_{ji}, b_{ji} ∈ C. The empty product in (1.3) is interpreted as unity. The existence conditions for the defining integral (1.1) are given below:

$$\psi_l>0,\quad |\arg(z)|<\frac{\pi}{2}\psi_l \quad l=\overline{1,r}; \qquad (2.3)$$

(4):

$$\psi_l\ge 0,\quad |\arg(z)|<\frac{\pi}{2}\psi_l \quad and\ \Re\{\varsigma_l\}+1<0, \qquad (2.4)$$

Where

$$\psi_l=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\tau_l\left(\sum_{j=n+1}^{p_l}A_{jl}+\sum_{j=m+1}^{q_l}B_{jl}\right) \qquad (2.5)$$

$$\varsigma_l=\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j+\tau_l\left(\sum_{j=m+1}^{q_l}b_{jl}-\sum_{j=n+1}^{p_l}a_{jl}\right)+\frac{1}{2}(p_l-q_l) \qquad l=\overline{1,r}. \qquad (2.6)$$

The general polynomials of R variables given by Srivastava [5] defined and represented as:

Ref.

5. H. M. Srivastava, A multilinear generating function for the Konhausar sets of biortogonal polynomials suggested by the Laguerre polynomials, pacific J.Math. 177 (1985), 183-191.

$$S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}[x_1,\ldots x_R] = \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \frac{(-n_1)_{m_1 s_1}}{\angle s_1} \cdots \frac{(-n_R)_{m_R s_R}}{\angle s_R} \quad (2.7)$$

$$A[n_1,s_1;\ldots;n_R s_R]\, x_1^{s_1} \ldots x_R^{s_R}$$

Notes

Where m_i is an arbitrary positive integer and coefficients $A[n_1,s_1;\ldots;n_R s_R]$ are arbitrary constants, real or complex.

Theorem1. With the set of sufficient conditions (2.3), (2.4), (2.5) and (2.6), let (η, u, $u_1,\ldots,u_R$, $\beta \in \mathbb{C}$), $\mathrm{Re}\left(1+\frac{\eta}{1-\alpha}\right) > 0$, $\alpha < 1$, Re(η, u, $u_1,\ldots,u_R$, β) > 0 and m_i is an arbitrary positive integer and coefficients $A[n_1,s_1;\ldots;n_R s_R]$ are arbitrary constants, real or complex, then

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots m_R}\left[x^{u_i},\ldots,x^{u_R}\right] \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[x^{\beta}\right]\right]$$

$$= \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \frac{(-n_1)_{m_1 s_1}}{\angle s_1} \cdots \frac{(-n_R)_{m_R s_R}}{\angle s_R} A[n_1,s_1;\ldots;n_R s_R]$$

$$\frac{x^{\eta+u+u_i k_i+\ldots+u_R k_R}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{u+u_i k_i+\ldots+u_R k_R}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;r}^{m,n+1}\left[\frac{x^{\beta}}{[a(1-\alpha)]^{\beta}}\left|\begin{matrix}[1-u-(u_i s_1+\ldots+u_R s_R);\beta],\,(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,pi}\\ (b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,qi},[-u-(u_i s_1+\ldots+u_R s_R);\beta]\end{matrix}\right.\right]$$

(2.8)

Proof: The Theorem 1 can be evaluated by using the definitions (1.1), (2.1) and (2.7) then by interchange the order of integrations and summations (which is permissible under the conditions stated above), evaluate the inner integral by making use of beta function formula, we arrive at the desired result.

Theorem2. Let (η, u, $u_1,\ldots,u_R$, $\beta \in \mathbb{C}$), $\mathrm{Re}\left(1+\frac{\eta}{1\ \alpha}\right) > 0$, $\alpha < 1$, Re(η, u, $u_1,\ldots,u_R$, β) $\overline{>}$ 0 and m is an arbitrary positive integer and coefficients $A[n_1,s_1;\ldots;n_R s_R]$ are arbitrary constants, real or complex, then

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1}S_{n_1,\dots,n_R}^{m_1,\dots m_R}\left[x^{u_i},\dots,x^{u_R}\right]H_{p,q,}^{m,n}\left[x^{\beta}\right]\right]$$

$$=\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\frac{(-n_1)m_1s_1}{\angle s_1}\cdots\frac{(-n_R)m_Rs_R}{\angle s_R}A[n_1,s_1;\dots;n_Rs_R]$$

$$\frac{x^{\eta+u+u_ik_i+\dots+u_Rk_R}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{u+u_ik_i+\dots+u_Rk_R}}$$

$$H_{p+1,q+1,}^{m,n+1}\left[\frac{x^{\beta}}{[a(1-\alpha)]^{\beta}}\middle|\begin{matrix}[1-u-(u_is_1+\dots+u_Rs_R);\beta],(a_j,A_j)_{1,n},\dots,[(a_j,A_j)]_{n+1,p}\\(b_j,B_j)_{1,m},\dots,[(b_j,B_j)]_{m+1,q},[-u-(u_is_1+\dots+u_Rs_R);\beta]\end{matrix}\right],$$

Where $H_{p,q,}^{m,n}[x]$ is the Fox's H- Function [4]. (2.9)

Proof: The result in (2.2) can be derived from Theorem 1 by taking $\tau_1 = \dots = \tau_r = 1$ and r = 1. We have the required result.

Theorem3. Suppose that the conditions corresponding to Theorem 2 are satisfied. Then

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1}x^{\frac{u_1n_1}{2}}H_{n_1}\left[\frac{1}{2\sqrt{x^{u_1}}}\right]\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[x^{\beta}\right]\right]=\sum_{s_1=0}^{[n_1/2]}\frac{(-n_1)2s_1}{\angle s_1}(-1)^{s_1}$$

$$\frac{x^{\eta+u+u_1s_1}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{u+u_1k_1}}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;r}^{m,n+1}\left[\frac{x^{\beta}}{[a(1-\alpha)]^{\beta}}\middle|\begin{matrix}[1-u-u_is_1);\beta],(a_j,A_j)_{1,n},\dots,[\tau_j(a_j,A_j)]_{n+1,pi}\\(b_j,B_j)_{1,m},\dots,[\tau_j(b_j,B_j)]_{m+1,qi},[-u-u_is_1;\beta]\end{matrix}\right],$$

where $H_n(x)$ is the Hermite polynomials. (2.10)

Ref.

4. Fox, C. The *G* and *H*-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.* 98, (1961), 395-429.

Proof: In Theorem 1, if we take R =1, m_1 =2 and $A_{n_1, s_1} = (-1)^{s_1}$, then we get the desired result.

III. Special Cases

1. Letting R =1 in the result (2.9), we get the result recently obtained by Chaurasia and Ghiya [9] for ρ, ρ_1, and $\rho_2 \to 0$.
2. Letting $n_{i'} \to 0$, i' = 1,…, R, in the result (2.9), we get the result obtained by Nair in [8].

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Generalizations of 2D-Canonical Sine-Sine Transform

By S.B.Chavhan
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Abstract - Integral transform, fractional integral transform is a flourishing filed of active research due to its wide range of application. Fourier transform, fractional Fourier transform is probably the most intensively studied among all fractional transforms, similarly 2D canonical sine-sine transforms, and 2D canonical cosine-cosine is a powerful mathematical tool for processing images. In this paper the canonical 2D sine-sine transform is define in generalized sense. And various testing functions spaces defined by using Gelfand-shilov technique. Also uniqueness theorem, modulation theorems are proved.

Keywords : 2D canonical sine-sine transform, testing function space, generalized functions, fourier transform.

GJSFR-F Classification : MSC 2010: 47D09

GENERALIZATIONS OF 2D-CANONICAL SINE-SINE TRANSFORM

Ref.

1. Chavhan S.B. BorkarV.C., "Operator of Two Dimensional Generalized Canonical Sine Transform. IJERD.Vol.4 Issue 2. (2012). 10-14

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S.B.Chavhan

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Keywords : 2D canonical sine-sine transform, testing function space, generalized functions, fourier transform.

I. Introduction

Integral transforms have been successfully used for almost two centuries is solving many problems in mathematical physical, applied mathematics and engineering science. Historically origin of the integral transform is P.S. Laplace and J.Fourier. Laplace transform is useful, for evaluating certain definite integral [2].

The definition of canonical sine-sine transform as follows [1].

$$\{2DCSST\ f(t,x)\}(s,w) = \langle f(t,x), K_{s_1}(t,x) K_{s_2}(x,w) \rangle$$

In the present paper, 2D sine-sine transform is extended in the distribution sense. The plan of the paper is as follows. The definitions are given in section 2. In section 3, testing functionspace is defined by Galfand-shilovtechnique [3],[4].Section 4 some result on countable union space are proved. In section5, inversion and uniqueness theorem are stated. In section 6, modulations theorems are given. The notations and terminology as per zemanian [5],[6].

II. Definition Two Dimensional Canonical Sine-Sine Transform

Let $E'(R\times R)$ denote the dual of $E(R\times R)$. Therefore the generalized canonical sine transform of $f(t,x)\in E'(R\times R)$ is defined as

$$\{2DCST\ f(t,x)\}(s,w)$$

$$= (-1)\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt$$

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Where $$\gamma_{E,k}\left\{K_{s_1}(t,s)K_{s_2}(x,w)\right\} = \sup_{\substack{-\infty<t<\infty\\ -\infty<x<\infty}} \left| D_t^k D_x^l K_{s_1}(t,s)K_{s_2}(x,w)\right| < \infty$$

III. Different S-Type Testing Function Spaces

In this section we have defined s-type testing function spaces by imposing conditions not only on the decreases of the fundamental functions at infinity, but also on the growth of their derivatives as the order of derivative increases. Clearly, $SS^{a,b}$ space will be extension of testing function space D, so that these spaces have been successfully, used in pseudo differential operator theory.

a) The space $SS_{\gamma}^{a,b}$:

It is given by

$$SS_{\gamma}^{a,b} = \left\{\phi : \phi \in E_+ \,/\, \sigma_{l,k,p}\phi(t,x) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x)\right| \le C_{kp} A^l . l^{l\gamma}\right\} \quad (3.1)$$

The constant $C_{k,p}$ and A depend on ϕ.

b) The space $SS^{ab,\beta}$:

$SS^{a,b,\beta}$ this space is given by

$$SS^{a,b,\beta} = \left\{\phi.\phi \in E_+ \,/\, \rho_{l,k,p}\phi(t,x) = \sup\left| t^l D_t^k D_x^p \phi(t,x)\right| \le C_{l,p} B^k k^{k\beta}\right\} \quad (3.2)$$

The constants $C_{l,p}$ and B depend on ϕ.

c) The space $SS_{\gamma}^{a,b,\beta}$

This space is formed by combining the condition (3.1) and (3.2)

$$SS_{\gamma}^{a,b,\beta} = \left\{\phi : \phi \in E_+ \,/\, \xi_{l,k,p}\phi(t,x) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x)\right| \le C\, A^l l^{l\gamma} B^k k^{k,\beta}\right\} \quad (3.3)$$

$l,k,p = 0,1,2...........$ Where A,B,C depend on ϕ.

d) The space $SS_{\gamma,m}^{a,b,\beta}$

It is defined as,

$$SS_{\gamma,m}^{a,b} = \left\{\phi : \phi \in E_+ \,/\, \sigma_{l,k,p}\phi(t,x) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x)\right| \le C_{k,p,\mu}(m+\mu)^l\, l^{l\gamma}\right\} \quad (3.4)$$

For any $\mu > 0$ where m is the constant, depending on the function ϕ.

e) The space $SS^{a,b,\beta,n}$

This space is given by

$$SS^{a,b,\beta,n} = \left\{\phi : \phi \in E_+ \,/\, \rho_{l,k,p}\phi(t,x) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x)\right| \le C_{l,p,\delta}(n+\delta)^k\, k^{k\beta}\right\} \quad (3.5)$$

For any $\delta > 0$ where n the constant is depends on the function ϕ.

Notes

Notes

f) The space $SS_{\gamma,m}^{a,b,\beta,n}$

This space is defined by combining the conditions in (3.4) and (3.5).

$$SS_{\gamma,m}^{a,b,\beta,n} = \Big\{\phi : \phi \in E_+ \,/\, \xi_{l,k,p}\phi(t,x) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x) \right| \\ \le C_{\mu\delta}(m+\mu)^l (n+\delta)^k \, . l^{l\gamma} k^{k\beta} \Big\} \quad (3.6)$$

IV. Results on Countable Unions-Type Space

Proposition 4.1: If $m_1 < m_2$ then $SS_{\gamma,m_1}^{a,b} \subset SS_{\gamma,m_2}^{a,b}$. The topology of $SS_{\gamma,m_1}^{a,b}$ is equivalent to the topology induced on $SS_{\gamma,m_1}^{a,b}$ by $SS_{\gamma,m_2}^{a,b}$

$$i.e \; T_{\gamma,m_1}{}^{a,b} \sim T_{\gamma,m_2}{}^{a,b} / SS_{\gamma,m_1}^{a,b}$$

Proof: For $\phi \in SS_{\gamma,m_1}^{a,b}$ and $\delta_{l,k,p}(\phi) \le C_{k,\mu}(m_1+\mu)^l l^{l\gamma}$

$$\le C_{k,\mu,p}(m_2+\mu)^l l^{l\gamma} \quad \text{Thus, } SS_{\gamma,m_1}^{a,b} \subset ss_{\gamma,m_2}^{a,b}$$

The space $SS_{\gamma,}^{a,b}$ can be expressed as union of countable normed spaces.

Proposition 4.2: $SS_{\gamma}^{a,b} = \overset{\infty}{\underset{i=1}{U}} SS_{\gamma,m_1}^{a,b}$ and if the space $SS_{\gamma}^{a,b}$ is equipped with strict inductive limit topology $S_{a,b,m}$ defined by injective map from $SS_{\gamma,m_1}^{a,b}$ to $SS_{\gamma}^{a,b}$ then the sequence $\{\phi_n\}$ in $SS_{\gamma}^{a,b}$ converges to zero.

Proof: we show that $SS_{\gamma}^{a,b} = \overset{\infty}{\underset{i=1}{U}} SS_{\gamma.m_1}^{a,b}$

Clearly $\overset{\infty}{\underset{i=1}{U}} SS_{\gamma.m_1}^{a,b} \subset SS_{k}^{a,b}$ for proving the other inclusion, let $\phi \in SS_{\gamma}^{a,b}$ then

$$\delta_{l,k,p}(\phi(t,x)) = \sup_{I_1} \left| t^l D_t^k D_x^p \; \phi_n(t,x) \right| \\ \le C_{k,p} A^l l^{l\gamma}, \quad (4.1)$$

where A is some positive constant, choose an integer $m = m_A$ and $\mu = 0$ such that $C_{k,p} A^l \le C_{k,p}(m+\mu)^l$.

Then (4.1) we get $\phi \in SS_{\gamma,m_1}^{a,b}$ implying that $SS_{\gamma}^{a,b} = \overset{\infty}{\underset{i=1}{U}} SS_{\gamma,m_1}^{a,b}$

Proposition 4.3: If $\gamma_1 < \gamma_2$ and $\beta_1 < \beta_2$ then $SS_{\gamma_1}^{a,b,\beta_1} \subset SS_{\gamma_2}^{a,b,\beta_2}$ and the topology of $SS_{\gamma_i}^{a,b,\beta_i}$ is equivalent to the topology induced on $SS_{\gamma_1}^{a,b,\beta_1}$ by $SS_{\gamma_2}^{a,b,\beta_2}$.

Proof: Let $\phi \in SS_{\gamma_1}^{a,b,\beta_1}$

$$\xi_{l,k,p}(\phi) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x) \right|$$

$$\leq CA^l l^{l\gamma_1} B^k k^{k\beta_1}$$

$$\leq CA^l l^{l\gamma_2} . B^p k^{p,\beta_2} \qquad \text{where} l,k,p = 0,1,2,3$$

Hence $\phi \in SS_{\gamma_2}^{a,b,\beta_2}$. Consequently, $SS_{\gamma_1}^{a,b,\beta_1} \subset SS_{\gamma_2}^{a,b,\beta_2}$. The topology of $SS_{\gamma_1}^{a,b,\beta_1}$

Is equivalent to the topology $T_{\gamma_2}^{a,b,\beta_2} / SS_{\gamma_2}^{a,b,\beta_2}$

It is clear from the definition of topologies of these spaces.

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Proposition 4.4: $SS^{a,b} = \overset{\infty}{\underset{\gamma_i \beta_i = 1}{U}} SS_{\gamma_i}^{a,b,\beta_i}$ and if the space $SS^{a,b}$ is equipped with the strict $SS^{a,b}$ inductive limit topology defined by the injective maps from $SS_{\gamma_i}^{a,b,\beta_i}$ to $SS^{a,b}$ then the sequence $\{\phi_n\}$ in $SS^{a,b}$ converges to zero iff $\{\phi_n\}$ is contained in some $SS_{\gamma_i}^{a,b,\beta_i}$ and converges to zero.

Proof: $SS^{a,b} = \overset{\infty}{\underset{\gamma_i \beta_i = 1}{U}} SS_{\gamma_i}^{a,b,,\beta_i}$

Clearly $$\overset{\infty}{\underset{\gamma_i \beta_i \;\; 1=}{U}} SS_{\gamma_i}^{a,b,\beta_i} \subset SS^{a}$$

For proving other inclusion, let $\phi(t,x) \in SS^{a,b}$ then

$$\eta_{l,k,p}(\phi) = \sup_{I_1} \left| t^l D_t^k D_x^p \phi(t,x) \right|,$$

is bounded by some number. We can choose integers γ_m and β_m such that

$$\eta_{l,k,p}(\phi) \leq C A^l l^{l,\gamma} B^{k,m}, k^{k,\beta,m}$$

$\therefore \phi \in SS_{\gamma_i}^{a,b,\beta_i}$ for some integer γ_i and β_i

Hence $SS^{a,b} \subset \overset{\infty}{\underset{\gamma_i \beta_i = 1}{U}} SS_{\gamma_i}^{a,b,\beta_i}$ Thus $SS^{a,b} = \overset{\infty}{\underset{\gamma_i \beta_i = 1}{U}} SS_{\gamma_i}^{a,b,\beta_i}$

V. Inversion and Uniqueness Theorems

Theorem 5.1: (Inversion) If $\{2DCSST\, f(t,x)\}(s,w)$ is canonical sine-sine transform of $f(t,x)$ then inverse of transform is given by

$$f(t,x) = -\sqrt{\frac{2\pi i}{b}} \sqrt{\frac{2\pi i}{b}}\, e^{\frac{-i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{-i}{2}\left(\frac{a}{b}\right)x^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(\frac{s}{b}t\right) \sin\left(\frac{w}{b}x\right) e^{\frac{-i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{-i}{2}\left(\frac{d}{b}\right)w^2} \{2DCSST\, f(t,x)\}(s,w)\, ds dw$$

Theorem 5.2: (Uniqueness) If $\{2DCSST\; f(t,x)\}(s,w)$ and $\{2DCSST\; g(t,x)\}(s,w)$ are 2D canonical sine-sine transform and $\sup pf \subset s_a$ and s_b also $\sup p\, g \subset s_a and \;\; s_b$

Where $s_a = \{t : t \in R^n, \;\; |t| \leq a, a > 0\}$ and $s_b = \{x : x \in R^n, \;\; |x| \leq b, b > 0\}$

If $\{2DCSST\; f(t,x)\}(s,w) = \{2DCSST\; g(t,x)\}(s,w)$

then, $f = g$ in the sense of equality in $D'(I)$

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Proof: By inversion theorem

$$f-g=$$

$$-e^{\frac{-i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{-i}{2}\left(\frac{a}{b}\right)x^2}\left[\left(\sqrt{\frac{2\pi i}{b}}\sqrt{\frac{2\pi i}{b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{-i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{-i}{2}\left(\frac{d}{b}\right)w^2}\sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)\{2DCSST\ f(t,x)\}(s,w)\,dsdw,\right)\right.$$

$$\left.-\left(\sqrt{\frac{2\pi i}{b}}\sqrt{\frac{2\pi i}{b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{-i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{-i}{2}\left(\frac{d}{b}\right)w^2}\sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)\{2DCSST\ g(t,x)\}(s,w)\,dsdw,\right)\right]$$

$$\therefore f-g=-\sqrt{\frac{2\pi i}{b}}\sqrt{\frac{2\pi i}{b}}e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{-\frac{i}{2}\left(\frac{a}{b}\right)x^2}\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}x\right)\sin\left(\frac{w}{b}x\right)$$

$$\left[\{2DCSST\,f(t,x)\}-\{2DCSST\,g(t,x)\}\right]dsdw$$

Thus $f=g$ in $D'(I)$

VI. Modulation Theorems for Canonical Sine-Sine Transform

Theorem 6.1: If $\{2DCSST\ f(t,x)\}(s,w)$ is canonical sine-sine transform of $f(t,x)$ then

$$\{2DCSST\cos\mu t\ f(t,x)\}(s,w)$$

$$=\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\{2DCSST\ f(t,x)\}(s+\mu b,w)+e^{i(\mu sd)}\{2DCSST\ f(t,x)\}(s-\mu b,w)\right]$$

Proof: Definition of two dimensional canonical sine-sine transform $f(t,x)$ is

$$\{2DCSST\ f(t,x)\}(s,w)$$

$$=-\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt$$

$$\{2DCSST\cos\mu t\ f(t,x)\}(s,w)$$

$$=-\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)\cos\mu t\,e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt$$

$$=-\frac{1}{2}\left[\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s+\mu b}{b}\right)t\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt\right.$$

$$\left.+\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s-\mu b}{b}\right)t\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt\right]$$

$$=\frac{1}{2}\left[e^{-i(s\mu d)}e^{\frac{-i}{2}(\mu^2bd)}\{2DCSST\ f(t,x)\}(s+ub,w)+e^{i(s\mu d)}e^{\frac{-i}{2}(\mu^2bd)}\{2DCSST\ f(t,x)\}(s-ub,w)\right]$$

$$\{2DCSST\cos\mu t\ f(t,x)\}(s,w)$$

$$=\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\{2DCSST\ f(t,x)\}(s+\mu b,w)+e^{i(\mu sd)}\{2DCSST\ f(t,x)\}(s-\mu b,w)\right]$$

Theorem 6.2 If $\{2DCSST\ f(t,x)\}(s,w)$ is canonical sine-sine transform of $f(t,x)$ then

$$\{2DCSST \sin\mu t\ f(t,x)\}(s,w) = \frac{ie^{-\frac{i}{2}(\mu^2 bd)}}{2}\left[e^{-i(s\mu d)}\{2DCCST\ f(t,x)\}(s+\mu b,w) - e^{i(\mu sd)}\{2DCCST\ f(t,x)\}(s-\mu b,w)\right]$$

Theorem 6.3 If $\{2DCSST\ f(t,x)\}(s,w)$ is canonical sine-sine transform of $f(t,x)$ then

$$\{2DCSSTe^{i\mu t}\ f(t,x)\}(s,w)$$

$$= \frac{e^{-\frac{i}{2}(\mu^2 bd)}}{2}\left[e^{-i(s\mu d)}\left(\{2DCSST\ f(t,x)\}(s+\mu b,w) - \{2DCCST\ f(t,x)\}(s+\mu b,w)\right)\right.$$

$$\left. + e^{i(s\mu d)}\left(\{2DCSST\ f(t,x)\}(s-\mu b,w) + \{2DCCST\ f(t,x)\}(s-\mu b,w)\right)\right]$$

Proof: Since $\{2DCSST\, e^{i\mu t}\ f(t,x)\}(s,w) = \{2DCSST(\cos\mu t + i\sin\mu t)\ f(t,x)\}(s,w)$

$$\{2DCSST\, e^{i\mu t}\ f(t,x)\}(s,w) = \{2DCSST \cos\mu t\ f(t,x)\}(s,w) + i\{2DCSST \sin\mu t\, f(t,x)\}(s,w)$$

$$-\frac{e^{-\frac{i}{2}(\mu^2 bd)}}{2}\left[e^{-i(s\mu d)}\{2DCCST\ f(t,x)\}(s+\mu b,w) - e^{i(s\mu d)}\{2DCCST\ f(t,x)\}(s-\mu b,w)\right]$$

$$\frac{e^{-\frac{i}{2}(\mu^2 bd)}}{2}\left[e^{-i(s\mu d)}\{2DCSST\ f(t,x)\}(s+\mu b,w) + e^{i(s\mu d)}\{2DCSST\ f(t,x)\}(s-\mu b,w)\right.$$

$$\left. -e^{-i(s\mu d)}\{2DCCST\ f(t,x)\}(s+\mu b,w) + e^{i(s\mu d)}\{2DCSCT\ f(t,x)\}(s-\mu b,w)\right]$$

$$\{2DCSSTe^{i\mu t}\ f(t,x)\}(s,w) = \frac{e^{-\frac{i}{2}(\mu^2 bd)}}{2}\left[e^{-i(s\mu d)}\left(\{2DCSST\ f(t,x)\}(s+\mu b,w) - \{2DCCST\ f(t,x)\}(s+\mu b,w)\right)\right.$$

$$\left. + e^{i(s\mu d)}\left(\{2DCSST\ f(t,x)\}(s-\mu b,w) + \{2DCCST\ f(t,x)\}(s-\mu b,w)\right)\right]$$

VII. Conclusion

In this paper 2D canonical sine–sine transform is generalized in the distributional sense. Uniqueness theorem is proved and various testing functions specs defined by using Gelfand-shilov technique, topology properties are discussed. And lastly modulation theorems are proved.

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Some Subclasses of P-Valent Analytic Functions

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Abstract - The object of the present paper is to derive the sufficient conditions for certain subclasses of p-valent analytic functions in the open unit disk. A number of known results would follow upon specializing the parameters involved in our main results. Also, sufficient conditions are found for function to be univalent.

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GJSFR-F Classification : MSC 2010: 11E45

SOME SUBCLASSES OF P-VALENT ANALYTIC FUNCTIONS

Strictly as per the compliance and regulations of :

Some Subclasses of P-Valent Analytic Functions

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Abstract - The object of the present paper is to derive the sufficient conditions for certain subclasses of p-valent analytic functions in the open unit disk. A number of known results would follow upon specializing the parameters involved in our main results. Also, sufficient conditions are found for function to be univalent.

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I. Introduction and Preliminaries

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic and p-valent in the open unit disk $\mathbb{U} := \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of analytic and univalent functions in $\mathbb{U}$. We note that $\mathcal{A}(1) \equiv \mathcal{S}$.

A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}^*(p, \alpha)$ of p-valently starlike of order α in $\mathbb{U}$ if and only if it satisfies the inequality

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < p).$$

On the other hand, a function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{K}(p, \alpha)$ of p-valently convex of order α in $\mathbb{U}$ if and only if it satisfies the inequality

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < p).$$

In particular, we write $\mathcal{S}^*(1, 0) := \mathcal{S}^*, \mathcal{K}(1, 0) := \mathcal{K}$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the usual subclass of $\mathcal{A}$, consisting of functions which are starlike and convex, respectively (see [1, 2]).

Ref.

[1] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer- Verlag, New York Berlin, Heidelberg, and Tokyo, (1983).

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The object of the present paper is to investigate various properties of the following classes of analytic and p-valent function defined as follows.

A function $f \in \mathcal{A}(p)$ is said to be a member of the class $\mathcal{B}(\gamma, \beta, p, \alpha)$ if and only if it satisfies the inequality

$$\left| \left(\frac{\beta\gamma z^3 f'''(z) + (2\beta\gamma + \beta - \gamma) z^2 f''(z) + z f'(z)}{\beta\gamma z^2 f''(z) + (\beta - \gamma) z f'(z) + (1 - \beta + \gamma) f(z)} \right) - p \right| < p - \alpha \tag{1}$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

for some α, for all $z \in \mathbb{U}$.

Note that the condition (1) implies that

$$\operatorname{Re} \left\{ \frac{\beta\gamma z^3 f'''(z) + (2\beta\gamma + \beta - \gamma) z^2 f''(z) + z f'(z)}{\beta\gamma z^2 f''(z) + (\beta - \gamma) z f'(z) + (1 - \beta + \gamma) f(z)} \right\} > \alpha,$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}).$$

We note that $\mathcal{B}(0, \beta, p, \alpha) \equiv \mathcal{T}_\beta(p; \alpha)$ is the class studied by Irmak and Raina in [3]. The important subclasses such as $\mathcal{S}^*(p, \alpha), \mathcal{K}(p, \alpha)$, $\mathcal{S}^*$ and $\mathcal{K}$ are seen to be easily identifiable with the aforesaid class.

In recent times, Irmak et al. [3] and Prajapat [9] investigated certain subclasses of multivalent analytic functions and obtained some sufficient conditions for these classes. In this paper, motivated by the aforementioned works, we obtained sufficient conditions for functions to be a member of the class $\mathcal{B}(\gamma, \beta, p, \alpha)$. We also indicate some special cases and consequences of the main result. The other results investigated include certain inequalities for p-valent functions which characterize the properties of starlikeness and convexity in the open unit disk. Furthermore our result unifies the result for a functions belonging to the class of p-valently starlike function of order α and p-valently convex function of order α.

In order to derive our main results, we need the following Lemmas.

Lemma 1. [4] *Let $w(z)$ be the non-constant and analytic function in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0) \tag{2}$$

where $k \geq 1$ is a real number.

Lemma 2. [5] *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(z)$ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \leq -n(1 + x^2)/2$. If the function $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ such that $\Phi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then* $\operatorname{Re} q(z) > 0$.

Lemma 3. [7] *Let δ be the complex number,* $\operatorname{Re} \delta > 0$, *and λ be a complex number, $|\lambda| \leq 1, \lambda \neq -1$ and let $h(z) = z + a_2 z^2 + \cdots$ be a regular function on $\mathbb{U}$. If*

$$\left| \lambda |z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{z h''(z)}{\delta h'(z)} \right| \leq 1$$

Ref.

[3] H. Irmak and R. K. Raina, The starlikeness and convexity of multivalent functions involving certain inequalities, *Rev. Mat. Comput.* (16) (2) (2003), 391–398.

for all $z \in \mathbb{U}$, *then the function*

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} h'(t)dt\right)^{1/\delta}$$
$$= z + \frac{2a_2}{\delta+1} z^2 + \left(\frac{3a_3}{\delta+2} + \frac{2\delta(1-\delta)a_2^2}{(\delta+1)^2}\right) z^3 + \cdots$$

is regular and univalent in $\mathbb{U}$.

Lemma 4. [8] *Let* δ *be a complex number,* $\operatorname{Re}\delta > 0$, *and* λ *a complex number,* $|\lambda| < 1$, *and* $h \in \mathcal{A}$. *If*

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1 - |\lambda|$$

for all $z \in \mathbb{U}$, *then for any complex number* $\eta, \operatorname{Re}\eta \geq \operatorname{Re}\delta$, *the function*

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} h'(t)dt\right)^{1/\eta}$$

is in the class $\mathcal{S}$.

Lemma 5. [6] *Let* $p(z)$ *be analytic in* $\mathbb{U}, p(0) = 1, p(z) \neq 0$ *in* $\mathbb{U}$ *and suppose that there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad for\ |z| < |z_0|, \ \ |\arg(p(z_0))| = \frac{\pi}{2}\alpha,$$

where $0 < \alpha \leq 1$, *then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \ \ when \quad \arg(p(z_0)) = \frac{\pi}{2}\alpha,$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \ \ when \quad \arg(p(z_0)) = -\frac{\pi}{2}\alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).$$

II. Main Results

By using Lemma 2, we first prove the following theorem.

Theorem 6. *Let* $f \in \mathcal{A}(p)$. *Define a function* $G_{\beta,\gamma}$ *by*

$$G_{\beta,\gamma}(z) := \beta\gamma z^2 f''(z) + (\beta-\gamma)zf'(z) + (1-\beta+\gamma)f(z), \quad (0 \leq \gamma \leq \beta \leq 1; z \in \mathbb{U}),$$

and if $G_{\beta,\gamma}(z)$ *satisfies*

$$\operatorname{Re}\left\{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(2 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right)\right\} > p\left(1 - \frac{n}{2}\right) + \frac{n}{2}\alpha$$

Ref.

[8] V. Pescar, Univalence criteria of certain integral operators, *Acta Cienc. Indica Math.*, (29) (1) (2003), 135–138.

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

Proof. Let $f \in \mathcal{A}(p)$. Define a function $w(z)$ in $\mathbb{U}$ by

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = p + (p-\alpha)w(z), (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}), \tag{3}$$

Notes

then the function $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$.
A computation using (3) shows that

$$\begin{aligned} \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left(2 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right) &= (p-\alpha)[zw'(z) + w(z)] + p \\ &= \Phi(w(z), zw'(z); z), \end{aligned}$$

where $\Phi(r, s; z) = (p-\alpha)[s + r] + p$.
For all real x, y satisfying $y \leq -n(1 + x^2)/2$, we have

$$\begin{aligned} \operatorname{Re} \Phi(ix, y; z) &= \operatorname{Re} \{(p-\alpha)[y + ix] + p\} \\ &\leq -\frac{n}{2}(p-\alpha)(1 + x^2) + p \\ &\leq -\frac{n}{2}(p-\alpha) + p \\ &= p\left(1 - \frac{n}{2}\right) + \frac{n}{2}\alpha. \end{aligned}$$

Let $\Omega = \left\{w : \operatorname{Re} w > p(1 - \frac{n}{2}) + \frac{n}{2}\alpha\right\}$. Then $\Phi(w(z), zw'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \leq -n(1 + x^2)/2,\ z \in \mathbb{U}$.

By using Lemma 2, we have $\operatorname{Re} w(z) > 0$, which implies that

$$\operatorname{Re}\left\{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right\} > \alpha, (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

and hence $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

By setting $\gamma = \beta = 0$ in Theorem 6, we have following corollary.

Corollary 7. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > p\left(1 - \frac{n}{2}\right) + \frac{n}{2}\alpha$$

$$(0 \leq \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in \mathcal{S}^*(p, \alpha)$.

Its further case when $\alpha = 0$ and $p = 1$, Corollary 7 reduces to Corollary 8.

Corollary 8. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\} > 1-\frac{n}{2}$$
$$(n \in \mathbb{N}),$$

then $f(z) \in \mathcal{S}^*$.

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By taking $\gamma = 0, \beta = 1$ in Theorem 6, we have the following corollary.

Corollary 9. *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\left(1+\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right)\right\} > p\left(1-\frac{n}{2}\right)+\frac{n}{2}\alpha$$
$$(0 \leq \alpha < p; p, n \in \mathbb{N}),$$

then $f(z) \in \mathcal{K}(p,\alpha)$.

A further case of Corollary 9, when $\alpha = 0, p = 1$ gives the following corollary.

Corollary 10. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\left(1+\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right)\right\} > 1-\frac{n}{2}$$
$$(n \in \mathbb{N}),$$

then $f(z) \in \mathcal{K}$.

Theorem 11. *Let $-1 < b < a \leq 1$, $0 \leq \alpha < p$, $p \in \mathbb{N}$ such that $p(1+\alpha)+a \leq 2p(p-b)+b$. If $G_{\beta,\gamma}(z)$ satisfies the inequality*

$$\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right| < \frac{p(a+b)}{(p+a)(p-b)} \quad (z \in \mathbb{U}), \tag{4}$$

then $f(z) \in \mathcal{B}(\gamma,\beta,p,\alpha)$.

Proof. Define a function $w(z)$ by

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{p+aw(z)}{p-bw(z)} \quad (z \in \mathbb{U}). \tag{5}$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. By the logarithmic differentiation of (5), we get

$$1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{p(a+b)zw'(z)}{(p+aw(z))(p-bw(z))}. \tag{6}$$

Now suppose that there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|}|w(z)| = |w(z_0)| = 1,$$

then from Lemma 1, we have (2). Letting $w(z_0) = e^{i\theta}$, from (6), we have

$$\left|1+\frac{z_0 G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)}-\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right| = \left|\frac{p(a+b)ke^{i\theta}}{(p+ae^{i\theta})(p-be^{i\theta})}\right| \geq \frac{p(a+b)}{(p+a)(p-b)}.$$

This contradicts our assumption (4). Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. Thus we conclude from (5) that

$$\begin{aligned}\left|\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}-p\right| &= \left|\frac{p+aw(z)}{p-bw(z)}-p\right| \\ &< \frac{p+a-p(p-b)}{p-b} \\ &\leq p-\alpha \quad (z\in\mathbb{U}),\end{aligned}$$

which implies that $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

Theorem 12. *Let $f \in \mathcal{A}(p)$. If $G_{\beta,\gamma}(z)$ satisfies anyone of the following conditions:*

$$\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right| < \frac{p-\alpha}{2p-\alpha}, \tag{7}$$

$$\left|\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right)\right| < p-\alpha, \tag{8}$$

$$\left|\frac{1+\dfrac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}}{\dfrac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}}-1\right| < \frac{p-\alpha}{(2p-\alpha)^2}, \tag{9}$$

$$\left|\frac{1+\dfrac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-p}{\dfrac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}-p}-1\right| < \frac{1}{(2p-\alpha)}, \tag{10}$$

$$\operatorname{Re}\left\{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\left(\frac{1+\dfrac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-p}{\dfrac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}-p}-1\right)\right\} < 1, \tag{11}$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

then $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

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Proof. Let $f \in \mathcal{A}(p)$. Define a function $w(z)$ in $\mathbb{U}$ by

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = p + (p-\alpha)w(z), (0 \le \gamma \le \beta \le 1; 0 \le \alpha < p; p \in \mathbb{N}), \tag{12}$$

then the function $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$.
It follows from (12) that

$$1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} = \frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}. \tag{13}$$

Hence, from (12) and (13), we have

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left(1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right) = (p-\alpha)zw'(z), \tag{14}$$

$$\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}} - 1 = \frac{(p-\alpha)zw'(z)}{[p+(p-\alpha)w(z)]^2}, \tag{15}$$

$$\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1 = \frac{zw'(z)}{w(z)} \frac{1}{p+(p-\alpha)w(z)}, \tag{16}$$

and

$$\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \left(\frac{1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - p}{\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p} - 1 \right) = \frac{zw'(z)}{w(z)}. \tag{17}$$

Now, suppose there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1,$$

then from Lemma 1, we have (2). Therefore, letting $w(z_0) = e^{i\theta}$ in each of (13)-(17), we obtain that

$$\left|1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right| = \left|\frac{(p-\alpha)ke^{i\theta}}{p+(p-\alpha)e^{i\theta}}\right| \ge \frac{p-\alpha}{2p-\alpha},$$

$$\left|\frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \left(1 + \frac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}\right)\right| = \left|(p-\alpha)ke^{i\theta}\right| \ge (p-\alpha),$$

$$\left| \frac{1+\dfrac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)}}{\dfrac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)}} - 1 \right| = \left| \frac{(p-\alpha)ke^{i\theta}}{(p+(p-\alpha)e^{i\theta})^2} \right| \geq \frac{p-\alpha}{(2p-\alpha)^2},$$

$$\left| \frac{1+\dfrac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - p}{\dfrac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} - p} - 1 \right| = \left| \frac{k}{p+(p-\alpha)e^{i\theta}} \right| \geq \frac{1}{(2p-\alpha)},$$

$$\mathrm{Re}\left\{ \frac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \left(\frac{1+\dfrac{z_0G''_{\beta,\gamma}(z_0)}{G'_{\beta,\gamma}(z_0)} - p}{\dfrac{z_0G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} - p} - 1 \right) \right\} = k \geq 1,$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

which contradict our assumption (7)-(11), respectively. Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. From (12), we have

$$\left| \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} - p \right| = |(p-\alpha)w(z)| < (p-\alpha), (0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

which implies that

$$\mathrm{Re}\left\{ \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} \right\} = \mathrm{Re}\left\{ \frac{\beta\gamma z^3 f'''(z) + (2\beta\gamma+\beta-\gamma)z^2 f''(z) + zf'(z)}{\beta\gamma z^2 f''(z) + (\beta-\gamma)zf'(z) + (1-\beta+\gamma)f(z)} \right\} > \alpha,$$

$$(0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < p; p \in \mathbb{N}),$$

and hence $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$.

Remark 1. *By taking* $\gamma = 0$; $\gamma = \beta = 0$; $\gamma = 0$ *and* $\beta = 1$; $\gamma = \beta = \alpha = 0$ *and* $p = 1$; $\gamma = \alpha = 0$ *and* $\beta = p = 1$ *in Theorem 12, we get the results of Irmak and Raina* [3, Theorem 1, Corollary 1-4].

Theorem 13. *Let* $0 \leq \gamma \leq \beta \leq 1; 0 \leq \alpha < 1, \delta$ *be a complex number,* $\mathrm{Re}\,\delta \geq \dfrac{3-2\alpha}{2-\alpha}$ *and* λ *be a complex number which satisfies the inequality*

$$|\lambda| \leq 1 - \frac{3-2\alpha}{\mathrm{Re}\,\delta(2-\alpha)}. \tag{18}$$

If $F_{\beta,\gamma}(z) := \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}$ *is regular in* $\mathbb{U}$ *and*

Ref.

[3] H. Irmak and R. K. Raina, The starlikeness and convexity of multivalent functions involving certain inequalities, *Rev. Mat. Complut.* (16) (2) (2003), 391–398.

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$$\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right|\leq\frac{1-\alpha}{2-\alpha}\quad(z\in\mathbb{U}),\tag{19}$$

then the function

$$F(z)=\left(\delta\int_0^z t^{\delta-1}\frac{G'_{\beta,\gamma}(t)}{G_{\beta,\gamma}(t)}dt\right)^{1/\delta}\tag{20}$$

is univalent in $\mathbb{U}$.

Proof. Define a function

$$h(z)=\int_0^z\frac{F_{\beta,\gamma}(t)}{t}dt,$$

then we have $h(0)=h'(0)-1=0$. Also a simple computation yields $h'(z)=\dfrac{F_{\beta,\gamma}(z)}{z}$ and

$$\frac{zh''(z)}{h'(z)}=\frac{zF'_{\beta,\gamma}(z)}{F_{\beta,\gamma}(z)}-1.\tag{21}$$

From (21), we have

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right|&\leq\left|\frac{zF'_{\beta,\gamma}(z)}{F_{\beta,\gamma}(z)}\right|+1\\&=\left|1+\frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)}-\frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right|+1.\end{aligned}\tag{22}$$

Hence, from (19) and (22), we have

$$\left|\frac{zh''(z)}{h'(z)}\right|\leq\frac{3-2\alpha}{2-\alpha}.\tag{23}$$

Using (23), we have

$$\begin{aligned}\left|\lambda\left|z\right|^{2\delta}+\left(1-\left|z\right|^{2\delta}\right)\frac{zh''(z)}{\delta h'(z)}\right|&\leq|\lambda|+\left|\frac{zh''(z)}{\delta h'(z)}\right|\\&\leq|\lambda|+\frac{1}{\mathrm{Re}\delta}\frac{3-2\alpha}{2-\alpha}.\end{aligned}$$

Again using (18), we have

$$\left|\lambda\left|z\right|^{2\delta}+\left(1-\left|z\right|^{2\delta}\right)\frac{zh''(z)}{\delta h'(z)}\right|\leq 1.$$

Applying Lemma 3, we obtain that the function $F(z)$ defined by (20) is univalent in $\mathbb{U}$.

We obtain Theorem 14 below, by using Lemma 4 and the same techniques as in the proof of Theorem 13.

Theorem 14. *Let δ be a complex number,* $\operatorname{Re}\delta > 0$, *$\lambda$ a complex number,* $|\lambda| < 1$, *and $f \in \mathcal{A}$. If*

$$\left|1 + \frac{zG''_{\beta,\gamma}(z)}{G'_{\beta,\gamma}(z)} - \frac{zG'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)}\right| \le \frac{1-\alpha}{2-\alpha} \quad (z \in \mathbb{U}; 0 \le \gamma \le \beta \le 1; 0 \le \alpha < 1),$$

then for any complex number η,

$$\operatorname{Re}\eta \ge \operatorname{Re}\delta \ge \frac{3-2\alpha}{(1-|\lambda|)(2-\alpha)},$$

the integral operator

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} \frac{G'_{\beta,\gamma}(t)}{G_{\beta,\gamma}(t)} dt\right)^{1/\eta}$$

is in the class $\mathcal{S}$.

Theorem 15. *Let $p(z)$ be an analytic function in $\mathbb{U}, p(z) \neq 0$ in $\mathbb{U}$ and suppose that*

$$\left|\arg\left(p(z) + \frac{z^2 G'_{\beta,\gamma}(z)}{G_{\beta,\gamma}(z)} p'(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}), \tag{24}$$

where $0 < \alpha < p, 0 \le \gamma \le \beta \le 1$ and $f(z) \in \mathcal{B}(\gamma, \beta, p, \alpha)$, then we have

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Proof. Suppose there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad for\ |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2}\alpha.$$

Then, applying Lemma 4, we have

$$\begin{aligned} \arg\left(p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} p'(z_0)\right) &= \arg\left(p(z_0)\left(1 + \frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\ &= \arg(p(z_0)) + \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right). \end{aligned} \tag{25}$$

When $\arg(p(z_0)) = \pi\alpha/2$, since

$$\operatorname{Re}\left(\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) > 0 \Rightarrow \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) > 0,$$

Eq. (25) becomes

$$\arg\left(p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} p'(z_0)\right) > \frac{\pi}{2}\alpha. \tag{26}$$

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Similarly, if $\arg(p(z_0)) = -\pi\alpha/2$, since

$$\operatorname{Re}\left(\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) < 0 \Rightarrow \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) < 0,$$

we obtain that

$$\arg\left(p(z_0) + \frac{z_0^2 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} p'(z_0)\right) = \arg(p(z_0)) + \arg\left(1 + i\frac{z_0 G'_{\beta,\gamma}(z_0)}{G_{\beta,\gamma}(z_0)} k\alpha\right) < -\frac{\pi}{2}\alpha. \tag{27}$$

Thus, we see that (26) and (27) contradict our assumption (24). Consequently, we conclude that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

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A Summation Formula Clung to Contiguous Relation

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Abstract - The main aim of the present paper is to evaluate a summation formula in the shadow of contiguous relation and recurrence relation.

Keywords : gaussian hypergeometric function, contiguous function, recurrence relation, bailey summation theorem and legendre duplication formula.

GJSFR-F Classification : MSC 2010: 33C60 , 33C70 , 33D15 , 33D50 , 33D60.

A SUMMATION FORMULA CLUNG TO CONTIGUOUS RELATION

Strictly as per the compliance and regulations of :

A Summation Formula Clung to Contiguous Relation

Salahuddin [α] & M. P. Chaudhary [σ]

Abstract - The main aim of the present paper is to evaluate a summation formula in the shadow of contiguous relation and recurrence relation.

Keywords : gaussian hypergeometric function, contiguous function, recurrence relation, bailey summation theorem and legendre duplication formula.

I. Introduction

Generalized Gaussian hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lll} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{lll} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lll} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$ (a-b)\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ \ \ c\ \ ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ll} a+1,\ b\ ; & \\ \ \ c\ \ ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ll} a,\ b+1\ ; & \\ \ \ c\ \ ; & z \end{array}\right] \tag{2} $$

Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\,\Gamma(z) \tag{3} $$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$ \sqrt{\pi}\,\Gamma(2z) = 2^{(2z-1)}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \tag{4} $$

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$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{5}$$

$$= \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{6}$$

Bailey summation theorem [Prud, p.491(7.3.7.8)]is defined as follows

$${}_2F_1\left[\begin{matrix} a, 1-a & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} \tag{7}$$

II. Main Result of Summation Formula

$${}_2F_1\left[\begin{matrix} a\ , & -a-50 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+50}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}\Big\{-34928139885605399750809119151620096000 0a$$

$$+4135768267261046395174249920112558080 00a^2$$

$$-1787359418883707931210034483807076352 00a^3$$

$$+356646499044269595131950037578039910 40a^4$$

$$-31155457766645392793468566675536291 84a^5$$

$$+2833446465662469238371437934356505 6a^6 + 1164904998949148764385393941775212 8a^7$$

$$-33899383724788152223788772718188 8a^8 - 2396789009667706671298440908056 0a^9$$

$$+57608139834188021547136979889 6a^{10} + 3582435594570356489340690378 8a^{11}$$

$$-21875396022431441793986218 8a^{12} - 30812261856226631564337865a^{13}$$

$$-294303555355218100065024a^{14}+9714972848474309029418a^{15}+233658656740810973892a^{16}$$

$$+66546794026691214 5a^{17} - 3378511960045886 4a^{18} - 45094814057471 2a^{19}$$

$$-1349964560868a^{20} + 14581955465a^{21} + 138711936a^{22} + 384578a^{23} + 12a^{24} - a^{25}$$

$$+349281398856054617956492924755640320000c$$

$$-107301280389995420369545074121624780800 0ac$$

$$+82693199264845062375514151945296773120 0a^2c$$

$$-26817888035228431000382086830266732544 0a^3c$$

$$+40676832460641070925386987323409864704a^4c$$

$$-2401490113928993197147231477443257856a^5c$$

$$-512158336652942331783233013400193 28a^6c+99981486234171070308068242556762 88a^7c$$

$$-24413055565924722735355620028320a^8c - 1914225699302339237611049471529 6a^9c$$

Notes

$$-232274039557158178140507496 88a^{10}c+210184656722763916906228484 88a^{11}c$$

$$+2471797320536397310206220 10a^{12}c-103062072108967249175657 76a^{13}c$$

$$-254358255448153386816468a^{14}c+323506520010497774808a^{15}c$$

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$$+26a^{24}c+659435977173851906965242147923558400000c^2$$

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$$+312a^{23}c^2+55246435938562021842840807485000908 8000c^3$$

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$$+32934395831879083071449170686301647 6672a^2c^3$$

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$$+277010882739343091655227615432461516800c^4$$

Notes

$$-2873760517765312949918066484135686307840ac^4$$

$$+1036676267812825613913782589915998453760a^2c^4$$

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$$-4313921375227166560a^{15}c^4 - 33558810621336000a^{16}c^4 + 45306739461984a^{17}c^4$$

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$$+9420627436165792745251737839816461516800c^5$$

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$$-1481727457514304a^{16}c^5 - 4456341489600a^{17}c^5 + 151205454400a^{18}c^5 + 951350400a^{19}c^5$$

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$$+383252528792830255890192550682610892800a^2c^6$$

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$$-92013545108210990074166259097600a^7c^6 + 269084999796825878210496107520a^8c^6$$

$$+82468385385445718215586265600a^9c^6 - 7307957274466037614641216000a^{10}c^6$$

$$-3750262082560487419685760a^{11}c^6 - 16237251367964842575360a^{12}c^6$$

$$+53279856667239634432a^{13}c^6 + 638921254888844928a^{14}c^6 + 463333171357056a^{15}c^6$$

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$$-27119794141440a^{16}c^6 - 134770796160a^{17}c^6 - 98978880a^{18}c^6 + 320320a^{19}c^6$$
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$$-249479792834281436764860534848539852 8ac^7$$
$$+486050047323646914801852260812849152a^2c^7$$
$$-358338803148453284795112566308208 64a^3c^7$$
$$+185780709999525528303843109634048a^4c^7$$
$$+7892916533772326801817468007219 2a^5c^7 - 1177109599312891745506779234304a^6c^7$$
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$$-304776543731945699522669706928455680ac^8$$
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$$+1917026424809845954644049920a^6c^8 - 5199988494853301193094901760a^7c^8$$
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$$+25157758400344258191360a^{10}c^8 - 260482121035239966720a^{11}c^8$$
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$$+37821073333897027811334467493560 32a^2c^9$$
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$$+2850641578081761835754884300 80a^5c^9 + 3765845101952647026736783360a^6c^9$$
$$-2019584461519864577772994 56a^7c^9 - 2986095861362649319691264a^8c^9$$
$$+4931546943537162584064 0a^9c^9 + 10682636024185104179 20a^{10}c^9$$
$$+2281972160200212 48a^{11}c^9 - 10197251259472076 8a^{12}c^9 - 584540190474240a^{13}c^9$$
$$+590920478720a^{14}c^9 + 9857802240a^{15}c^9 + 12446720a^{16}c^9$$
$$+715603528748546837854090690560 0000c^{10}$$

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$$-23508047138326125897557496299520000ac^{10}$$
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$$+190513152a^{11}c^{14}+31764516878150060069617664 00c^{15}$$

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$$-3688574120239732003574906 88ac^{15} + 692145504817981649688985 6a^2c^{15}$$

$$+29332147035959662215168 0a^3c^{15} - 46815224839340032000 00a^4c^{15}$$

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$$-19931332608ac^{23} - 201326592a^2c^{23} + 20132659200c^{24}$$

$$-16777216ac^{24} + 33554432c25\Big\}$$

$$+\frac{1}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+51}{2})}\Big\{+19607814681608194157031720804679680 00000$$

$$-3661026774131950114663981678663434240000a$$

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$$-3043255274124936913351196421079574016a^5$$

$$-229179332123522872538863190221839744a^6$$

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\begin{gathered}
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+276944498519675 36a^{18} - 209906546918288a^{19} - 4290754509118a^{20} - 23145910865a^{21} \\
-7772764a^{22} + 379822a^{23} + 1262a^{24} + a^{25} + 5080828409738663152845985129137438720000c \\
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+32142393315032a^{19}c - 382686073770a^{20}c - 3536396864a^{21}c - 9614228a^{22}c \\
+312a^{23}c + 26a^{24}c + 538308198504070379869768104668037120000 0c^2 \\
-600463487254115047865710931881073049600 0ac^2 \\
+23658072409661423488083141668465187225 60a^2c^2 \\
-4132017256050339287343444801001721364 48a^3c^2 \\
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+6892565247568a^{19}c^2 + 5758672920a^{20}c^2 - 108628520a^{21}c^2 - 393432a^{22}c^2 \\
-312a^{23}c^2 + 326450586299244027981274919219075481 6000c^3
\end{gathered}
$$

Notes

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$$-5823830370208328913615653906572902 4ac^{9}$$

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$$+531805897684928490673213788730163\!2a^2c^9$$

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$$+6204214476800c^{23} - 201326592ac^{23} - 201326592a^{2}c^{23} + 20971520000c^{24}$$

$$+16777216ac^{24} + 33554432c^{25}\Big\}\Big] \tag{8}$$

Derivation of main result (8):

Substituting $b = -a - 50, z = \frac{1}{2}$ in given result (2),we get

$$(2a+50)\ {}_2F_1\left[\begin{matrix} a \ , & -a-50 & ; \ \frac{1}{2} \\ c & & ; \end{matrix}\right]$$

$$= a\ {}_2F_1\left[\begin{matrix} a+1 \ , & -a-50 & ; \ \frac{1}{2} \\ c & & ; \end{matrix}\right] + (a+50)\ {}_2F_1\left[\begin{matrix} a \ , & -a-49 & ; \ \frac{1}{2} \\ c & & ; \end{matrix}\right]$$

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Now using same parallel method which is used in Ref[6], we can prove the main result.

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A Logit Regression Analysis of Homeowners in Nigeria

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Abstract - This paper studied the application of logit regression analysis to homeownership in Ado-Ekiti area of Ekiti State (Nigeria). The performance of the logit model in terms of classification of homeowners with respect to the average monthly income of some individuals was examined. The data of homeownership and income was fitted to the model by the WLS techniques. Result showed that the odds ratio in favour of owning a house by an individual whose average monthly income is 0.158 (N Million) was 1.0387. Also the probability of owning a house by the individual was 0.51.

Keywords : logit regression, home ownership, average monthly income, odds ratio, WLS-weighted least squares, OLS-ordinary least squares.

GJSFR-F Classification : MSC 2010: 62G08

A LOGIT REGRESSION ANALYSIS OF HOMEOWNERS IN NIGERIA

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A Logit Regression Analysis of Homeowners in Nigeria

O.Y. Halid [α] & F.I. Akinnitire [σ]

Abstract - This paper studied the application of logit regression analysis to homeownership in Ado-Ekiti area of Ekiti State (Nigeria). The performance of the logit model in terms of classification of homeowners with respect to the average monthly income of some individuals was examined. The data of homeownership and income was fitted to the model by the WLS techniques. Result showed that the odds ratio in favour of owning a house by an individual whose average monthly income is 0.158 (N Million) was 1.0387. Also the probability of owning a house by the individual was 0.51.

Keywords : logit regression, home ownership, average monthly income, odds ratio, WLS-weighted least squares, OLS-ordinary least squares.

I. Introduction

Feeding, clothing and shelter constitute the fundamental human needs. Among the three, shelter is the most complex, challenging and expensive. Apart from providing shelter, housing is a sure way of creating jobs, eradicating poverty, ensuring social security and propagating sustainable economic development.

In Nigeria, the housing sector is the second highest employer of labour next only to agriculture. Research has consistently shown the importance of the housing sector on the economy and the long social and financial benefits to individual homeowners.

Homeownership brings substantial social benefits for families, communities and the country as a whole. Because of these benefits, policy makers have promoted homeownership through a number of channels. Homeownership has been an essential element of the Nigerian dream for decades and continues to be even today.

Homeownership has significant impact on social outcomes, specifically educational achievement, civil participation, health benefits, public assistance, property maintenance and improvement.

In general, research supports the view that homeownership brings substantial social benefits. Because of these extensive social benefits, policies that support homeownership are well justified.

Apart from all these advantages, homeownership also provides one with pride of ownership, freedom of control, privacy, strong credit base, financial stability, appreciating asset to mention a few.

The immense economic benefits of homeownership are also well documented. For instance, United Nations Centre for Human Settlement estimate Nigeria's current housing deficit at 16 million units. Considering that an average household is between five and six persons, it is inferred by experts that 80 - 96 million housing units will be achieved within

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a period 30 - 50 years provided construction of 200,000 - 250,000 new housing units is carried out yearly. This is disheartening in the view of the fact that current national housing production figures stand at less than 10,000 per year, not to mention the estimated cost which runs into tens of trillions of Naira.

Inspite of the growing trend of homeownership in Nigeria, certain impediments such as high unemployment rate and moderate GDP growth of between 7% and 9%, inflation and high interest rate which all degenerate from unwholesome macro-economic environment and lack of financing systems are still lingering.

Land use acts, high cost of building materials, high cost of land in urban areas, poor quality of construction are also impediments to homeownership.

II. The Logit Regression Model

The logit regression analysis is a technique which allows for estimating the probability that an event occur or not by predicting a binary dependent outcome from a set of independent variable(s). The applications of the logit model to different areas had been previously seen in sources such as Ojo (1989), Gujarati (2003), Ogunleye and Fagbohun (2009) and others too many to mention.

Let x_i be a random variable (say income) and $y=1$ indicating an individual owns a house $y=0$ if otherwise, then

$$P_i = E(Y=1 \mid X_i) = \frac{1}{1+e^{-(\beta_1+\beta_2 X_i)}} \quad (1)$$

for simplicity, we set $Z_i = \beta_1 + \beta_2 X_i$,

$$P_i = \frac{1}{1+e^{-Z_i}} = \frac{e^{Z_i}}{1+e^{Z_i}} \quad (2)$$

(2) is called the cumulative logistic distribution function,

where Z_i ranges from $-\infty$ to $+\infty$, P_i ranges from 0 to 1, called the probability of owning a house.

P_i is non linearity related to Z_i and β_i's are the model coefficients.

Since P_i is the probability of owning a house, then

$$1-P_i = \frac{1}{1+e^{Z_i}} \quad (3)$$

called the probability of not owning a house.

Then, we write

$$\frac{P_i}{1-P_i} = \frac{1+e^{Zi}}{1+e^{-Zi}} = e^{Z_i} \quad (4)$$

called the odds ratio in favour of owning a house (the ratio of the probability that an individual will own a house to the probability of not owning a house).

Taking the natural logarithm of (4), we get

$$L_i = ln\left(\frac{P_i}{1-P_i}\right) = Z_i = \beta_1 + \beta_2 X_i + \mu_i \quad (5)$$

Notes

called logit model. L_i is called the logit and μ_i is the stochastic error term.

III. Estimation

The estimate of model coefficient of logit regression models depends on the data at hand. This is categorized as follows.

Notes

Case I

Suppose the data at hand is on individuals, the OLS technique becomes infeasible since if we have $P_i=1$, if an individual owns a house, and $P_i=0$, if he does not, then the logit

$$L_i = \begin{cases} ln\left(\frac{1}{0}\right), \text{ if an individual owns a house} \\ ln\left(\frac{0}{1}\right), \text{ if an individual does not own a house} \end{cases} \tag{6}$$

Clearly, these expressions are meaningless and hence such data cannot be used in estimation of (5) by the OLS method.

As a result of this, the maximum likelihood method may be used.

Case II

For a grouped data on several individuals grouped according to income level and number of individuals owning a house at each income level X_i, there are N_i individuals, n_i among whom are homeowner $(n_i \le N_i)$ so that

$$\hat{P}_i = \frac{n_i}{N_i} \tag{7}$$

This is the relative frequency which can be used as the true P_i corresponding to each X_i.

If N_i is fairly large, $\hat{P}_i$ will be a reasonably good estimate of P_i using the estimated $\hat{P}_i$, the logit estimate in (5) can be obtained by

$$\hat{L}_i = ln\left(\frac{\hat{P}_i}{1-\hat{P}_i}\right) = Z_i = \hat{\beta}_1 + \hat{\beta}_2 X_i \tag{8}$$

which is a fairly good estimate of the true logit L_i assuming the N_i at each X_i is reasonably large.

Since the properties of the stochastic error term μ_i is unknown, and N_i is fairly large, X_i is independently distributed binomial variable so that

$$\mu_i \sim N\left[0, \frac{1}{N_i P_i (1-P_i}\right] \tag{9}$$

which implies that μ_i follows the normal distribution with mean zero and variance $\frac{1}{N_i P_i (1-P_i)}$.

Consequently, the logit is estimated using the weighted least square (WLS) procedure to resolve the problem of heteroscedasticity.

This gives rise to

$$\sqrt{w_i}\,L_i = \beta_1\sqrt{w_i} + \beta_2\sqrt{w_i}\,X_i + \sqrt{w_i}\,\mu_i \tag{10}$$

which can be written as

$$L_i^* = \beta_1\sqrt{w_i} + \beta_2 X_i^* + \upsilon_i \tag{11}$$

where weights $w_i = N_i P_i(1-P_i)$, L_i^* is the weighted L_i, X_i^* weighted X_i and υ_i weighted μ_i.

The odds ratio in favour of owning a house by an individual with average income X_i is given by

$$\frac{\hat{P}_i}{1-\hat{P}_i} \tag{12}$$

where $\hat{P}_i$ is the estimated probability of owning a house, while the estimated logit is given by

$$ln\left(\frac{\hat{P}_i}{1-\hat{P}_i}\right) \tag{13}$$

The probability of an individual with average monthly income X_i owning a house is

$$\hat{P}_i = \frac{e^{-L_i^*}}{1+e^{-L_i^*}} \tag{14}$$

where L_i^* is as defined in (11).

IV. Analysis

A questionnaire was administered to 100 inhabitants of different areas of Ado-Ekiti, Ekiti State.

These 100 respondents were classified into 5 groups of 20 individuals, each based on their average monthly income.

Out of each (N_i) 20 individual, 14, 7, 5, 6 and 9 were homeowners (n_i) giving rise to respective relative frequencies 0.70, 0.35, 0.25, 0.30 and 0.45.

The respective weights (w_i) are 4.20, 4.55, 3.75, 4.20 and 4.95 with corresponding average monthly income 0.19, 0.15, 0.15, 0.18 and 0.12 (N million).

These data was fitted to the logit model in (11) by the WLS technique using SAS 9.3 so that

$$L_i^* = -2.4979\sqrt{w_i} + 4.6432\,X_i^* \tag{15}$$

is the estimated regression curve with coefficient of determination (R^2) value 0.9228.

Notes

The odds ratio in favour of owning a house by an individual whose average income is 0.158(N million) is 1.0387 while the probability of owning a house by such individual is 0.51. Also, an estimated logit of 0.038 was also obtained.

V. Conclusion

Notes

The coefficient of determination value 0.9228 was an indication of a goodness of fit. This also indicates a strong relationship between the income and the probability of owning a house.

The calculated t-value of 32.44 is hugely in excess of the tabulated value of 5.01 and hence this leads to the conclusion that income of individuals will influence the probability of being homeowners.

The probability 0.51 of owning a house by an individual whose average monthly income is 0.158 (N million) was obtained. This shows that an increase in monthly income may increase the probability of owning a house.

The odds ratio of 1.0387 in favour of owning a house gives a slight advantage over the chance of not owning a house in the state.

VI. Recommendation

Government should make some amendments on the land use act to make more land available for residential purposes in certain 'newly created' states such as Ekiti. This will enhance rapid development in the state.

Government through its agencies should control the activities of land owners, middlemen, estate valuers and other play makers involved in land issues.

This will control the land prices and make more land available for residential purposes. This will also reduce cases of land disputes.

Non-governmental organisations, mortgage banks and other private investors should invest more into housing schemes in the state and the country at large.

This will make more people to become homeowners and make them enjoy the full benefits of home ownership.

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The Development of Discrete Version of Laplace Transformation (Sigma (σ) Transformation) Obtained from the Relationship between Laplace and Fourier Transformations

By Dr. Umana Thompson Itaketo
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Abstract - The relationships between Laplace and Fourier transformations are studied very closely. It is observed that Fourier transformation can be obtained from Laplace transformation but the reverse is not true. Based on this, a generic mathematical analysis leads to an expression relating Laplace transformation to Fourier transformation. Further mathematical analysis from that expression leads to something quite new: The Discrete Version of Laplace transformation, which the author calls sigma (σ) transformation.

Keywords : laplace transformation, fourier transformation, relationships, analysis, discrete, sigma (σ) transformation.

GJSFR-F Classification : MSC 2010: 44A10

THE DEVELOPMENT OF DISCRETE VERSION OF LAPLACE TRANSFORMATION SIGMA TRANSFORMATION OBTAINED FROM THE RELATIONSHIP BETWEEN LAPLACE AND FOURIER TRANSFORMATIONS

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The Development of Discrete Version of Laplace Transformation (Sigma (σ) Transformation) Obtained from the Relationship between Laplace and Fourier Transformations

Dr. Umana Thompson Itaketo

Abstract - The relationships between Laplace and Fourier transformations are studied very closely. It is observed that Fourier transformation can be obtained from Laplace transformation but the reverse is not true. Based on this, a generic mathematical analysis leads to an expression relating Laplace transformation to Fourier transformation. Further mathematical analysis from that expression leads to something quite new: The Discrete Version of Laplace transformation, which the author calls sigma (σ) transformation.

Keywords : laplace transformation, fourier transformation, relationships, analysis, discrete, sigma (σ) transformation.

I. Introduction

It is a common knowledge in Mathematics that Laplace transformation is about converting any time-dependent function, f(t), to a complex domain function, F(s), (where s = σ + jw) (Ogata, 2010) and vice-versa. Also, it is equally known in Mathematics that Fourier transformation is about converting a time-dependent function, f(t), to a frequency-dependent function, F(w), and vice-versa. Mathematical expressions representing these two statements are given by:

$$F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt \qquad (1)$$

(Stephenson, 2011, 2nd ed.)

(Equation for Laplace transformation).

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$$F(t) = \mathcal{L}^{-1}\ F(s) = \frac{1}{2\pi e^{st}} \int_{\sigma\ jw}^{\sigma + jwt} F(s)\ ds \qquad (2)$$

(Stephenson, 2011, 2nd ed.)
(Equation for inverse Laplace transformation).

$$F(jw) = \int_{-\infty}^{\infty} e^{-jwt}\ f(t)\ dt \qquad (3)$$

(Gabel, 2010)
(Equation for Fourier transformation).

$$F(t) = \mathcal{L}^{-1}\ F(jw) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw) e^{jwt}\ dw \qquad (4)$$

(Gabel, 2010)
(Equation for inverse Fourier transformation).

Taking a hard look at those four (4) equations, one would have some impressions that there are some relationships between Laplace and Fourier transformations which have not yet been explored. This paper attempts to uncover, link and exploit such relationships.

II. Establishing a Relationship Between Laplace and Fourier Transformation

Looking at equations (1) and (3), it will immediately be noticed that Fourier transformation, F(jw), is a special case of Laplace transformation F(s), when S = jw, covering the entire time spectrum -∞ to +∞. (Recall that in the complex plane, S = σ + jw. Hence when the real component σ = 0, S = 0 + jw, and F(s) = F(jw)). This is demonstrated below:

Refer to equation (1),

$$F(s) = \int_{-\infty}^{\infty} e^{-st}\ f(t)\ dt \qquad \text{(Eqn. (1) recalled).}$$

But S = σ + jw

$$\therefore F(\sigma + jw) = \int_{0}^{\infty} e^{-(\sigma + jw)t}\ f(t)\ dt \qquad (5)$$

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If σ = 0, equation (5) becomes:

$$F(jw) = \int_0^{\infty} e^{-jwt} f(t)\, dt \qquad (6)(a)$$

Equation (6)(a) is a one-sided Fourier transformation in the right half of the time axis. Equation (3) equally expresses Fourier transformation but across the entire time axis (-∞ to +∞). It can hence be confirmed that Fourier transformation is a special case of Laplace transformation when σ = 0. Now, if in equation (5),

$$jw = 0,\ F(\sigma) = \int_0^{\infty} e^{-(\sigma)t} f(t)\, dt \qquad (6)(b).$$

It should be noted that equation (6)(b) does not represent Laplace transformation. Hence, the reverse of how equation (6)(a) was obtained is not true. This will further be explained later.

III. THE LAW OF LINEARITY APPLIED TO LAPLACE AND FOURIER TRANSFORMATIONS

If σ and jw are linearly-related, and the law of linearity applied to equation (5), F(σ+ jw) in equation (5) becomes:

$$F(s) = F(\sigma + jw) = F(\sigma) + F(jw) \text{ (Kreyszig, 2005)} \qquad (7)$$

But from (5),

$$F(\sigma + jw) = \int_0^{\infty} e^{-(\sigma + jw)t} f(t)\, dt \qquad \text{(eqn. (5) recalled).}$$

$$:.\ F(\sigma) + F(jw) = \int_0^{\infty} e^{-(\sigma + jw)} f(t)\, dt \qquad (8)$$

Also, from equation (3),

$$F(jw) = \int_{-\infty}^{\infty} e^{-jwt} f(t)\, dt \qquad \text{(eqn. (3) recalled)}$$

Substituting equation (3) in (8), we have:

$$F(\sigma) + \int_{-\infty}^{\infty} e^{-jwt} f(t)\, dt = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t)\, dt \qquad (9)$$

Equation (9) then becomes:

$$F(\sigma) + \left(\int_{0}^{\infty} e^{-jwt} f(t)\, dt + \int_{\infty}^{0} e^{-jwt} f(t) dt \right) = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t)\, dt$$

$$:.\ F(\sigma) = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t)\, dt - \int_{0}^{\infty} e^{-jwt} f(t)\, dt - \int_{-\infty}^{0} e^{-jw} f(t)\, dt$$

$$F(\sigma) = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t)\, dt - \int_{0}^{\infty} e^{-jwt} f(t)\, dt - \int_{-\infty}^{0} e^{-jwt} f(t)\, dt$$

$$F(\sigma) = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t)\, dt - \left(\int_{0}^{\infty} e^{-jwt} f(t)\, dt + \int_{-\infty}^{0} e^{-jwt} f(t)\, dt \right)$$

$$F(\sigma) = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t)\, dt - \left(| \tfrac{1}{2}F(jw)| + | \tfrac{1}{2}F(jw) | \right)$$ (Ejimanya, 2005).

$$F(\sigma) = \int_{0}^{\infty} e^{-(\sigma + jw)t} f(t) - F(jw)$$

$$F(\sigma) = F(S) - F(jw) \qquad (10)$$

Equation (10) goes to prove the linearity property of the complex spectrum, s(=σ+ jw) earlier postulated in equation (7), on which Laplace transformation is based.

However, it is pertinent to mention here that Laplace transformation is expressed by F(s), not F(σ). What then does F(σ) signify?

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IV. The Significance of F(σ)

The expression F(σ) can be viewed as the real axis component of Laplace transformation while F(jw) (Fourier transformation), can be viewed as the imaginary component of Laplace transformation. This implies that F(σ) and F(jw) are complementary to each other to produce F(σ). Hence making use of equation (10), a table could be developed for F(σ) from the knowledge of F(s) and F(jw).

Now, given a function F(t), its F(s) and F(jw) can be determined or obtained from tables. A difference of that, that is F(s) – F(jw) will give F(σ) which is the real axis component of Laplace transformation of the given expression. It should be noted that as at today, a table of F(σ) does not exist.

As earlier mentioned above, it would strike readers compulsively that Fourier transformation, F(jw) of any function, is the imaginary component of Laplace transformation because it can be obtained by setting σ = 0 in s = σ + jw as earlier discussed. Fortunately, this has been developed and presented in tables as Fourier transformation. However, by setting jw = 0 in s = σ + jw, the tables for the resulting expression, F(s) = F(σ + 0) = F(σ), has never been developed nor interpretation given to its meaning.

A word of caution in Laplace transformation: One may argue that the claim that F(σ) has never been developed is not true. That is, the individual could express F(σ) as:

$$F(\sigma) \quad = \int_0^{\infty} e^{-\sigma t} \; f(t)\,dt \qquad (11)$$

(11) (Nagrath *et al.*, 2009 2nd ed.)

It should be noted that equation (11)does not express Laplace transformation. The right hand side of equation (11) only resembles Laplace transformation. Reason: The right hand side of any Laplace transformation MUST contain e^{-st}. It should be noted that $e^{-\sigma t}$ in equation (11) is not the same as e^{-st}. In every Laplace expression, the "s" component (signifying a complex spectrum: s = σ + jw), must be there, otherwise it is not Laplace transformation.

V. Applications of F(σ)

As already mentioned above, F(σ) signifies the real axis component of Laplace expression. It has been shown in equation (10) that this component can be obtained from the expression F(σ) = F(s) – F(jw). It will not be mathematically true that F(σ) for any function can be obtained from a Laplace transformation of that function simply by setting jw = 0. The reason for this claim is that Laplace transformation of a function has an "s" (a complex function given by s = σ + jw), embedded in it. The "s" is usually applied in all the processes of obtaining the final result of F(s). It will therefore be mathematically untrue to go to the final expression of any Laplace transformation expression and simply put jw = 0, obtain an expression, and claim that expression to represent F(σ).

The following example will justify this argument:

Let's take a function $f(t) = te^{-at}$.

From Laplace tables, the Laplace transform, F(s) of that function is

$$F(s) = \left(\frac{1}{s + a}\right)^2 = \frac{1}{(s + a)^2} \qquad \text{(Distefanno III, } et\ al.\text{, 2009)} \qquad (12)$$

Also, from Fourier tables, the Fourier transform F(jw) of that function is

$$F(jw) = \left(\frac{1}{jw + a}\right)^2 = \frac{1}{(a + jw)^2} \qquad \text{(Distefanno III } et\ al.\text{, 2009)} \qquad (13)$$

Bye-the-way, equation (13) goes to confirm the earlier postulation made in equation (6) that F(jw) is a special case of Laplace transformation when σ is set to zero, that is when σ = 0.

Now, back to the argument. In equation (12), it is known that s = σ + jw. Hence

$$F(s) = F(\sigma + jw) = \frac{1}{(s + a)^2} = \frac{1}{[(\sigma + jw) + a]^2}$$

$$= \frac{1}{[(\sigma + a) + jw]^2} \qquad (14)$$

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If jw is set to zero in both sides of equation (14) in order to determine F(σ), we will obtain:

$$F(\sigma) = \frac{1}{(\sigma + a)^2} \qquad (15)$$

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Now, let's carry out the operation in equation (10), that is F (σ) = F(s) – F(jw) on the same function, $f(t) = te^{-at}$.

From Laplace transform tables, F(s) for that expression $= \frac{1}{(s+a)^2}$,

(already stated in equation (12)). Also, from Fourier transform tables,

$F(jw) = \frac{1}{(a+jw)}$, (equally stated in equation (13).

From equation (10),

$$F(\sigma) = F(s)$$

$$= \frac{1}{(s+a)^2} - \frac{1}{(a+jw)} \qquad (16)$$

Substituting for s = σ

$$F(s) = \frac{1}{(\sigma + jw + a)^2} - \frac{1}{(a+jw)^2}$$

$$F(s) = \frac{(a+jw)^2 - (\sigma + jw + a)^2}{(\sigma + jw + a)^2\,(a+jw)^2}$$

$$F(s) = \frac{(a^2 + 2jwa - w^2) - [(\sigma + a) + jw]^2}{[(\sigma + a) + jw]^2\,[a + jw]}$$

$$F(s) = \frac{[(a^2 - w^2) + j\,(2wa)] - [(\sigma + a)^2 + 2jw(\sigma + a) + w^2]}{[(\sigma + a) + jw]^2\,[a + jw]}$$

$$F(s) = \frac{(a^2 - w^2 + j2wa) - [(\sigma^2 + 2\sigma a + a^2) + 2jw\sigma + 2jwa + w^2]}{[(\sigma + a) + jw]^2\,[a + jw]^2}$$

$$F(s) = \frac{\cancel{a^2} - w^2 + j\,2\cancel{wa} - \sigma^2 - 2\sigma a - \cancel{a^2} - 2jw\sigma - 2\cancel{jwa} - w^2}{[(\sigma + a) + jw]^2\,(a + jw)}$$

$$F(s) = \frac{-w^2 - \sigma^2 - 2\sigma a - 2jw\sigma - w^2}{[(\sigma + a) + jw]^2\,[a + jw]}$$

$$F(s) = \frac{-[2w^2 + \sigma\,(\sigma + 2a)] - j2\sigma w}{[(\sigma + a) + jw]^2\,[a + jw]^2}$$

But along the real axis, w = 0.

$$:. F(s) = F(\sigma + jw) = F(\sigma) = \frac{-\sigma(\sigma + 2a)}{(\sigma + a)^2 (a)^2} \tag{17}$$

It can be seen that the RHS of equation (17) and that of (15) are not the same. Equation (17) was obtained from first principles whereas (15) was not.

Similar procedure can be applied to develop what the author chooses to call "sigma (σ) transformation", which could otherwise be called "Real axis translation of Laplace transformation, F(s)." From such approach, a comprehensive table, like Laplace and Fourier tables, could be developed for sigma (σ) transformation, F(σ), by considering various f(t)'s and their respective F(s) and F(jw).

Sigma (σ) has values through the entire spectrum of real numbers, that is from $-\infty$ to $+\infty$. The value for "a" can be obtained from a given expression, such as f(t) = te^{-at}. In the expression f(t) = te^{-at}, if a = 2, for instance, f(t) = te^{-2t}. At a point along the real axis, say σ= 3, using equation (17), we have:

$$F(\sigma) = F(3) = \frac{-3(3 + 2(2)}{(3 + 2)^2 (2)^2} = \frac{-21}{100}$$

$$F(3) = \frac{-21}{100},$$

can be interpreted as the "Discrete Laplace transformation" of the expression, f(t) = te^{-2t} at a point σ = 3 along the real axis.

The development of Discrete Laplace transformation, as presented, may not have applications now but since Science and Technology are continuously developing, it is likely to have in the near future.

VI. Discussion

Fourier transformation, F(jw), is frequency transformation of a continuous – time signal, f(t). Also, it will be recalled that **z**-transformation, by formulation and definition, is a discrete transformation process applied to discrete events. The **z**-transformation expression is given by:

Notes

$$F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k} \text{ (Ogata, 2009)} \qquad (18)$$

where z is defined by $z = e^{-ST}$.

Notes

A very similar analogy here is that just as the Laplace transformation is the transformation of a time-based signal to the complex (s) plane, the sigma (σ) transformation shall one day provide information about the discrete components of Laplace transformation along the real axis.

VII. Conclusion

Knowledge in all spheres of human endeavour, including Science and Technology, is evolving day by day. In this particular adventure into the intricacies and properties of Laplace and Fourier transformations, a new expression has been established through the relationship between the two; the sigma (σ) transformation. It is hoped that sooner or later, this new expression shall be put to use in Science, Engineering and Technology for the benefit of mankind.

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Analytic and Numeric Solution of Linear Partial Differential Equation of Fractional Order

By M. A. Abdou, M. M. El-Kojok & S. A. Raad
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Abstract - The existence and uniqueness solution of the Cauchy problem are discussed and proved in a Banach space E due to Bielecki method and Picard method depending on the properties we expect a solution to possess. Moreover, some properties concerning the stability of solution are obtained. The product Nyström method is used as a numerical method to obtain a linear system of algebraic equations. Also, many important theorems related to the existence and uniqueness solution of the algebraic system are derived. Finally, an application is given and numerical results are obtained.

Keywords : linear partial differential equation of fractional order, semigroup, linear algebraic system, product nyström method.

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Ref.

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Analytic and Numeric Solution of Linear Partial Differential Equation of Fractional Order

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Abstract - The existence and uniqueness solution of the Cauchy problem are discussed and proved in a Banach space *E* due to Bielecki method and Picard method depending on the properties we expect a solution to possess. Moreover, some properties concerning the stability of solution are obtained. The product Nyström method is used as a numerical method to obtain a linear system of algebraic equations. Also, many important theorems related to the existence and uniqueness solution of the algebraic system are derived. Finally, an application is given and numerical results are obtained.

Keywords : linear partial differential equation of fractional order, semigroup, linear algebraic system, product nyström method.

I. Introduction

The use of semigroups methods for partial differential equations has had a long history starting with the works of Feller [1], Hille [2], and Yosida [3]. The basic results of the semigroup theory may be considered as a natural generalization of theorem of M. Stone on one-parameter group of unitary operators in a Hilbert space (see Yosida [4]). Also, the semigroups play a special role in applications, for example they describe how densities of initial states evolve in time. Moreover, there are equations which generate semigroups. These equations appear in such diverse areas as astrophysics-fluctuations in the brightness of the Milky Way [5], population dynamics [6,7], and in the theory of jump processes.

In [8], Mijatovie and Pilipovie introduced and analyzed α-times integrated semigroups for $\alpha \in (\frac{1}{2}, 1)$. In [9], El-Borai studied the Cauchy problem in a Banach space E for a linear fractional evolution equation. In his paper, the existence and uniqueness of the solution of the Cauchy problem were discussed and proved. Also, the solution was obtained in terms of some probability densities. In [10], El-Borai discussed the existence and uniqueness solution of the nonlinear Cauchy problem.

In this work, we treat the following Cauchy problem of the fractional evolution equation

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$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = Au(x,t) + B(t)u(x,t) \ ,$$

with the initial condition: $u(x,0) = u_0(x)$,

In a Banach space *E*. Here $u(x,t)$ is an *E*-valued function on $E \times [0,T]$, $T < \infty$, A is a linear closed operator defined on a dense set S_1 in *E* into *E*, $\{B(t), t \in [0,T]\}$ is a family of linear closed operators defined on a dense set $S_2 \supset S_1$ in E into *E*, $u_0(x) \in E$ and $0 < \alpha \leq 1$.

II. Linear Fractional Evolution Equation

Consider the Cauchy problem of the fractional evolution equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = Au(x,t) + B(t)u(x,t) \ ; \ 0 < \alpha \leq 1 \ , \tag{2.1}$$

with the initial condition:

$$u(x,0) = u_0(x) \ , \tag{2.2}$$

in a Banach space *E,* where $u(x,t)$ is an *E*-valued function on $E \times [0,T]$, $T < \infty$, A is a linear closed operator defined on a dense set S_1 in E into E, $\{B(t), 0 \leq t \leq T\}$ is a family of linear closed operators defined on a dense set $S_2 \supset S_1$ in E into *E,* and $u_0(x) \in E$.

It is assumed that *A* generates an analytic semigroup $Q(t)$. This condition implies:

$$\|Q(t)\| \leq k \quad for \ \ t \geq 0 \ , and \ \|AQ(t)\| \leq \frac{k}{t} \quad for \quad t > 0 \ , \tag{2.3}$$

where $\|\cdot\|$ is the norm in *E* and *k* is a positive constant (Zaidman [11]).

Let us suppose that $B(t)g$ is uniformly Hölder continuous in $t \in [0,T]$, for every $g \in S_1$; that is

$$\| B(t_2) g - B(t_1) g \| \leq k_1 (t_2 - t_1)^{\beta} \ , \tag{2.4}$$

for all $t_2 > t_1$, $t_1, t_2 \in [0,T]$, where k_1 and β are positive constants, $\beta \leq 1$.

We suppose also that there exists a number $\gamma \in (0,1)$, such that

$$\| B(t_2) Q(t_1) h \| \leq \frac{k_2}{t_1^{\gamma}} \| h \| \ , \tag{2.5}$$

where $t_1 > 0$, $t_2 \in [0,T]$, $h \in E$ and k_2 is a positive constant (El-Borai [9,12,13]). (Notice that $Q(t) h \in S_1$ for each $h \in E$ and each $t > 0$) .

Following Gelfand and Shilov ([14],[15]) , we can define the integral of order $\alpha > 0$ by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta) d\theta \ .$$

If $0 < \alpha < 1$, we can define the derivative of order α by

$$\frac{d^{\alpha} f(t)}{d t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^{\alpha}} d\theta \quad , \quad \left(f'(\theta) = \frac{d f(\theta)}{d\theta} \right)$$

where f is an abstract function with values in E.

Ref.

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Notes

Let $C_E(E\times[0,T])$ be the set of all continuous functions $u(x,t)\in E$. We define on $C_E(E\times[0,T])$ a norm by $\|u(x,t)\|_{C_E(E\times[0,T])}=\max_{x,t}\|u(x,t)\|_E$, $\forall\, t\in[0,T]$, $x\in E$. By a solution of the Cauchy problem (2.1), (2.2), we mean an abstract function $u(x,t)$ such that the following conditions are satisfied:

(a) $u(x,t)\in C_E(E\times[0,T])$ and $u(x,t)\in S_1$ for all $t\in[0,T], x\in E$.

(b) $\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ exists and is continuous on $E\times[0,T]$, where $0<\alpha<1$.

(c) $u(x,t)$ satisfies (2.1) with the initial condition (2.2) on $E\times[0,T]$.

Now, it is suitable to rewrite the Cauchy problem (2.1), (2.2), in the form

$$u(x,t)=u_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}Au(x,\theta)\,d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}B(\theta)u(x,\theta)\,d\theta\,, \tag{2.6}$$

where the solution of (2.6) is equivalent to the solution of Cauchy problem (2.1), (2.2)

III. The Existence and Uniqueness Solution of Linear Fractional Evolution Equation

In this section , the existence and uniqueness solution of (2.6) and consequently its equivalent Cauchy problem (2.1), (2.2), will be discussed and proved in a Banach E by two different ways. The first way is due to "Bielecki method", and the second is "Picard method" depending on the properties we expect a solution to possess.

a) Modified Bielecki Method

Here, we will generalized the technique of Bielecki method to obtain the existence and uniqueness solution of (2.6) in *E*, for $0<\alpha\le 1$.
It's suitable to consider the following lemma.

Lemma 1:

If $\lambda>1$ and $0<\delta<1$, then

$$\int_0^t (t-\eta)^{\delta-1}d\eta \le (\frac{1}{\lambda})^{\delta-1}t\ , \tag{3.1}$$

and

$$\int_0^t e^{\lambda\eta}(t-\eta)^{\delta-1}\,d\eta \le (\frac{1}{\lambda})^{\delta}[1+\frac{1}{\delta}]e^{\lambda t}. \tag{3.2}$$

Theorem 1 :

If *A* and *B* are linear bounded operators in a Banach space *E*, and $0<\alpha\le 1$, then (2.6) has a unique solution in *E*.

Proof:

Let K be an operator defined by

$$K\,u(x,t)=u_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1} Au(x,\theta)\,d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1} B(\theta)\,u(x,\theta)\,d\theta\ . \tag{3.3}$$

Hence, we have

$$\|K\,u(x,t)\| \le \|u_0(x)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\|Au(x,\theta)\|\,d\theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\|B(\theta)\,u(x,\theta)\|\,d\theta\ . \tag{3.4}$$

Notes

Since A and B are bounded operators, there exists positive constants L and M such that

$$\|Au(x,t)\| \le L\,\|u(x,t)\|\ ,\text{and}\ \|B(t)u(x,t)\| \le M\,\|u(x,t)\|. \tag{3.5}$$

In the light of (3.5), (3.4) takes the form

$$\|K\,u(x,t)\| \le \|u_0(x)\| + \frac{(L+M)}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}\|u(x,\theta)\|\,d\theta\ . \tag{3.6}$$

Using (3.1) in (3.6), we get

$$\|K\,u(x,t)\| \le \|u_0(x)\| + \frac{(L+M)}{\Gamma(\alpha)}\left(\frac{1}{\lambda}\right)^{\alpha-1} T\,\|u(x,t)\|\ ,\ (T=\max_{0\le t\le T} t)\ . \tag{3.7}$$

Inequality (3.7) shows that, the operator K maps the ball $B_r \subset E$ into itself, where

$$r=\frac{\sigma}{1-\delta},\quad (\sigma=\|u_0(x)\|,\ \delta=\frac{(L+M)}{\Gamma(\alpha)}\left(\frac{1}{\lambda}\right)^{\alpha-1}T)$$

Since $r>0$, $\sigma>0$, therefore $\delta<1$. Also, the inequality (3.7) involves the boundedness of the operator K.

For the two functions $u(x,t)$ and $v(x,t)$ in E, the formula (3.3) leads to

$$\|K\,u(x,t)-K\,v(x,t)\| \le \frac{1}{\Gamma(\alpha)} \left\{\int_0^t (t-\theta)^{\alpha-1}\|A(u(x,\theta)-v(x,\theta))\|\,d\theta + \int_0^t (t-\theta)^{\alpha-1}\|B(\theta)\,(u(x,\theta)-v(x,\theta))\|\,d\theta\right\}. \tag{3.8}$$

Using (3.5) in (3.8), we have for $\lambda>1$

$$\|K\,u(x,t)-K\,v(x,t)\| \le \frac{(L+M)}{\Gamma(\alpha)} \max_{x,t}\left\{e^{-\lambda(t+x)}\|u(x,t)-v(x,t)\|\right\}\int_0^t (t-\theta)^{\alpha-1} e^{\lambda(\theta+x)}\,d\theta.$$

Using (3.2), the above inequality becomes

$$\max_{x,t}\left\{e^{-\lambda(t+x)}\left\| Ku(x,t)-Kv(x,t)\right\|\right\}\le \sigma\max_{x,t}\left\{e^{-\lambda(t+x)}\left\| u(x,t)-v(x,t)\right\|\right\}, \tag{3.9}$$

where
$$\sigma=\frac{(L+M)}{\Gamma(\alpha)}\left(\frac{1}{\lambda}\right)^{\alpha}\left[1+\frac{1}{\alpha}\right].$$

Inequality (3.9) can be adapted in the form

$$d(Ku(x,t),Kv(x,t))\le\sigma d(u(x,t),v(x,t)).$$

If we choose λ sufficiently large, then $\sigma<1$, and d is a contraction mapping. By Banach fixed point theorem, K has a unique fixed point which is the unique solution of (2.6).

b) Semigroup Method

To obtain the solution of the Cauchy problem (2.1), (2.2) in the dense set S_1 in E, we can follow the work of El-Borai [9]. Hence, the Cauchy problem (2.1), (2.2), and (2.6) are equivalent to the following integral equation

$$u(x,t)=\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)Q((t-\eta)^\alpha\theta)w(x,\eta)\,d\theta\,d\eta+\int_0^\infty\zeta_\alpha(\theta)Q(t^\alpha\theta)u_0(x)\,d\theta, \tag{3.10}$$

where $\zeta_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$, and $w(x,t)=B(t)u(x,t),\ \ (x,t)\in(E\times[0,T])$.

The integral equation (3.10) represents a Volterra equation of the second kind with Abel kernel, where the first term of the R.H.S is known and continuous. The integral equation will be solved numerically in the next section.

Now, we will prove that (3.10) has a unique solution which represents the required solution of the Cauchy problem (2.1), (2.2).

Theorem 2:
The Cauchy problem (2.1), (2.2) has a unique solution in $C_E(E\times[0,T])$.
The proof of this theorem depends on the following lemmas.

Lemma 2:
Under the condition (2.5), (3.10) has a solution in the space $C_E(E\times[0,T])$.

Proof:
Consider the following integral equation

$$w(x,t)=\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)B(t)Q((t-\eta)^\alpha\theta)w(x,\eta)\,d\theta\,d\eta+\int_0^\infty\zeta_\alpha(\theta)\,B(t)\,Q(t^\alpha\theta)u_0(x)\,d\theta. \tag{3.11}$$

Ref.

9. M.M.EL-Borai, *Some probability densities and fundamental solutions of fractional evolution equation*, Chaos, Solitons & Fractals 14(2002), 433-440.

Using the method of successive approximations, we set

$$w_{n+1}(x,t)=\int_0^\infty \zeta_\alpha(\theta)\, B(t)Q(t^\alpha\theta)u_0(x)\,d\theta+$$

$$\alpha\int_0^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)B(t)Q((t-\eta)^\alpha\theta)w_n(x,\eta)\,d\theta\,d\eta\,.$$

Thus, we have

$$\|w_2(x,t)-w_1(x,t)\|\le$$

$$\alpha\int_0^t\int_0^\infty\theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)\left\|B(t)Q((t-\eta)^\alpha\theta)[w_1(x,\eta)-w_0(x,\eta)]\right\|d\theta\,d\eta\,,$$

where $w_0(x,t)$ is the zero element in E.
In view of the condition (2.5), we get

$$\|w_2(x,t)-w_1(x,t)\|\le$$

$$\alpha\int_0^t\int_0^\infty\theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)\frac{k_2}{((t-\eta)^\alpha\theta)^\gamma}\|w_1(x,\eta)-w_0(x,\eta)\|\,d\theta\,d\eta.$$

The above inequality for $u_0(x)\in S_1$ can be adapted in the form

$$\|w_2(x,t)-w_1(x,t)\|\le\frac{\mu t^\nu}{\nu},\tag{3.12}$$

where,

$$\mu=\alpha k_2\int_0^\infty\theta^{1-\gamma}\zeta_\alpha(\theta)\sup_{t,\theta}\left\|\int_0^\infty\zeta_\alpha(\theta)B(\eta)Q(\eta^\alpha\theta)u_0(x)\,d\theta\right\|d\theta\,,\qquad \nu=\alpha(1-\gamma).\tag{3.13}$$

By induction, we obtain

$$\|w_{n+1}(x,t)-w_n(x,t)\|\le\frac{\mu^n t^{n\nu}(\Gamma(\nu))^n}{\Gamma(n\nu+1)}\,.$$

Thus, the series $\sum_{i=0}^{\infty}\|w_{i+1}(x,t)-w_i(x,t)\|$ converges uniformly on $E\times[0,T]$, under the condition, $\mu t^\nu\Gamma(\nu)<1$.

Since $w_{n+1}(x,t)=\sum_{i=0}^{n}[w_{i+1}(x,t)-w_i(x,t)]$, it follows that the sequence $\{w_n(x,t)\}$ converges uniformly in the space $C_E(E\times[0,T])$ to a continuous function $w(x,t)$ which satisfies (3.11), consequently $u(x,t)\in C_E(E\times[0,T])$.

Lemma 3:
Under the condition (2.5), (3.10) has a unique solution in the space $C_E(E\times[0,T])$.

Proof:
For the two functions $w_1(x,t)$ and $w_2(x,t)$ in the space $C_E(E\times[0,T])$, the formula (3.11) with the aid of condition (2.5), leads to

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$$\| w_2(x,t) - w_1(x,t) \| \le \mu \int_0^t (t-\eta)^{\nu-1} \| w_2(x,t) - w_1(x,t) \| d\eta \,.$$

Consequently,

$$\| w_2(x,t) - w_1(x,t) \| \le \mu \, \rho \int_0^t e^{\lambda(\eta+x)} (t-\eta)^{\nu-1} \, d\eta \,, \tag{3.14}$$

where, $\rho = \max_{x,t} \, [e^{-\lambda(t+x)} \| w_2(x,t) - w_1(x,t) \|]$, and $\lambda > 1$.

Using (3.2) in (3.14), we get

$$\| w_2(x,t) - w_1(x,t) \| \le \mu \, \rho (\frac{1}{\lambda})^{\nu} [1 + \frac{1}{\nu}] e^{\lambda(t+x)} \,.$$

Thus, we have

$$\max_{x,t} [e^{-\lambda(t+x)} \| w_2(x,t) - w_1(x,t) \|] \le \mu \, \rho \, (\frac{1}{\lambda})^{\nu} \, [1 + \frac{1}{\nu}] \,.$$

We can choose λ sufficiently large such that

$$\mu \, (\frac{1}{\lambda})^{\nu} \, [1 + \frac{1}{\nu}] = \mu_1 < 1 \,.$$

Thus,

$$\rho = \max_{x,t} [e^{-\lambda(t+x)} \| w_2(x,t) - w_1(x,t) \|] = 0 \,.$$

This completes the proof of the lemma.

Lemma 4:

Under the conditions (2.4) and (2.5), the solution $w(x,t)$ of (3.11) satisfies a uniform Hölder condition. (El-Borai [9])

Proof of Theorem 2:

By virtue of lemmas (3) and (4), we deduce that, the function $u(x,t) \in S_1$ and represents the unique solution of Cauchy problem (2.1), (2.2) in the space $C_E \, (E \times [0,T])$.

Corollary 1:

The integral equation (3.10) has a unique solution in the Banach space $C_{\Re}(\Re \times [0,T])$. Now, we will prove the stability of the solutions of the Cauchy problem (2.1), (2.2). In other words, we will show that the Cauchy problem (2.1), (2.2) is correctly formulated.

Theorem 3:

Let $\{u_n(x,t)\}$ be a sequence of functions, each of which is a solution of (2.1) with the initial condition $u_n(x,0) = g_n(x)$, where $g_n(x) \in S_1 \, (n = 1,2,\ldots)$. If the sequence $\{g_n(x)\}$ converges to an element $u_0(x) \in S_1$, the sequence $\{A g_n(x)\}$ converges and the sequence $\{B(t) g_n(x)\}$ converges uniformly on $E \times [0,T]$. Then, the sequence of solutions $\{u_n(x,t)\}$ converges uniformly on $E \times [0,T]$ to a limit function $u(x,t)$, which is the solution of the Cauchy problem (2.1), (2.2).

Proof:

Consider the sequences $\{z_n(x,t)\}$ and $\{u_n^*(x,t)\}$, where

$$\frac{\partial^{\alpha} u_n^*(x,t)}{\partial t^{\alpha}} - A u_n^*(x,t) = z_n(x,t) \,,$$

Ref.

9. M.M.EL-Borai, *Some probability densities and fundamental solutions of fractional evolution equation*, Chaos, Solitons & Fractals 14(2002), 433-440.

$$u_n^*(x,t) = u_n(x,t) - g_n(x) \ , \ \ u_n(x,0) = g_n(x) \ ,$$

$$u_n^*(x,t) = \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) Q((t-\eta)^\alpha \theta) z_n(x,\eta) d\theta \, d\eta \ ,$$

and

$$z_n(x,t) = B(t) u_n^*(x,t) + B(t) g_n(x) + A g_n(x).$$

In view of the conditions (2.5) and (3.13), we get

$$\| z_n(x,t) - z_m(x,t) \| \le \mu \int_0^t (t-\eta)^{\nu-1} \| z_n(x,\eta) - z_m(x,\eta) \| d\eta$$
$$+ \| B(t) g_n(x) - B(t) g_m(x) \| + \| A g_n(x) - A g_m(x) \|.$$

Given $\varepsilon > 0$, we can find a positive integer $N = N(\varepsilon)$, such that

$$\| z_n(x,t) - z_m(x,t) \| \le$$
$$\mu \int_0^t (t-\eta)^{\nu-1} \| z_n(x,\eta) - z_m(x,\eta) \| d\eta + (1-\mu_1)\varepsilon \ ,$$

for all $n \ge N, m \ge N$ and $(x,t) \in E \times [0,T]$.

Using (3.2), the above inequality takes the form

$$(1-\mu_1) e^{-\lambda(t+x)} \| z_n(x,t) - z_m(x,t) \| \le (1-\mu_1) e^{-\lambda(t+x)} \varepsilon \ .$$

Thus, for sufficiently large λ, we get

$$\max_{x,t} [e^{-\lambda(t+x)} \| z_n(x,t) - z_m(x,t) \|] \le \varepsilon \ .$$

Since E is a complete space, it follows that the sequence $\{z_n(x,t)\}$ converges uniformly on $E \times [0,T]$ to a continuous function $z(x,t)$, so the sequence $\{u_n^*(x,t)\}$ converges uniformly on $E \times [0,T]$ to a continuous function $u^*(x,t)$. It can be proved that $z(x,t)$ satisfies a uniform Hölder condition on $[0,T]$, thus $u^*(x,t) \in S_1$.

IV. The Numerical Solution of Linear Fractional Evolution Equation

In this section, we will use the product Nyström method (Linz [17], and Dzhuraev [18]), to obtain numerically, the solution of the Cauchy problem (2.1), (2.2), in the Banach space $C_{\Re}(\Re \times [0,T])$, where $\| u(x,t) \|_{C_{\Re}(\Re \times [0,T])} = \max_{x,t} | u(x,t) |$, $\forall t \in [0,T], -\infty < x < \infty$. For this, the integral equation (3.10) can be written in the form

$$u(x,t) = f^*(x,t) + \alpha \int_0^t p(t,\eta) \, Q^*(t,\eta) \, B(\eta) u(x,\eta) \, d\eta, \qquad (4.1)$$

Ref.

17. A. Dzhuraev, *Methods of Singular Integral Equations*, London, New York, 1992.

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where,

$$f^*(x,t)=\int_0^\infty \zeta_\alpha(\theta)\,Q(t^\alpha\theta)u_0(x)\,d\theta\ , \tag{4.2}$$

$$Q^*(t,\eta)=\int_0^\infty \theta\,\zeta_\alpha(\theta)\,Q((t-\eta)^\alpha\,\theta)\,d\theta\ , \tag{4.3}$$

and the bad kernel

$$p(t,\eta)=(t-\eta)^{\alpha-1}\quad ,\ (0<\alpha<1,\ 0\le\eta\le t\le T\ ;\ T<\infty\ ). \tag{4.4}$$

Here, the unknown function $u(x,t)\in C_{\Re}(\Re\times[0,T])$, while $f^*(x,t)$, $Q^*(t,\eta)$ and $p(t,\eta)$ are known functions and satisfy the following conditions:

(1) $f^*(x,t)$ is a continuous function in $(\Re\times[0,T])$.

(2) $Q^*(t,\eta)$ with its partial derivatives are continuous functions in $[0,T]$.

(3) $p(t,\eta)$ is a badly behaved function of its arguments such that:

(a) for each continuous function $u(x,t)$ and $0\le t_1\le t_2\le t$, the integrals

$$\int_{t_1}^{t_2} p(t,\eta)Q^*(t,\eta)\,B(\eta)u(x,\eta)\,d\eta\ ,$$

and

$$\int_0^t p(t,\eta)Q^*(t,\eta)B(\eta)u(x,\eta)\,d\eta\ ,$$

are continuous functions in $(\Re\times[0,T])$.

(b) $p(t,\eta)$ is absolutely integrable with respect to η for all $0\le t\le T$.

Remark 1:

By virtue of corollary (1), the integral equation (4.1) has a unique solution in the Banach space $C_{\Re}(\Re\times[0,T])$.

Now, we will apply the product Nyström method, to obtain numerically, the solution of (4.1). Therefore, putting $t=t_i=\eta_i=x_i=x$, $t_i=ih$, $h=t_{i+1}-t_i$ $(i=0,1,\ldots,N$ and N is even$)$, and using the following notations

$$u_{i,i}=u(t_i,x_i)\ ,\quad Q^*_{i,j}=Q^*(t_i,\eta_j)\ ,\quad f^*_{i,i}=f^*(t_i,x_i),$$
$$B_i=B(\eta_i),$$

we get the following linear algebraic system

$$u_{i,i}=f^*_{i,i}+\alpha\sum_{j=0}^{N} w_{i,j}\,Q^*_{i,j}\,B_j\,u_{j,j}\ ,\quad (\ i=0,1,2,\ldots,N) \tag{4.6}$$

where,

$$w_{i,0}=\beta_1(t_i)\ ,\quad w_{i,2j+1}=2\gamma_{j+1}(t_i)$$
$$w_{i,2j}=\alpha_j^*(t_i)+\beta_{j+1}(t_i)\ ,\quad w_{i,N}=\alpha^*_{\frac{N}{2}}(t_i). \tag{4.7}$$

And,

$$\alpha_j^{*}(t_i)=\frac{1}{2h^2}\int_{t_{2j-2}}^{t_{2j}} p(t_i,\eta)(\eta-\eta_{2j-2})(\eta-\eta_{2j-1})\,d\eta \ ,$$

$$\beta_j(t_i)=\frac{1}{2h^2}\int_{t_{2j-2}}^{t_{2j}} p(t_i,\eta)(\eta_{2j-1}-\eta)(\eta_{2j}-\eta)\,d\eta \ ,$$

$$\gamma_j(t_i)=\frac{1}{2h^2}\int_{t_{2j-2}}^{t_{2j}} p(t_i,\eta)(\eta-\eta_{2j-2})(\eta_{2j}-\eta)\,d\eta \ . \tag{4.8}$$

Evaluating the integrals of (4.8), where $p(t,\eta)=(t-\eta)^{\alpha-1}$, and introducing the results in the values of $w's$, we get

$$w_{i,0}=\frac{-h^{\alpha}}{2\alpha(\alpha+1)(\alpha+2)}\{[2|i-2|+\alpha+2]|i-2|^{\alpha+1}- -[2|i|^2-3(2+\alpha)|i|+2(\alpha+1)(\alpha+2)]|i|^{\alpha}\},$$

$$w_{i,2j+1}=\frac{2h^{\alpha}}{\alpha(\alpha+1)(\alpha+2)}\{(\alpha+2)[|i-2j-2|^{\alpha+1} +|i-2j|^{\alpha+1}]+|i-2j-2|^{\alpha+2}-|i-2j|^{\alpha+2}\},$$

$$w_{i,2j}=\frac{-h^{\alpha}}{2\alpha(\alpha+1)(\alpha+2)}\{(\alpha+2)|i-2j+2|^{\alpha+1} +(\alpha+2)|i-2j-2|^{\alpha+1}+6(\alpha+2)|i-2j|^{\alpha+1} +2|i-2j-2|^{\alpha+2}-2|i-2j+2|^{\alpha+2}\},$$

and

$$w_{i,N}=\frac{-h^{\alpha}}{2\alpha(\alpha+1)(\alpha+2)} \{2(\alpha+1)(\alpha+2)|i-N|^{\alpha}+3(\alpha+2)|i-N|^{\alpha+1}+ (\alpha+2)|i-N+2|^{\alpha+1}+2|i-N|^{\alpha+2}-2|i-n+2|^{\alpha+2}\}. \tag{4.9}$$

The linear algebraic system (4.6) represents $(N+1)$ equations in $u_{i,i}$. Therefore, the approximate solution of $u(x,t)$ can be written in the vector form

$$(I-\alpha W)U=F^{*} \ , \tag{4.10}$$

where,

$$W=\begin{bmatrix} 1-\alpha\, w_{0,0}Q_{0,0}^{*}\,B_0 & -\alpha w_{0,1}Q_{0,1}^{*}\,B_1 & \dots & -\alpha w_{0,N}Q_{0,N}^{*}\,B_N \\ -\alpha\, w_{1,0}Q_{1,0}^{*}\,B_0 & 1-\alpha w_{1,1}Q_{1,1}^{*}\,B_1 & \dots & -\alpha w_{1,N}Q_{1,N}^{*}\,B_N \\ \vdots & \vdots & \vdots & \vdots \\ . & . & . & . \\ -\alpha\, w_{N,0}Q_{N,0}^{*}\,B_0 & -\alpha w_{N,1}Q_{N,1}^{*}\,B_1 & \dots & 1-\alpha w_{N,N}Q_{N,N}^{*}\,B_N \end{bmatrix},$$

Notes

$$U = \begin{bmatrix} u_{0,0} \\ u_{1,1} \\ . \\ . \\ . \\ u_{N,N} \end{bmatrix}, \text{ and } F^* = \begin{bmatrix} f^*_{0,0} \\ f^*_{1,1} \\ . \\ . \\ . \\ f^*_{N,N} \end{bmatrix}$$

Notes

When $det(W) \neq 0$, the algebraic system (4.6) has a unique solution in the form

$$U = [I - \alpha W]^{-1} \; F^*, \tag{4.11}$$

where I is the identity matrix.
Theorem 4:

The algebraic system (4.6) has a unique solution in the Banach space ℓ^∞, under the following conditions

$$\sup_i \left| f^*_{i,i} \right| \leq q \,, \quad (q \text{ is a constant}) \;. \tag{4.12}$$

$$\sup_i \sum_{j=0}^{N} \left| w_{i,j} \, Q^*_{i,j} \right| \leq q^* \,, \quad (q^* \text{ is a constant}) \;. \tag{4.13}$$

$$\sup_i \left| B_i \, u_{i,i} \right| \leq M \, \sup_i \left| u_{i,i} \right| \,, \quad (M \text{ is a constant}) \;. \tag{4.14}$$

Proof :

Let Y be the set of all functions $U = \{u_{i,i}\} \, in \; \ell^\infty$ such that $\| U \|_{\ell^\infty} \leq \rho^*$, ρ^* is a constant. Define the operator $\tilde{T}$ by

$$\tilde{T} \, U = F^* + \alpha \, W \, U \,, \tag{4.15}$$

where , $\left\| \tilde{T} \, U \right\|_{\ell^\infty} = \sup_i \left| \tilde{T} \; u_{i,i} \right|$, $\quad \forall \; i = 0,1,2 ,...$.

The formulas (4.6) and (4.15) lead to

$$\left| \tilde{T} \, u_{i,i} \right| \leq \sup_i \left| f^*_{i,i} \right| + \alpha \sup_i \sum_{j=0}^{N} \left| w_{i,j} \; Q^*_{i,j} \right| \sup_j \left| B_j u_{j,j} \right| \;, \forall \, i = 0,1,2,... \;.$$

In view of the conditions (4.12) and (4.14), the above inequality takes the form

$$\left\| \tilde{T} \, U \right\|_{\ell^\infty} \leq q + \lambda^* \left\| U \right\|_{\ell^\infty} \,, \; (\lambda^* = \alpha \, q^* M) \;. \tag{4.16}$$

Inequality (4.16) shows that, the operator $\tilde{T}$ maps the set Y into itself, where

$$\rho^* = \frac{q}{1 - \lambda^*} \;.$$

Since $\rho^* > 0$, $q > 0$, therefore $\lambda^* < 1$. Also, the inequality (4.16) involves the boundedness of operator $\tilde{T}$.

For the two functions U and V in ℓ^{∞}, the formulas (4.6) and (4.15) lead to

$$\left|\tilde{T}\,u_{i,i}-\tilde{T}\,v_{i,i}\right|\le\alpha\sup_{i}\sum_{j=0}^{N}\left|w_{i,j}\,Q_{i,j}^{*}\right|.\sup_{j}\left|B_j(u_{j,j}-v_{j,j})\right| .$$

The above inequality, with the aid of conditions (4.13) and (4.14), can be adapted in the form

$$\left\|\tilde{T}\,U-\tilde{T}\,V\right\|_{\ell^{\infty}}\le\lambda^{*}\left\|U-V\right\|_{\ell^{\infty}} .$$

Therefore, $\tilde{T}$ is a continuous operator in ℓ^{∞}, then under the condition $\lambda^{*}<1$, $\tilde{T}$ is contractive. Hence, by Banach fixed point theorem, $\tilde{T}$ has a unique fixed point which is the unique solution of the linear algebraic system in the Banach space ℓ^{∞}.

Theorem 5:

If the conditions (4.13) and (4.14) of Theorem (4) are verified, and the sequence of functions $\left\{F_m^{*}\right\}=\left\{(f_{i,i}^{*})_m\right\}$ converges uniformly to the function $F^{*}=\left\{f_{i,i}^{*}\right\}$ in the Banach space ℓ^{∞}. Then, the sequence of approximate solutions $\left\{U_m\right\}=\left\{(u_{i,i})_m\right\}$ converges uniformly to the exact solution $U=\left\{u_{i,i}\right\}$ of the linear algebraic system (4.6) in ℓ^{∞}.

Proof:

In the light of (4.6), we get

$$\left|u_{i,i}-(u_{i,i})_m\right|\le\alpha\sup_{i}\sum_{j=0}^{N}\left|w_{i,j}\,Q_{i,j}^{*}\right|\sup_{j}\left|B_j(u_{j,j}-(u_{j,j})_m)\right|$$
$$+\sup_{i}\left|f_{i,i}-(f_{i,i})_m\right| \quad ,\ \forall\, i=0,1,2,\ldots \; .$$

Using the conditions (4.13) and (4.14), we have

$$\left\|U-U_m\right\|_{\ell^{\infty}}\le\frac{1}{1-\lambda^{*}}\left\|F^{*}-F_m^{*}\right\|_{\ell^{\infty}} \quad ; \qquad (\lambda^{*}<1).$$

Since $\left\|F^{*}-F_m^{*}\right\|_{\ell^{\infty}}\to 0$ as $m\to\infty$, so that $\left\|U-U_m\right\|_{\ell^{\infty}}\to 0$.

This complete the prove of the theorem.

When $N\to\infty$, it is natural to expect that the sum $\sum_{j=0}^{N} w_{i,j}\,Q_{i,j}^{*}\,B_j\,u_{j,j}\,;\ 0\le i,j\le N,$ becomes $\int_0^t p(t,\eta)\,Q^{*}(t,\eta)\,B(\eta)u(x,\eta)\,d\eta$. Consequently, the solution of the algebraic system (4.6) is the same solution of the integral equation (4.1).

Theorem 6:

If the sequence of continuous functions $\left\{f_n^{*}(x,t)\right\}$ converges uniformly to the function $f^{*}(x,t)$, and the functions $Q^{*}(t,\eta)$, $p(t,\eta)$ satisfy, respectively, the conditions (2) and (3-b). Then, the sequence of approximate solutions $\left\{u_n(x,t)\right\}$ converges uniformly to the exact solution of (4.1) in the Banach space $C_{\Re}(\Re\times[0,T])$.

Notes

Notes

Proof:
The formula (4.1) with its approximate solution give

$$\max_{x,t}\left|u(x,t)-u_n(x,t)\right| \le \max_{x,t}\left| f^*(x,t)-f_n^*(x,t)\right|$$

$$+\alpha\int_0^t \left|p(t,\eta)\right|\left|Q^*(t,\eta)\right|.\max_{x,\eta}\left|B(\eta)\,(u(x,\eta)-u_n(x,\eta))\right| d\eta\,, \tag{4.17}$$

$$\forall\, 0\le\eta\le t\le T\,,\quad -\infty<x<\infty\,.$$

In view of the conditions (2) and (3-b), there exist two constants c_1 and c_2, such that

$$\left|Q^*(t,\eta)\right|\le c_1 \quad ,and \int_0^t \left|p(t,\eta)\right| d\eta \le c_2\,. \tag{4.18}$$

Hence, the inequality (4.17) with the aid of (4.18) and (3.5), takes the form

$$\left\|u(x,t)-u_n(x,t)\right\|_{C_{\Re}(\Re\times[0,T])} \le \frac{1}{(1-c^*)}\left\| f^*(x,t)-f_n^*(x,t)\right\|_{C_{\Re}(\Re\times[0,T])}\,,$$

$$\left(c^*=\alpha\, c_1 c_2\, M\right).$$

Since

$$\left\| f^*(x,t)-f_n^*(x,t)\right\|_{C_{\Re}(\Re\times[0,T])}\to 0 \quad as \quad n\to\infty,$$

$$\text{hence}\quad \left\|u(x,t)-u_n(x,t)\right\|_{C_{\Re}(\Re\times[0,T])}\to 0.$$

Definition 1:

The product Nyström method is said to convergent of order r in [a,b] if and only if for sufficiently large N, there exists a constant $c>0$ independent of N, such that

$$\left\|\phi(x)-\phi_N(x)\right\|_\infty \le c\, N^{-r}\,.$$

Definition 2:

The consistency error R_N of the product Nyström method is determined by the following equation

$$R_N=\left|\int_0^{t_N} p(t,\eta)Q^*(t,\eta)B(\eta)u(x,\eta)\,d\eta-\sum_{j=0}^{N} w_{i,j}\,Q_{i,j}^*\,B_j u_{j,j}\right|. \tag{4.19}$$

Also , (4.19) gives

$$u(x,t)-u_N(x,t)=\sum_{j=0}^{N} w_{i,j}\,Q^*(t_i,\eta_j)[B_j(u(x_j,\eta_j)-u_N(x_j,\eta_j))]+R_N\,, \tag{4.20}$$

where $u_N(x,t)$ is the approximate solution of (4.1).

Theorem 7:
Assume that, the hypothesis of Theorem (5) are verified, then

$$\lim_{N\to\infty} R_N=0\,. \tag{4.21}$$

Notes

Proof:
The formula (4.20) leads to

$$|R_N| \le \sup_i |u_{i,i} - (u_{i,i})_N| + \sup_i \sum_{j=0}^{N} |w_{i,j} Q^*_{i,j}| \sup_j |B_j(u_{j,j} - (u_{j,j})_N)|.$$

In view of the conditions (4.13) and (4.14), the above inequality takes the form

$$\|R_N\|_{\ell^\infty} \le \|U - U_N\|_{\ell^\infty} + q^* M \|U - U_N\|_{\ell^\infty} \quad , \ \forall\ N = 1,2,\ldots .$$

Since $\|U - U_N\|_{\ell^\infty} \to 0$ as $N \to \infty$ (see Theorem (5), it follows that $\|R_N\|_{\ell^\infty} \to 0$.

Application I:

In (4.1), let $0 < \alpha < 1$, $Q^*(t,\eta) = 1$, $B(\eta) = I$, where I is the identity operator. Hence, we get a linear Volterra integral equation of the second kind with Abel kernel

$$u(x,t) = x\,(1 - t^\alpha) + t\,(1 - \frac{t^\alpha}{\alpha + 1}) + \alpha \int_0^t (t - \eta)^{\alpha - 1} u(x,\eta)\, d\eta, \tag{4.22}$$

where the exact solution $u(x,t) = x + t$.

The results are obtained numerically in the flowing Table which lists various values of $x, t \in [0,0.8]$ together with the values of the exact and approximate solutions and the error of (4.22). Also, we can see from this table that:

1. The exact and approximate solutions are coincident for $x = t = 0$.
2. As x and t are increasing through [0,0.8], the error is also increasing for $\alpha = 0.98$, $\alpha = 0.8$ and $\alpha = 0.4$.
3. The maximum value of the error is 0.421056 which occurs at $x = t = 0.8$ for $\alpha = 0.8$.

		$\alpha = 0.4$		$\alpha = 0.8$				$\alpha = 0.98$	
x=t	*Exact*	Appr. Sol.	Error	Appr. Sol.	Error	*x=t*	*Exact*	Appr. Sol.	Error
0	0	0	0	0	0	0	0	0	0
0.08	0.16	0.154992	0.005008	0.155284	0.004716	0.08	0.16	0.156646	0.003354
0.16	0.32	0.306929	0.013071	0.302817	0.017183	0.16	0.32	0.306367	0.013633
0.24	0.48	0.456241	0.023759	0.442848	0.037152	0.24	0.48	0.448667	0.031333
0.32	0.64	0.602977	0.037023	0.575147	0.064853	0.32	0.64	0.582967	0.057033
0.4	0.8	0.747068	0.052932	0.699325	0.100675	0.4	0.8	0.708623	0.091377
0.48	0.96	0.88839	0.07161	0.814898	0.145102	0.48	0.96	0.824933	0.135067
0.56	1.12	1.02678	0.09322	0.921307	0.198693	0.56	1.12	0.93113	0.18887
0.64	1.28	1.162041	0.117959	1.017924	0.262076	0.64	1.28	1.02638	0.25362
0.72	1.44	1.293951	0.146049	1.104056	0.335944	0.72	1.44	1.10978	0.33022
0.8	1.6	1.422257	0.177743	1.178944	0.421056	0.8	1.6	1.180347	0.419653

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Multipliers of Distributions Spaces

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Abstract - In this paper we consider multipliers of distributions spaces. We obtain some characterizations of various multipliers spaces similar to those of multipliers of Banach algebras.

Keywords : *distribution, multiplier, convolution.*

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Multipliers of Distributions Spaces

V. S. K. Assiamoua [α], Y. Mensah [σ] & Y. M. Awussi [ρ]

Abstract - In this paper we consider multipliers of distributions spaces. We obtain some characterizations of various multipliers spaces similar to those of multipliers of Banach algebras.

Keywords : distribution, multiplier, convolution.

I. Preliminaries

Let G be a locally compact group and let $L_1(G)$ denote the Lebesgue space with respect to the Haar measure on G. An $L_1(G)-$multiplier is any continuous linear transformation from $L_1(G)$ into itself which commutes with translations. Many investigations have been carried out to characterize the $L_1(G)-$multipliers. The theory has been extended to other group algebras, general Banach algebras, some locally convex linear spaces and $\mathcal{C}^*-$algebras. See [8], [1] and references therein. For further informations on the theory of multipliers we refer to [8]. In this paper we study multipliers of distributions. Some results in this area involving temperate distributions can be found in [5] and [7]

a) *Definition and notations of distributions spaces*

In this section we recall some facts about distributions. For more details we refer to [2], [3] and [4]. As usual $\mathcal{D}$ denotes the set of infinitely differentiable complex valued functions on $\mathbb{R}^n$ having compact support. The set $\mathcal{D}$ is topologized in the following way. The subset $\mathcal{D}_K$ of $\mathcal{D}$ whose elements have their supports in the compact subset K of $\mathbb{R}^n$ is a locally convex vector space with the semi-norms $P_{m,K}$ defined by

$$P_{m,K}(\phi) = \sup_{x\in K,|q|\leq m} |D^q(\phi)| \tag{1}$$

where $m \in \mathbb{N}$, $q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, $|q| = \sum_{i=1}^{n} q_i$, $x = (x_1, \ldots, x_n)$ and $D^q = \frac{\partial^{|q|}}{\partial x_1^{q_1} \ldots \partial x_n^{q_n}}$. Then $\mathcal{D}$ is the strict inductive limit of the $\mathcal{D}_K$'s, when K runs over an increasing sequence of compact sets whose union is $\mathbb{R}^n$. Thus a net

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$\{\phi_\alpha\}$ converges to 0 in $\mathcal{D}$ means that all the ϕ_α's have their supports in a fixed K and for any $q \in \mathbb{N}^n$ the net $\{D^q\phi_\alpha\}$ converges to 0 uniformly in K.

A *distribution* on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{D}$. The vector space of distributions is naturally denoted by $\mathcal{D}'$. Let U be a distribution. We say that U vanishes on a subset Ω of $\mathbb{R}^n$ if $U(\phi) = 0$ whenever $supp\phi \subset \Omega$. The support of U is by definition the smallest closed subset of Ω of $\mathbb{R}^n$ such that U vanishes off Ω.

Let $\mathcal{E}$ denotes the set of infinitely differentiable complex functions on $\mathbb{R}^n$. It is endowed with the locally convex topology with semi-norms $P_{K,m}$ defined as in (1) where K now is no longer fixed. It was shown that the topological dual $\mathcal{E}'$ is identical with the space of distributions with compact support [2, page 38].

Let $\mathcal{S}$ be the subset of $\mathcal{E}$ which contains the functions ϕ such that

$$\lim_{\|x\|\to\infty} |x^k D^q\phi(x)| = 0, \forall k \in \mathbb{N} \tag{2}$$

with $x^k = x_1^k x_2^k \ldots x_n^k$. $\mathcal{S}$ is called the space of rapidly decreasing infinitely differentiable functions whereas its topological dual $\mathcal{S}'$ is called the space of slowly increasing distributions or temperate distributions.

Finally denote by $\mathcal{O}_M$ the subset of $\mathcal{E}$ such that

$$\phi \in \mathcal{O}_M \text{ if and only if } \lim_{\|x\|\to\infty} |\varphi(x)D^q\phi(x)| = 0,\ \forall\varphi \in \mathcal{S} \tag{3}$$

and by $\mathcal{O}'_C$ the subset of $\mathcal{D}'$ such that

$$U \in \mathcal{O}'_C \text{ if and only if } (1+\|x\|^2)^{\frac{k}{2}}U \text{ is bounded in } \mathbb{R}^n, k \in \mathbb{N}, \tag{4}$$

which means $(1+\|x\|^2)U^{\frac{k}{2}}(\phi_\alpha)$ converges to 0 whenever ϕ_α and all its derivatives belong to $L_\infty(\mathbb{R}^n)$ and (ϕ_α) such as its derivatives converges to 0 in $L_\infty(\mathbb{R}^n)$. For the various topologies see [4].

In the sequel $\mathcal{D}'$ and $\mathcal{S}'$ will be endowed with their strong dual topologies and $\mathcal{O}'_C$ (which is not the dual of $\mathcal{O}_M$) will carry its usual topology except otherwise stated.

b) *Operations on Distributions*

The translation $\tau_h U$ is defined by

$$\tau_h U(\phi) = U(\tau_{-h}\phi),\ U \in \mathcal{D}', \phi \in \mathcal{D}. \tag{5}$$

The partial derivative $\dfrac{\partial U}{\partial x_k}$ is defined by

$$\frac{\partial U}{\partial x_k}(\phi) = -U(\frac{\partial\phi}{\partial x_k}),\ U \in \mathcal{D}', \phi \in \mathcal{D}. \tag{6}$$

It is shown in [4, pages 77 and 88] that

$$\frac{\partial U}{\partial x_k} = \lim_{h_k\to 0} \frac{\tau_{-h}U - U}{h_k} \tag{7}$$

Ref.

[2] Grubb, G. *Distributions and operators*. Springer, New York, (2009).

where $h = (0, \ldots, 0, h_k, 0, \ldots, 0) \in \mathbb{R}^n$ and also that

$$\frac{\partial \tau_h U}{\partial h_k} = -\frac{\partial \tau_h U}{\partial x_k}. \tag{8}$$

The multiplication αU is defined by

$$\alpha U(\phi) = U(\alpha\phi),\ U \in \mathcal{D}', \alpha \in \mathcal{E}, \phi \in \mathcal{D}. \tag{9}$$

The Fourier transform $\mathcal{F}U$ is defined by

$$\mathcal{F}U(\phi) = U(\mathcal{F}\phi),\ U \in \mathcal{S}', \phi \in \mathcal{S}. \tag{10}$$

The convolution $U * V$ is defined by

$$U * V(\phi) = U(V(\tau_{-y}\phi)) \tag{11}$$

where V acts on ϕ as a function of x and U on the result as a function of y.

Convolution is not always meaningful, but it makes sense for instance if one the distributions at least has compact support. When it is defined, convolution is bilinear, commutative and the mapping $(U, V) \mapsto U * V$ is hypocontinuous in the sense that if one of U or V varies in a bounded set of $\mathcal{D}'$ or $\mathcal{E}'$ and the other converges to 0, then $U * V$ converges to 0. Moreover

$$\tau_h(U * V) = \tau_h U * V = U * \tau_h V \tag{12}$$

$$\text{and} \quad \frac{\partial(U * V)}{\partial x_k} = \frac{\partial U}{\partial x_k} * V = U * \frac{\partial V}{\partial x_k}. \tag{13}$$

We recall the following well-known lemma from [4, page 268].

Lemma 1.1 *The Fourier transform is an isomorphism between $\mathcal{O}'_C$ and $\mathcal{O}_M$, and changes convolution into multiplication in $\mathcal{S}'$ i.e. if $U \in \mathcal{S}'$, $W \in \mathcal{O}'_C$ then $\mathcal{F}U \in \mathcal{S}'$, $\mathcal{F}W \in \mathcal{O}_M$ and*

$$\mathcal{F}(W * U) = \mathcal{F}W\mathcal{F}U. \tag{14}$$

II. Main Results

Definition 2.1 *Let $\mathcal{V}$ or $\mathcal{W}$ be any one of $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{O}'_C$ or $\mathcal{E}'$. We call $(\mathcal{V}, \mathcal{W})-$multiplier any continuous linear transformation $T : \mathcal{V} \to \mathcal{W}$ that commutes with translations.*

We denote by $\mathcal{L}(\mathcal{V}, \mathcal{W})$ the space of continuous linear transformation from $\mathcal{V}$ to $\mathcal{W}$ and by $\mathcal{M}(\mathcal{V}, \mathcal{W})$ the subset of $\mathcal{L}(\mathcal{V}, \mathcal{W})$ consisting of the $(\mathcal{V}, \mathcal{W})-$multipliers. We write $\mathcal{L}(\mathcal{V})$ and $\mathcal{M}(\mathcal{V})$ for $\mathcal{L}(\mathcal{V}, \mathcal{W})$ and $\mathcal{M}(\mathcal{V}, \mathcal{W})$ when $\mathcal{V} = \mathcal{W}$ respectively.

Ref.

[4] Schwartz, L. , *Théorie des distributions.* Hermann, Paris, (1966).

Theorem 2.2 *Let T be a linear continuous operator in $\mathcal{D}'$. Then the following statements are equivalent:*

i) $T \in \mathcal{M}(\mathcal{D}')$.

ii) *T commutes with partial derivatives.*

iii) *There exists a unique W in $\mathcal{E}'$ such that*

$$TU = W * U \text{ for all } U \text{ in } \mathcal{D}'. \tag{15}$$

Proof.

1. i)⇒ ii)
   Suppose $T \in \mathcal{M}(\mathcal{D}')$. Then $T(\tau_h U) = \tau_h(TU)$, $\forall\, U \in \mathcal{D}'$, $\forall\, h \in \mathbb{R}^n$.
   We have

$$\begin{aligned} T(\frac{\partial U}{\partial x_k}) &= T(\lim_{h_k \to 0} \frac{\tau_{-h}U - U}{h_k}) \\ &= \lim_{h_k \to 0} T(\frac{\tau_{-h}U - U}{h_k}) \\ &= \lim_{h_k \to 0} \frac{\tau_{-h}TU - TU}{h_k} = \frac{\partial TU}{\partial x_k}. \end{aligned}$$

   Hence T commutes with partial derivatives.

2. ii) ⇒ i). This implication can be found in [4, page 163]. For the convenience of the reader, we reproduce it here.
   Assume that ii) holds and consider the function ψ define by

$$\psi(h) = [T(\tau_h U)](\tau_h \phi). \tag{16}$$

   We are going to prove that for $U \in \mathcal{D}'$ and $\phi \in \mathcal{D}$ fixed, the function ψ is independent of $h = (h_1, h_2, \dots, h_n)$ in $\mathbb{R}^n$. For $k = 1, 2, \dots, n$, we have

$$\begin{aligned} \frac{\partial \psi}{\partial h_k}(h) &= \frac{\partial}{\partial h_k}\{[T(\tau_h U)](\tau_h \phi)\} \\ &= [\frac{\partial}{\partial h_k} T(\tau_h U)](\tau_h \phi) + [T(\tau_h U)](\frac{\partial}{\partial h_k}(\tau_h \phi)) \\ &= T[\frac{\partial}{\partial h_k}(\tau_h U)](\tau_h \phi) + [T(\tau_h U)](\frac{\partial}{\partial h_k}(\tau_h \phi)) \\ &= T[-\frac{\partial}{\partial x_k}(\tau_h U)](\tau_h \phi) - [T(\tau_h U)](\frac{\partial}{\partial x_k}(\tau_h \phi)) \end{aligned}$$

Ref.

[4] Schwartz, L. , *Théorie des distributions*. Hermann, Paris, (1966).

$$= -\frac{\partial}{\partial x_k}[T(\tau_h U)](\tau_h \phi) - [T(\tau_h U)](\frac{\partial}{\partial x_k}(\tau_h \phi))$$

$$= [T(\tau_h U)](\frac{\partial}{\partial x_k}(\tau_h \phi)) - [T(\tau_h U)](\frac{\partial}{\partial x_k}(\tau_h \phi)) = 0,$$

Notes

using successively the fact that T commutes with partial derivatives and is linear, and formulas (8) and (6). We conclude from $\frac{\partial \psi}{\partial h_k}(h) = 0$ for $k = 1, 2, \ldots, n$ that ψ is a constant equal to $\psi(0)$. That means that $[T(\tau_h U)](\tau_h \phi) = TU(\phi)$. Now, $T(\tau_h U)(\tau_h \phi) = \tau_{-h} T(\tau_h U)(\phi)$ by definition. Thus $\tau_{-h} T(\tau_h U) = TU$ and $T(\tau_h U) = \tau_h(TU)$. Hence i) holds.

3. iii) $\Rightarrow$ i).
   For any $W \in \mathcal{E}'$, the mapping $U \mapsto T_W(U) = W * U$ defines a continuous operator in $\mathcal{D}'$ which commutes with translations according to (12) and because $(U, W) \mapsto W * U$ is hypocontinuous.

4. i) $\Rightarrow$ iii).
   Let T be an element of $\mathcal{M}(\mathcal{D}')$. We shall show first that if $T(U * V)$ and $TU * V$ are defined then they are equal. In fact, for $\phi \in \mathcal{D}$, we have $(TU * V)(\phi) = (V * TU)(\phi) = V(TU(\tau_{-y}\phi)) = V(\tau_y TU(\phi)) = V(T\tau_y U(\phi)) = V * U(T^*\phi) = T(U * V(\phi))$ where T^* is the adjoint of T. Consequently $TU * V = T(U * V)$. Applying this equality to $U = \delta * U$ where δ is the Dirac distribution at the origin, we obtain $TU = T\delta * U$. Put $T\delta = W$ to have $TU = W * U$. Indeed W belongs to $\mathcal{E}'$. (See the remark in [4, page 163]).

Examples.

1. The identity operator T is a multiplier; the element W of $\mathcal{E}'$ associated with it is δ since $U = \delta * U$.

2. Translations τ_h are multipliers; the W of $\mathcal{E}'$ corresponding is δ_h, the point mass at $h \in \mathbb{R}^n$, since $\tau_h U = \delta_h * U$.

3. Distributional derivatives are continuous. They are multipliers because they commute with one another. Then we have $\frac{\partial U}{\partial x_k} = \frac{\partial \delta}{\partial x_k} * U$.

Theorem 2.3 *Let the set $\mathcal{L}(\mathcal{D}')$ be endowed with its strong topology i.e. the topology of convergence on bounded sets of $\mathcal{D}$. Then*

1. *The mapping $T \mapsto W = T\delta$ from $\mathcal{M}(\mathcal{D}')$ into $\mathcal{E}'$ equipped with its strong topology is a bicontinuous isomorphism in the topology induced by $\mathcal{L}(\mathcal{D}')$ on $\mathcal{M}(\mathcal{D}')$ i.e. $\mathcal{M}(\mathcal{D}')$ is isomorphic to $\mathcal{E}'$.*

2. $\mathcal{M}(\mathcal{E}', \mathcal{D}')$ *is isomorphic to* $\mathcal{D}'$*.*

3. $\mathcal{M}(\mathcal{O}'_C, \mathcal{S}')$ *is isomorphic to* $\mathcal{S}'$*.*

4. $\mathcal{M}(\mathcal{S}')$ *is isomorphic to* $\mathcal{O}'_C$*.*

Proof.

Let us prove the assertion 1.
The mapping $\mathcal{M}(\mathcal{D}') \to \mathcal{E}'$, $T \mapsto T\delta$ is obviously linear. It is surjective because each $W \in \mathcal{E}'$ defines a multiplier T_W such that $T_W\delta = W * \delta = W$. It is injective, for if $T\delta = S\delta$, $T, S \in \mathcal{M}(\mathcal{D}')$, then for any U in $\mathcal{D}'$, we have $TU = T\delta * U = S\delta * U = SU$, so $T = U$.
Let us show now that it is continuous. Suppose (T_α) tends to 0 in $\mathcal{M}(\mathcal{D}')$. Then $(T_\alpha\delta * U)$ tends to 0 in $\mathcal{D}'$ uniformly on $U \in B'$ where B' is a bounded set in $\mathcal{D}'$. Put $T_\alpha\delta = W_\alpha$. Then for every bounded set B in $\mathcal{D}$, $\sup_{\phi \in B, U \in B'} |W_\alpha * U(\phi)|$ tends to 0 in $\mathbb{R}$. Now $W_\alpha * U(\phi) = W_\alpha(U(\tau_{-y}\phi))$ and $U(\tau_{-y}\phi)$ belongs to $\mathcal{E}$. Moreover $(U(\tau_{-y}\phi))_{\phi \in B, U \in B'}$ is bounded in $\mathcal{E}$ i.e. is bounded on every compact subset K of $\mathbb{R}^n$. Hence for every bounded set B_0 in $\mathcal{E}$ of the form $B_0 = (U(\tau_{-y}\phi))$, $\sup_{B_0 \ni \psi} |W_\alpha(\psi)|$ tends to 0, with respect to α. Now let B_1 be an arbitrary bounded set in $\mathcal{E}$. We are going to prove that $\sup_{\psi \in B_1} |W_\alpha(\psi)|$ tends to 0. For any compact $K \subset \mathbb{R}^n$, put $B_K = \phi_K B_1 = \{\phi_K\, \psi \colon \psi \in B_1\}$ where $\phi_K \in \mathcal{D}$ with $supp\phi_K \supset K$ and $\phi_K \equiv 1$ on K. The set B_K is bounded in $\mathcal{D}$. Then $\sup_{\psi_K \in B_K} |W_\alpha(\psi_K)| = \sup_{\psi_K \in B_K} |W_\alpha * \delta(\psi_K)| = \sup_{\psi_K \in B_K} |W_\alpha(\delta(\tau_{-y}\psi_K))|$ which tends to 0 according to what is pointed out above.
If $\sup_{\psi \in B_1} |W_\alpha(\psi)|$ does not converge to 0, then there would exist $\varepsilon > 0$ such that for every α there would exist $\beta > \alpha$ such that $|W_\beta(\psi)| > \varepsilon$ for every $\psi \in B_1$. But for every $\varepsilon > 0$ there exists α_0 such that $\alpha > \alpha_0$ implies $|W_\alpha(\psi_K)| < \varepsilon$ for every compact $K \subset \mathbb{R}^n$ and every $\psi_K \in B_K$. Now $W_\beta(\psi) = W_\beta(\psi_{K_\beta})$ where $K_\beta = suppW_\beta$ is compact. Let β be such that $\beta > \alpha > \alpha_0$, so that $|W_\beta(\psi_{K_\beta})| > \varepsilon$. This is a contradictory. We conclude that (W_α) tends to 0 in $\mathcal{E}'$. Then $T \mapsto T\delta$ is continuous.

The inverse mapping is also continuous. In fact, let (W_α) be a net in $\mathcal{E}'$ which converges to 0. Then for U in a bounded set of $\mathcal{D}'$, $((W_\alpha * U)$ converges to 0 in $\mathcal{D}'$ (hypocontinuity of the convolution). That is the mapping $W \mapsto T_W$ from $\mathcal{E}'$ into $\mathcal{M}(\mathcal{D}')$ is continuous.
The assertion 1 is completely proved.
The assertions 2., 3. and 4. are proved similarly.

Corollary 2.4 $T \in \mathcal{M}(\mathcal{S}')$ *if and only if there exists a unique* $\phi \in \mathcal{O}_M$ *such that* $\mathcal{F}(TU) = \phi\mathcal{F}(U)$*. Moreover the mapping* $T \mapsto \phi$ *is a topological isomorphism between* $\mathcal{M}(\mathcal{S}')$ *and* $\mathcal{O}_M$*.*

Proof. The corollary follows from Theorem 2.3, assertion 4. and Lemma 1.1 with $\mathcal{F}W = \phi$.

Notes

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Notes

Remark.

Some writers define a $(\mathcal{V},\mathcal{W})-$multiplier as a function ϕ such that $\phi V \in \mathcal{W}$ for every $V \in \mathcal{V}$. Let us denote in this case the multipliers spaces by $M(\mathcal{V},\mathcal{W})$. It was pointed out in [4, page 246] that $M(\mathcal{S}') = M(\mathcal{S}',\mathcal{S}')$ is precisely $\mathcal{O}_M$. Then according to Corollary 2.4, we can say that $M(\mathcal{S}')$ is topologically isomorphic to $\mathcal{M}(\mathcal{S}')$. See [6] for some extension results.

Theorem 2.5 *1. $T \in \mathcal{M}(\mathcal{D}')$ and $U \in \mathcal{E}'$ imply $TU \in \mathcal{E}'$. Moreover, every $T \in \mathcal{M}(\mathcal{E}')$ has a unique extension to an element of $\mathcal{M}(\mathcal{D}')$.*

2. $T \in \mathcal{L}(\mathcal{E}')$ belongs to $\mathcal{M}(\mathcal{E}')$ if and only if $TU*V = T(U*V)$, $U,V \in \mathcal{E}'$.

Proof.

1. If $T \in \mathcal{M}(\mathcal{D}')$ then by Theorem 2.2, $TU = W * U$ for some $W \in \mathcal{E}'$. Hence $TU \in \mathcal{E}'$ whenever $U \in \mathcal{E}'$ because $supp(W*U)$ is closed and included in $suppW + suppU$ which is compact.
   Since $\mathcal{E}'$ is a dense subspace of $\mathcal{D}'$ then $T \in \mathcal{M}(\mathcal{E}')$ has a unique continuous extension $\hat{T}$ to $\mathcal{D}'$ and $\hat{T} \in \mathcal{M}(\mathcal{D}')$.

2. $T\tau_h = \tau_h T$ implies $T(U*V) = TU*V,\ U,V \in \mathcal{E}'$; this is contained in the proof of Theorem 2.2 above.
   Conversely, suppose $T(U*V) = TU*V,\ U,V \in \mathcal{E}'$. Then $T(\tau_h U) = T(\delta * \tau_h U) = T\delta * \tau_h U = \tau_h(T\delta * U) = \tau_h(TU)$ i.e. $T \in \mathcal{M}(\mathcal{E}')$.

One has the following immediate consequence.

Corollary 2.6 *A continuous linear operator belongs to $\mathcal{M}(\mathcal{D}')$ if and only if its restriction to $\mathcal{E}'$ commutes with convolution, i.e.*

$$T \in \mathcal{M}(\mathcal{D}') \Leftrightarrow T_{|\mathcal{E}'}(U*V) = T_{|\mathcal{E}'}U * V.$$

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Before start writing a good quality Computer Science Research Paper, let us first understand what is Computer Science Research Paper? So, Computer Science Research Paper is the paper which is written by professionals or scientists who are associated to Computer Science and Information Technology, or doing research study in these areas. If you are novel to this field then you can consult about this field from your supervisor or guide.

TECHNIQUES FOR WRITING A GOOD QUALITY RESEARCH PAPER:

1. Choosing the topic: In most cases, the topic is searched by the interest of author but it can be also suggested by the guides. You can have several topics and then you can judge that in which topic or subject you are finding yourself most comfortable. This can be done by asking several questions to yourself, like Will I be able to carry our search in this area? Will I find all necessary recourses to accomplish the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

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15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

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18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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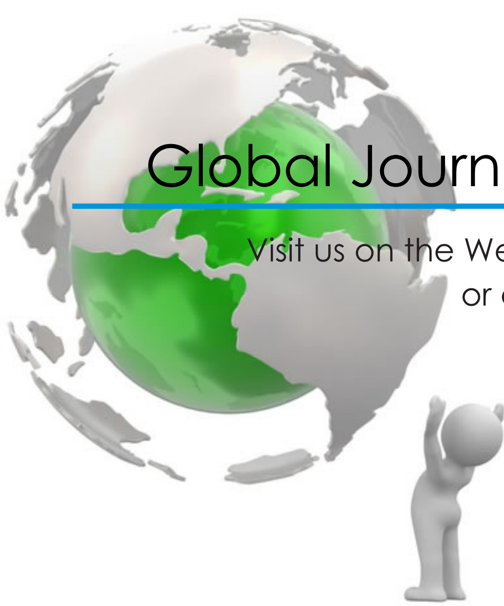

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