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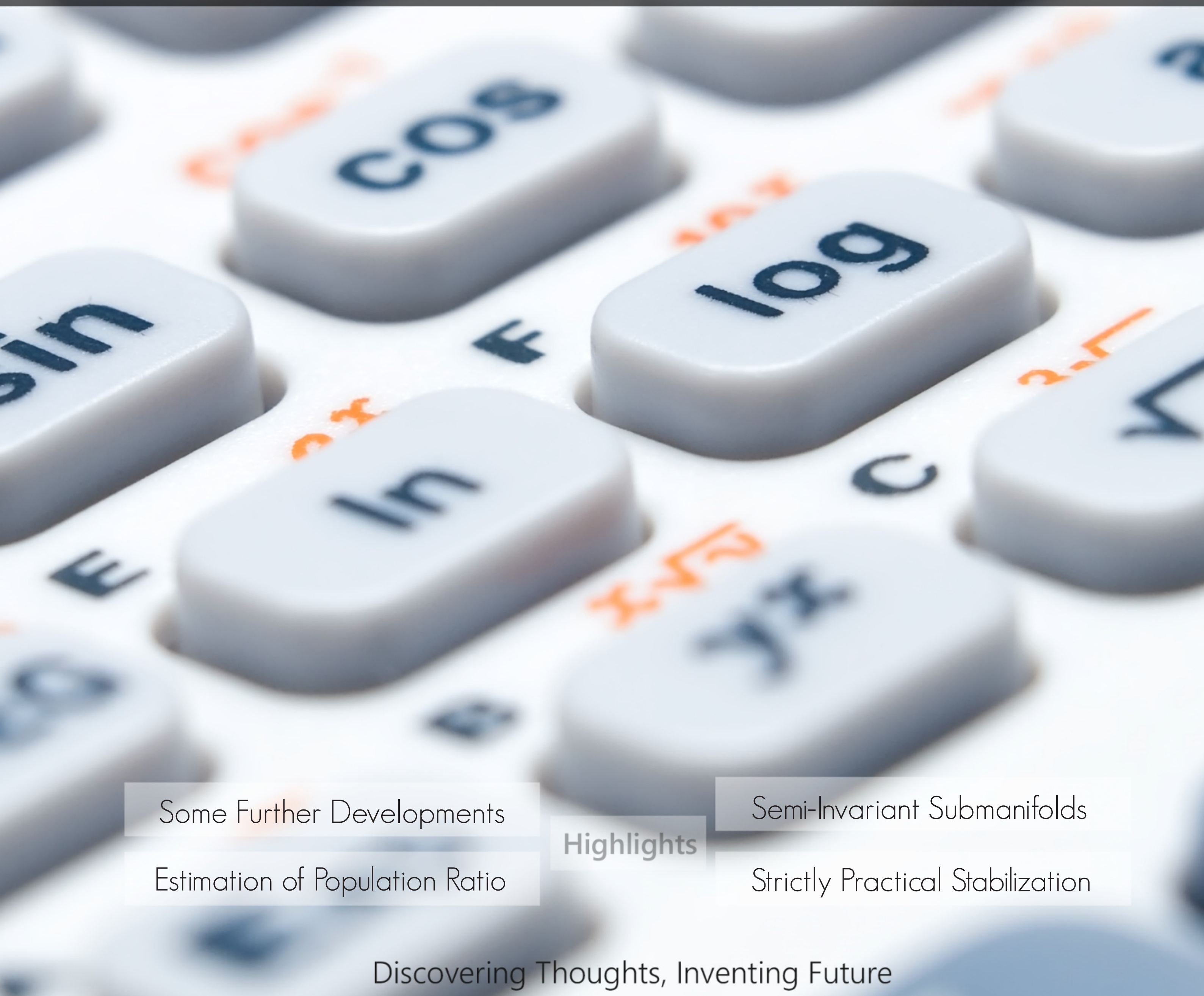

Highlights

Some Further Developments

Estimation of Population Ratio

Semi-Invariant Submanifolds

Strictly Practical Stabilization

Discovering Thoughts, Inventing Future

VOLUME 13 ISSUE 4 VERSION 1.0

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Some Further Developments in the Infinite Product Representation of Elementary Functions

By Viktor Reshniak
Middle Tennessee State University

Abstract - An innovatory approach has been recently proposed for the derivation of infinite product representation of elementary functions. The approach is based on the comparison of different alternative forms of Green's functions for boundary-value problems stated for the two-dimensional Laplace equation. A number of new infinite product representations of elementary functions was actually derived within the scope of that approach. The present study continues the trend: it aims at an analysis of the approach and exploring ways for its extending to some other problem statements that might also be efficiently treated.

Keywords : *green's functions; infinite products; elementary functions.*

GJSFR-F Classification : *MSC 2010: 08B25 , 26A09*

SOME FURTHER DEVELOPMENTS IN THE INFINITE PRODUCT REPRESENTATION OF ELEMENTARY FUNCTIONS

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Viktor Reshniak

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I. Introduction

In a series of recent works (see, for example, [6] and [7]), an innovatory approach was proposed to the derivation of infinite product representation of elementary functions. The approach is based on the multiplicity of forms of Green's functions for some boundary-value problems. To introduce the key idea of the approach, we turn to one of the simplest problem settings of that kind. That is the Dirichlet problem

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0, \qquad (x,y)\in\Omega \tag{1}$$

$$u(x,0)=u(x,b)=0, \qquad \lim_{x\to\pm\infty} u(x,y)<\infty \tag{2}$$

posed in the infinite strip $\Omega=\{-\infty<x<\infty, 0<y<b\}$.
The classical representation

$$G(x,y;\xi,\eta)=\frac{1}{4\pi}\ln\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}, \qquad \omega=\frac{\pi}{b} \tag{3}$$

of the Green's function for the setting in (1)-(2) can be obtained by one of the two standard methods available in the field. Indeed, either the conformal mapping method [1, 2] algorithm, or the method of eigenfunction expansion [4, 5] procedure, with subsequent summation of trigonometric series that represents the Green's function, appear successful.

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Ref

1. C. Caratheodory, Conformal representation, Cambridge university press, 1969.

In [6], it was recalled that if the method of images routine (that is also considered workable, but efficient in a limited number of problem settings) is used instead, then we arrive at an alternative to (3) expression

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}} \tag{4}$$

Since the forms in (3) and (4) are equivalent, we obtain the multi-variable identity

$$\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}} = \prod_{n=-\infty}^{\infty}\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2} \tag{5}$$

which can be used as a starting point for the derivation of a number of infinite product representations of elementary functions.

Upon some trivial algebra, the identity in (5) can, for example, be transformed in the infinite product representation

$$\sin t = \frac{2t}{\pi}\prod_{k=1}^{\infty}\left[1+\frac{4t^2-\pi^2}{(1-4k^2)\pi^2}\right]$$

of the trigonometric sine function. The latter can be considered as an equivalent alternative form to the classical Euler infinite product expansion

$$\sin t = t\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2\pi^2}\right)$$

of the sine function.

A score of new as well as alternative to existing infinite product representations of elementary functions was presented in [7] by the approach just described. Note, that the author of [7] has claimed that research in the area is still open. Accepting the challenge, we will show in the next section, that the approach also appears efficient in a few other cases.

II. Further Extension of the Approach

a) *Semi-infinite Strip Region*

We start with the mixed boundary-value problem

$$u(0,y) = \frac{\partial u(a,y)}{\partial x} = 0 \tag{6}$$

$$u(x,0) = 0, \qquad \lim_{y\to\infty} u(x,y) < \infty \tag{7}$$

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega = \{0 < x < a, 0 < y < \infty\}$.

Following the technique proposed in [6] and [7], two equivalent representations of the Green's function for the problem in (6)-(7) can be found. The classical approach of eigenfunction expansion leads to a closed analytical form of its solution. On the other

Ref.

6. Yu. A. Melnikov, A new approach to the representation of some trigonometric and hyperbolic functions by infinite products, J. Math. Anal. Appl., 344, 1, 2008, 521-534.

hand, another classical approach – the method of images - provides an infinite product expression of the required Green's function.

Method of eigenfunction expansion. Let the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=-f(x,y), \qquad (x,y)\in\Omega \tag{8}$$

Ref

be subject to the boundary conditions in (6)-(7).
Once the solution of the stated problem is found as the integral

$$u(x,y)=\iint_{\Omega} G(x,y;\xi,\eta)f(\xi,\eta)d\Omega(\xi,\eta), \qquad (x,y)\in\Omega \tag{9}$$

the kernel function $G(x,y;\xi,\eta)$ of the above represents the Green's function to the boundary value problem in (6)-(7).

According to the theory of Fourier series, when the eigenfunctions are known, the solution of the given boundary-value problem can be obtained by the superposition [5]

$$u(x,y)=\sum_{k=1}^{\infty} Y_k(y)X_k(x) \tag{10}$$

where X_k is a complete system of eigenfunctions, orthogonal in [0,a]

$$X_k=\sin\lambda_k x,$$
$$\lambda_k=\frac{\pi(2k-1)}{2a}, \quad k=1,2,\ldots$$

Functions $Y_k(y)$ are to be determined from the Fourier series expansion of the right-hand side part in (8)

$$f(x,y)=\sum_{k=1}^{\infty} f_k(y)X_k(x), \qquad f_k(y)=\frac{2}{a}\int_0^a f(\xi,y)X_k(\xi)d\xi \tag{11}$$

resulting in

$$\frac{d^2Y_k(y)}{dy^2}-\lambda_k^2Y_k^2(y)=-\frac{2}{a}\int_0^a f(\xi,y)\sin\lambda_k\xi\, d\xi \tag{12}$$

$$Y_k(0)=0 \tag{13}$$

$$\lim_{y\to\infty}|Y_k(y)|<\infty \tag{14}$$

Clearly, solution of the problem in (12)-(14) has the form

$$Y_k(y)=\int_0^\infty f_k(\eta)g_k(y;\eta)d\eta \tag{15}$$

5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.

where $g_k(y;\eta)$ is the Green's function for the homogeneous equation corresponding to (12) with boundary conditions in (13) and (14) ([4],[7])

$$g_k(y;\eta) = \frac{1}{2\lambda_k}\begin{cases} e^{\lambda_k(y-\eta)} - e^{-\lambda_k(y+\eta)}, & y \le \eta \\ e^{\lambda_k(\eta-y)} - e^{-\lambda_k(y+\eta)}, & y \ge \eta \end{cases}$$

Therefore the series in (10), along with (15) solves the problem posed in (6)-(8)

$$u(x,y) = \int_0^a \int_0^\infty \left[\frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\, \sin\lambda_k\xi\, \sin\lambda_k x\right] f(\xi,\eta)d\xi d\eta$$

In light of (9) the kernel function of the above expression represents the Green's function to the problem in (6)-(7) posed for the Laplace equation in Ω

$$G(x,y;\xi,\eta) = \frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\, \sin\lambda_k\xi\, \sin\lambda_k x \tag{16}$$

Upon the trigonometric identity

$$\sin\lambda_k\xi\, \sin\lambda_k x = \frac{1}{2}\left[\cos\lambda_k(\xi - x) - \cos\lambda_k(\xi + x)\right]$$

and (see, for example [3]) the summation formula

$$\sum_{k=1}^{\infty}\frac{p^{2k-1}\cos(2k-1)x}{2k-1} = \frac{1}{4}\ln\frac{1+2p\cos x + p^2}{1-2p\cos x+p^2}, \qquad 0 < x < 2\pi, \quad p^2 \le 1$$

the expression in (16) transforms into

$$\begin{aligned} G(x,y;\xi,\eta) = \frac{1}{4\pi}\Bigg(&\ln\frac{1+2e^{\omega(y-\eta)}\cos\,\omega(\xi-x)+e^{2\omega(y-\eta)}}{1-2e^{\omega(y-\eta)}\cos\,\omega(\xi-x)+e^{2\omega(y-\eta)}} \\ &-\ln\frac{1+2e^{\omega(y-\eta)}\cos\,\omega(\xi+x)+e^{2\omega(y-\eta)}}{1-2e^{\omega(y-\eta)}\cos\,\omega(\xi+x)+e^{2\omega(y-\eta)}} \\ &-\ln\frac{1+2e^{-\omega(y+\eta)}\cos\,\omega(\xi-x)+e^{-2\omega(y+\eta)}}{1-2e^{-\omega(y+\eta)}\cos\,\omega(\xi-x)+e^{-2\omega(y+\eta)}} \\ &+\ln\frac{1+2e^{-\omega(y+\eta)}\cos\,\omega(\xi+x)+e^{-2\omega(y+\eta)}}{1-2e^{-\omega(y+\eta)}\cos\,\omega(\xi+x)+e^{-2\omega(y+\eta)}}\Bigg) \end{aligned} \tag{17}$$

where

$$\omega = \frac{\pi}{2a}$$

After introducing the variables

$$\alpha = \frac{\pi}{2a}(y-\eta); \quad \beta = -\frac{\pi}{2a}(y+\eta); \quad \gamma = \frac{\pi}{2a}(x-\xi); \quad \theta = \frac{\pi}{2a}(\xi+x) \tag{18}$$

Ref.

4. Dean G. Duffy, Green's Functions with Applications, Chapman and Hall/CRC, 2001.

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the form in (17) reads

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\left(\frac{\left(1+2e^{\alpha}\cos\,\gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\,\gamma+e^{2\alpha}\right)}\frac{\left(1+2e^{\beta}\cos\,\theta+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\,\theta+e^{2\beta}\right)}\right.$$
$$\left.\times\frac{\left(1-2e^{\alpha}\cos\,\theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\,\theta+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\,\gamma+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\,\gamma+e^{2\beta}\right)}\right) \tag{19}$$

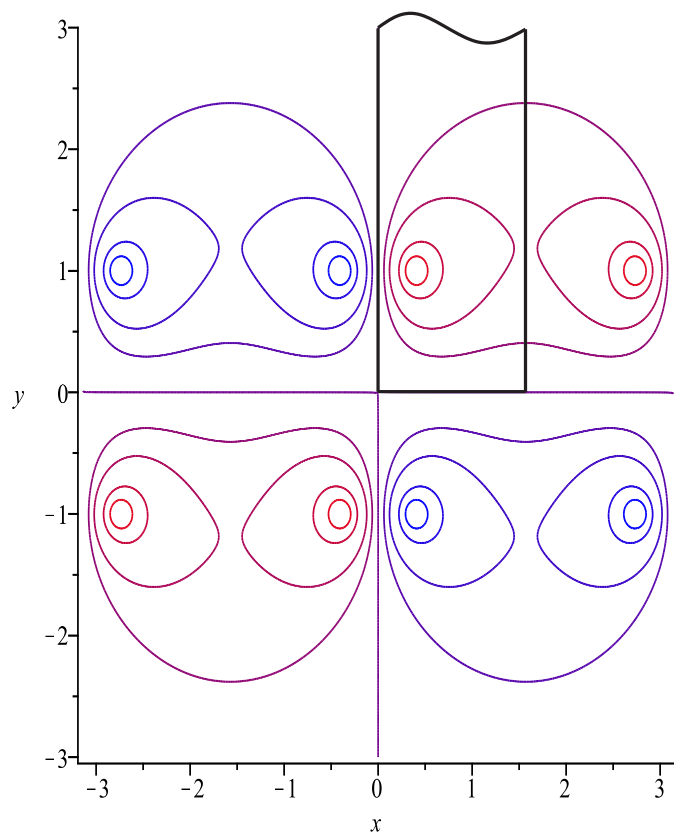


Figure 1 : Field generated in the semi-infinite strip $0 \le x \le \pi/2,\ y \ge 0$ by a source located at $\xi = \pi/8,\ \eta = 1$

By problem statement, the observation point (x, y) and the source point (ξ, η) are interior to the semi-infinite strip Ω. It means that parameters $\alpha, \beta, \gamma, \theta$ range as

$$-\infty<\alpha<\infty,\qquad -\infty<\beta<0,\qquad -\frac{\pi}{2}\le\gamma\le\frac{\pi}{2},\qquad 0\le\theta\le\pi \tag{20}$$

given that α and γ are not equal to zero at the same time.

But on the other hand the function in (19) is analytic everywhere except, of course, at the points of singularity, as it can be seen in Figure 1. Hence, the parameter ranges in (20) can be extended to the entire region of analyticity of (19).

Method of images. The Green's function for the two-dimensional Laplace equation has the form ([5, 7])

Ref

5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln r(x, y, \xi, \eta) + \mu(x, y; \xi, \eta)$$

where $r(x, y, \xi, \eta)$ is the distance between the field point and the source point, and $\mu(x, y; \xi, \eta)$ is a harmonic in Ω function.

Notes

According to the method of images, the regular component $\mu(x, y; \xi, \eta)$ is sought as a superposition of responses at a field point P from singularities Q_j^* placed outside the region Ω

$$\mu(x, y) = \sum_{j=1}^{m} \pm \frac{1}{2\pi} \ln |P - Q_j^*|$$

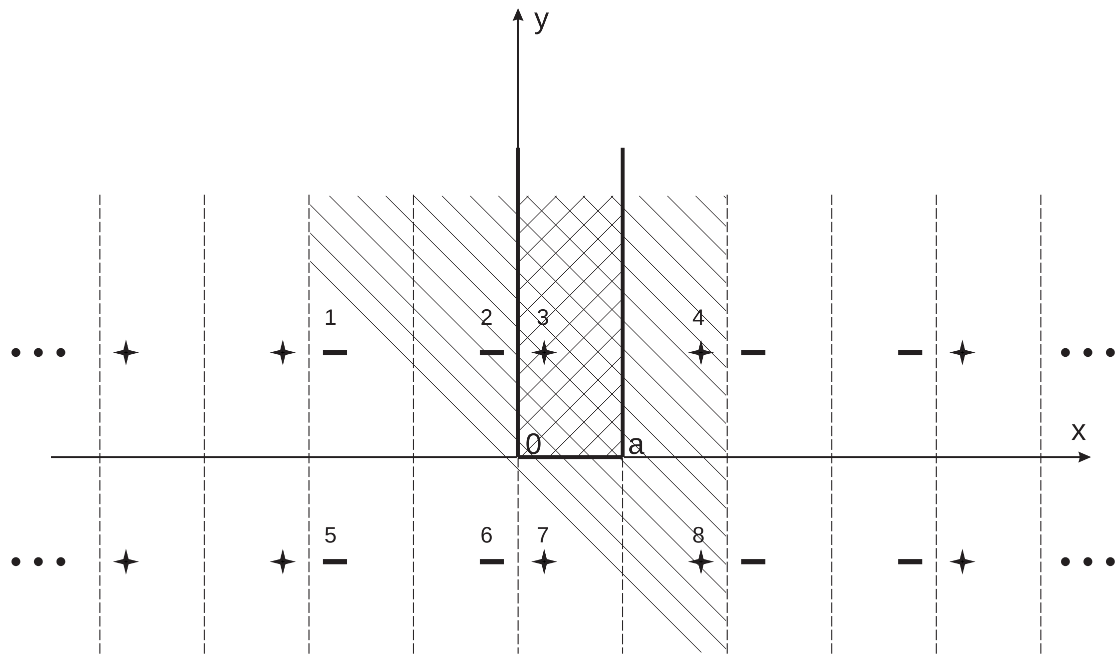


Figure 2 : Sequence of sources and sinks that arises in the method of images applied to the problem in (6)-(7)

The sign of each term in the above as well as the positions of the corresponding singularities are chosen to satisfy boundary conditions of the given boundary-value problem.

Figure 2 illustrates this approach to the problem in (6)-(7). Unit sources are labeled with the plussign and sinks with the minus sign. Geometry of Ω leads to the infinite sequence of singularities, where the shaded strip of width $4a$ is repeated periodically in the direction of the x-axis. The singularities have the following coordinates.

$$(\xi_1^0 = -(2a - \xi);\ \eta_1^0 = \eta) \qquad (\xi_5^0 = -(2a - \xi);\ \eta_5^0 = -\eta)$$

$$(\xi_2^0 = -\xi;\ \eta_2^0 = \eta) \qquad (\xi_6^0 = -\xi;\ \eta_6^0 = -\eta)$$

$$(\xi_3^0 = \xi;\ \eta_3^0 = \eta) \qquad (\xi_7^0 = \xi;\ \eta_7^0 = -\eta)$$

$$(\xi_4^0 = 2a - \xi;\ \eta_4^0 = \eta) \qquad (\xi_8^0 = 2a - \xi;\ \eta_8^0 = -\eta)$$

Hence, the total effect on the field point P from the infinite sequence of suitably chosen sinks and sources Q_j^* is described by the following Green's function

$$G(x,y;\xi,\eta)=\sum_{k=-\infty}^{\infty}G^k(x,y;\xi,\eta)$$

Notes

$$=\frac{1}{2\pi}\sum_{k=0}^{\infty}\ln\sqrt{\frac{\left((x-\xi_1^k)^2+(y-\eta_1^k)^2\right)\left((x-\xi_2^k)^2+(y-\eta_2^k)^2\right)}{\left((x-\xi_3^k)^2+(y-\eta_3^k)^2\right)\left((x-\xi_4^k)^2+(y-\eta_4^k)^2\right)}}$$

$$\times\sqrt{\frac{\left((x-\xi_7^k)^2+(y-\eta_7^k)^2\right)\left((x-\xi_8^k)^2+(y-\eta_8^k)^2\right)}{\left((x-\xi_5^k)^2+(y-\eta_5^k)^2\right)\left((x-\xi_6^k)^2+(y-\eta_6^k)^2\right)}} \tag{21}$$

with ξ_n^k representing the coordinates of singularities

$$\xi_n^k=\xi_n^0+4ak,\qquad k=\pm1,\pm2,\ldots$$

The form in (21) reads, in terms of the variables introduced in (18), as

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\prod_{k=-\infty}^{\infty}\left(\frac{\left((\gamma+\pi(1-2k))^2+\alpha^2\right)\left((\theta-2\pi k)^2+\alpha^2\right)}{\left((\gamma+\pi(1-2k))^2+\beta^2\right)\left((\theta-2\pi k)^2+\beta^2\right)}\right.$$

$$\left.\times\frac{\left((\theta-\pi(1+2k))^2+\beta^2\right)\left((\gamma-2\pi k)^2+\beta^2\right)}{\left((\theta-\pi(1+2k))^2+\alpha^2\right)\left((\gamma-2\pi k)^2+\alpha^2\right)}\right) \tag{22}$$

Infinite Products and Elementary Functions. As it follows directly from the equivalence of the representations of (19) and (22), the identity

$$\frac{\left(1+2e^{\alpha}\cos\gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\gamma+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\gamma+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\gamma+e^{2\beta}\right)}$$

$$\times\frac{\left(1+2e^{\beta}\cos\theta+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\theta+e^{2\beta}\right)}\frac{\left(1-2e^{\alpha}\cos\theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\theta+e^{2\alpha}\right)}$$

$$=\prod_{k=-\infty}^{\infty}\left(\frac{\left((\gamma+\pi(1-2k))^2+\alpha^2\right)\left((\theta-2\pi k)^2+\alpha^2\right)}{\left((\gamma+\pi(1-2k))^2+\beta^2\right)\left((\theta-2\pi k)^2+\beta^2\right)}\right.$$

$$\left.\times\frac{\left((\theta-\pi(1+2k))^2+\beta^2\right)\left((\gamma-2\pi k)^2+\beta^2\right)}{\left((\theta-\pi(1+2k))^2+\alpha^2\right)\left((\gamma-2\pi k)^2+\alpha^2\right)}\right) \tag{23}$$

is valid at the region of analiticity of the function in (19).

The above identity involves four arbitrary parameters, and by a suitable choice of those some interesting representations of elementary functions can be obtained. To start with, we write its left-hand side in the form

$$\frac{(\cos\gamma + A_2)(\cos\gamma - A_1)(\cos\theta + A_1)(\cos\theta - A_2)}{(\cos\gamma - A_2)(\cos\gamma + A_1)(\cos\theta - A_1)(\cos\theta + A_2)}, \tag{24}$$

where

$$A_1 = \frac{1+e^{2\beta}}{2e^{\beta}}, \quad A_2 = \frac{1+e^{2\alpha}}{2e^{\alpha}}$$

This shows that it cannot be converted to an expression consisting of a single trigonometric function for any real value of the parameters α, β, γ, θ. It's also obvious from (24), that to avoid trivial results we should put some constraints on these parameters, specifically $\alpha \neq \pm\beta$, $\gamma \neq \pm\theta$.

It is worth noting that the trigonometric terms in the expression (24) depend only on the parameters γ and θ, and their successful choice allows that expression to be dependent only on the exponential terms.

As an illustration, let us consider the following parameter values: $\gamma = 0$, $\theta = \frac{\pi}{2}$, $\beta = 2u$, $\alpha = 2v$, which converts the expression in (23) into

$$\frac{\tanh^2 u}{\tanh^2 v} = \prod_{k=-\infty}^{\infty} \frac{\left(\pi^2(1+2k)^2 + 4v^2\right)\left(\pi^2k^2 + u^2\right)}{\left(\pi^2(1+2k)^2 + 4u^2\right)\left(\pi^2k^2 + v^2\right)} \tag{25}$$

Note that the above representation has already been derived in [6] and [7] upon considering a different problem.

If the parameter v in (25) is taken to infinity then we arrive at

$$\tanh^2 u = \prod_{k=-\infty}^{\infty} \frac{4\left(\pi^2k^2 + u^2\right)}{\pi^2(1+2k)^2 + 4u^2} \tag{26}$$

When the parameter u is taken to infinity, the following expansion arises

$$\coth^2 v = \prod_{k=-\infty}^{\infty} \frac{\pi^2(1+2k)^2 + 4v^2}{4\left(\pi^2k^2 + v^2\right)} \tag{27}$$

Relations in (26) and (27) have already been derived in [7] and their convergence has already been investigated.

Recalling the interrelation between hyperbolic and trigonometric functions

$$\tanh(x) = -i\ \tan(ix)$$

$$\coth(x) = i\ \cot(ix)$$

Ref.

7. Yuri A. Melnikov, Green's functions and Infinite Products: Bridging the Divide, Birkhauser-Springer, New York - London, 2011.

the new infinite product representations

$$\tan^2 u = \prod_{k=-\infty}^{\infty} \frac{4\Big(\pi^2 k^2 - u^2\Big)}{4u^2 - \pi^2(1+2k)^2} \tag{28}$$

Notes

$$\cot^2 v = \prod_{k=-\infty}^{\infty} \frac{4v^2 - \pi^2(1+2k)^2}{4\Big(\pi^2 k^2 - v^2\Big)} \tag{29}$$

can be obtained.

To prove the convergence of the identity in (28), we rewrite it in the form

$$\begin{aligned} \tan^2 u &= \frac{4u^2}{4u^2-\pi^2} \prod_{k=1}^{\infty} \frac{16\Big(\pi^2 k^2 - u^2\Big)^2}{\Big(4u^2 - \pi^2(1+2k)^2\Big)\Big(4u^2 - \pi^2(1-2k)^2\Big)} \\ &= \frac{4u^2}{4u^2-\pi^2} \prod_{k=1}^{\infty} \left(1 + \pi^2 \frac{\pi^2(8k^2-1)+8u^2}{\Big(4u^2 - \pi^2(1+2k)^2\Big)\Big(4u^2 - \pi^2(1-2k)^2\Big)}\right) \end{aligned} \tag{30}$$

Since the numerator in the second additive component in the product is a second degree polynomial in k, while the denominator is a polynomial of degree four the expansion in (30) converges uniformly for all $u \neq (1 \pm 2n)\pi/2$ at the rate of $1/k^2$.

A similar procedure shows the uniform convergence of the representation in (29)

$$\cot^2 v = \frac{\pi^2 - 4v^2}{4v^2} \prod_{k=1}^{\infty} \left(1 - \frac{\pi^2}{16}\left[\frac{\pi^2(8k^2+1)+8v^2}{(\pi^2k^2 - v^2)^2}\right]\right)$$

for all $v \neq \pm\pi n$ at the rate of $1/k^2$.

The representations in (28) and (29) deliver

$$\tan u = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4\Big(\pi^2 k^2 - u^2\Big)}{4u^2 - \pi^2(1+2k)^2}} \tag{31}$$

where the minus sign is valid for $u \in (-\dfrac{\pi}{2}; 0)$ while the plus sign holds for $u \in [0; \dfrac{\pi}{2})$.

$$\cot v = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4v^2 - \pi^2(1+2k)^2}{4\Big(\pi^2 k^2 - v^2\Big)}} \tag{32}$$

where the plus sign is valid for $u \in (0; \dfrac{\pi}{2})$ while the minus corresponds to $u \in [\dfrac{\pi}{2}; \pi)$.

Figure 3 provides a sense of the convergence of the representations in (31).

Two other expansions can be obtained using the standard identities

$$\sec u = \pm\sqrt{1+\tan^2 u} = \pm\sqrt{1+\prod_{k=-\infty}^{\infty}\frac{4\left(\pi^2k^2-u^2\right)}{4u^2-\pi^2(1+2k)^2}} \tag{33}$$

where the plus sign is valid for $u \in [-\frac{\pi}{2}; \frac{\pi}{2})$, while the minus sign stays for $u \in [\frac{\pi}{2}; \frac{3\pi}{2}]$, and

$$\csc v = \pm\sqrt{1+\cot^2 v} = \pm\sqrt{1+\prod_{k=-\infty}^{\infty}\frac{4v^2-\pi^2(1+2k)^2}{4\left(\pi^2k^2-v^2\right)}} \tag{34}$$

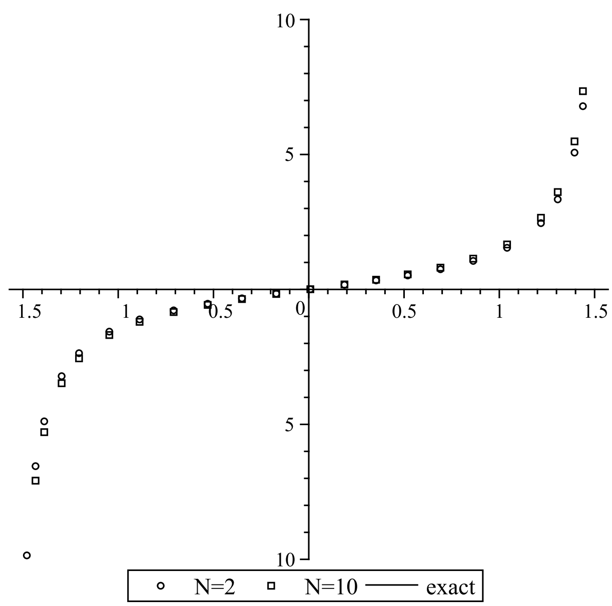
where the plus sign stays for $u \in [0; \pi)$ and the minus sign stays for $u \in [\pi; 2\pi]$.

The convergence rate of the representation in (33) is illustrated in Figure 4.

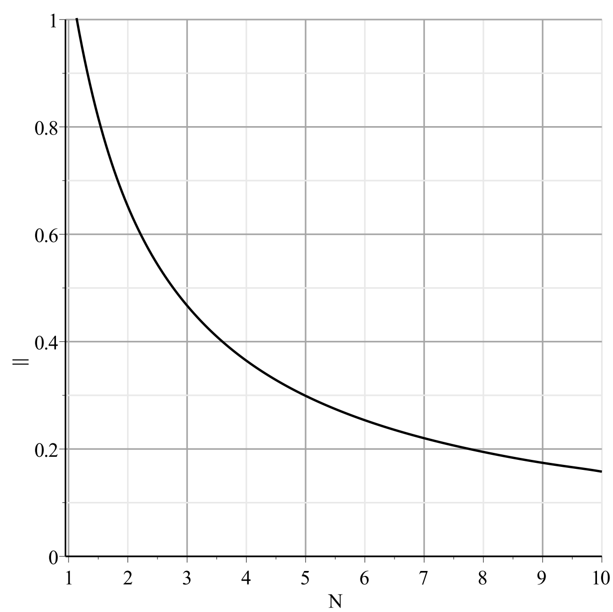
Similarly to the work done for the problem stated in (6)-(7), we will consider another mixed boundary-value problem

$$u(0,y) = \frac{\partial u(a,y)}{\partial x} = 0 \tag{35}$$

$$\frac{\partial u(x,0)}{\partial x} = 0, \qquad \lim_{y\to\infty} u(x,y) < \infty \tag{36}$$



(a) Convergence with the 2nd and 10th partial products.



(b) L^2-norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}]$; N is a number of partial products

Figure 3 : Convergence of the representation in (31)

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega = \{0 < x < a, 0 < y < \infty\}$.

Notes

Notes

Method of eigenfunction expansion. Upon implementing the routine applied to the problem in (6)-(7) and using the same sequence of eigenfunctions, one obtains the Green's function for the problem in (35)-(36) in the form

$$G(x,y;\xi,\eta)=\frac{2}{a}\sum_{k=1}^{\infty}g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x$$

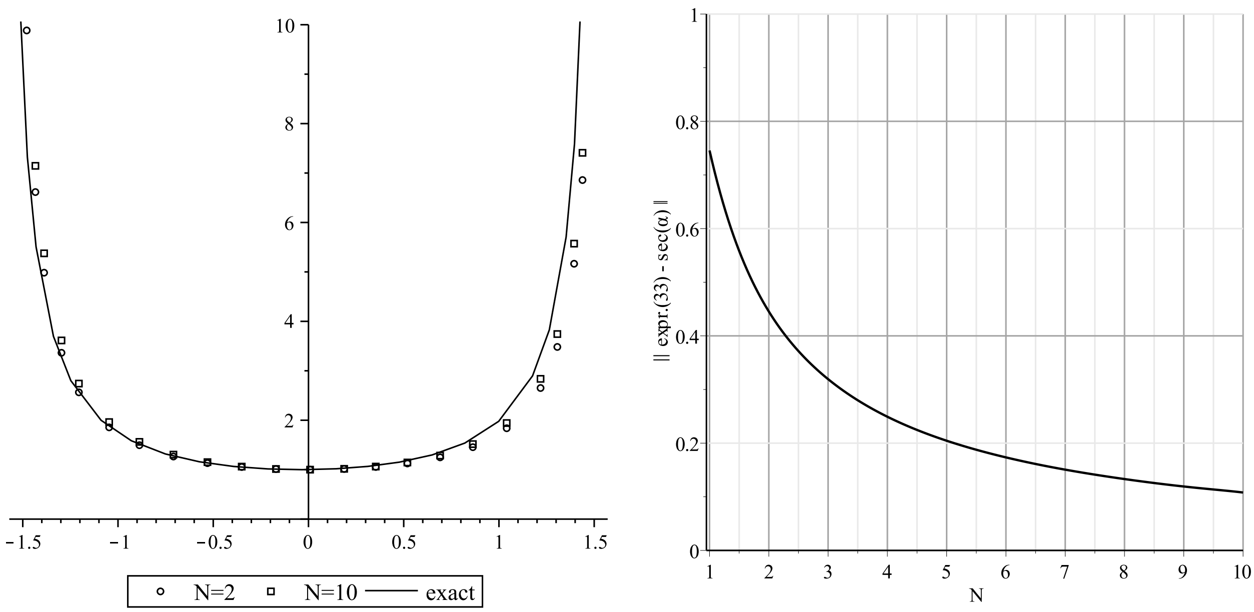
$$g_k(y;\eta)=\frac{1}{2\lambda_k}\begin{cases}e^{\lambda_k(y-\eta)}+e^{-\lambda_k(y+\eta)}, & y\le\eta\\ e^{\lambda_k(\eta-y)}+e^{-\lambda_k(y+\eta)}, & y\ge\eta\end{cases}$$

which reduces to

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\ \frac{\left(1+2e^{\alpha}\cos\ \gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\ \gamma+e^{2\alpha}\right)}\frac{\left(1+2e^{\beta}\cos\ \gamma+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\ \gamma+e^{2\beta}\right)}$$

$$\times\frac{\left(1-2e^{\alpha}\cos\ \theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\ \theta+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\ \theta+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\ \theta+e^{2\beta}\right)}\Bigg) \qquad (37)$$

where we use the variables introduced earlier in (18).



(a) Convergence with the 2nd and 10th partial products.

(b) L^2-norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}\ ]$; N is a number of partial products

Figure 4 : Convergence of the representation in (33)

Method of images. Distribution of singularities used in the method of images for the problem in (35)-(36) is shown in Figure 5. This gives rise to the infinite product version of the Green's function, written in terms of the variables introduced in (18), as

$$G(x,y;\xi,\eta)=\frac{1}{4\pi}\ln\prod_{k=-\infty}^{\infty}\frac{\Big((\gamma+\pi(1-2k))^2+\alpha^2\Big)\Big((\theta-2\pi k)^2+\alpha^2\Big)}{\Big((\theta-\pi(1+2k))^2+\beta^2\Big)\Big((\gamma-2\pi k)^2+\beta^2\Big)}$$

$$\frac{\Big((\gamma+\pi(1-2k))^2+\beta^2\Big)\Big((\theta-2\pi k)^2+\beta^2\Big)}{\Big((\theta-\pi(1+2k))^2+\alpha^2\Big)\Big((\gamma-2\pi k)^2+\alpha^2\Big)}$$

Notes

which being compared with (37) constitutes another identity valid for $\alpha\neq\pm\beta,\ \gamma\neq\pm\theta$. Assuming $\gamma=0,\ \theta=\frac{\pi}{2},\ \beta=2u,\ \alpha=2v$, one arrives at

$$\coth^2 u\ \coth^2 v=\prod_{k=-\infty}^{\infty}\frac{\Big(\pi^2(1+2k)^2+4v^2\Big)\Big(\pi^2(1+2k)^2+4u^2\Big)}{\Big(\pi^2k^2+u^2\Big)\Big(\pi^2k^2+v^2\Big)}$$

Note that it has already been obtained earlier (see (25)) in this presentation.

b) Exterior of Circles

The key idea of the developments in the previous sections is to obtain an alternative to the classical expression for a Green's function as an appropriate arrangement of sinks and sources. For some regions such arrangements are periodic and described by infinite sequences of symmetrically placed images.

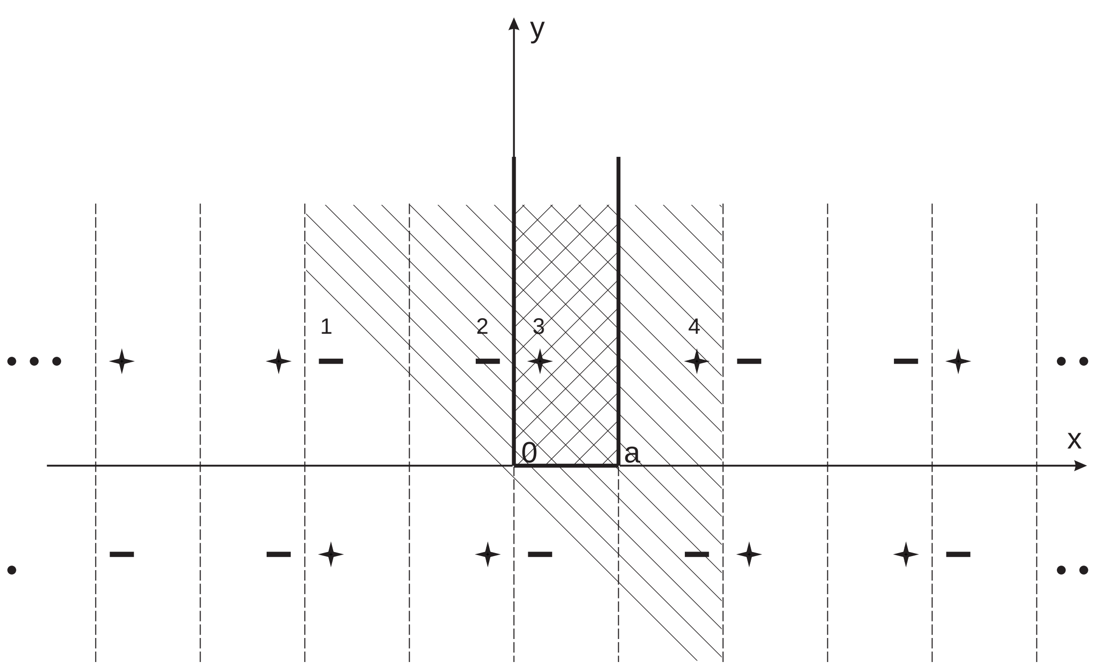


Figure 5 : Sequence of sources and sinks that arises in the method of images for the problem in (35)-(36)

This approach was applied, for instance, in [8] to construct the potential due to a line charge in the infinite strip or in [2, 8] to construct the potential induced by a single line charge in the rectangular prism or in the circular ring. The same approach was used in [6, 7] to find a potential field induced by the Green's function to the boundary-value problem posed for the Laplace equation in infinite and semi-infinite strip region with different combinations of boundary conditions. Results obtained in the earlier works served as a basis for construction of infinite product representations of elementary functions. The present work is the logical continuation of [6, 7], and the expansions (31) - (32) and (33) - (34) complement the results derived there.

Working on further developments of the discussed approach, we have probably to turn to different geometries or different equations. One of such cases is described below.

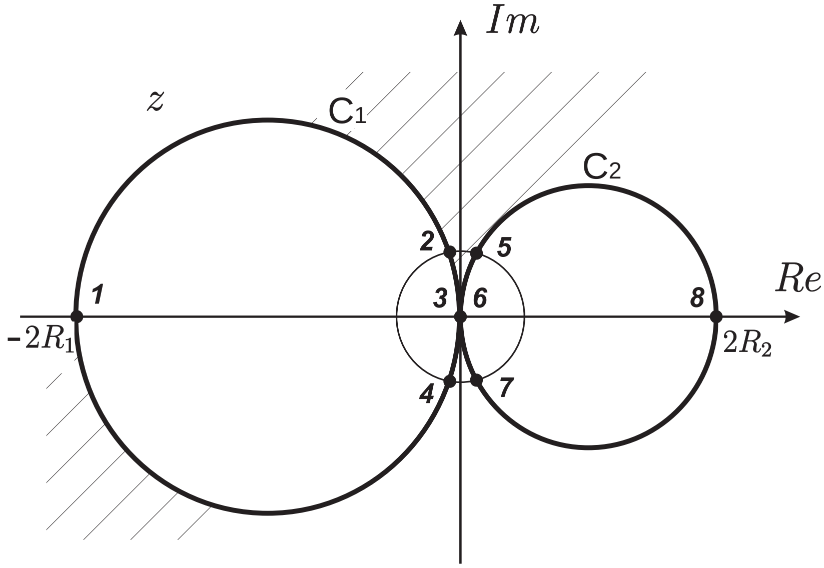
Conformal mapping. Conformal mapping gives us a tool to derive Green's functions of the Dirichlet problem posed for the Laplace equation in a simply-connected region. If there is known a function $w(z,\zeta)$ which conformally maps the given region in the z-plane onto the interior of the unit disc in the w-plane with point ζ mapped on the center of the disc, the corresponding Green's function is presented ([1, 2, 7]) in terms of $w(z,\zeta)$ as

$$G(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln|\omega(z,\zeta)| \tag{38}$$

where

$$z = x + iy$$
$$\zeta = \xi + i\eta$$

As an example, consider the exterior of two circles having external contact (Figure 6). Function $f(z) = 1/z$ transforms this region into the infinite strip ([1, 2]) while the latter can be mapped onto the interior of a unit circle. Omitting cumbersome but trivial algebra, we just present the function that maps the exterior of two circles with external contact onto the interior of a unit circle as



Ref

2. Z. Nehari, Conformal mapping, McGraw-Hill, New York, 1952.

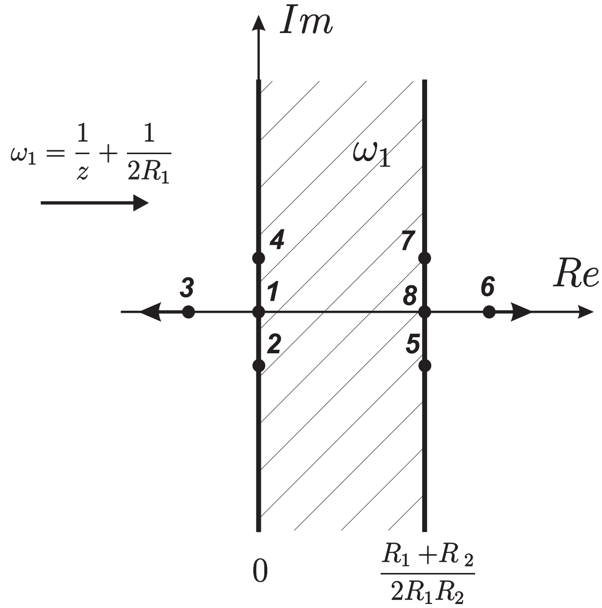


Figure 6 : Conformal mapping of the exterior of two circles with external contact onto the infinite strip

$$\omega(z,\zeta) = \frac{\exp\left(i\lambda\left(\frac{1}{z}+\frac{1}{2R_1}\right)\right) - \exp\left(i\lambda\left(\frac{1}{\zeta}+\frac{1}{2R_1}\right)\right)}{\exp\left(i\lambda\left(\frac{1}{z}+\frac{1}{2R_1}\right)\right) - \exp\overline{\left(i\lambda\left(\frac{1}{\zeta}+\frac{1}{2R_1}\right)\right)}}$$

where

$$\lambda = 2\pi\frac{R_1R_2}{R_1+R_2}$$

The modulus of the above reads as

$$|\omega(z,\zeta)| = \sqrt{\frac{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\lambda(\alpha-\beta)}{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\lambda\left(\alpha+\beta+\frac{1}{R_1}\right)}}$$

where

$$\alpha = \frac{x}{x^2+y^2}, \quad \beta = \frac{\xi}{\xi^2+\eta^2}, \quad \gamma = \frac{y}{x^2+y^2}, \quad \theta = \frac{\eta}{\xi^2+\eta^2} \tag{39}$$

Hence, in light of (38), the Green's function to the Dirichlet problem for the region depicted in Figure 6 appears as

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda\left[\alpha+\beta+\frac{1}{R_1}\right]\right)}{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda(\alpha-\beta)\right)} \tag{40}$$

Notes

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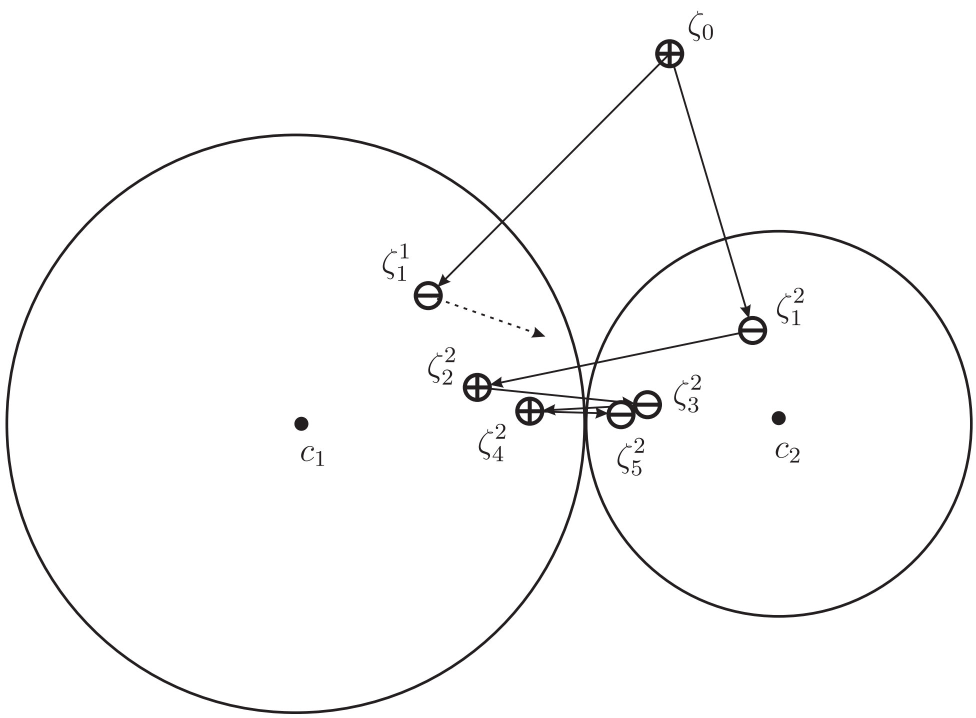


Figure 7: Sequence of sources and sinks that arises in the method of images

Method of images. In the case when boundaries of the region are formed by straight lines, images are placed symmetrically with respect to boundaries in a straightforward manner. But every straight line can be considered as a circumference with infinite radius. Thus the method of images can also be generalized to regions formed by circular arcs. In order to implement this idea, recall that an inversion with respect to a circle with radius R centered at a point c is given by the formula

$$z_1 = f(z_0) = \frac{R^2}{\overline{\overline{z_0 - c}}} + c \tag{41}$$

where the points c, z_0, z_1 lie in the same plane.

Thus the Green's function to the Dirichlet problem posed for the Laplace equation in the exterior of a circle is given by the expression

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \frac{|z - z_0|}{|z - z_1|} + \mu(x, y; \xi, \eta) \tag{42}$$

The compensatory function $\mu(x, y; \xi, \eta)$ can easily be derived from the fact that the argument of the logarithm in (42) has a constant value on the circumference of a circle and is equal to $\frac{|z_0 - c|}{R}$.

Thus the formula (42) converts to

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|z - z_1||z_0 - c|}{|z - z_0|R}$$

In the case of two circles, each inversion of the form in (41) with respect to the first circle perturbs the potential on the circumference of the second circle and thus, should be compensated by another inversion with respect to the perturbed circle and so on. Consequently to satisfy the Dirichlet boundary conditions we should use the infinite sequence of inversions (Figure 7). This generates the potential field

$$F(z,\zeta_n^j)=\frac{1}{2\pi}\left(\ln\frac{|\zeta_1^1-z||\zeta_0^1-c_1|}{|\zeta_0^1-z|R_1}+\ln\frac{|\zeta_1^2-z||\zeta_0^2-c_2|}{|\zeta_0^2-z|R_2}\right.$$

$$-\ln\frac{|\zeta_2^1-z||\zeta_1^1-c_2|}{|\zeta_1^1-z|R_2}-\ln\frac{|\zeta_2^2-z||\zeta_1^2-c_1|}{|\zeta_1^2-z|R_1}$$

$$\left.+\ln\frac{|\zeta_3^1-z||\zeta_2^1-c_1|}{|\zeta_2^1-z|R_1}+\ln\frac{|\zeta_3^2-z||\zeta_2^2-c_2|}{|\zeta_2^2-z|R_2}-\ldots\right)$$

Clearly, all the singularities appear twice in the above expression and hence the Green's function has the following infinite product form

$$\begin{aligned}G(x,y;\xi,\eta)&=\frac{1}{2}F(z;\zeta_n^j)\\&=\frac{1}{4\pi}\ln\prod_{k=0}^{\infty}\frac{|\zeta_{2k+1}^1-z|^2|\zeta_{2k+1}^2-z|^2|\zeta_{2k}^1-c_1||\zeta_{2k}^2-c_2|}{|\zeta_{2k}^1-z||\zeta_{2k}^2-z||\zeta_{2k+2}^1-z||\zeta_{2k+2}^2-z||\zeta_{2k+1}^1-c_2||\zeta_{2k+1}^2-c_1|}\end{aligned}\tag{43}$$

where $z= x+iy$ is a field point, c_1, c_2, R_1, R_2 are centres and radii of the circles, $\zeta_0^1=\zeta_0^2=z_0=\xi+i\eta$ is a source point and ζ_k^j are points at which singularities are placed

$$\zeta_k^1=\begin{cases}\dfrac{R_2^2}{\overline{\zeta_{k-1}^1-c_2}}+c_2, & \text{if } k \text{ is even}\\[2ex] \dfrac{R_1^2}{\overline{\zeta_{k-1}^1-c_1}}+c_1, & \text{if } k \text{ is odd}\end{cases}$$

$$\zeta_k^2=\begin{cases}\dfrac{R_1^2}{\overline{\zeta_{k-1}^2-c_1}}+c_1, & \text{if } k \text{ is even}\\[2ex] \dfrac{R_2^2}{\overline{\zeta_{k-1}^2-c_2}}+c_2, & \text{if } k \text{ is odd}\end{cases}\tag{44}$$

In our case, $c_1=-R_1, c_2=R_2$ and the expressions in (44) transform to

$$\zeta_k^1=\begin{cases}\dfrac{z_0}{\dfrac{k}{2}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)z_0+1}, & \text{if } k \text{ is even}\\[4ex] \dfrac{-\overline{z}_0}{\dfrac{1}{2}\left(\dfrac{k+1}{R_1}+\dfrac{k-1}{R_2}\right)\overline{z}_0+1}, & \text{if } k \text{ is odd}\end{cases}$$

Notes

$$\zeta_k^2 = \begin{cases} \dfrac{-z_0}{\dfrac{k}{2}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right) z_0 - 1}, & \text{if } k \text{ is even} \\[2ex] \dfrac{\overline{z}_0}{\dfrac{1}{2}\left(\dfrac{k-1}{R_1}+\dfrac{k+1}{R_2}\right) \overline{z}_0 - 1}, & \text{if } k \text{ is odd} \end{cases}$$

Notes

When k approaches infinity both values ζ_k^1 and ζ_k^2 tend to zero, implying that the limit of the general term in (43) equals unity. That is

$$\lim_{k\to\infty}\left(\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 + R_1||\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - R_2||\zeta_{2k+1}^2 + R_1|}\right) = 1$$

and thus the infinite product in (43) is absolutely and uniformly convergent.

Infinite Products and Elementary Functions. Obviously, in (40) and (43) one finds just alternative representations of the same Green's function, and we arrive at the multivariable identity

$$\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda\left[\alpha+\beta+\dfrac{1}{R_1}\right]\right)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda(\alpha-\beta)\right)}$$

$$= \prod_{k=0}^{\infty}\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 + R_1||\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - R_2||\zeta_{2k+1}^2 + R_1|} \tag{45}$$

Complex variables z and z_0 can be written in terms of real parameters α, β, γ, θ, which were introduced earlier in (39), as

$$z_0 = \xi + i\eta = \frac{\beta + i\theta}{\beta^2+\theta^2}$$

$$z = x + iy = \frac{\alpha + i\gamma}{\beta^2+\theta^2}$$

Thus, the expression in (45) has six arbitrary parameters: α, β, γ, θ, R_1 and R_2. If $\lambda = 1$, then $R_2 = R_1/(2\pi R_1 - 1)$ and the number of independent parameters reduces to five.

Trigonometric functions. For any fixed value of the parameters γ and θ the expression in (45)involves only trigonometric functions. Particularly, when $\gamma = 0$ and $\theta = 0$ the left-hand side of (45) represents the following elementary function

$$\frac{1-\cos\left(\alpha+\beta+\dfrac{1}{R_1}\right)}{1-\cos(\alpha-\beta)} \tag{46}$$

If $R_1 = \dfrac{1}{n\pi}$ $\left(R_2 = \dfrac{1}{\pi(2-n)}, n = 0, 1, ...\right)$ the above expression converts to

$$\frac{1-(-1)^n \cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} \tag{47}$$

Similarly, when $R_1 = \dfrac{2}{\pi(1+2n)}$ $\left(R_2 = \dfrac{2}{\pi(3-2n)}, n = 0, 1, ...\right)$, the function (46) converts to

$$\frac{1+(-1)^n \sin(\alpha+\beta)}{1-\cos(\alpha-\beta)}$$

So the functions (47) and (48) have the following infinite product representations

$$\frac{1-(-1)^n \cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} = \prod_{k=0}^{\infty} \frac{\Big[\alpha+\beta+\pi(2k+n)\Big]^2 \Big[\alpha+\beta-\pi(2(k+1)-n)\Big]^2}{\Big|(\alpha-\beta)^2-4\pi^2k^2\Big|\Big|(\alpha-\beta)^2-4\pi^2(k+1)^2\Big|} \tag{49}$$

and

$$\frac{1+(-1)^n \sin(\alpha+\beta)}{1-\cos(\alpha-\beta)} = \prod_{k=0}^{\infty} \frac{\Big[2(\alpha+\beta)+\pi(1+4k+2n)\Big]^2 \Big[2(\alpha+\beta)-\pi(3+4k-2n)\Big]^2}{16\Big|(\alpha-\beta)^2-4\pi^2k^2\Big|\Big|(\alpha-\beta)^2-4\pi^2(k+1)^2\Big|} \tag{50}$$

Different interrelations between the parameters α and β are possible. For instance, when $\beta = \alpha \pm (2p+1)\pi$ $(p = 0, 1, 2, \dots)$, the expressions in (49) and (50) read

$$\frac{1+(-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\Big(2\alpha+\pi\big(2p+n\big)\Big)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|}, \tag{51}$$

$$\frac{1+(-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\Big(2\alpha-\pi(2(p+1)-n)\Big)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|} \tag{52}$$

and

$$\frac{1-(-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\Big(4\alpha+\pi\big(2(2p+n)+1\big)\Big)^2-4\pi^2(2k+1)^2\right]^2}{16\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|},$$

$$\frac{1-(-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\Big(4\alpha-\pi\big(2(2p-n)+3\big)\Big)^2-4\pi^2(2k+1)^2\right]^2}{16\pi^4\Big|(2p+1)^2-4k^2\Big|\Big|(2p+1)^2-4(k+1)^2\Big|}$$

which, in turn, provide us with infinite product representations of the sine and cosine functions

Notes

$$\cos\alpha = (-1)^n\left(-1+2\prod_{k=0}^{\infty}\frac{\left[\left(\alpha+\pi\left(2p+n\right)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{53}$$

Notes

$$\cos\alpha = (-1)^n\left(-1+2\prod_{k=0}^{\infty}\frac{\left[\left(\alpha-\pi(2(p+1)-n)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{54}$$

and

$$\sin\alpha = (-1)^n\left(1-2\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha+\pi\left(2(2p+n)+1\right)\right)^2-4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{55}$$

$$\sin\alpha = (-1)^n\left(1-2\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha-\pi\left(2(2p-n)+3\right)\right)^2-4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{56}$$

where $n, p = 0, 1, 2, \ldots$ For example, in the case of $n=1$ and $p=0$ the identities in (53)-(56) reduce to

$$\cos\alpha = 1+2\prod_{k=0}^{\infty}\frac{\left((\alpha+\pi)^2-\pi^2(2k+1)^2\right)^2}{\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{57}$$

$$\cos\alpha = 1+2\prod_{k=0}^{\infty}\frac{\left((\alpha-\pi)^2-\pi^2(2k+1)^2\right)^2}{\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{58}$$

and

$$\sin\alpha = -1-2\prod_{k=0}^{\infty}\frac{\left((2\alpha+3\pi)^2-4\pi^2(k+1)^2\right)^2}{16\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{59}$$

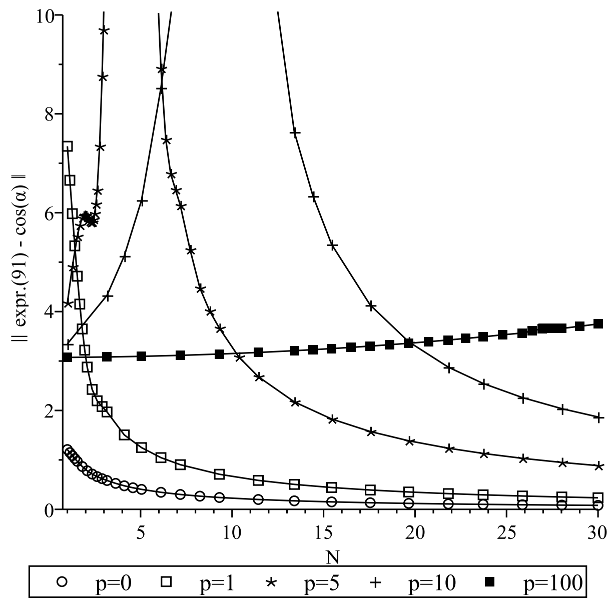
$$\sin\alpha = -1-2\prod_{k=0}^{\infty}\frac{\left((2\alpha-\pi)^2-4\pi^2(k+1)^2\right)^2}{16\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{60}$$

Figures 8 and 9 illustrate the sensitivity of the convergence of the representation in (53) to the choice of the parameters n and p. It is seen that the parameter values have the dramatical influence on the rate of convergence. The identities in (54)-(56) exhibit a similar behaviour.

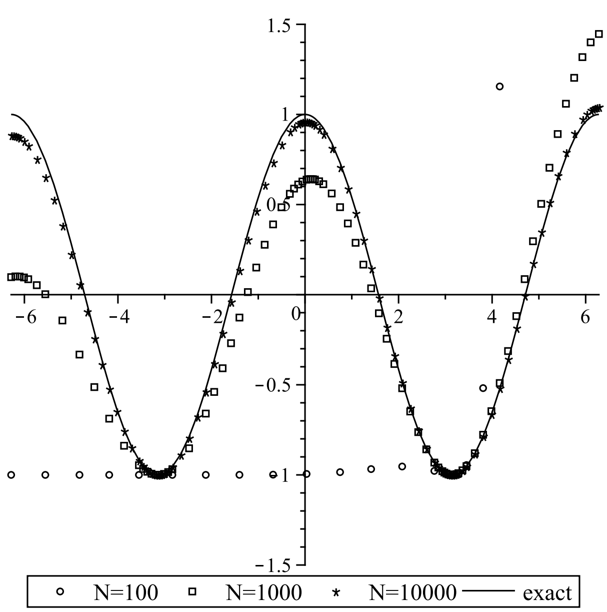
By applying the power reduction formula to the expressions in (51) and (52) several new infinite product expansions for the sine and cosine functions can be obtained.

If n in (51) and (52) is an even number, i.e. $n = 2n$, then the expansions of the cosine function can be written as

$$\cos\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|4\left(\alpha+\pi\left(p+n\right)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{61}$$



(a) L^2-norm of the error in $[0, 2\pi]$; N is a number of partial products.



(b) Convergence of the infinite product expansion when $n = 0$, $p = 100$

Figure 8 : Convergence of the representation in (53) for different values of the parameter p; parameter $n = 0$ is fixed

Notes

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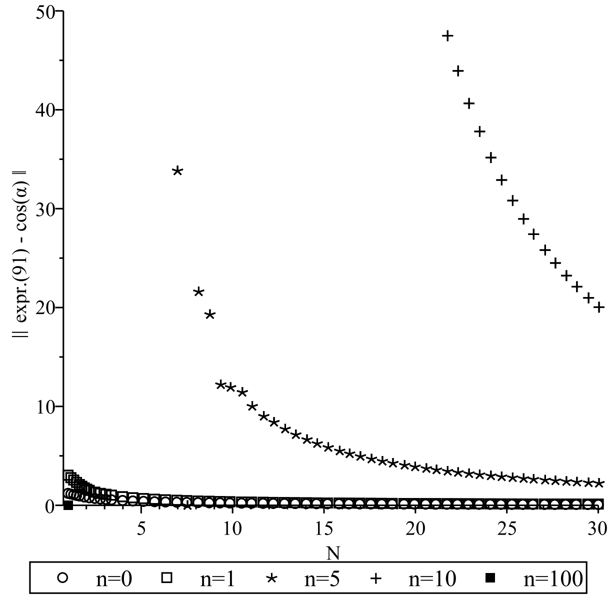
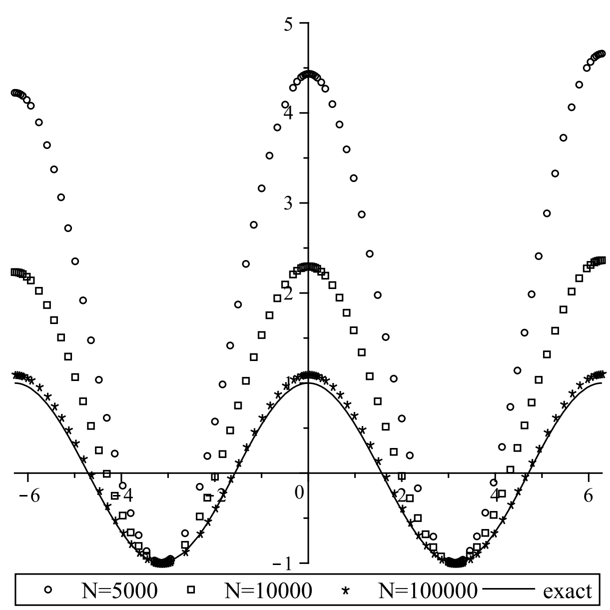


(a) L^2-norm of the error in $[0, 2\pi]$; N is a number of partial products



(b) Convergence of the infinite product expansion when $p=0$, $n=100$

Figure 9: Convergence of the representation in (53) for different values of the parameter n; parameter $p=0$ is fixed

$$\cos\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{62}$$

where the plus sign corresponds to $\alpha\in\left[-\dfrac{\pi}{2};\dfrac{\pi}{2}\right)$ and the minus sign corresponds to $\alpha\in\left[\dfrac{\pi}{2};\dfrac{3\pi}{2}\right]$.

Similarly, when n in (51) and (52) is an odd number, i.e. $n= 2n+ 1$, then the expansion of the sine function can be written as

$$\sin\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{63}$$

$$\sin\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi\big(2(p-n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{64}$$

where the plus sign corresponds to $\alpha\in[0;\pi)$ and the minus sign corresponds to $\alpha\in[\pi;2\pi]$.

The infinite product representations of the tangent function can also be obtained from the expansions in (61)-(64). Those are

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha-\pi\big(p-n+1\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

and

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi\big(2(p-n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi\big(2(p-n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha-\pi\big(p-n+1\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

where the plus sign corresponds to $\alpha\in\left[-\frac{\pi}{2};0\right)$ and the minus sign corresponds to $\alpha\in\left[0;\frac{\pi}{2}\right]$.

So far we assumed $\beta=\alpha\pm(2p+1)\pi$. Different representations can be derived for the relation between α and β as

$$\beta=\alpha\pm(2p+1)\frac{\pi}{2},\qquad p=0,1,2,\ldots$$

yielding in the following

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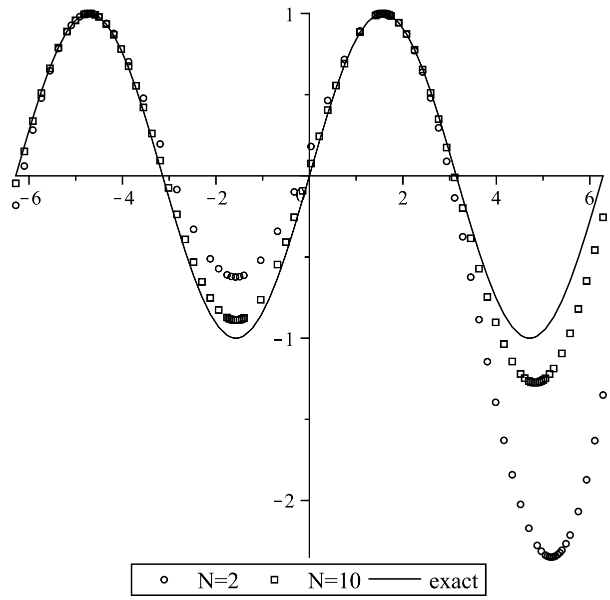
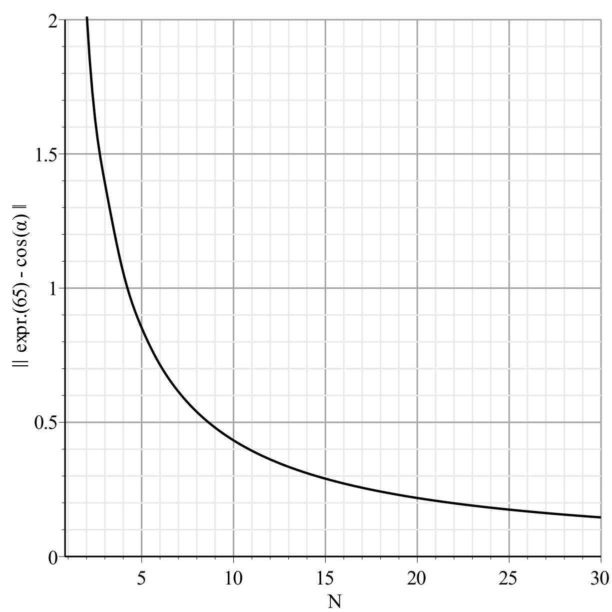


(a) Convergence with the 2nd and 10th partial products.



(b) L_2-norm of the error in $[0, 2\pi]$; N is a number of partial products

Figure 10 : Convergence of the representation in (65)

$$\sin\alpha = (-1)^{n+p}\left(-1+\prod_{k=0}^{\infty}\frac{\left[\Big(2\alpha+\pi(2(p+n)-1)\Big)^2-4\pi^2(2k+1)^2\right]^2}{\pi^4\Big|(1+2p)^2-16k^2\Big|\Big|(1+2p)^2-16(k+1)^2\Big|}\right) \tag{65}$$

$$\sin\alpha = (-1)^{n+p}\left(1-\prod_{k=0}^{\infty}\frac{\left[\Big(2\alpha-\pi(2(p-n)+3)\Big)^2-4\pi^2(2k+1)^2\right]^2}{\pi^4\Big|(1+2p)^2-16k^2\Big|\Big|(1+2p)^2-16(k+1)^2\Big|}\right) \tag{66}$$

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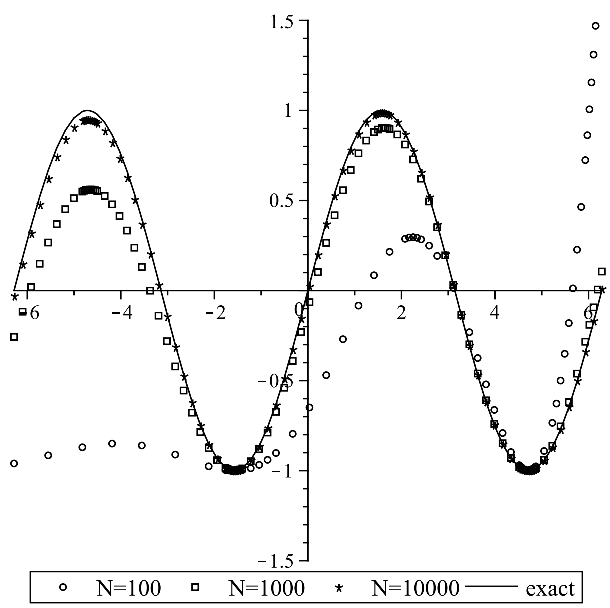
and

$$\cos\alpha = (-1)^{n+p}\left(-1+\prod_{k=0}^{\infty}\frac{16\left[\left(\alpha+\pi(p+n)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{67}$$

$$\cos\alpha = (-1)^{n+p}\left(1-\prod_{k=0}^{\infty}\frac{16\left[\left(\alpha-\pi(p-n+1)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{68}$$

The convergence rate of the expansion in (65) with $n=1$, $p=0$ is illustrated in Figure 10. The sensitivity of the convergence in (65) to the choice of the parameters n and p is illustrated in Figure 11. The identities in (66)-(68) exhibit the similar behaviour.

By applying the power reduction formula to the identities in (67)-(68) one can derive alternative representations of the sine and cosine functions. When n and p are even numbers, i.e. $n=2n$ and $p=2p$, the expansions of the cosine and sine functions can be obtained in the form



(a) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

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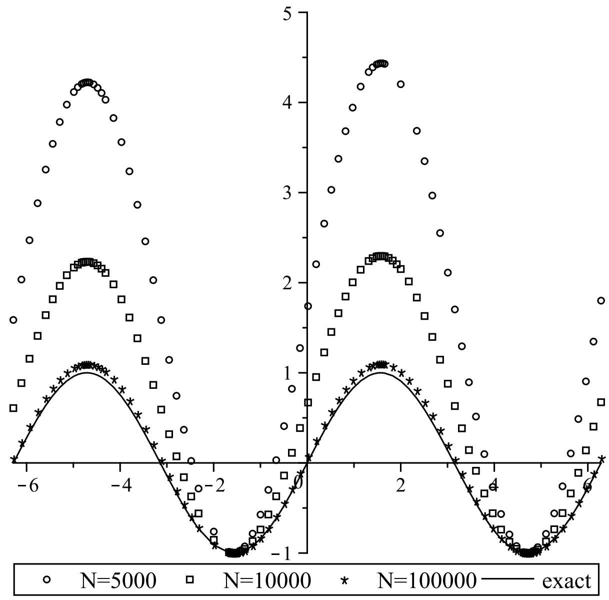

(b) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

Figure 11 : Convergence of the representation in (65) for different values of the parameters n and $p=0$

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{69}$$

and

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{70}$$

When n is odd and p is even, i.e. $n=2n+1$ and $p=2p$, one arrives at

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{71}$$

and

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{72}$$

When n is even and p is odd, i.e. $n=2\,n$ and $p=2p+1$, we arrive at

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{73}$$

and

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-2\pi\left((p-n)+1\right)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{74}$$

Finally, if both n and p are odd, i.e. $n= 2n+1$ and $p= 2p+1$, the expressions in (67)-(68) convert to

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+2\pi(p+n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{75}$$

and

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{76}$$

In the above representations, the sign is selected similarly to those in (61)-(64)

Representations for the tangent function follow directly from (69)-(76). This provide us with the total number of sixteen expansions. Four of those are shown below.

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}$$

while the others can be derived in a similar manner.

Notes

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Comparison of Numerical Schemes for Shallow Water Equation

By Md.Saiduzzaman & Sobuj Kumar Ray
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In this paper we will present different numerical schemes such as Lax-Friedrich scheme, Lax-Wendroff scheme, Leap-Frog scheme for the shallow water equation and implement the numerical schemes by computer programming. Next we will compare these different schemes with respect to their efficiency and the quality of the solution indeed. Here we apply linear advection equation in order to test the accuracy of these schemes for shallow water equation.

Keywords : equation of continuity, linear advection equation, lax-friedrich scheme, lax-wendroff scheme, leap-frog scheme, shallow water equations. stencil.

GJSFR-F Classification : MSC 2010: 32W50

COMPARISON OF NUMERICAL SCHEMES FOR SHALLOW WATER EQUATION

Strictly as per the compliance and regulations of :

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Md.Saiduzzaman[α] & Sobuj Kumar Ray[σ]

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I. Introduction

The shallow water equations (SWEs) describe the evolution of a hydrostatic homogeneous (constant density), incompressible fluid in response to gravitational and rotational accelerations and they are derived from the principles of conservation of mass and conservation of momentum. The SWEs (also called Saint–Venant equations) are one of the simplest form of the equations of motion that can be used to describe the horizontal structure of an atmosphere and ocean that model the propagation of disturbances in fluids. They are widely used to model the free surface water flows such as periodic (tidal) flows, transient wave phenomena [1] (tsunamis, flood waves, and dam-break waves) etc. The shallow water equations are only relevant when the horizontal scale of the flow is much smaller than the depth of the fluid.

In fluid dynamics the flow of the fluid is known as the Navier – Stocks equation. The shallow water equations are good approximation to the fluid motion equation when fluid density is homogeneous and depth is small in comparison to characteristic horizontal distance. So the most attention will be given to the flow of shallow water equation [2].

Shallow water equation is a system of first order partial differential equation. It is linear in derivative but non-linear in unknowns and so is called quasi-linear partial differential equation [3], [4].

Ref

1. Pilar García-Navarro & Pilar Brufau, "One Dimensional Dam Break Flow Modeling: Some results".

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II. Mathematical Models for Fluid Motions

a) Symbols and Notations

Let $\Omega \subset \boldsymbol{R}^d$, $d \in -1,2''$ be a region occupied by a fluid flow and let $[t_0, T)$ be time interval with $0 \leq t_0 \leq T$. An arbitrary point in Ω is denoted by $\mathbf{x}=(x_1,\ldots, x_d)^{\mathrm{T}}$. For the description of a general unsteady compressible fluid flow, we introduce the quantities:

The density $\rho = \rho\,(\mathbf{x}\;,t)$, the velocity vector $\mathbf{v} = \mathbf{v}(\mathbf{x}\;,t) = (v_1\,(\mathbf{x}\;,t),..,v_d\,(\mathbf{x}\;,t))^{\mathrm{T}}$,the pressure $p = p(x\;,t)$, the energy density $E = E\,(\mathbf{x}\;,t)$.We denote the external forces by $\mathbf{f} = \mathbf{f}(\mathbf{x}\;,t) = (f_1\,(x\;,t)\;,\ldots,\;f_d\;(x\;,t))^{\mathrm{T}}$, the mass flux by $\mathbf{q} = \mathbf{q}\,(\mathbf{x}\;,t)$. For the description of the viscous flow, let λ and μ denote the coefficient of viscosity and coefficient of kinematic viscosity respectively.

All the above quantities describing fluid flows are functions of space and time. The density of such a quantity is generally written as a function of $\boldsymbol{\Phi}=\boldsymbol{\Phi}(\mathbf{x},\,t)$, where for given any time instant $t \in [t_0,\,T)$, $\mathbf{x} \in\, G(t)$ denote the points of set $G(t) \subset R^d$ occupied by the fluid at time t. So the domain of definition of the function ϕ is the set

$$G= -(\mathbf{x},\mathrm{t});\mathbf{x}\in G(t), t\in[t_0,T)'' \in R^{d+1} \tag{1}$$

In particular $G\,(t_0)$ represents the domain occupied by the fluid at the initial time t_0. For the motion of a particular fluid particle, the trajectory can be described by an application $\mathbf{x} = \mathbf{X}(x_0\,,\;t_0\,;\;t)$ where $\mathbf{x}_0 \in G\,(t_0)$ represents the initial position of the particle at time t_0. Any bounded domain

$$V\,(t) =-\mathbf{x}=\mathbf{X}(x_{0,}\;t_0\,;\,t) — \mathbf{x}_0 \in V\,(t_0) \subset G\,(t_0)'' \subset G(t) \tag{2}$$

occupied by the fluid at any instant t is called a control volume. The total amount of a quantity with density $\phi(\mathbf{x},\,\mathrm{t})$ contained in the volume V (t) at time t is given by the integral

$$\phi(t) = \int_{V(t)} \phi(\mathbf{x},t)\,d\mathbf{x} \tag{3}$$

In what follows, we are interested in the rate of change of ϕ(t).for the analysis of derivative

$$\frac{d}{dt}\,\phi(t) = \frac{d}{dt}\int_{V(t)} \phi(x,t)\,dx \tag{4}$$

The so-called Reynolds transport theorem plays a crucial rule:

b) The Equation of Continuity

The mass m(V(t)) of the fluid contained in an arbitrary control volume V(t) at some given time t is given by:

$$m(V(t)= \int_{V(t)} \rho(\mathbf{x},t)d\mathbf{x}, \tag{5}$$

where $\rho(\mathbf{x},t)$ is the mass density. As the domain V(t) contains the same particles at each time instant t (i.e. no mass is generated or destroyed), the law of conservation of mass can be written as:

The mass of a amount of fluid within the control volume $V(t)$ is independent of time, i.e.

Notes

$$\frac{dmV(t)}{dt} = 0, \text{ for all t} \in [t_0, T)$$

Assuming $\rho(\mathbf{x}, t)$ is smooth, one obtain from the Reynolds transport theorem the integral of the mass conservation law

$$\int_{V(t)} \left[\frac{\partial\rho}{\partial t}(\mathbf{x}, t) + \nabla.\left(\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)\right)\right] d\mathbf{x} = 0 \tag{6}$$

Notes

For all $t \in [t_0, T)$

As the control volume can be chosen arbitrary, whenever the integrand is continuous we obtain the differential form of the law of conservation of mass:

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0 \tag{7}$$

which is known as the continuity equation.

c) The Linear Advection Equation

In fluid dynamics the flow of the fluid is known as the Navier- Stokes equation. Shallow water equations are good approximation to the fluid motion equation when the flow is one dimensional and incompressible unsteady fluid and derived from the water wave by assuming that the water depth is sufficiently small compared to the water wave length. Now we will discuss the mathematical description of the shallow water equation for flood modeling.

We now consider the linear advection equation, which is given by

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \tag{8}$$

The Cauchy problem is defined by this equation on the domain $-\infty < x < \infty$, $t \geq 0$ together with initial conditions $U(x, 0) = u_0(x)$

Here we assume that the velocity c is constant. For simplicity we assume that the velocity c is positive.

$$c > 0$$

If u_0 is differentiable, then the solution is simply

$$u(t, x) = u_0(x + ct) \tag{9}$$

for $t \geq 0$ (which is based on [5]). As time involved, the initial data simply propagates unchanged to the right ($c > 0$), The solution $u(t, x)$ is constant along each ray $x - ct = x_0$, which are known as characteristics of the equation.

The characteristics are curves in the x−t plane satisfying the ordinary differential equations $x'(t) = c$, $x(0) = x_0$.

If we differentiate $u(x, t)$ along one of this curves to find the rate of change of u along the characteristic, we find that

$$\frac{d}{dt}u(t, x(t)) = \frac{\partial}{\partial t}u(t, x(t)) + \frac{\partial}{\partial x}u(t, x(t))\,x' = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

Confirming that u is constant along these characteristics.

More generally, we might consider a variable coefficient advection equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial(c(x))u}{\partial x} = 0 \tag{10}$$

Where $c(x)$ is a function, then we have

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = - c'(x)$$

$$\text{or,} \quad \left(\frac{\partial}{\partial t} + c(x)\frac{\partial}{\partial x}\right) u(x, t) = -c'(x)\, u(x, t)$$

It follows that the evaluation of u along any curve x(t) satisfying

$$x'(t) = c(x(t))$$

$$x(0) = x_0$$

Satisfies a simple ordinary differential equation:

$$\frac{d}{dt} u(t, x(t)) = -c'(x(t))\, u(x(t), t)$$

The curves determined by (11) are again called characteristics.

In this case the solution is not constant along these curves, but can be easily determined by solving two sets of ordinary differential equations.

It can be shown that if $u_0(x)$ is a smooth function, say $u_0 \in c^k(-\infty, \infty)$ then the solution $u(x, t)$ is equally smooth in space and time, $u \in c^k((-\infty, \infty).\ (0, \infty))$

In the above discussion we also mention the model of Linear advection equation as well as its solution. [6]

d) Shallow Water Equation

Shallow water waves which arise if the water height is much smaller than the wave length of the water. The flow of water is distributed process because the flow rate, velocity and depth in space and time. To derive the one dimensional equation, we consider,

1. The velocity parallel to x- direction and almost independent of y: $v(t, x)\, \underline{e}_x$
2. The fluid is incompressible, so the density ρ is constant.

The shallow water equations are based on the laws of conservation of mass and the laws of conservation of momentum [6]. Now we will show the mathematical description of the shallow water equation which shows the direction of the propagation of the wave length as well as the flow of the fluid.

e) Mathematical Description of the Model

Conservation of mass, simply state that the mass of the volume at a given time t equal to the mass of source inside it.

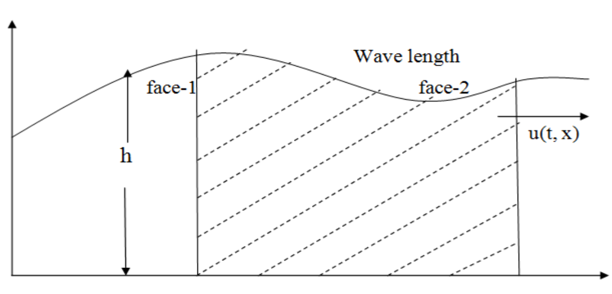

Figure 1: Shallow Water

Ref.

6. Randall J. Le Veque, "Numerical Methods for Conservation Laws".

Notes

Now the change of mass inside the volume $((\Delta x\, h(t,x)))$ is

$$\frac{\partial}{\partial t}(\rho \Delta x h(t,x) \tag{12}$$

Mass flow in/out the volume:

Mass flux vector $\rho v\ \underline{e}_x$ then integrate $\rho v\ \underline{e}_x$ at face-1 we get influx: $= q_1$

$$\text{So } q_1 = \rho\ v\ h(t\ ,x_1)$$

Integrate $\rho v\ \underline{e}_x$ at face-2 we get influx: $= q_2$

$$\text{So } q_2 = -\rho\ v\ h(t\ ,x_2)$$

So the change of mass flux $= \rho v h(t,x_1) - \rho v h(t,x_2)$ (2.2)

Hence we obtain from (2.1) and (2.2),

$$\frac{\partial}{\partial t}(\rho \Delta x h(t,\ x)) = \rho v h(t,x_1) - \rho v h(t,x_2)$$

$$\rho \frac{\partial}{\partial t}\ (h(t,\ x))\ = \rho\ \text{“}v h(t,x_1) - v h(t,x_2)/\Delta x\text{”}$$

Taking $\Delta x \rightarrow 0$ we have

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} \tag{13}$$

Conservation of momentum state that the rate of change of the total momentum of a volume of fluid formed by the same particles at any time and occupying the control volume $V(t)$ at the time instant t is equal to the resultant of the forces acting on the fluid in $V(t)$.

Now the change of momentum inside the volume $((\Delta x h(t,\ x)))$ is

$$\frac{\partial}{\partial t}\ (\rho v\ \Delta x\ h(t,\ x)) \tag{14}$$

Net flux of momentum:

Momentum flux vector $(\rho v) v\ \underline{e}_x$, then the net flux momentum at$[x_1,\ x_2]$ is

$$\rho v^2\ h\ (t,\ x_1) - \rho\ v^2 h(t\ ,\ x_2) \tag{15}$$

The pressure P is determined from a hydrostatic law, stating that the pressure at depth y is $\rho g h$, where g is gravitational force. Then pressure force for dy is,

$$P(t,\ x,\ y) = \rho\ g\ (h\ (t\ ,\ x) - y) + p_0$$

Where p_0 is atmospheric pressure. Consider $P_1 = P(t,\ x_1\ ,\ y)$ and $P_2 = P(t,\ x_2\ ,\ y)$.

Now force acting at $[x_1,\ x_2]$ is

$$P_1\, dy - p_2\, dy = \rho g (h(t,\ x_1) - h(t,\ x_1))\ dy$$

Integrating from $y=0$ to $y = h(t,\ x)$ we have total forces

$$\rho g\ (h(t,\ x_1) - h(t\ ,\ x_1))\ h(t\ ,\ x) \tag{16}$$

using (14) , (15) and (16) we have,

$$\rho \ \frac{\partial}{\partial t}\left(\rho \Delta x h(t,\ x)\right) = \rho v^2\ h(t,\ x_1) - \rho v^2 h(t,\ x_2)\ + \rho g\ \left(h(t,\ x_1) - h(t,\ x_1)\right)\ h(t,\ x)$$

$$\frac{\partial}{\partial t}\,(hv) = \rho\ -vh(t, x_1)\ -vh(t\ , x_2)/\Delta x''\ + \rho g\ h(t,\ x) - h(t,\ x_1) - \ h(t,\ x_2)/\Delta x''$$

Taking $\Delta x \to 0$ and dividing by ρ then we have,

$$\frac{\partial}{\partial t}\,(hv)\ \ + \frac{\partial}{\partial t}\,(hv^2)\ +\ gh(\frac{\partial h}{\partial x}) = 0$$

$$h\,\frac{\partial v}{\partial t}\ \ + v\,\frac{\partial h}{\partial t}\ \ +\ hv\,\frac{\partial v}{\partial x}\ \ +\ v\,\frac{\partial (hv)}{\partial x} + \ gh(\frac{\partial h}{\partial x}) = 0$$

$$v\,(\frac{\partial h}{\partial t}\ + \frac{\partial (hv)}{\partial x}\ ) \ +\ h(\frac{\partial v}{\partial t}\ \ +\ v\,\frac{\partial v}{\partial x})\ + gh\ (\frac{\partial h}{\partial x}) = 0$$

$$h\ (\frac{\partial v}{\partial x}\ \ + g\frac{\partial h}{\partial x}\ \ +\ v\,\frac{\partial v}{\partial x}) = 0 \qquad \text{[using (13)]}$$

Hence Shallow water equation become,

$$\frac{\partial h}{\partial t}\ \ + \frac{\partial (vh)}{\partial x} = 0$$

$$\frac{\partial v}{\partial t}\ + g\ \frac{\partial h}{\partial x}\ +\ v\,\frac{\partial v}{\partial x} = 0 \tag{17}$$

If we denote $u{:} = \binom{h}{v}$ and A $(u){:} = \begin{pmatrix} v & h \\ g & v \end{pmatrix}$

Then we have

$$\frac{\partial u}{\partial t} + A(u)\,\frac{\partial u}{\partial x}\ = 0$$

This is a quasi-linear system of partial differential equations and this is the mathematical description of shallow water equation.

Now we construct two dimensional shallow water model and also add with source or rainfall term (based on [7]). Next we write vector form of shallow water equation. We are considering water flow into the system. Flood is caused by overflow of water. The heavy rainfall is one of the cause of it. So we have

$$\frac{\partial h}{\partial t}\ \ +\ h\ \frac{\partial u}{\partial x}\ \ +\ \ u\ \frac{\partial h}{\partial x}\ \ \ +\ h\frac{\partial v}{\partial y}\ \ +\ v\,\frac{\partial h}{\partial y} = f_s$$

$$\frac{\partial u}{\partial t} +\ u\ \frac{\partial u}{\partial x}\ \ +\ v\frac{\partial u}{\partial y}\ \ + g\,\frac{\partial h}{\partial x}\ \ + g\ \frac{\partial z}{\partial x}\ \ =\ \ a_{fx} \tag{18}$$

$$\frac{\partial v}{\partial t}\ \ +\ u\ \frac{\partial v}{\partial x}\ \ +\ v\frac{\partial v}{\partial y}\ \ + g\,\frac{\partial h}{\partial y}\ \ +\ g\ \frac{\partial z}{\partial y}\ \ = a_{fy}$$

And in vector form

$$\frac{dh}{dt}\ \ + \nabla.(Vh) = f_s$$

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Ref.

7. C. B. Vreugdenhil, "Numerical Method for Shallow Water Flow" Kluwer Academic Publisher, Dordsecht 1994.

$$\frac{dh}{dt} + V.\nabla V + g\nabla h + g\nabla z - a_f = 0$$

Where

V(x, t)[m/s]: velocity vector, h(x, t) [m]: water level above ground, f_s: source term (Rainfall), z[m]: topographical height, g[m/s^2]: acceleration due to gravity, c_f [m]: Manning coefficient, a_f [m]: friction slop.

Where $a_f = V|V| \,/\, c_f$

III. Numerical Methods

a) Finite Difference Methods

We will develop approximations to solutions of general nonlinear scalar conservation laws on a bounded domain,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \text{ for a} < \text{x} < \text{b, } 0 < \text{t} \tag{19}$$

Initially we will assume that the characteristic speeds satisfy $\lambda = f'(u) > 0$ for all states that occur in the solution of (19). Thus we also provide initial data

$$u(x, 0) = u_0(x)$$

and data at the left hand boundary

$$u(t, a) = u_a(t) \tag{20}$$

The numerical techniques have been applied to find their approximate solution of (19), i.e density $u(x, t)$ and flux $f(u(x, t))$ at a certain number of points in the time space domain , in such way as to satisfy the basic equations as far as possible. This procedure is called the discretization(based on [8]). Each point is called a grid point and the set of grid points is called a grid. We will discretize space

$$a = x_{-1/2} < x_{1/2} < x_{3/2} < \ldots\ldots\ldots. < x_{I-1/2} = b$$

and time

$$0 = t^0 < t^1 < t^2 < \ldots\ldots\ldots. < t^N = T$$

We will define the componential grid cells to be the intervals $(x_{i+1/2}, x_{i-1/2})$ with cell weights

$$\Delta x_i \equiv (x_{i+1/2} - x_{i-1/2})$$

We will also define the time steps to be

$$\Delta t^{n+1/2} \equiv t^{n+1} - t^n$$

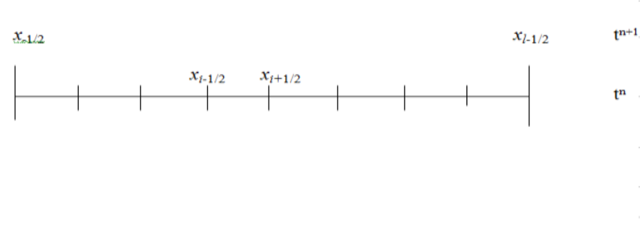

Figure 2 : Spatial and temporal discretization

Ref
8. L.S. Andallah, "Finite Difference Method-Explicit Upwind Difference scheme", Lecture note, Department of Mathematics, Jahangirnagar University, 2007.

The main numerical technique applied to the hyperbolic system of partial differential equations is the finite difference method (FDM). Its foundation is the following functions of continuous arguments which describe the state of the flow are replaced by functions defined on a finite number of the grid points within the considered domain. The derivates are then replaced by divided differences. Thus the differential equations are replaced by algebraic finite difference relationships. The different ways in which derivatives and integrals are expressed by discrete functions are called finite difference schemes or methods [9].

b) The CFL Condition

One of the first papers on finite difference methods for PDEs was written in 1928 by Courant, Friedrich and Levy [6]. They use finite difference methods as an analytic tool for proving existence of solutions of certain PDEs. The idea is to define a sequence of approximate solutions (via finite difference equation), prove that they converge as the grid is refined, and then show that the limit function must satisfy the PDE, giving existence of a solution.

In the course of proving convergence of this sequence, they recognized that a necessary stability condition for any numerical method of is that the domain of dependence of the finite difference method should be the domain of dependence of the PDEs, at least in the limit as k, $h \to 0$. This condition is known as the CFL, condition after Courant, Friedrich and Lewy.

The domain of dependence D $(\tilde{x}, \tilde{t})$ for the PDE has already been described in the previous section. The set $D(\tilde{x}, \tilde{t})$ consists of the points $\tilde{x} - c\tilde{t}$, since only initial data at these points can affect the solution at $(\tilde{x}, \tilde{t})$.

c) Lax-Friedrich Scheme

We now describe the Lax-Friedrich scheme for example we take linear advection equation [4].For discretization we use

1. Forward difference for time derivative

$$\text{i.e. } \frac{\partial u}{\partial t} \approx (u_i^n - u_i^n) / \Delta t^{n+1/2} \tag{21}$$

2. Central difference for spatial derivative

$$\text{i.e. } \frac{\partial u}{\partial x} \approx (u_{i+1}^n - u_{i-1}^n)/2\Delta x^{n+1/2} \tag{22}$$

Substituting these values in (8) we get

$$(u_i^n - u_i^n) / \Delta t^{n+1/2} + c(u_{i+1}^n - u_{i-1}^n)/2\Delta x^{n+1/2} = 0 \tag{23}$$

$$u_i^{n+1} = u_i^n - c\Delta t^{n+1/2}/2\Delta x_i (u_{i+1}^n - u_{i-1}^n) \tag{24}$$

Unfortunately, despite the quite natural derivation of this method, it suffers from severe stability problem and is useless in practice. Now we replace u_i^n by 1/2 $(u_{i+1}^n + u_{i-1}^n)$ and is stable provided $\Delta t^{n+1/2}/\Delta x$ is sufficiently small. Hence we have

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - c\Delta t^{n+1/2}/2\Delta x_i (u_{i+1}^n - u_{i-1}^n) \tag{25}$$

Ref.

9. Hanh ngwyen Van, "Numerical Methods for Water Flow Models in Water Resources"1996.

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Stencil:

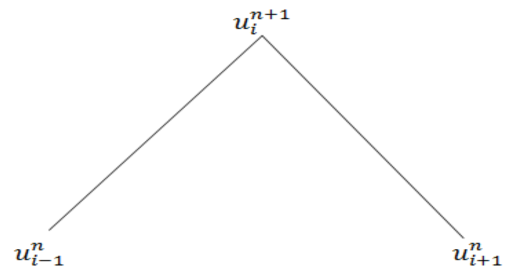

Figure 3 : Stencil of Lax-Friedrich Scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following CFL (The Courant - Friedrichs - Lewy) condition

$$\gamma_i^{n+1} \equiv \text{—}c\ \Delta t^{n+1/2}/\Delta x_i\text{—} \leq 1$$

If we take equidistance grid and denote $\Delta x = h$ and $\Delta t = k$ then we have,

$$\left|\frac{ck}{h}\right| \leq 1 \tag{26}$$

This is the stability restriction for this method.

d) Lax – wendroff scheme for linear system

In a search for stable and more accurate shock capturing numerical schemes, P. Lax and B. Wendroff proposed the idea of combining the spatial and temporal discretization in order to globally achieve second order. Lax-Wendroff's scheme is an explicit second order method [10].

We consider the time dependent Cauchy problem in one space dimension [6],

$$u_t + Au_x = 0, -\infty < x < \infty,\ t \geq 0 \tag{27}$$

$$u\ (x,\ 0) = u_0\ (x). \tag{28}$$

We discretize that x-t plane by choosing a mesh width $h \equiv \Delta x$ and a time step $k \equiv \Delta t$.

A wide variety of methods can be devised for the linear system (27) by using different finite difference approximations. Most of these are based directly on finite difference approximations to the PDE. An exception is the Lax-Wendoff scheme [10], which is based on the Taylor series expansion

$$u(x,\ t+k) = u\ (x,\ t) + k\ u_t\ (x,\ t) + \frac{1}{2}\ k^2\ u_{tt}\ (x,\ t) + \tag{29}$$

The Lax-Wendroff scheme is just the first three terms of (29).
Hence the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{k}{2h}\ A(U_{j+1}^n - U_{j-1}^n) + \frac{k^2}{2h^2}\ A^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \tag{30}$$

Ref

10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

Stencil:

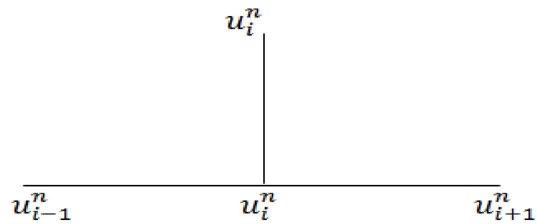

Figure 4 : Stencil of Lax-Wendroff Scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following relation:
It turn out that the resulting scheme is second-order in both space and time and stable for

$$\left| \frac{\partial f}{\partial u} \right| \Delta t \leq \Delta x$$

e) Lax – wendroff scheme for non-linear system

Our intension is to apply the Lax-Wendoff scheme for shallow water equation which is a quasi-linear equation i.e. non-linear [10]. It was further simplified by some authors in order to avoid evaluation of the celerity (Jacobian) at an intermediate position. In the case of homogeneous systems of equations, the same guidelines lead to the following scheme [11]

$$U_j^{n+1} = U_j^n - \Delta t\ /2\ \Delta x\ (F_{i+1}^n - F_{i-1}^n) + \frac{1}{2}\,(\ \Delta t/\Delta x)^2$$

$$[J_{i+1/2}^n\ (F_{i+1}^n - F_i^n - \ J_{i+1/2}^n\ (F_i^n - F_{i-1}^n)] \tag{31}$$

where now $J = \frac{\partial F}{\partial U}$ is the Jacobian matrix of the system. If the system is linear, the matrix is constant F = JU with J =constant and in the non – linear case JU must be evaluated at an intermediate position $J_{i+1/2} = J(U_{i+1/2})$ .The numerical flux can be written as

$$F_{i+1/2}^n = \frac{1}{2}\,(F_{i+1}^n - F_i^n) + \frac{1}{2}\,\frac{\Delta t}{\Delta x}\,(J_{i+1/2}^n)^2\ \ (F_{i+1}^n - F_{i+1}^n) \tag{32}$$

The scheme is non-dissipative for J = constant, and displays oscillations near strong gradients (shocks). It can also lead to numerical difficulties near critical or sonic points. Several authors have recommended the addition of extra dissipative terms (pseudo viscosity) in these cases [12]. Lax-Wendroff's scheme is one of the most frequently encountered in the literature related to classical shock-capturing schemes. Difficulties have been reported when trying to include source terms in the discretization and to keep second order of accuracy at the same time.
Hence finally for hyperbolic system the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{1}{2}\,A(U_{j+1}^n - U_{j-1}^n)(\frac{\Delta t}{\Delta x}) + \frac{1}{2}\,A^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)(\frac{\Delta t}{\Delta x}) \tag{33}$$

Ref.
10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

f) Leap-Frog scheme for linear system

Leap-Frog scheme is a special form of Lax-Wendroff scheme. It is obtained from Lax-Wendroff scheme. It is exceptional to the above schemes [10]. It is also based on the Taylor series expansion as mentioned above. The Leap-Frog scheme is just the first two terms of (31).

Hence the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{k}{2h} A(U_{j+1}^n - U_{j-1}^n) \tag{34}$$

Stencil:

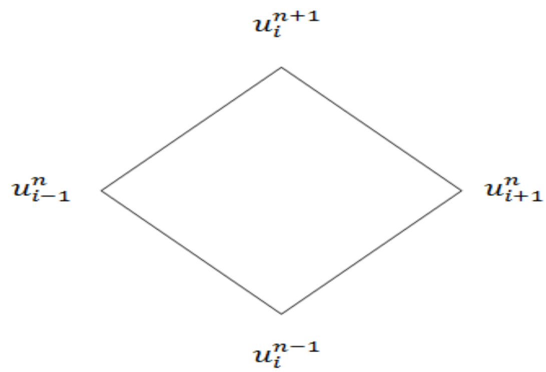

Figure 5 : Stencil of Leap-Frog scheme

Stability condition: The method is stable provided that Δt and Δx are related by the following relation:
It turn out that the resulting scheme is second-order in both space and time and stable for

$$— \frac{\partial f}{\partial u} — \Delta t \leq \Delta$$

g) Leap-Frog scheme for non-linear system

For hyperbolic system the scheme can be written by the following equation

$$U_j^{n+1} = U_j^n - \frac{1}{2} A(U_{j+1}^n - U_{j-1}^n)(\frac{\Delta t}{\Delta x}) \tag{35}$$

h) Finite Difference schemes for shallow water equation

In the following, we specify the finite difference methods for solving the shallow water equation [15].

Here $h(t^n, x_i)$ is abbreviated by h_i^n and $v(t^n, x_i)$ is abbreviated by v_i^n. We denote flux $f(t^n, x_i)$ by f_i^n for first equation and flux $f1(t^n, x_i)$ by $f1_i^n$ for second equation.

In Lax-Friedrich scheme, the shallow water equation can be written as

$$h_i^n = \frac{1}{2}\ (h_{i+1}^{n-1} + h_{i-1}^{n-1}) - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1})\ \frac{\Delta t}{\Delta x}$$

$$v_i^n = \frac{1}{2}\ (v_{i+1}^{n-1} + v_{i-1}^{n-1}) - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1})\ \frac{\Delta t}{\Delta x}$$

Where $f = h * v$

$$f1 = g * h + \frac{1}{2}\, v^2$$

In Lax-Wendroff scheme, the shallow water equation can be written as

Ref

10. John A. Trangenstein "Numerical Solution for Partial Differential Equations".

$$h_i^n = h_i^{n-1} - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1})\frac{\Delta t}{\Delta x} + \frac{1}{2}[(\frac{\Delta t}{\Delta x})^2\{\frac{1}{2}(v_{i+1}^{n-1} + v_i^{n-1})(f_{i+1}^{n-1} - f_i^{n-1})\} - \{\frac{1}{2}(v_{i-1}^{n-1} + v_i^{n-1})(f_i^{n-1} - f_{i-1}^{n-1})\}]$$

$$v_i^n = v_i^{n-1} - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1})\frac{\Delta t}{\Delta x} + \frac{1}{2}[(\frac{\Delta t}{\Delta x})^2\{\frac{1}{2}(v_{i+1}^{n-1} + v_i^{n-1})(f1_{i+1}^{n-1} - f1_i^{n-1})\} - \{\frac{1}{2}(v_{i-1}^{n-1} + v_i^{n-1})(f1_i^{n-1} - f1_{i-1}^{n-1})\}]$$

Where

$$f = h * v$$

$$f1 = g * h + \frac{1}{2}v^2$$

In Leap-Frog scheme, the shallow water equation can be written as

$$h_i^n = h_i^{n-1} - \frac{1}{2}(f_{i+1}^{n-1} - f_{i-1}^{n-1})\frac{\Delta t}{\Delta x}$$

$$v_i^n = v_i^{n-1} - \frac{1}{2}(f1_{i+1}^{n-1} - f1_{i-1}^{n-1})\frac{\Delta t}{\Delta x}$$

Where

$$f = h * v$$

$$f1 = g * h + \frac{1}{2}v^2$$

Here we consider $t_n = n.\Delta t$, $x_i = i.\Delta x$, where $n = 0, \ldots\ldots\ldots\ldots, \Delta t + 1$, $i = 0, \ldots\ldots\ldots, \Delta x + 1$.

IV. Result

Shallow water equation solved by analytical method is too complex that's why in this paper we apply two numerical schemes known as Lax-Friedrich scheme and Lax-Wendroff scheme. To Test the accuracy of the implementation of the numerical schemes we consider the linear advection equation whose analytical solution is known to us.

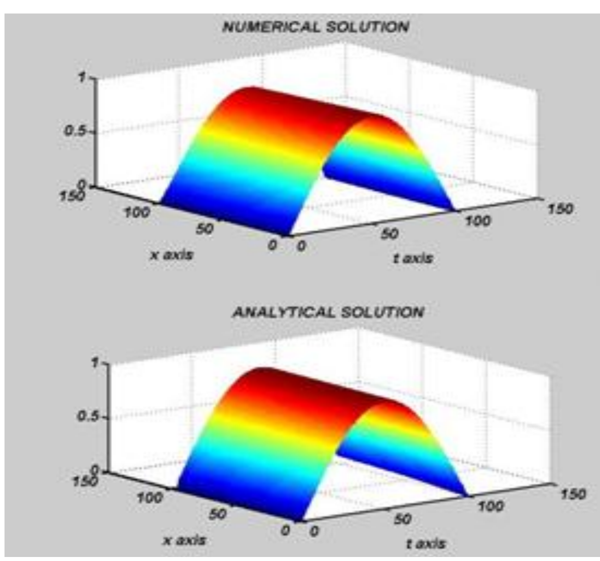

Figure 6 : Numerical and Analytical solution of linear advection equation using Lax-Friedrich scheme

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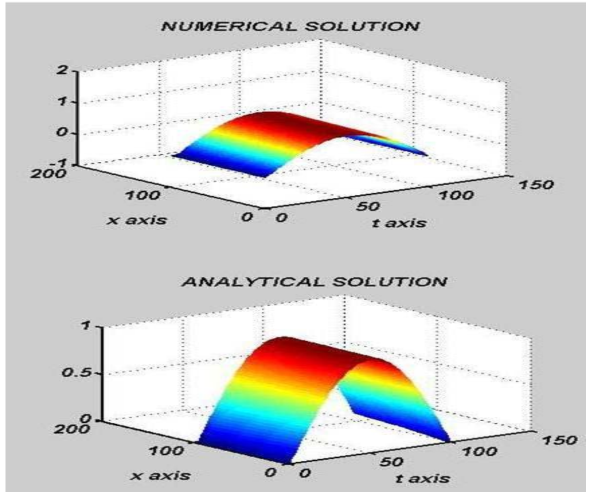

Figure 7 : Numerical and Analytical solution of linear advection equation using Lax-Wendroff scheme

Fig. 6 and fig.7 shows the analytical and numerical solution of linear advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme respectively. From these figure we observe that there is no difference between the analytical and numerical solution. Now we define error term as follows:

$$\text{Error} = \left(\frac{\text{Analytical solution} - \text{numerical solution}}{\text{Analytical solution}}\right) \times 100.$$

From the program we obtain, the error of the numerical solution of linear advection equation is 0.05, 0.048 and .99 using Lax-Friedrich scheme and Lax-Wendroff scheme and Leap-Frog scheme respectively. So we can say the error between them is very negligible and if we take $\Delta x \rightarrow 0$ then the error $\rightarrow 0$.

Figure 6. and 7 shows numerical and analytical solution of advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme, respectively.

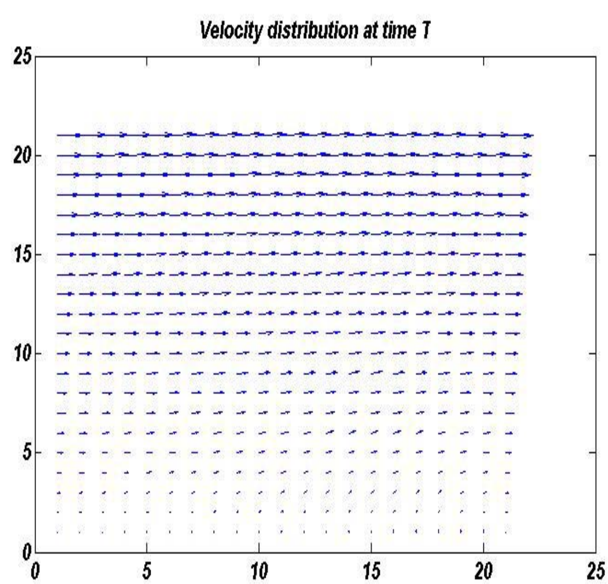

Figure 8 : Velocity distribution for numerical solution of linear advection equation using Lax-Friedrich scheme

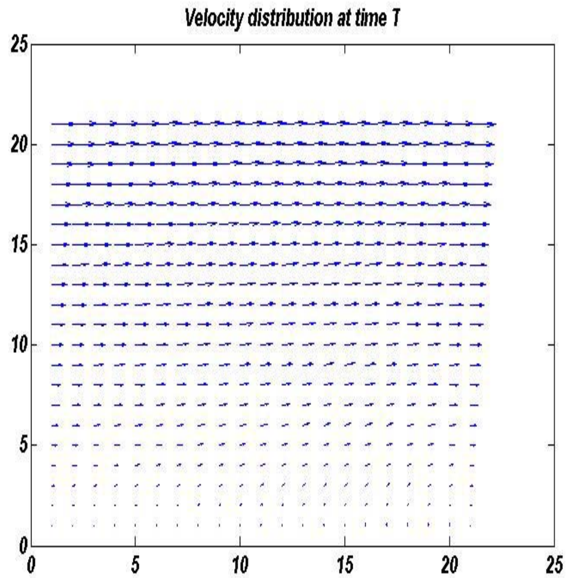

Figure 9 : Velocity distribution for numerical solution of linearadvection equation using Lax-Wendroff scheme

Fig. 8. and Fig. 9. shows water velocity distribution for solution of advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme, respectively.

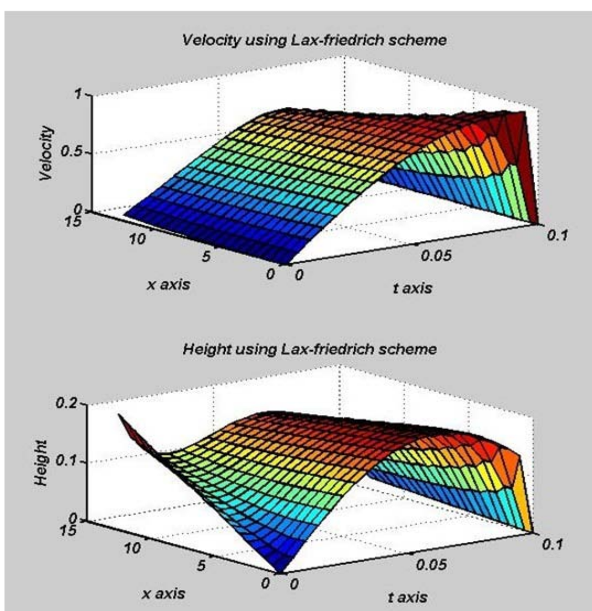

Figure 10 : Water velocity and height of shallow water equation using Lax-Friedrich scheme

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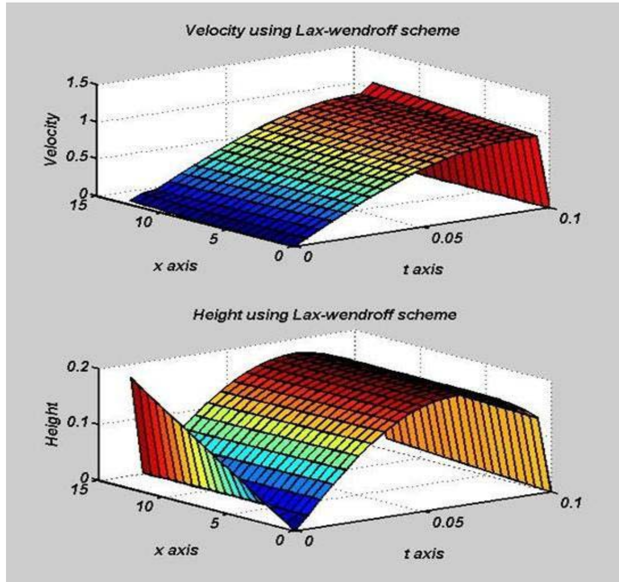

Figure 11 : Water velocity and height of shallow water equation using Lax-Wendroff scheme

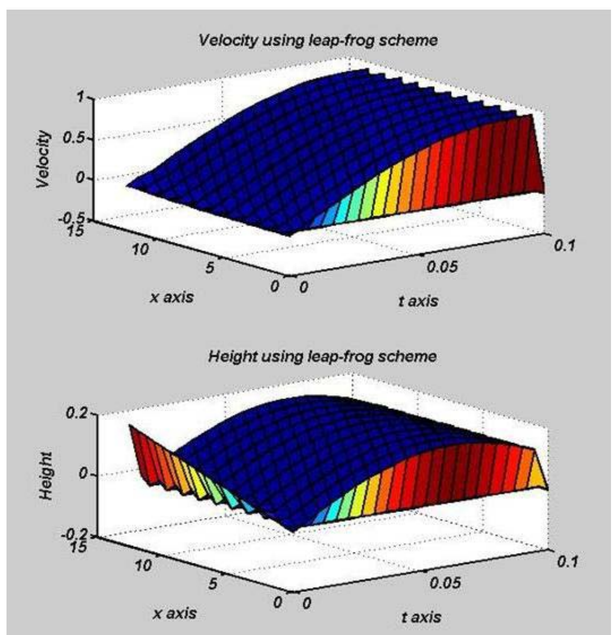

Figure 12 : Water velocity and height of shallow water equation using Leap-Frog scheme

Fig. 10 and 11 and 12 shows water velocity and height of shallow water equation using Lax-Friedrich scheme, Lax-Wendroff scheme and Leap-Frog scheme, respectively.

Notes

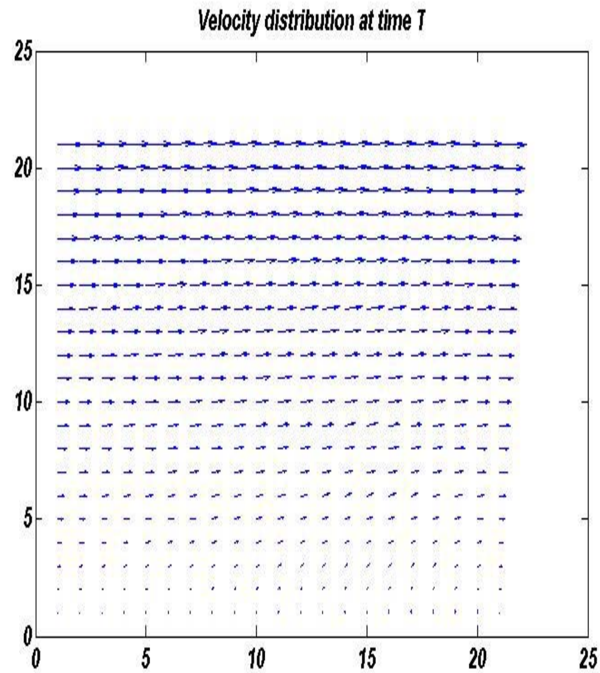

Figure 13 : Velocity distribution for numerical solution of shallow water equation using Lax-Friedrich scheme

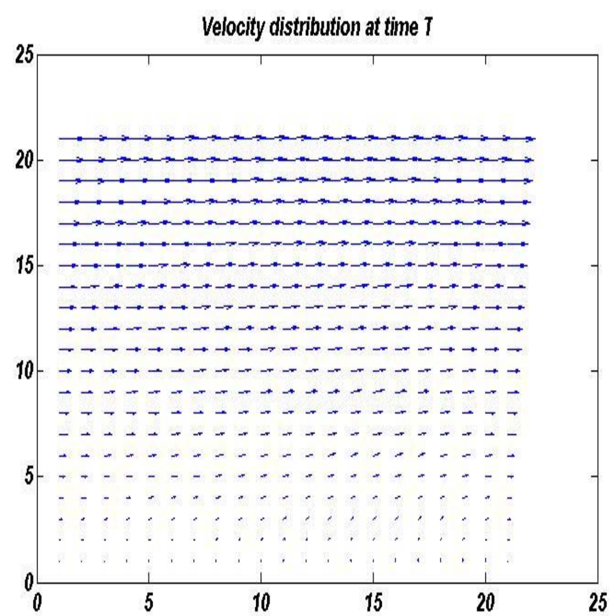

Figure 14 : Velocity distribution for numerical solution of shallow water equation using Lax-Wendroff scheme

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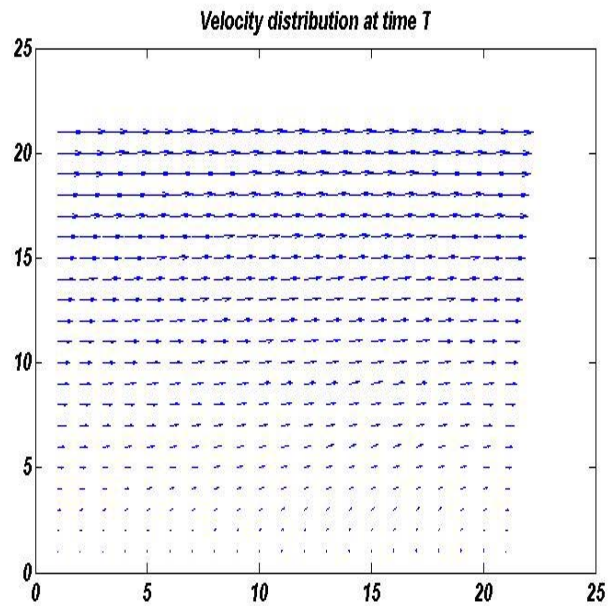

Figure 15 : Velocity distribution for numerical solution of shallow water equation using Leap-Frog scheme

Fig. 13, 14 and 15 shows velocity distribution for numerical solution of shallow water equation using Lax-Friedrich scheme and Lax-Wendroff scheme and Leap-Frog scheme respectively in forms of mesh, which we have simulated from our numerical model.

Fig. 8, and 9 implies that the velocity distribution for numerical solution of linear advection equation using Lax-Friedrich scheme and Lax-Wendroff scheme is identical.

We also observe from the scheme of Leap-Frog scheme that it is obtain from Lax-Wendroff scheme.

V. Conclusion

From the above experiment it has revealed clearly that Lax-Wendroff scheme is more efficient than Lax-Friedrich scheme and Leap-Frog scheme because the error term in Lax-Wendroff scheme is less than the Lax-Friedrich scheme and Leap-Frog scheme.

In this paper we have consider only the 1D shallow water equation. First we have shown the numerical and analytical solution of linear advection equation using Lax-Friedrich scheme, Lax-Friedrich scheme and their qualitative graphs where we have obtained the error terms of these schemes which is essential to find the acceptability of these schemes. Finally we have shown the Lax-Friedrich scheme, Lax-Wendroff scheme and Leap-Frog scheme and their respective graphs for shallow water equation. It may be more effective experiment for flood modeling using higher order shallow water equation and we left this as our future work.

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Helical-One, Two, Three-Revolutional Cyclical Surfaces

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Abstract - This paper describes method for modelling of helical-*n*-revolutional cyclical surfaces. The axis of the cyclical surface Φ_1 is the helix s_1 created by revolving the point about n each other revolving axes o_n ($n = 1,2,3$), that move together with Frenet-Serret moving trihedron along the cylindrical helix s. Particular evolutions are determined by its angular velocity and orientation. The moving circle along the helix s or s_1, where its center lies on the helix and circle lies in the normal plane of the helix creates the cyclical surface.

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GJSFR-F Classification : *MSC 2010: 05B20*

HELICAL-ONE, TWO, THREE-REVOLUTIONAL CYCLICALSURFACES

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Helical-One, Two, Three-Revolutional Cyclical Surfaces

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Abstract - This paper describes method for modelling of helical-*n*-revolutional cyclical surfaces. The axis of the cyclical surface Φ_1 is the helix s_1 created by revolving the point about *n* each other revolving axes o_n (n = 1,2,3), that move together with Frenet-Serret moving trihedron along the cylindrical helix *s*. Particular evolutions are determined by its angular velocity and orientation. The moving circle along the helix *s* or s_1, where its center lies on the helix and circle lies in the normal plane of the helix creates the cyclical surface.

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I. Introduction

Let three-dimensional Euclidean space E^3 is determined by Cartesian coordinate system $(0,x,y,z)$. In this space is given cylindrical helix s with axis identical with coordinate axis z determined by vector function (Fig.1)

$$\mathbf{r}(v)=(x_s,y_s,z_s,1)=(a\cos mv, sg\, a\sin mv, bv, 1),\ v\in\langle 0,2\pi\rangle, \tag{1}$$

where parameter a is radius of the helix, b is the reduced pitch, sg determined orientation of the helix, ($sg=+1$ for right-handed and $sg=-1$ for left-handed revolution), m is number of pitches. Let $(0',n,b,t)$ be Frenet-Serret moving trihedron of the cylindrical helix s represented by regular square matrix

$$\mathbf{M}(v)=\begin{pmatrix} n_x(v) & n_y(v) & n_z(v) & 0 \\ b_x(v) & b_y(v) & b_z(v) & 0 \\ t_x(v) & t_y(v) & t_z(v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where the matrix elements are the coordinates of unit vectors of the principle normal n, binormal b and tangent t of the helix s in the point $0'\in s$ in the coordinate system $(0,x,y,z)$

$$\mathbf{t}(v)=(t_x(v),t_y(v),t_z(v))=\frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|}, \tag{3}$$

$$\mathbf{b}(v)=(b_x(v),b_y(v),b_z(v))=\frac{\mathbf{r}'(v)\times\mathbf{r}''(v)}{|\mathbf{r}'(v)\times\mathbf{r}''(v)|}, \tag{4}$$

$$\mathbf{n}(v)=(n_x(v),n_y(v),n_z(v))=\mathbf{b}(v)\times\mathbf{t}(v). \tag{5}$$

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Transformations of revolutions about coordinate axes x, y, z are represented by matices $\mathbf{T}_x(\varphi, \psi)$, $\mathbf{T}_y(\varphi, \psi)$, $\mathbf{T}_z(\varphi, \psi)$, where φ is angle and ψ is orientation of the revolution, transformation of translation is represented by matrix $\mathbf{T}(\pm d_x, \pm d_y, \pm d_z)$, where $(\pm d_x, \pm d_y, \pm d_z)$ is translation vector determined by its coordinates (6), (7):

$$\mathbf{T}_x(\varphi, \psi)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \psi \sin\varphi & 0 \\ 0 & -\psi \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{T}_y(\varphi, \psi)=\begin{pmatrix} \cos\varphi & 0 & \psi \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\psi \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

$$\mathbf{T}_z(\varphi, \psi)=\begin{pmatrix} \cos\varphi & \psi \sin\varphi & 0 & 0 \\ -\psi \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{T}(\pm d_x, \pm d_y, \pm d_z)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm d_x & \pm d_y & \pm d_z & 1 \end{pmatrix}. \tag{7}$$

The moving circle $c=(0', r)$ along the helix s, where its center $0'$ lies in the normal plane determined by principal normal n and binormal b of the helix in the point $0' \in s$ creates the cyclical surface Φ. The vector function of this surface is

$$\mathbf{P}(u,v)=\mathbf{r}(v)+\mathbf{c}(u).\mathbf{M}(v), \ u \in \langle 0, 2\pi \rangle, \ v \in \langle 0, 2\pi \rangle, \tag{8}$$

where $\mathbf{r}(v)$ is vector function of the helix s expressed in equation (1), $\mathbf{M}(v)$ is transformation matrix of the coordinate system $(0', n, b, t)$ into coordinate system $(0, x, y, z)$ (2) and $\mathbf{c}(u)=(r\cos u, r\sin u, 0, 1), u \in \langle 0, 2\pi \rangle$ is vector function of the circle c determined by its center $0'$ and radius r (Fig.2). In Fig.3 there are displayed two screws of the right-handed cyclical surface Φ together with the cylindrical surface on which helix s is wound.

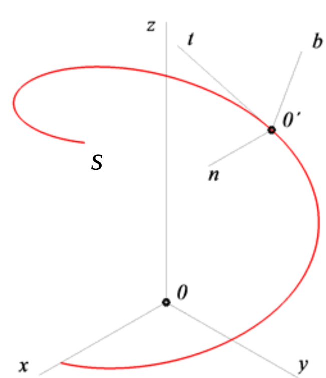

Fig. 1 : Helix s with Trihedron

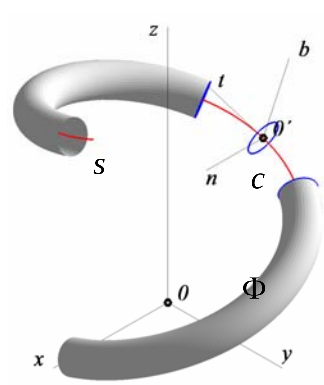

Fig. 2 : Cyclical Surface Φ

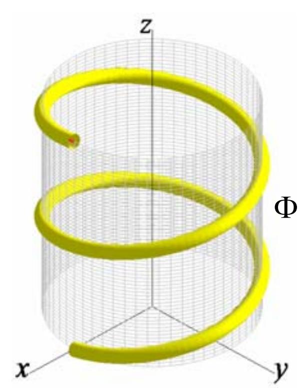

Fig. 3 : Surface Φ and Cylinder

II. Cyclical Helical Surface Created by One Revolution

The helix s_1 created by revolution of the point $P(x_0, y_0, z_0, 1)$ about the axis o_1 connected to the moving trihedron of the helix s, is represented by vector function

$$\mathbf{r}_1(v)=\mathbf{r}(v)+(x_0, y_0, z_0, 1).\mathbf{T}_1(m_1 v, sg_1).\mathbf{M}(v) \tag{9}$$

Notes

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and cyclical surface Φ_1 created in a similar way as surface Φ by vector function

$$\mathbf{P}_1(u,v)=\mathbf{r}_1(v)+\mathbf{c}_1(u).\mathbf{M}_1(v),\ u\in\langle 0,2\pi\rangle,\ v\in\langle 0,2\pi\rangle, \tag{10}$$

where $\mathbf{r}_1(v)$ is vector function of the helix s_1 expressed in equation (9), $\mathbf{M}_1(v)$ is transformation matrix of the coordinate system $(0'',n',b',t')$ into coordinate system $(0,x,y,z)$ (11), $\mathbf{c}_1(u)=(r_1\cos u,\ r_1\sin u,0,1),u\in\langle 0,2\pi\rangle$ is vector function of the circle c_1 determined by center $0''\in s_1$ and radius r_1

$$\mathbf{M}_1(v)=\begin{pmatrix} n'_x(v) & n'_y(v) & n'_z(v) & 0\\ b'_x(v) & b'_y(v) & b'_z(v) & 0\\ t'_x(v) & t'_y(v) & t'_z(v) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{11}$$

Elements of this matrix are coordinates of unit vectors of the principle normal n', binormal b' and tangent t' of the helix s_1 in the point $0''\in s_1$ in the coordinate system $(0',n,b,t)$

$$\mathbf{t}'(v)=\left(t'_x(v),t'_y(v),t'_z(v)\right)=\frac{\mathbf{r}'_1(v)}{|\mathbf{r}'_1(v)|} \tag{12}$$

$$\mathbf{b}'(v)=\left(b'_x(v),b'_y(v),b'_z(v)\right)=\frac{\mathbf{r}'_1(v)\times\mathbf{r}''_1(v)}{|\mathbf{r}'_1(v)\times\mathbf{r}''_1(v)|}, \tag{13}$$

$$\mathbf{n}'(v)=\left(n'_x(v),n'_y(v),n'_z(v)\right)=\mathbf{b}'(v)\times\mathbf{t}'(v). \tag{14}$$

a) Revolution about tangent t of the helix s

The helix created by the revolution of the point P about the axis $o_1=t$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1v,sg_1)=\mathbf{T}_z(m_1v,sg_1)$. In Fig.4 is displayed cyclical surface Φ, whose axis is helix s with parameters $m=2,sg=+1$ and surface Φ_1, whose axis is helix s_1 created by revolution of the point $P=(d,0,0,1)$ about tangent t of the helix s with parameters $m_1=8m,sg_1=+1$.

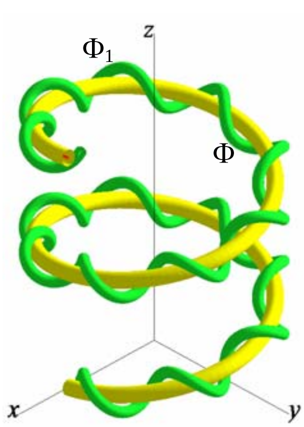

Fig. 4 : Cyclical Surfaces Φ, Φ_1

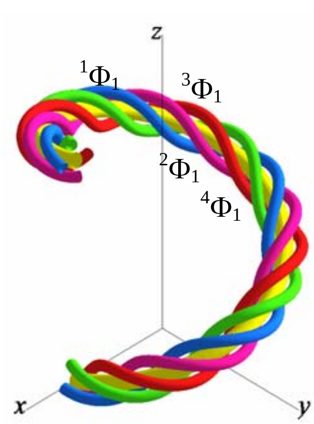

Fig. 5 : 4 Surfaces ${}^i\Phi_1$

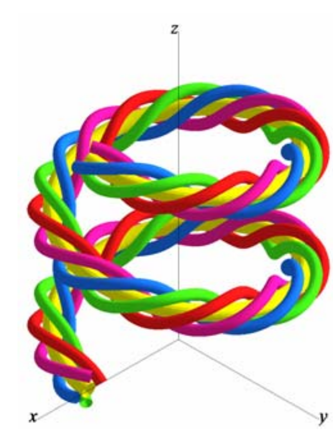

Fig. 6 : Left-handed Surfaces Φ, ${}^i\Phi_1$

In Fig.5 are displayed $k=4$ surfaces ${}^{i}\Phi_1$, whose axes are helix ${}^{i}s_1$, $i=1,...,k$ created by revolution of the points ${}^{i}P=(d\cos i\alpha, d\sin i\alpha, 0, 1)$, $\alpha=2\pi/k$ about tangent t of the helix s with parameters $m_1=4m, sg_1=+1$, in Fig.6 are displayed the same surfaces with altered orientation of the revolution $m=2, sg=-1$, $m_1=4m, sg_1=-1$.

b) Revolution about principal normal n of the helix s

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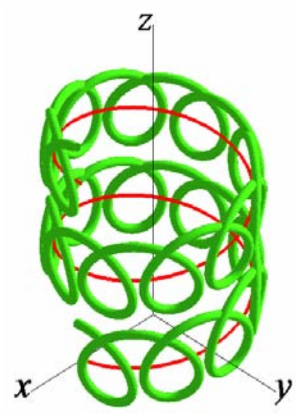

Fig. 7 : Normal Surface Φ_1

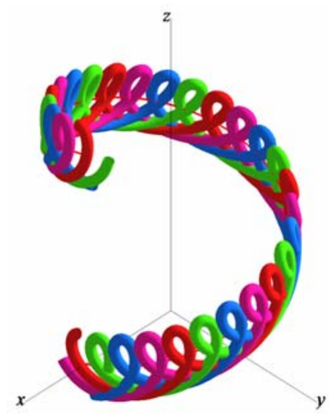

Fig. 8 : 4 Surfaces ${}^{i}\Phi_1$

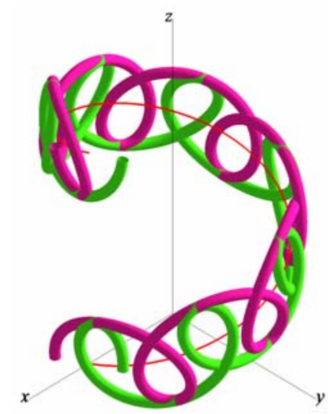

Fig. 9 : Normal Surfaces ${}^{1}\Phi_1$, ${}^{3}\Phi_1$

The helix s_1 created by the revolution of the point P about the axis $o_1=n$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1 v, sg_1)=\mathbf{T}_x(m_1 v, sg_1)$. In Fig.7 is displayed helix s with parameters $m=2, sg=+1$ and normal surface Φ_1, whose axis is helix s_1 created by revolution of the point $P=(0, d, 0, 1)$ about normal n of the helix s with parameters $m_1=10m, sg_1=+1$, in Fig.8 are displayed $k=4$ normal surfaces ${}^{i}\Phi_1$, whose axes are helix ${}^{i}s_1$, $i=1,...,k$ created by revolution of the points ${}^{i}P=(0, d\cos i\alpha, d\sin i\alpha, 1)$, $\alpha=2\pi/k$ about normal n of the helix s with parameters $m_1=7m, sg_1=-1$, in Fig.9 are displayed surfaces ${}^{1}\Phi_1$, ${}^{3}\Phi_1$ with altered orientation of the revolution $sg_1=\pm 1$.

c) Revolution about binormal b of the helix s

The helix created by the revolution of the point P about axis $o_1=b$ is expressed by vector function (9), in which matrix $\mathbf{T}_1(m_1 v, sg_1)=\mathbf{T}_y(m_1 v, sg_1)$. In Fig.10 is displayed helix s with parameters $m=2, sg=+1$ and binormal surface Φ_1, whose axis is helix s_1 created by revolution of the point $P=(d, 0, 0, 1)$ about binormal b of the helix s with parameters $m_1=10m, sg_1=+1$, in Fig.11 is displayed binormal surface with parameters $m_1=8m, sg_1=-1$, in Fig.12 are surfaces ${}^{1}\Phi_1$, ${}^{3}\Phi_1$ created by revolution of the points ${}^{i}P=(d\cos i\alpha, d\sin i\alpha, 0, 1)$, $\alpha=2\pi/k$ about binormal b of the helix with altered orientation of the revolution $sg_1=\pm 1$.

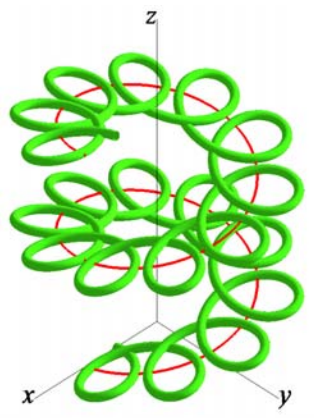

Fig. 10 : Binormal Surface Φ_1

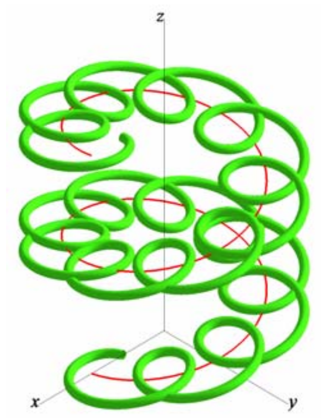

Fig. 11 : 4 Surfaces ${}^{i}\Phi_1$

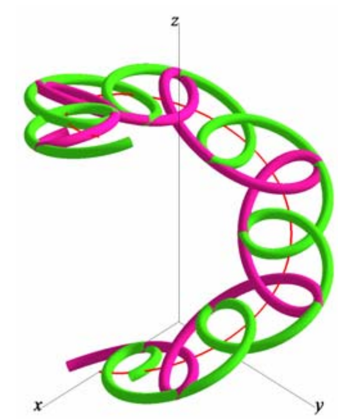

Fig. 12 : Binormal Surfaces ${}^{1}\Phi_1, {}^{3}\Phi_1$

III. Cyclical Helical Surface Created by Two Revolutions

The helix s_1 created by revolution of the point $P(x_0, y_0, z_0, 1)$ about axis o_2, which revolves about the axis o_1 identical with one edge of the moving trihedron of the helix s is represented by vector function

$$\mathbf{r}_1(v) = \mathbf{r}(v) + (x_0, y_0, z_0, 1).\mathbf{T}_2(m_2 v, sg_2).\mathbf{T}_1(m_1 v, sg_1).\mathbf{M}(v), \tag{15}$$

where matrix $\mathbf{T}_2(m_2 v, sg_2)$ represents revolution of the point P about the axis o_2 and matrix $\mathbf{T}_1(m_1 v, sg_1)$ represents revolution of the axis o_2 about the axis o_1.

a) Revolution about two parallel axes

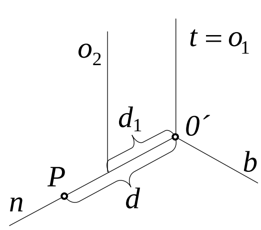

Fig. 13

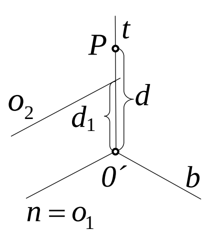

Fig. 14

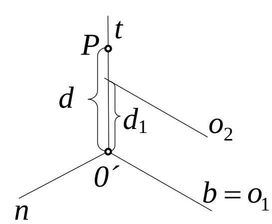

Fig. 15

If the helix s_1 is created by revolution of the point P about two parallel axes $o_2 \parallel o_1$ and $o_1 = t$, where $d_1 = |o_1 o_2|$ is the distance between them, then (Fig.13)

$$\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(-d_1, 0, 0).\mathbf{T}_z(m_2 v, sg_2).\mathbf{T}(+d_1, 0, 0),\ \mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1). \tag{16}$$

In Fig.16 is displayed this surface Φ_1 with parameters $m_1 = 8m, sg_1 = +1, m_2 = 4m_1, sg_2 = +1$.

If the helix s_1 is created by revolution of the point P about parallel axes $o_2 \parallel o_1$ and $o_1 = n$, where $d_1 = |o_1 o_2|$, then (Fig.14)

$$\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(0, 0, -d_1).\mathbf{T}_x(m_2 v, sg_2).\mathbf{T}(0, 0, +d_1),\ \mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_x(m_1 v, sg_1). \tag{17}$$

Notes

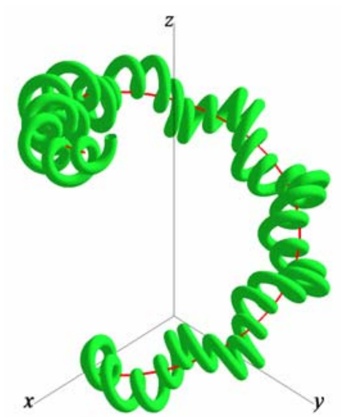

Fig. 16 : $o_2 \parallel o_1,\ o_1 = t$

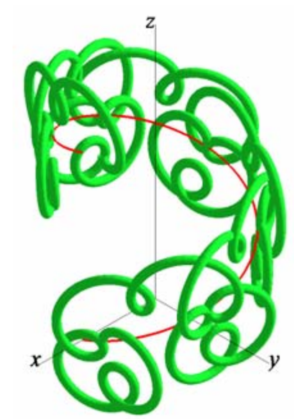

Fig. 17 : $o_2 \parallel o_1,\ o_1 = n$

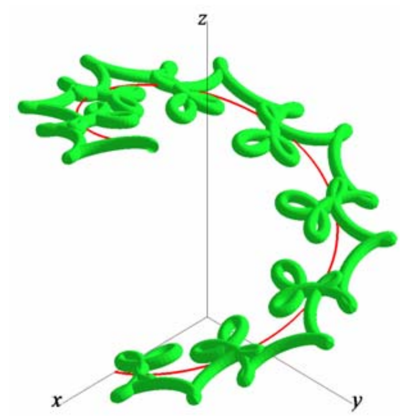

Fig. 18 : $o_2 \parallel o_1,\ o_1 = b$

In Fig.17 is displayed this surface Φ_1 with parameters $m_1 = 6m, sg_1 = +1, m_2 = 4m_1, sg_2 = +1$.

If the helix s_1 is created by revolution of the point $P = (0,0,d,1)$ about parallel axes $o_2 \parallel o_1$ and $o_1 = b$, where $d_1 = |o_1 o_2|$, then (Fig.15)

$$\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(0,0,-d_1).\mathbf{T}_y(m_2 v, sg_2).\mathbf{T}(0,0,+d_1),\ \mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_y(m_1 v, sg_1). \tag{18}$$

In Fig.18 is displayed this surface Φ_1 with parameters $m_1 = 8m, sg_1 = -1, m_2 = 5m_1, sg_2 = +1$.

b) Revolution about two intersecting axes

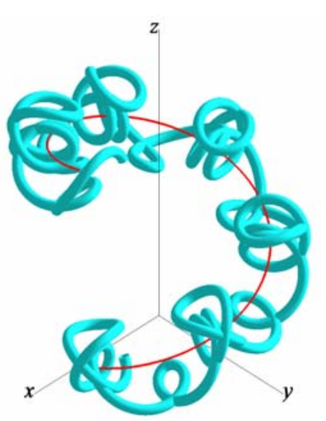

Fig. 19 : $(o_2 = t) \perp (o_1 = n)$

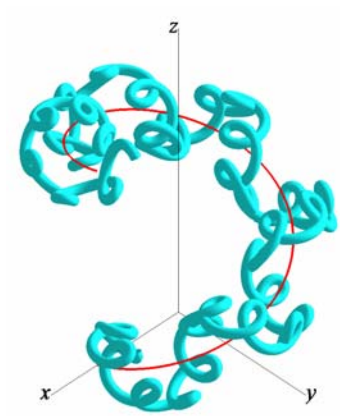

Fig. 20 : $(o_2 = n) \perp (o_1 = t)$

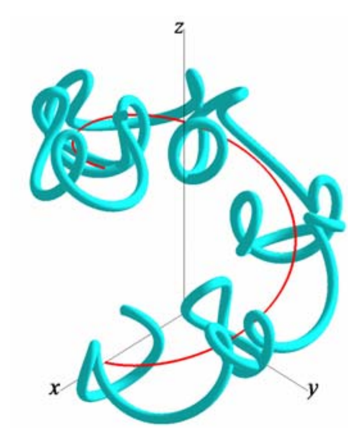

Fig. 21 : $(o_2 = n) \perp (o_1 = b)$

In Fig.19 is displayed surface created by revolution of the point $P = (2,2,0,1)$ about mutually perpendicular axes $(o_2 = t) \perp (o_1 = n)$ determined by the parameters $m_1 = 6m, sg_1 = -1$, $m_2 = 4m_1, sg_2 = -1$, where matrices $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_z(m_2 v, sg_2)$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_x(m_1 v, sg_1)$, in Fig.20 is displayed surface created by revolution of the point $P = (2.2,1.2,0,1)$ about mutually perpendicular axes $(o_2 = n) \perp (o_1 = t)$ determined by parameters $m_1 = 6m, sg_1 = +1$, $m_2 = 6m_1, sg_2 = +1$, and matrices $\mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}_x(m_2 v, sg_2)$, $\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}_z(m_1 v, sg_1)$, here we see action of changing the order of the revolutions to form of the surfaces. In Fig.21 is displayed surface created by revolution of the point $P = (2.5,2.5,0,1)$ about mutually perpendicular axes $(o_2 = n) \perp (o_1 = b)$ determined by parameters

Notes

$m_1 = 5m, sg_1 = -1$, $m_2 = 3m_1, sg_2 = +1$, $\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}_x(m_2v, sg_2)$, $\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_y(m_1v, sg_1)$.In Figs.22,23 is displayed surface created by revolution of the point $P = (d,0,0,1)$ about intersecting axes $o_2 \times (o_1 = t)$ determined by the parameters $m_1 = 6m, sg_1 = +1$, $m_2 = 6m_1, sg_2 = -1$, $\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_z(m_1v, sg_1)$, $\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}_y(\alpha,+1).\mathbf{T}_x(m_2v, sg_2).\mathbf{T}_y(\alpha,-1)$.

c) Revolution about two skew axes

In Figs.24,25 is displayed surface created by revolution of the point $P = (d,0,0,1)$ about mutually skew axes $(o_2 \parallel n) / (o_1 = t)$, determined by parameters $m_1 = 4m, sg_1 = +1$, $m_2 = 8m_1, sg_2 = +1$, where transformation matrices of two revolutions are

$$\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_z(m_1v, sg_1),\ \mathbf{T}_2(m_2v, sg_2) = \mathbf{T}(0,0,-d_1).\mathbf{T}_x(m_2v, sg_2).\mathbf{T}(0,0,+d_1).$$

In Figs.26,27 is displayed surface created by revolution of the point $P = (d,0,0,1)$ about mutually skew axes $(o_2 \times t, o_2 \times n)$ / $(o_1 = t)$ determined by parameters $m_1 = 6m, sg_1 = +1$, $m_2 = 4m_1, sg_2 = -1$, and transformation matrices $\mathbf{T}_2(m_2v, sg_2) = \mathbf{T}_y(\alpha,+1).\mathbf{T}_x(m_2v, sg_2).\mathbf{T}_y(\alpha,-1)$, $\mathbf{T}_1(m_1v, sg_1) = \mathbf{T}_y(m_1v, sg_1)$.

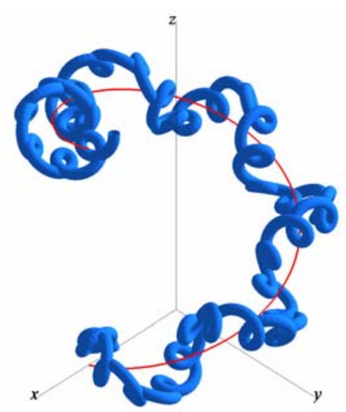

Fig. 22: $o_2 \times (o_1 = t)$

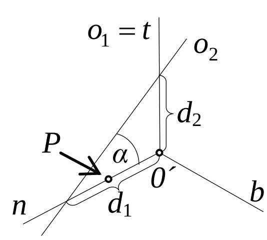

Fig. 23

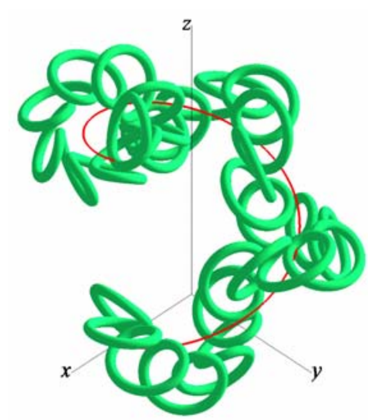

Fig. 24: $(o_2 \parallel n)$ / $(o_1 = t)$

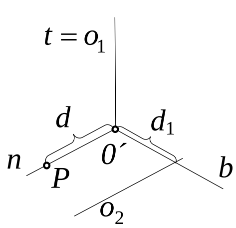

Fig. 25

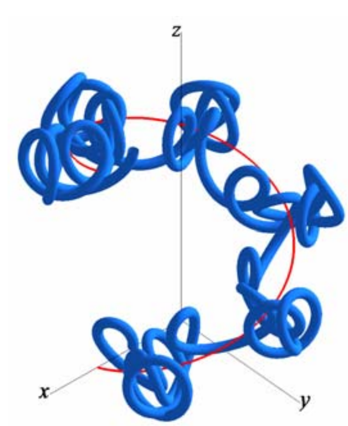

Fig. 26: $(o_2 \times t, o_2 \times n)$ / $(o_1 = t)$

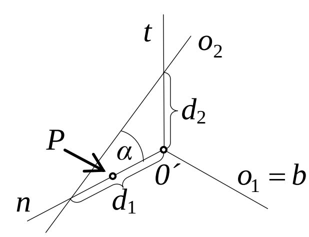

Fig. 27

IV. Cyclical Helical Surface Created by Three Revolutions

The helix s_1 created by the revolution of the point $P=(x_0, y_0, z_0, 1)$ about the axis o_3, which revolves about the axis o_2 and this revolves about the axis o_1 identical with any edge of the moving trihedron of the helix *s* is represented by vector function

$$\mathbf{r}_1(v)=\mathbf{r}(v)+(x_0, y_0, z_0, 1)\,.\,\mathbf{T}_3(m_3 v, sg_3)\,.\,\mathbf{T}_2(m_2 v, sg_2).\mathbf{T}_1(m_1 v, sg_1).\mathbf{M}(v), \tag{19}$$

Notes

where matrix $\mathbf{T}_3(m_3 v, sg_3)$ represents revolution of the point *P* about the axis o_3, matrix $\mathbf{T}_2(m_2 v, sg_2)$ represents revolution of the axis o_3 about the axis o_2 and matrix $\mathbf{T}_1(m_1 v, sg_1)$ represents revolution of the axis o_2 about the axis o_1.

a) Revolution about three parallel axes

In Fig.28 is displayed surface created by revolution about three parallel axes $o_3 \,||\, o_2 \,||\, o_1 = t$ determined by parameters $m_1 = 4m, sg_1 = +1$, $m_2 = 4m_1, sg_2 = +1$, $m_3 = 3m_2, sg_1 = +1$, matrices $\mathbf{T}_3(m_3 v, sg_3)=\mathbf{T}(-d_2,0,0).\mathbf{T}_z(m_3 v, sg_3).\mathbf{T}(+d_2,0,0)$, $\mathbf{T}_2(m_2 v, sg_2)=\mathbf{T}(-d_1,0,0).\mathbf{T}_z(m_2 v, sg_2).\mathbf{T}(+d_1,0,0)$, $\mathbf{T}_1(m_1 v, sg_1)=\mathbf{T}_z(m_1 v, sg_1)$. In Fig.29 is displayed surface created by revolution about three parallel axes $o_3 \,||\, o_2 \,||\, o_1 = n$ determined by parameters $m_1 = 4m, sg_1 = +1$, $m_2 = 4m_1, sg_2 = +1$, $m_3 = 4m_2, sg_1 = +1$ and by transformation matrices $\mathbf{T}_1(m_1 v, sg_1)=\mathbf{T}_x(m_1 v, sg_1)$ $\mathbf{T}_2(m_2 v, sg_2)=\mathbf{T}(0,0,-d_1).\mathbf{T}_z(m_2 v, sg_2).\mathbf{T}(0,0,+d_1)$, $\mathbf{T}_3(m_3 v, sg_3)=\mathbf{T}(0,0,-d_2).\mathbf{T}_x(m_3 v, sg_3).\mathbf{T}(0,0,+d_2)$.

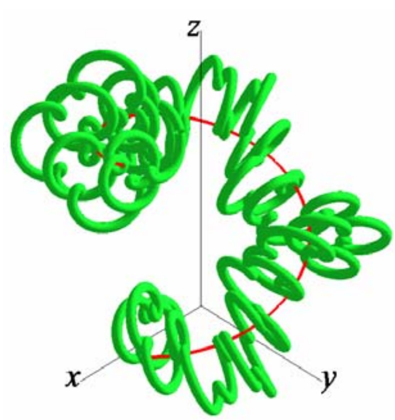

Fig. 28 : $o_3 \,||\, o_2 \,||\, o_1 = t$

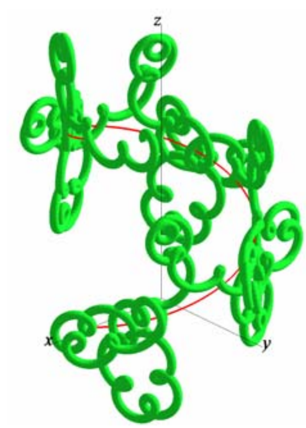

Fig. 29 : $o_3 \,||\, o_2 \,||\, o_1 = n$

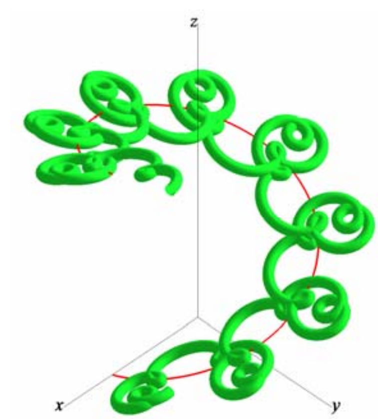

Fig. 30 : $o_3 \,||\, o_2 \,||\, o_1 = b$

In Fig.30 is displayed surface created by revolution about three parallel axes $o_3 \,||\, o_2 \,||\, o_1 = b$ determined by parameters $m_1 = 3m, sg_1 = +1$, $m_2 = 3m_1, sg_2 = +1$, $m_3 = 3m_2, sg_1 = +1$ and transformation matrices $\mathbf{T}_3(m_3 v, sg_3)=\mathbf{T}(0,0,-d_2).\mathbf{T}_y(m_3 v, sg_3).\mathbf{T}(0,0,+d_2)$, $\mathbf{T}_2(m_2 v, sg_2)=\mathbf{T}(0,0,-d_1).\mathbf{T}_y(m_2 v, sg_2).\mathbf{T}(0,0,+d_1)$, $\mathbf{T}_1(m_1 v, sg_1)=\mathbf{T}_y(m_1 v, sg_1)$.

b) Revolution about three perpendicular axes

In Figs.31,32,33 are displayed surfaces created by revolution of the point $P=(d, d, 0, 1)$ about three perpendicular axes with common point $o_3 \perp o_2 \perp o_1$, which are identical with edges of the trihedron of the helix *s*, where parameters are the same $m_1 = 3m, sg_1 = +1$, $m_2 = 3m_1, sg_2 = +1$, $m_3 = 3m_2, sg_1 = +1$, but the order of the revolutions changes.

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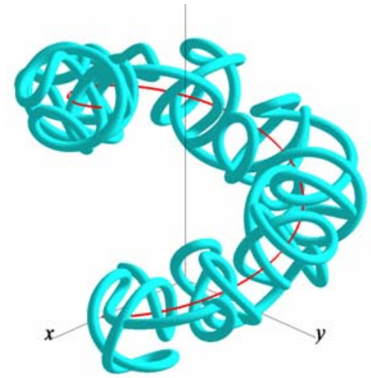

Fig. 31 : $o_3 = n, o_2 = b, o_1 = t$

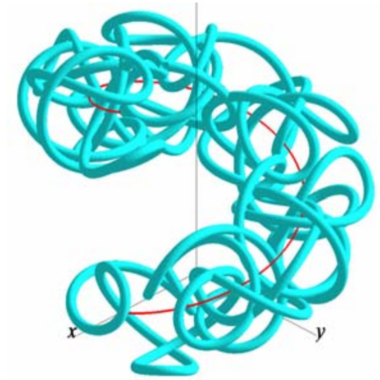

Fig. 32 : $o_3 = n, o_2 = t, o_1 = b$

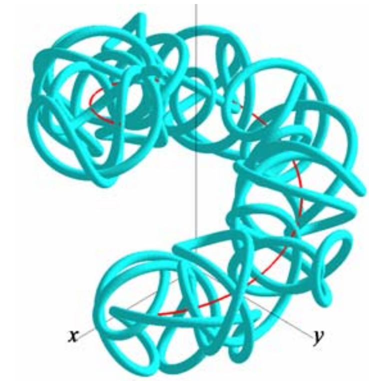

Fig. 33 : $o_3 = b, o_2 = n, o_1 = t$

c) Revolution about three skew axes

In Fig.34 are displayed surfaces created by revolution of the point $P = (0,0,0,1)$ about three skew axes $o_3 / o_2 / o_1$, which are parallel with edges of the trihedron of the helix s, $o_3 \parallel n$, $o_2 \parallel b$, $o_3 \parallel t$, where parameters are $m_1 = 4m, sg_1 = +1$, $m_2 = 2m_1, sg_2 = +1$, $m_3 = 6m_2, sg_1 = +1$, transformation matrices of three revolutions are

$$\mathbf{T}_3(m_3 v, sg_3) = \mathbf{T}(0,0,-d_3).\mathbf{T}_x(m_3 v, sg_3).\mathbf{T}(0,0,+d_3),\ \mathbf{T}_2(m_2 v, sg_2) = \mathbf{T}(-d_2,0,0).\mathbf{T}_y(m_2 v, sg_2).\mathbf{T}(+d_2,0,0),$$
$$\mathbf{T}_1(m_1 v, sg_1) = \mathbf{T}(0,-d_1,0).\mathbf{T}_z(m_1 v, sg_1).\mathbf{T}(0,+d_2,0).$$

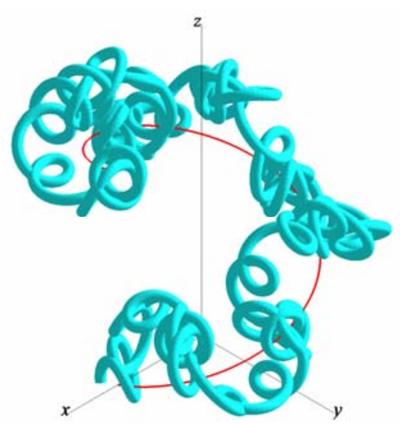

Fig. 34 : $o_3 \parallel n,\ o_2 \parallel b,\ o_3 \parallel t$

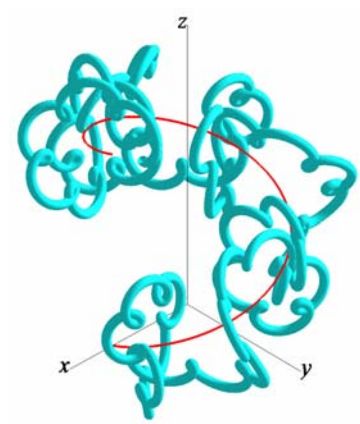

Fig. 35 : $o_3 \parallel n, o_2 = n, o_1 = b$

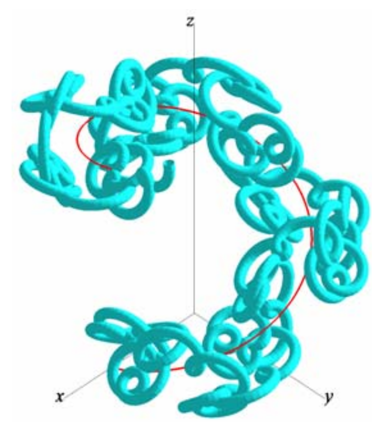

Fig. 36 : $o_3 \parallel n, o_2 = n, o_1 = t$

In Figs.35,36 are displayed surfaces created by revolution about the axes $o_3 \parallel n, o_2 = n, o_1 = b$ or axes $o_3 \parallel n, o_2 = n, o_1 = t$.

V. Conclusion

The described method of modeling of the helical-*n*-revolutional cyclical surfaces makes it possible to model different interest surfaces simply by changing the parameters.

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Estimation of Population Ratio in Simple Random Sampling using Variable Transformation

By Onyeka, A.C., Nlebedim, V.U. & Izunobi, C.H.
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Abstract - This paper proposes six new estimators of the population ratio (R) of the population means of two variables (y and x) in Simple Random Sampling (SRS) scheme, using a variable transformation of the auxiliary variable, x. Properties of the proposed estimators, including optimality conditions, are derived up to first order approximation, and conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) are also obtained. The results are supported with empirical illustrations, which show that some of the proposed estimators have relatively large gains in efficiency over the customary ratio estimator, $\hat{R}$ for the data set considered.

Keywords : *variable transformation, ratio, product and regression-type estimators, mean squared error.*

GJSFR-F Classification : *MSC 2010: 62D05*

ESTIMATION OF POPULATION RATIO IN SIMPLE RANDOM SAMPLING USINGVARIABLE TRANSFORMATION

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Abstract - This paper proposes six new estimators of the population ratio (R) of the population means of two variables (y and x) in Simple Random Sampling (SRS) scheme, using a variable transformation of the auxiliary variable, x. Properties of the proposed estimators, including optimality conditions, are derived up to first order approximation, and conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) are also obtained. The results are supported with empirical illustrations, which show that some of the proposed estimators have relatively large gains in efficiency over the customary ratio estimator, $\hat{R}$ for the data set considered.

Keywords : variable transformation, ratio, product and regression-type estimators, mean squared error.

I. Introduction

It is a common practice, in sample surveys, to use information obtained on an auxiliary variable to improve the efficiency of estimates of the population mean and total of the study variable. In some studies, however, the ratio of the population means (or totals) of the study and auxiliary variables might be of great significance, hence the need to estimate such ratios. For instance, one might be interested in the unemployment rate, which is usually obtained as the ratio of the number of unemployed people to the number of employed individuals in a country's labour force. Other parameters that could be obtained as a ratio of two parameters include income per capita, which is the ratio of the total income of a household to the total number of members of the household; the average salary of workers in a given establishment or company, which is usually obtained as the ratio of the total salary funds of the establishment to the company's total number of employees; and the employment sex ratio of a country, which is the ratio of the number of employed men and women in the country's labour force.

The usual or customary estimator of the population ratio, $R = \bar{Y}/\bar{X}$, of the population means of two variables, y and x, under the simple random sampling scheme, is given as $\hat{R} = \bar{y}/\bar{x}$, which is the ratio of the sample means of the two variables [Cochran(1977)]. However, several authors have contributed to the problem of estimating the population ratio of two means, often utilizing information on single or more auxiliary variables. These include Singh (1965), Srivastava (1967), Srivastava et al. (1988), Upadhyaya et al. (2000), Khare and Sinha (2007), and Khare et al. (2012). In using information on one or more auxiliary characters to

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estimate the population ratio, $R = \bar{Y}/\bar{X}$, the two variables, y and x, are considered as the study variables, while other variables, say $z_i\ (i = 1, 2, \cdots, k)$, are considered as auxiliary variables, known to have some strong correlation with the variables, y and x. This implies that after observing the variables, y and x, more funds would be required to obtain information on the auxiliary variables, z_i's. If the variable y is the study variable, as it is often the case in most practical surveys, then the variable x, together with the variables, z_i's, would all be considered as auxiliary variables, which require extra funds in order to be observed. In the present study, we restrict observations, and consequently, funding costs, to only two variables, y and x, taking the variable x as an auxiliary variable having strong correlation with the variable, y. The parameter of interest still remains the population ratio, $R = \bar{Y}/\bar{X}$, and the objective of the study is to estimate R using variable transformation of the (auxiliary) variable, x, on the assumption that the population mean ($\bar{X}$) of x is known.

Notes

II. The Proposed Estimators

Let $y_i(x_i)$ be observations on two variables, and let a random sample of size n be drawn from a population of N units using simple random sampling without replacement (SRSWOR) scheme. Consider the variable transformation,

$$x_i^* = \frac{N\bar{X} - nx_i}{N - n}, \quad i = 1, 2, \cdots, N, \tag{2.1}$$

The transformation (2.1) has been used by authors like Srivenkataramana (1980), Singh and Tailor (2005), Tailor and Sharma (2009), and Sharma and Tailor (2010) to improve estimates of the population mean, $\bar{Y}$, under the simple random sampling scheme, and Onyeka (2013) under the poststratified sampling scheme. The associated sample mean is obtained as:

$$\bar{x}^* = (1+\pi)\bar{X} - \pi\bar{x}, \qquad \pi = \frac{n}{N - n} \tag{2.2}$$

Using the (sample) means, $\bar{y}$, $\bar{x}$, and $\bar{x}^*$, and assuming knowledge of the population mean, $\bar{X}$, of the (auxiliary) variable, x, we propose the following six estimators of the population ratio, $R = \bar{Y}/\bar{X}$, in simple random sampling without replacement (SRSWOR) scheme:

$$\hat{R}_1 = \frac{\bar{y}}{\bar{x} - b(\bar{x}^* - \bar{X})} \quad \text{(regression-type estimator of sample mean, } \bar{x}) \tag{2.3}$$

$$\hat{R}_2 = \frac{\bar{y}}{\left(\frac{\bar{x}}{\bar{x}^*}\bar{X}\right)} = \frac{\bar{y}\bar{x}^*}{\bar{x}\bar{X}} \quad \text{(ratio-type estimator of sample mean, } \bar{x}) \tag{2.4}$$

$$\hat{R}_3 = \frac{\bar{y}}{\left(\frac{\bar{x}\bar{x}^*}{\bar{X}}\right)} = \frac{\bar{y}\bar{X}}{\bar{x}\bar{x}^*} \quad \text{(product-type estimator of sample mean, } \bar{x}) \tag{2.5}$$

$$\hat{R}_4 = \frac{\bar{y}}{\bar{x}^*} \quad \text{(transformed mean estimator, } \bar{x}^*) \tag{2.6}$$

$$\hat{R}_5 = \frac{\bar{y}}{\bar{x}^* - b(\bar{x} - \bar{X})} \quad \text{(regression-type estimator of transformed mean, } \bar{x}^*) \tag{2.7}$$

$$\hat{R}_6 = \frac{\bar{y}}{\left(\frac{\bar{x}^*}{\bar{x}}\bar{X}\right)} = \frac{\bar{y}\bar{x}}{\bar{x}^*\bar{X}} \quad \text{(ratio-type estimator of transformed mean, } \bar{x}^*\text{)} \tag{2.8}$$

where b is a suitable constant, often chosen to be very close to the population regression coefficient of y on x.

Notes

It is worth mentioning here, that Adewara et al. (2012) proposed some modified estimators of the population mean, $\bar{Y}$, involving the transformed (sample) means, $\bar{x}^*$ and $\bar{y}^*$, having the relationships:

$$\bar{X} = f\bar{x} + (1-f)\bar{x}^* \text{ and } \bar{Y} = f\bar{y} + (1-f)\bar{y}^* \quad , \; f = n/N \tag{2.9}$$

This is quite worrisome in view of the fact that the transformed mean, $\bar{y}^*$, is a function of the population mean, $\bar{Y}$, which is usually unknown. If the population mean ($\bar{Y}$) of the study variable, y, is already known, then there is no need constructing estimators to estimate what is already known. Consequently, there seems to be no justification for the use of the transformed mean, $\bar{y}^*$, in estimating the population mean, $\bar{Y}$, as well as the population ratio, $R = \bar{Y}/\bar{X}$. In estimating the population ratio, in particular, the much one could do is to assume that one of the population means, (say $\bar{X}$), is known, hence the justification for the use of the transformed mean, $\bar{x}^*$. To use both the transformed means, $\bar{x}^*$ and $\bar{y}^*$, in constructing estimators of the population ratio is to suggest or assume that both the population means, $\bar{X}$ and $\bar{Y}$, are already known, which implies that the population ratio is equally known and needs not to be estimated in the first place.

Let
$$e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}} \text{ and } e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}} . \tag{2.10}$$

Then,

$$E(e_0) = E(e_1) = 0 \tag{2.11}$$

$$E(e_0^2) = \frac{V(\bar{y})}{\bar{Y}^2} = \left(\frac{1-f}{n}\right)\frac{S_y^2}{\bar{Y}^2} \tag{2.12}$$

$$E(e_1^2) = \frac{V(\bar{x})}{\bar{X}^2} = \left(\frac{1-f}{n}\right)\frac{S_x^2}{\bar{X}^2} \tag{2.13}$$

and

$$E(e_0 e_1) = \frac{Cov(\bar{y}, \bar{x})}{\bar{Y}\bar{X}} = \left(\frac{1-f}{n}\right)\frac{S_{yx}}{\bar{Y}\bar{X}} \tag{2.14}$$

where $S_y^2(S_x^2)$ is the variance of $y(x)$ and S_{yx} is the covariance of y and x.

To obtain the properties of the proposed estimator, $\hat{R}_1$, we first rewrite (2.3) in terms of e_0 and e_1 and expand up to first order in expected values, to obtain:

$$(\hat{R}_1 - R) = R[e_0 - (1+\pi b)e_1 - (1+\pi b)e_0 e_1 + (1+\pi b)^2 e_1^2] \tag{2.15}$$

and

$$(\hat{R}_1 - R)^2 = R^2[e_0^2 + (1+\pi b)^2 e_1^2 - 2(1+\pi b)e_0 e_1] \tag{2.16}$$

Taking the expectations of (2.15) and (2.16), and using (2.11) – (2.14) to make the necessary substitutions, gives the bias and mean squared error of the proposed estimators, $\hat{R}_1$, up to first order approximation, respectively as:

$$B(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi b)\left[(1+\pi b)RS_x^2 - S_{yx}\right] \tag{2.17}$$

and

$$MSE(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi b)^2 R^2 S_x^2 - 2(1+\pi b)RS_{yx}\right] \tag{2.18}$$

Following similar procedure, we obtain the biases and mean squared errors of the six proposed estimators, together with those of the usual or customary ratio estimator, $\hat{R} = \bar{y}/\bar{x}$, up to first order approximations as:

$$B(\hat{R}) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[RS_x^2 - S_{yx}\right] \tag{2.19}$$

$$B(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi b)\left[(1+\pi b)RS_x^2 - S_{yx}\right] \tag{2.20}$$

$$B(\hat{R}_2) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi)\left[RS_x^2 - S_{yx}\right] \tag{2.21}$$

$$B(\hat{R}_3) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[(1-\pi+\pi^2)RS_x^2 - (1-\pi)S_{yx}\right] \tag{2.22}$$

$$B(\hat{R}_4) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\pi\left[\pi RS_x^2 + S_{yx}\right] \tag{2.23}$$

$$B(\hat{R}_5) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(\pi+b)\left[(\pi+b)RS_x^2 + S_{yx}\right] \tag{2.24}$$

$$B(\hat{R}_6) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1+\pi)\left[\pi RS_x^2 + S_{yx}\right] \tag{2.25}$$

and,

$$MSE(\hat{R}) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + R^2 S_x^2 - 2RS_{yx}\right] \tag{2.26}$$

$$MSE(\hat{R}_1) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi b)^2 R^2 S_x^2 - 2(1+\pi b)RS_{yx}\right] \tag{2.27}$$

$$MSE(\hat{R}_2) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1+\pi)^2 R^2 S_x^2 - 2(1+\pi)RS_{yx}\right] \tag{2.28}$$

$$MSE(\hat{R}_3) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1-\pi)^2 R^2 S_x^2 - 2(1-\pi)RS_{yx}\right] \tag{2.29}$$

$$MSE(\hat{R}_4) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + \pi^2 R^2 S_x^2 + 2\pi RS_{yx}\right] \tag{2.30}$$

Notes

$$\operatorname{MSE}(\hat{R}_5) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (\pi + b)^2 R^2 S_x^2 + 2(\pi + b) R S_{yx}\right] \tag{2.31}$$

$$\operatorname{MSE}(\hat{R}_6) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + (1 + \pi)^2 R^2 S_x^2 + 2(1 + b) R S_{yx}\right] \tag{2.32}$$

Notes

Generally, the mean squared errors of the estimators could be written as:

$$\operatorname{MSE}(\hat{R}_d) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)\left[S_y^2 + \theta_d^2 R^2 S_x^2 - 2\theta_d R S_{yx}\right] \tag{2.33}$$

where

$$\theta_1 = (1 + \pi b),\ \theta_2 = (1 + \pi),\ \theta_3 = (1 - \pi),\ \theta_4 = (-\pi),\ \theta_5 = [-(\pi + b)],\ \theta_6 = [-(1 + \pi)] \tag{2.34}$$

III. Efficiency Comparison

The efficiencies of the six proposed estimators, $\hat{R}_d$, d = 1, 2, …, 6 are first compared to that of the customary ratio estimator, $\hat{R}$, in estimating the population ratio, R, of two population means in simple random sampling scheme. The performance of the proposed estimators among themselves is also considered. Further consideration is also given to the optimum or best estimators among the proposed set of estimators.

a) Efficiency of Proposed Estimators over the Customary Ratio Estimator, $\hat{R}$

Using (2.26) and (2.33), the proposed estimators, $\hat{R}_d$, d = 1, 2, …, 6 would perform better than the customary ratio estimator, $\hat{R}$, in terms of having a smaller mean squared error if:

$$\left.\begin{array}{ll} (1) & \theta_d < 1 \text{ and } B < R \\ \text{or} & \\ (2) & \theta_d > 1 \text{ and } B > R \end{array}\right\} \tag{3.1}$$

where θ_d is as given in (2.34), and $B = S_{yx}/S_x^2$ is the population regression coefficient of y on x. This shows that the proposed estimators are not always more efficient than the customary ratio estimator, $\hat{R}$. For instance, the proposed estimator, $\hat{R}_2$, in (2.4) would only be more efficient than the estimator, $\hat{R}$, if and only if B > R, since, from (2.2) and (2.34), the quantity, $\theta_2 = 1 + \pi$ is greater than unity. This means that the customary ratio estimator, $\hat{R}$, would be more efficient than the proposed estimator, $\hat{R}_2$ for data sets in which the value of the population regression coefficient, B, is smaller than the value of the population ratio, R. However, it would be shown later in this study that the proposed estimators, under certain general optimality conditions, always perform better than the customary ratio estimator, $\hat{R}$.

b) Efficiency Comparison among the Proposed Estimators

Let $\hat{R}_j$ and $\hat{R}_k$, $j \neq k$, and $j, k = 1, 2, \cdots, 6$ be any two particular estimators from the six proposed estimators in (2.3) to (2.8). Then using (2.33), the estimator, $\hat{R}_j$ would be more efficient than the estimator, $\hat{R}_k$, in terms of having a smaller mean squared error, if:

$$\text{or} \qquad \left.\begin{array}{ll} (1) & \theta_j < \theta_k \ \text{ and } \ B < R\theta_k \\ & \\ (2) & \theta_j > \theta_k \ \text{ and } \ B > R\theta_k \end{array}\right\} \tag{3.2}$$

where θ_j and θ_k are obtained from θ_d, as given in (2.34). For instance, in comparing the proposed estimators, $\hat{R}_2$ and $\hat{R}_3$, we observe, from (2.34) that $\theta_2 = 1+\pi$ and $\theta_3 = 1-\pi$, indicating that the quantity, θ_2, is greater than θ_3, since $\pi = \frac{n}{N-n}$ is always positive. Consequently, and by using condition (2) of (3.2), the proposed estimator, $\hat{R}_2$ would be more efficient than the proposed estimator, $\hat{R}_3$ if and only if, $B > R(1-\pi)$. Comparison of the efficiencies of the remaining proposed estimators could be carry out in a similar manner, using the efficiency conditions in (3.2).

c) Optimum Estimators

The optimum estimators, among the six proposed estimators, could be obtained by minimizing the mean squared error of the proposed estimators, $\hat{R}_d$, in (2.33) with respect to the quantity, θ_d defined in (2.34). Applying the least square method, this gives the optimum value of θ_d, say θ_d^0, as

$$\theta_d^0 = \frac{B}{R} \tag{3.4}$$

with the associated minimum mean squared error of $\hat{R}_d$ obtained as:

$$MSE_{opt}(\hat{R}_d) = \frac{1}{\bar{X}^2}\left(\frac{1-f}{n}\right)(1-\rho^2)S_y^2 \tag{3.5}$$

where $\rho = S_{yx}/(S_y S_x)$ is the population correlation coefficient of the variables y and x.

Notice, from (2.3) and (2.7), that the proposed estimators, $\hat{R}_1$ and $\hat{R}_5$, particularly give us the opportunity to choose suitable values of the constant, b. This means that with both estimators, it is possible and a lot easier to meet the optimality condition (3.4) by making appropriate choice of the constant, b. Using (3.4) and (2.34), it follows that the proposed estimator, $\hat{R}_1$ would be an optimum estimator if we choose the value of b, say b_1^0, as:

$$b_1^0 = \frac{B-R}{\pi R} \tag{3.6}$$

Similarly, the proposed estimator, $\hat{R}_5$ would be an optimum estimator if we choose the value of b, say b_5^0, as:

$$b_5^0 = -\left(\frac{B+\pi R}{R}\right) \tag{3.7}$$

The associated minimum mean squared errors of the estimators, $\hat{R}_1$ and $\hat{R}_5$ are the same as already given in (3.5). Comparing (3.5) and (2.26), we obtain the difference between the

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mean squared error of the optimum estimators and that of the customary ratio estimator, $\hat{R}$, as:

$$\Delta = MSE(\hat{R}) - MSE_{opt}(\hat{R}_d) = \frac{1}{\overline{X}^2}\left(\frac{1-f}{n}\right)(\rho S_y - RS_x)^2, \tag{3.8}$$

which is always greater than zero. Hence the optimum estimators, using the optimality condition (3.4), are always more efficient than the customary ratio estimator, $\hat{R}$, for the purpose of estimating the population ratio, R, of two population means, under the simple random sampling scheme.

IV. Numerical Illustration

The theoretical results obtained in the present study are illustrated here numerically, using the data given on page 171 of Johnston (1982). The data set is summarized as follows:

y = Percentage of hives affected by disease

x = Date of flowering of a particular summer species (number of days from January 1)

$N = 10$, $n = 4$, $\overline{Y} = 52$, $\overline{X} = 200$, $S_y^2 = 65.97338$, $S_x^2 = 84.01556$, $S_{yx} = -69.98292$

We assume, for illustration purposes, that we are interested in the ratio of the percentage of hives affected by the disease to the number of days of flowering. That is, $R = \overline{Y}/\overline{X}$. Then, the computed percentage relative efficiencies (PRE) of the six proposed estimators, $\hat{R}_d$, $d = 1, 2, \cdots, 6$, and the optimum estimators, $\hat{R}_d^0$, over the customary ratio estimator, $\hat{R} = \overline{y}/\overline{x}$ are displayed in Table 1.

Table 1 shows that apart from the estimator, $\hat{R}_2$, the remaining five proposed estimators are more efficient than the customary ratio estimator, $\hat{R}$, for the data under consideration, and the gains in efficiency of some of the estimators, like $\hat{R}_6$ and $\hat{R}_4$, are relatively large. Notice that the values of B and R are respectively obtained as $B = -0.83298$ and $R = 0.26$, showing that B is smaller than R. That is, $B < R$. Consequently, and using the efficiency condition (1) of (3.1), the proposed estimators would be more efficient than $\hat{R}$ only if the value of the associated θ_d is less than unity. Table 1 confirms this efficiency condition, since all the estimators whose values of θ_d are less than unity are found to perform better than the estimator, $\hat{R}$.

Table 1 : PRE of Proposed Estimators over the estimator, R

d	Estimator	π	$b = B$	θ_d	MSE	PRE
-	$\hat{R}$	0.6667		1	0.0004052	100
1	$\hat{R}_1$	0.6667	-0.83298	0.44465	0.0003123	130
2	$\hat{R}_2$	0.6667		1.66670	0.0005340	76
3	$\hat{R}_3$	0.6667		0.33330	0.0002953	137
4	$\hat{R}_4$	0.6667		-0.66670	0.0001659	244
5	$\hat{R}_5$	0.6667	-0.83298	0.16628	0.0002707	150
6	$\hat{R}_6$	0.6667		-1.66670	0.0000791	512
-	$\hat{R}_d^0$	**0.6667**		**-3.20377**	**0.0000288**	**1407**

Also to be observed from Table 1 is the fact that the first three proposed estimators, $\hat{R}_1$, $\hat{R}_2$ and $\hat{R}_3$ have smaller gains in efficiency than the last three proposed estimators, $\hat{R}_4$, $\hat{R}_5$ and $\hat{R}_6$. Notice, from (2.3) to (2.8), that while the first three proposed estimators have the sample mean, $\bar{x}$, as the lead statistic in the denominator, the transformed mean, $\bar{x}^*$, is the lead statistic in the denominator of the last three proposed estimators. Consequently, Table 1 suggests that estimators with the transformed mean, $\bar{x}^*$, as the lead statistic in the denominator are likely to be more efficient than those with the sample mean, $\bar{x}$, as the lead statistic in the denominator, when there is a strong negative correlation between the two variables, like we presently have in the data under consideration. The optimum estimators, as expected, are the most efficient estimators, in terms of having the smallest mean squared error when compared with the customary ratio estimator as well as all the proposed estimators.

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V. Concluding Remarks

Here, we have proposed and considered six new estimators of the population ratio (R) of two population means in SRSWOR scheme, using a variable transformation of the auxiliary variable, x. The biases and mean squared errors of the proposed estimators were obtained up to first order approximation. Conditions under which the proposed estimators perform better than the customary ratio estimator ($\hat{R} = \bar{y}/\bar{x}$) were derived. Also obtained were the optimality conditions under which some of the proposed estimators could become the best (optimum) estimators. The results of the study were supported and illustrated numerically. The empirical illustration confirmed, among other things, both the optimality and efficiency conditions, which we had earlier obtained theoretically in the study. The empirical study revealed that relatively large gains in efficiency over the customary ratio estimator could be obtained by using some of the new estimators proposed in the present study. Again, the direction (positive or negative) of the linear relationship between the two variables plays a role in identifying some of the proposed estimators that are likely to be more efficient than the others, for a given set of data. The first three proposed estimators make use of the sample mean, $\bar{x}$ as the lead statistic in the denominator, and are likely to perform better than the last three proposed estimators, when there is a strong positive linear relationship between the two variables. When there is a strong negative correlation between the variables, the last three proposed estimators, which incidentally make use of the transformed sample mean, $\bar{x}^*$, as the lead statistic in the denominator are likely to be more efficient than the first three proposed estimators. However, the best estimators to use for any given set of data could be obtained by using the optimality conditions given in (3.4).

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On Certain Summation Formulae Involving Gauss Theorem

By Salahuddin & Intazar Husain
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Abstract - The main object of present paper is to obtain some summation formuale involving Contiguous relation, Recurrence relation, Gauss second summation theorem and Legendre duplication formula.

Keywords : contiguous relation, recurrence relation, gauss second summation theorem, legendre duplication formula.

GJSFR-F Classification : MSC 2010: 34M30, 11T24

ON CERTAIN SUMMATION FORMULAE INVOLVING GAUSS THEOREM

Strictly as per the compliance and regulations of :

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On Certain Summation Formulae Involving Gauss Theorem

Salahuddin [α] & Intazar Husain [σ]

Abstract - The main object of present paper is to obtain some summation formuale involving Contiguous relation, Recurrence relation, Gauss second summation theorem and Legendre duplication formula.

Keywords and Phrases : contiguous relation, recurrence relation, gauss second summation theorem, legendre duplication formula.

I. Introduction

The Pochhammer's symbol is defined by

$$(\alpha,\ k) = (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 & ; \text{ if } k=0 \\ k! & ; \text{ if } \alpha = 1 \end{cases} \tag{1}$$

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ & & z \\ b_1, b_2, \cdots, b_B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$$ {}_AF_B\left[\begin{array}{lll} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lll} (a_j)_{j=1}^A & ; & \\ & & z \\ (b_j)_{j=1}^B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{2}$$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation is defined by

[Andrews p.367(8), E. D. p.52(19)]

$$c\ (1-z)\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ c\ \ ; & z \end{array}\right] = c\ {}_2F_1\left[\begin{array}{ll} a-1,\ b\ ; & \\ c\ \ ; & z \end{array}\right] - (c-b)\ z\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ c+1\ ; & z \end{array}\right] \tag{3}$$

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Recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z) \tag{4}$$

Legendre's duplication formula

$$\sqrt{\pi}\;\Gamma(2z) = 2^{(2z-1)}\;\Gamma(z)\;\Gamma\left(z+\frac{1}{2}\right) \tag{5}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\;\Gamma(\frac{b}{2})\;\Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{6}$$

$$= \frac{2^{(a-1)}\;\Gamma(\frac{a}{2})\;\Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{7}$$

Gauss second summation theorem [Prudnikov., 491(7.3.7.8)]

$${}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b+1}{2} & ; \end{array}\ \frac{1}{2}\right] = \frac{\Gamma(\frac{a+b+1}{2})\;\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\;\Gamma(\frac{b+1}{2})} \tag{8}$$

$$= \frac{2^{(b-1)}\;\Gamma(\frac{b}{2})\;\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\;\Gamma(\frac{a+1}{2})} \tag{9}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$${}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b-1}{2} & ; \end{array}\ \frac{1}{2}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\;\Gamma(\frac{b+1}{2})} + \frac{2\;\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\;\Gamma(b)}\right] \tag{10}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$${}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b-1}{2} & ; \end{array}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\;\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\;\Gamma(\frac{a}{2})\;\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{11}$$

II. Main Results of Summation Formulae

For $a < 1$ and $a > 9$

$${}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b-9}{2} & ; \end{array}\ \frac{1}{2}\right] = \frac{2^{(b-1)}\;\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\left\{\frac{(128a+384b-1152)}{(a-9)} - \right.\right.$$

$$- \frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)} + \frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)} -$$

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$$-\frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-$$

$$-\frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)}+\frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg] \tag{12}$$

For $a<1$ and $a>10$

$${}_2F_1\left[\begin{matrix} a, & b & ; & \frac{1}{2} \\ \frac{a+b-10}{2} & ; & & 2\end{matrix}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-10}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-10}{2})}\Bigg\{\frac{(256a+768b-2560)}{(a-10)}-$$

$$-\frac{(a-b-10)(576a+1472b-4608)}{(a-10)(a-8)}+\frac{(a-b-10)(a-b-8)(432a+912b-2592)}{(a-10)(a-8)(a-6)}-$$

$$-\frac{(a-b-10)(a-b-8)(a-b-6)(120a+200b-480)}{(a-10)(a-8)(a-6)(a-4)}+$$

$$+\frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(9a+11b-18)}{(a-10)(a-8)(a-6)(a-4)(a-2)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(512a+512b-4096)}{(a-9)}-\frac{(a-b-10)(960a+832b-5888)}{(a-9)(a-7)}+$$

$$+\frac{(a-b-10)(a-b-8)(560a+400b-2400)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-10)(a-b-8)(a-b-6)(104a+56b-256)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-10)(a-b-8)(a-b-6)(a-b-4)(3a+b-2)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}\Bigg] \tag{13}$$

For $a<1$ and $a>11$

$${}_2F_1\left[\begin{matrix} a, & b & ; & \frac{1}{2} \\ \frac{a+b-11}{2} & ; & & 2\end{matrix}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-11}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}\Bigg\{\frac{(512a+1536b-5632)}{(a-11)}-$$

$$-\frac{(a-b-11)(1280a+3328b-11520)}{(a-11)(a-9)}+\frac{(a-b-11)(a-b-9)(1120a+2464b-7840)}{(a-11)(a-9)(a-7)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(400a+720b-2000)}{(a-11)(a-9)(a-7)(a-5)}+$$

$$+\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(50a+70b-150)}{(a-11)(a-9)(a-7)(a-5)(a-3)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-10}{2})}\Bigg\{\frac{(1024a+1024b-9216)}{(a-10)}-\frac{(a-b-11)(2176a+1920b-15488)}{(a-10)(a-8)}+$$

$$+\frac{(a-b-11)(a-b-9)(1536a+1152b-8064)}{(a-10)(a-8)(a-6)}-$$

$$-\frac{(a-b-11)(a-b-9)(a-b-7)(400a+240b-1360)}{(a-10)(a-8)(a-6)(a-4)}+$$

$$+\frac{(a-b-11)(a-b-9)(a-b-7)(a-b-5)(28a+12b-44)}{(a-10)(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg] \qquad (14)$$

The above summation theorems can be easily verified by using computer algebra system programming languages, like Maple, Matlab, or Mathematica.

III. Derivations of Summation Formulae (12) to (14)

Derivation of (12): Substituting $c=\frac{a+b-9}{2}$ and $z=\frac{1}{2}$ in equation (3), we get

$$\left(\frac{a+b-9}{4}\right){}_2F_1\left[\begin{matrix}a,\ b & ; & \frac{1}{2}\\ \frac{a+b-9}{2} & ; & \end{matrix}\right]=\left(\frac{a+b-9}{2}\right){}_2F_1\left[\begin{matrix}a-1,\ b & ; & \frac{1}{2}\\ \frac{a+b-9}{2} & ; & \end{matrix}\right]-$$

$$-\left(\frac{a-b-9}{4}\right){}_2F_1\left[\begin{matrix}a,\ b & ; & \frac{1}{2}\\ \frac{a+b-7}{2} & ; & \end{matrix}\right]$$

$${}_2F_1\left[\begin{matrix}a,\ b & ; & \frac{1}{2}\\ \frac{a+b-9}{2} & ; & \end{matrix}\right]=2.\ {}_2F_1\left[\begin{matrix}a-1,\ b & ; & \frac{1}{2}\\ \frac{a+b-9}{2} & ; & \end{matrix}\right]-\left(\frac{a-b-9}{a+b-9}\right){}_2F_1\left[\begin{matrix}a,\ b & ; & \frac{1}{2}\\ \frac{a+b-7}{2} & ; & \end{matrix}\right]$$

Now involving the derived from Asish et all , we get

$${}_2F_1\left[\begin{matrix}a,\ b & ; & \frac{1}{2}\\ \frac{a+b-9}{2} & ; & \end{matrix}\right]=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(128a+384b-1152)}{(a-9)}-$$

$$-\frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)}+\frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)}\Bigg\}+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-$$

$$-\frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)}+\frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)}-$$

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$$-\frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]-$$

$$-\left(\frac{a-b-9}{a+b-9}\right)\frac{2^{(b-1)}\,\Gamma(\frac{a+b-7}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-7}{2})}\left\{\frac{(32a+96b-224)}{(a-7)}-\frac{(a-b-7)(48a+112b-240)}{(a-7)(a-5)}+\right.\right.$$

Notes

$$+\frac{(a-b-7)(a-b-5)(18a+30b-54)}{(a-7)(a-5)(a-3)}-\frac{(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-6}{2})}\left\{\frac{(64a+64b-320)}{(a-6)}-\frac{(a-b-7)(72a+56b-232)}{(a-6)(a-4)}+\right.$$

$$\left.\left.+\frac{(a-b-7)(a-b-5)(16a+8b-24)}{(a-6)(a-4)(a-2)}\right\}\right]$$

$$=\frac{2^{(b-1)}\,\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\left\{\frac{(128a+384b-1152)}{(a-9)}-\frac{(a-b-9)(224a+544b-1568)}{(a-9)(a-7)}+\right.\right.$$

$$+\frac{(a-b-9)(a-b-7)(112a+208b-560)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(14a+18b-42)}{(a-9)(a-7)(a-5)(a-3)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\left\{\frac{(256a+256b-1792)}{(a-8)}-\frac{(a-b-9)(352a+288b-1824)}{(a-8)(a-6)}+\right.$$

$$+\frac{(a-b-9)(a-b-7)(120a+72b-408)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(6a+2b-10)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]-$$

$$-\frac{2^{(b-1)}\,\Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\left\{\frac{(a-b-9)(32a+96b-224)}{(a-9)(a-7)}-\right.\right.$$

$$-\frac{(a-b-9)(a-b-7)(48a+112b-240)}{(a-9)(a-7)(a-5)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(18a+30b-54)}{(a-9)(a-7)(a-5)(a-3)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\left\{\frac{(a-b-9)(64a+64b-320)}{(a-8)(a-6)}-\frac{(a-b-9)(a-b-7)(72a+56b-232)}{(a-8)(a-6)(a-4)}+\right.$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(16a+8b-24)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

$$=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-9}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-9}{2})}\Bigg\{\frac{(128a+384b-1152)}{(a-9)}-$$

$$-\frac{(a-b-9)(256a+640b-1792)}{(a-9)(a-7)}+\frac{(a-b-9)(a-b-7)(160a+320b-800)}{(a-9)(a-7)(a-5)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(32a+48b-96)}{(a-9)(a-7)(a-5)(a-3)}+$$

$$+\frac{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a+b-1)}{(a-9)(a-7)(a-5)(a-3)(a-1)}\Bigg\}+$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-8}{2})}\Bigg\{\frac{(256a+256b-1792)}{(a-8)}-\frac{(a-b-9)(416a+352b-2144)}{(a-8)(a-6)}+$$

$$+\frac{(a-b-9)(a-b-7)(192a+128b-640)}{(a-8)(a-6)(a-4)}-$$

$$-\frac{(a-b-9)(a-b-7)(a-b-5)(22a+10b-34)}{(a-8)(a-6)(a-4)(a-2)}\Bigg\}\Bigg]$$

Thus , we prove the result (12)

Similarly, we can prove the other results.

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Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

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GJSFR-F Classification : MSC 2010: 20F67, 90C34

SEMI-INVARIANT SUBMANIFOLDS OF NEARLY HYPERBOLIC COSYMPLECTIC MANIFOLD

Semi-Invariant Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

Mobin Ahmad [α] & Kashif Ali [σ]

Abstract - We consider a nearly hyperbolic cosymplectic manifold and study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. We also study parallel distributions on nearly hyperbolic cosymplectic manifold and find the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold.

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I. Introduction

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was initiated by A. Bejancu in [7]. A semi-invariant submanifold is the extension of the concept of a CR-submanifold of a Kaehler manifold to submanifolds of almost contact manifolds. The study of Semi-invariant submanifolds of Sasakian manifolds was initiated by Bejancu-Papaghuic in [9]. The same concept was studied under the name contact CR-submanifold by Yano-Kon in [16] and K. Matsumoto in [14]. The study of semi-invariant submanifolds in almost contact manifold was enriched by several geometers (see, [1], [2], [3], [4], [5], [6], [11], [14]). On the other hand, almost hyperbolic (f,g,η,ξ)-structure was defined and studied by Upadhyay and Dube in [15]. Joshi and Dube studied semi-invariant submanifolds of an almost r-contact hyperbolic metric manifolds in [12]. In this paper, we study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure-(ϕ,ξ,η,g), where a tensor ϕ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the followings

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \eta(\xi) = -1, \tag{2.2}$$

$$\phi(\xi) = 0, \ \ \eta o \phi = 0, \tag{2.3}$$

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$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

for any X, Y tangent to $\bar{M}$ [15]. In this case

$$g(\phi X, Y) = -g(\phi Y, X). \tag{2.5}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure-(ϕ, ξ, η, g) is said to be nearly hyperbolic cosymplectic manifold [10] if

$$(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y X) = 0, \tag{2.6}$$

$$\bar{\nabla}_X \xi = 0 \tag{2.7}$$

for all X, Y tangent to $\bar{M}$.

The Nijenhuis tensor $N(X, Y)$ of a nearly hyperbolic cosymplectic manifold $\bar{M}$ is defined as

$$N(X, Y) = (\bar{\nabla}_{\phi X} \phi)Y - (\bar{\nabla}_{\phi Y} \phi)X - \phi(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_Y \phi)X \tag{2.8}$$

$for\ any\ X, Y \in T\bar{M}$.

From (2.6), we have

$$(\bar{\nabla}_{\phi X} \phi)Y = -(\bar{\nabla}_Y \phi)\phi X. \tag{2.9}$$

Also, we have

$$(\bar{\nabla}_Y \phi)\phi X = (\bar{\nabla}_Y \eta)(X)\xi + \phi(\bar{\nabla}_Y \phi)X. \tag{2.10}$$

From (2.9) and (2.10), we get

$$(\bar{\nabla}_{\phi X} \phi)Y = (\bar{\nabla}_Y \eta)(X)\xi + \phi(\bar{\nabla}_Y \phi)X. \tag{2.11}$$

Using (2.11) in (2.8), we get

$$N(X, Y) = 4\phi(\bar{\nabla}_Y \phi)X + 2g(\phi X, Y)\xi \tag{2.12}$$

$for\ X, Y \in TM$.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold. In section 3, we study some properties semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold. In section 4, we discuss the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold. In section 5, we study parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold.

III. Semi-Invariant Submanifolds

Let M be a submanifold immersed in $\bar{M}$. We assume that the vector field ξ is tangent to M. Denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M. Then M is called a semi-invariant submanifold [8] of $\bar{M}$ if there exist two differentiable distributions D and $D^\perp$ on M satisfying.

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D, $D^\perp$ and $\{\xi\}$ are mutually orthogonal to each other.

(ii) The distribution D is invariant by ϕ, that is, $\phi D_X = D_X$ for each $X \epsilon M$,

(iii) The distribution $D^\perp$ is anti-invariant by ϕ,that is, $\phi D_X^\perp \subset T_X M^\perp$ for each $X \epsilon M$,

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where TM and $T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M respectively. Let the Riemannian metric induced on M is denoted by the same symbol g and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{3.1}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{3.2}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M, h$ is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X,Y), N) \tag{3.3}$$

$for\ any\ x \in M\ and\ X \in T_x M$. We write

$$X = PX + QX, \tag{3.4}$$

where $PX \in D$ and $QX \in D^{\perp}$.
Similarly, for N normal to M we have

$$\phi N = BN + CN, \tag{3.5}$$

where $BN(resp. CN)$ is the tangential component $(resp. normal\ component)$ of $\emptyset N$.

Lemma 3.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$, then

$$2(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]$$

$for\ all\ X, Y \in D$.

Proof. By Gauss formula (3.1), we have

$$\overline{\nabla}_X Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]. \tag{3.6}$$

Also, by covariant differentiation we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X + \phi[X,Y]. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$(\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]. \tag{3.8}$$

Adding (2.6) and (3.8), we get

$$2(\overline{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]$$

$for\ all\ X, Y \in D$.

Hence Lemma is proved.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$, then

$$2(\overline{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X - \phi[X,Y] \quad (3.9)$$

$for\ all\ X,Y \in D^{\perp}$.

Proof. By Weingarten formula (3.2), we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X$$

$for\ any\ X,Y \in D^{\perp}$.

Also, by covariant differentiation, we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y]. \quad (3.10)$$

From (3.9) and (3.10), we get

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X - \phi[X,Y]. \quad (3.11)$$

Adding (2.6) and (311), we get

$$2(\overline{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X - \phi[X,Y]$$

$for\ all\ X,Y \in D^{\perp}$.

Hence Lemma is proved.

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$. Then

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

$for\ all\ X \in D\ and\ Y \in D^{\perp}$.

Proof. Using Gauss and Weingarten formulas, we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X). \quad (3.12)$$

Also, by covariant differentiation, we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y]. \quad (3.13)$$

From (3.12) and (3.13), we get

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]. \quad (3.14)$$

Adding (2.6) and (3.14), we obtain

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

$for\ all\ X \in D\ and\ Y \in D^{\perp}$.

Hence Lemma is proved.

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Lemma 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y = \phi P(\nabla_X Y) + \phi P(\nabla_Y X), \tag{3.15}$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y = 2Bh(X,Y), \tag{3.16}$$

$$h(Y,\phi PX) + h(X,\phi PY) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX = -2\eta(Y)\phi QX - 2\eta(X)\phi QY + \phi Q(\nabla_X Y)$$

$$+\phi Q(\nabla_Y X) + 2Ch(X,Y), \tag{3.17}$$

$$\eta(\nabla_X \phi PY) + \eta(\nabla_Y \phi PX) - \eta\big(A_{\phi QY}X\big) - \eta\big(A_{\phi QX}Y\big) = 0 \tag{3.18}$$

$for\ any\ X, Y \in T\bar{M}.$

Proof. Differentiating (3.4) covariantly and using (3.1) and (3.2), we get

$$\bar{\nabla}_Y \phi X = (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(X,Y).$$

Also,

$$\bar{\nabla}_Y \phi X = P\nabla_X \phi PY + Q\nabla_Y \phi PX + \eta(\nabla_Y \phi PX)\xi + h(Y,\phi PX) + \nabla_{\phi QX}^\perp Y$$

$$-PA_{\phi QX}Y - QA_{\phi QX}Y - \eta\big(A_{\phi QX}Y\big)\xi.$$

Thus, we have

$$(\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(X,Y) = P\nabla_X \phi PY + Q\nabla_Y \phi PX + \eta(\nabla_Y \phi PX)\xi$$

$$+h(Y,\phi PX) + \nabla_{\phi QX}^\perp Y - PA_{\phi QX}Y - QA_{\phi QX}Y - \eta\big(A_{\phi QX}Y\big)\xi. \tag{3.19}$$

Interchanging $X\ and\ Y$, we have

$$(\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(Y,X) = P\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_X \phi PY)\xi$$

$$+h(X,\phi PY) + \nabla_{\phi QY}^\perp X - PA_{\phi QY}X - QA_{\phi QY}X - \eta\big(A_{\phi QY}X\big)\xi. \tag{3.20}$$

Adding (3.19) and (3.20), we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi(\nabla_X Y) + 2\phi h(X,Y) = P\nabla_X \phi PY + P\nabla_Y \phi PX$$

$$+Q\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_Y \phi PX)\xi + \eta(\nabla_X \phi PY)\xi + h(Y,\phi PX) + h(X,\phi PY)$$

$$+\nabla_{\phi QX}^\perp Y + \nabla_{\phi QY}^\perp X - PA_{\phi QX}Y - PA_{\phi QY}X - QA_{\phi QX}Y - QA_{\phi QY}X - \eta\big(A_{\phi QX}Y\big)\xi$$

$$-\eta\big(A_{\phi QY}X\big)\xi. \tag{3.21}$$

By virtue of (2.6) and (3.21), we obtain

$$\phi P(\nabla_Y X) + \phi P(\nabla_X Y) + \phi Q\nabla_Y X + \phi Q\nabla_X Y + 2Bh(X,Y) + 2Ch(X,Y) = P\nabla_Y \phi PX$$

$$+P\nabla_X \phi PY + Q\nabla_Y \phi PX + Q\nabla_X \phi PY + \eta(\nabla_Y \phi PX)\xi + \eta(\nabla_X \phi PY)\xi + h(Y,\phi PX)$$

$$+h(X,\phi PY) + \nabla^{\perp}_{\phi QX} Y + \nabla^{\perp}_{\phi QY} X - PA_{\phi QX} Y - PA_{\phi QY} X - QA_{\phi QX} Y - QA_{\phi QY} X$$

$$-\eta\left(A_{\phi QX} Y\right)\xi - \eta\left(A_{\phi QY} X\right)\xi.$$

Comparing horizontal, vertical and normal components we get the desired result.
Hence the Lemma is proved.

Definition 3.5. The horizontal distribution D is said to be parallel [10] on $M if \nabla_X Y \in D$ for all vector field $X, Y \in D$.

Theorem 3.6. Let M be a semi-invariant submanifoldof a nearly hyperbolic cosymplectic manifold $\bar{M}$. If the horizontal distribution D is parallel, then

$$h(X,\phi Y) = h(Y,\phi X)$$

$for\ all\ X, Y \in D.$

Proof. Let $X, Y \in D$ and D is parallel then $\nabla_X \phi Y \in D\ and\ \nabla_Y \phi X \in D$. From (3.12), we have

$$h(Y,\phi X) + h(X,\phi Y) = 2\phi h(X,Y). \tag{3.22}$$

Replacing X by ϕX in (3.22) and using (2.1), we get

$$h(Y,X) + h(\phi X,\phi Y) = 2\phi h(\phi X,Y). \tag{3.23}$$

Again replacing $Y\ by\ \phi Y$ in (3.22) and using (2.1), we get

$$h(\phi Y,\phi X) + h(X,Y) = 2\phi h(X,\phi Y). \tag{3.24}$$

By virtue of (3.23) and (3.24), we have

$$\phi h(\phi X,Y) = \phi h(X,\phi Y). \tag{3.25}$$

Operating ϕ on both sides of (3.25), we get

$$h(\phi X,Y) = h(X,\phi Y).$$

Hence the theorem is proved.

Definition 3.7. A semi-invariant submanifold is said to be mixed totally geodesic [8] if $h(X,Y) = 0\ \ for\ all\ X \in D\ and\ \ Y \in D^{\perp}$.

Theorem 3.8. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then M is a mixed totally geodesic if and only if $A_{NX} \in D\ for\ all\ X \in D$.

Proof. Let $A_{NX} \in D\ for\ all\ X \in D$.

Now, $g(h(X,Y),N) = g(A_{NX},Y) = 0\ \ for\ Y \in D^{\perp}$, which is equivalent to $h(X,Y) = 0$.
Hence M is totally mixed geodesic.

Conversely, Let M is totally mixed geodesic, that is $h(X,Y) = 0\ for\ X \in D\ and\ \ Y \in D^{\perp}$.

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Now, $g(h(X,Y),N) = g(A_{NX},Y)$ gives that $g(A_{NX},Y) = 0$. Consequently, we have $A_{NX} \in D$ for all $Y \in D^{\perp}$.
Hence the theorem is proved.

IV. Integrability Conditions for Distributions

Notes

Theorem 4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then the distribution $D\oplus\langle\xi\rangle$ is integrable if

$$h(\phi X, Z) + h(\phi Z, X) = 0 \tag{4.1}$$

$for\ any\ X,Y,Z \in D\oplus\langle\xi\rangle.$

Proof. The torsion tensor $S(X,Y)$ of an almost contact structure (ϕ,ξ,η,g)
Using Gauss formula in (2.12), we have

$$N(X,Y) = 4\phi\big(\nabla_Y \emptyset X - \phi\nabla_Y X - \phi h(Y,X) + h(Y,\phi X)\big). \tag{4.2}$$

If $D\oplus\langle\xi\rangle$ is integrable, then $N(X,Y) = 0$. Hence

$$4\phi\nabla_Y\phi X - 4\nabla_Y X - 4\eta(\nabla_Y X)\xi - 4h(Y,X) + 4\phi h(Y,\phi X) = 0. \tag{4.3}$$

Comparing normal parts from both sides of (4.3), we get

$$\phi Q(\nabla_Y\phi X) - h(Y,X) = Ch(Y,\phi X) = 0. \tag{4.4}$$

Replacing $Y\ by\ \phi Z, Z \in D$ in (4.4), we have

$$\phi Q\big(\nabla_{\phi z}\phi X\big) - h(\phi Z, X) + Ch(\phi Y,\phi X) = 0. \tag{4.5}$$

Interchanging $X\ and\ Z$ in (4.5), we obtain

$$\phi Q\big(\nabla_{\phi X}\phi Z\big) - h(\phi X, Z) + Ch(\phi X,\phi Y) = 0. \tag{4.6}$$

Subtracting (4.6) from (4.5), we get

$$\phi Q[\phi X,\phi Z] - h(\phi X, Z) + h(\phi Z, X) = 0. \tag{4.7}$$

Since $D\oplus\langle\xi\rangle$ is integrable so that $[\phi X,\phi Z] \in D\oplus\langle\xi\rangle\ for\ X,Z \in D.$
Consequently (4.7) gives

$$h(\phi X, Z) + h(\phi Z, X) = 0.$$

Hence the theorem is proved.

Proposition 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosy mplectic manifold $\bar{M}$. Then

$$A_{\phi Y}Z - A_{\phi Z}Y = \phi P[Y,Z]$$

$for\ any\ Y,Z \in D^{\perp}.$

Proof. Let $X \in \chi(M)$ and $Y, Z \in D^{\perp}$. From (3.1) and (3.2), we have

$$2g(A_{\phi Z}Y, X) = g(h(Y,X), \phi Z) + g(h(X,Y), \phi Z).$$

Using (3.3) and (3.1), we get

$$2g(A_{\phi Z}Y, X) = -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_Y \phi)X + (\bar{\nabla}_X \phi)Y, Z).$$

Using (2.6) in above equation, we have

$$2g(A_{\phi Z}Y, X) = -g(\phi \bar{\nabla}_Y Z, X) + g(A_{\phi Y}Z, X). \tag{4.8}$$

Transvecting X from both sides of (4.8), we have

$$2A_{\phi Z}Y = \phi \bar{\nabla}_Y Z + A_{\phi Y}Z.$$

Interchanging $Y and\ Z$, we have

$$2A_{\phi Y}Z = \phi \bar{\nabla}_Z Y + A_{\phi Z}Y.$$

Subtracting above two equations, we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi[Y, Z].$$

Comparing the tangential parts from both sides of above equation, we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi P[Y, Z], \tag{4.9}$$

where $[Y, Z]$ is Lie bracket.
Hence the proposition is proved.

Theorem 4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$.Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \tag{4.10}$$

$for\ all\ Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable. Then $[Y, Z] \in D^{\perp}\ for\ any\ Y, Z \in D^{\perp}$. Therefore, $P[Y, Z] = 0$. From (4.9), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = 0.$$

Conversely, let (4.10) holds good. Then by virtue of (4.9), we get

$$\phi P[Y, Z] = 0$$

$for\ all\ Y, Z \in D^{\perp}$. Since $rank\ \phi = 2n$, therefore, either $P[Y, Z] = 0\ or\ P[Y, Z] = k\xi$. But $P[Y, Z] = k\xi$ is not possible as P being a projection operator on D. Hence $P[Y, Z] = 0$. This implies that $[Y, Z] \in D^{\perp} for\ all\ Z \in D^{\perp}$. Thus $D^{\perp}$ is integrable.
Hence the theorem is proved.

Notes

Notes

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Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

By Sapna Rani & Dilbaj Singh
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Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

Keywords : *strict practical stability, impulsive differential equations, lyapunov function.*

GJSFR-F Classification : *MSC 2010: 37L45, 12H20*

STRICTLY PRACTICAL STABILIZATION OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS BY USING LYAPUNOV FUNCTIONS

Strictly as per the compliance and regulations of :

Strictly Practical Stabilization of Impulsive Functional Differential Equations by using Lyapunov Functions

Sapna Rani [α] & Dilbaj Singh [σ]

Abstract - In this paper, we extend the concept of strict practical stability to impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. As practical stability does not give us much information about the rate of decay of solution so we develop the idea for strict practical stability of functional differential equations with impulsive effect and obtained some conditions for strict practical uniform stability for functional differential equations with impulse by using piecewise continuous Lyapunov functions and Razumikhin technique.

Keywords : strict practical stability, impulsive differential equations, lyapunov function.

I. Introduction

Various physical processes undergo abrupt changes of state at certain moments of time between intervals of continuous equation. The duration of these changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well approximated in terms of the instantaneous changes of state i.e. Impulses [7]. On the other hand Functional differential equations are important in scientific and technical professions and they are used to represent a rate of change of time varying phenomenon [5]. When both functional differential equations and Impulses are Involved, Impulsive functional differential system becomes a natural framework of mathematical modelling of varying physical phenomenon [5]. Impulsive functional differential systems are different from ordinary differential systems in the sense that the state undergo abrupt changes at certain moments and the derivation of the state variable depends not only on present state but also on past state. Stability is highly important in all physical application [1,2,3]. A stable equilibrium represents a behaviour usually which cannot be changed. Several stability criteria are obtained by many authors which shows that impulses do contribute to the stabilization of functional differential equations [2,3,5].

Theory of stability in the sense of Lyapunov is now well known and is widely used in concrete problems of real world. The desirable feature is to know the size of region of stability so that we can judge whether or not a given system is sufficiently stable to function properly and may be able to see how to improve its stability. On the other hand the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. So we need a notion of stability that is more suitable than stability and such concept is practical stability [4,6,8,9].

Ref

1. Dilbaj Singh, S.K. Srivastava, Stability criteria of impulsive differential equations with comparison results, Advances in Differential Equations and Control Processes, Vol. 10, pp. 171-182, 2012.

In this paper we establish stability result which provides sufficient conditions to maintain uniform strict practical stability of the trivial solution of a functional differential equation with impulse. The Lyapunov's second method of functions called Lyapunov function is employed in this work.

This paper is organized as: In section 2, we introduce some basic definitions and notations. In section 3, based upon Lyapunov functions and Razumikhin method, some conditions for strict uniform practical stability are obtained. Finally in section 4, some concluding remarks are given.

Notes

II. Preliminaries

Consider the following impulsive functional differential system:

$$\dot{x}(t)=f(t, x_t), \ t\geq t_0, \ t\neq\tau_k \tag{2.1}$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^-) = I_k(x(\tau_k^-)), \quad t=\tau_k, \ k= 1,2,....$$

Where $f\in C([R^+\times D, R^n]$, D is an open set in PC([-τ, 0], R^n), where τ>0, PC([-τ, 0], R^n) = {φ :([-τ, 0]→R^n), φ(t) is continuous everywhere except a finite number of points $\hat{t}$ at which φ($\hat{t}^+$) and φ($\hat{t}^-$) exist and φ($\hat{t}^+$) = φ($\hat{t}$)}. f(t,0) = 0 for all t∈ R^+,$I_k(0)$ = 0, for all k∈Z, 0 = $\tau_0 < \tau_1 < \tau_2 < \ldots\ldots\ldots\ldots \tau_k \rightarrow \infty$ for k → ∞ and $x(t^+) = \lim_{s\to t^+} x(s)$, $x(t^-) = \lim_{s\to t^-} x(s)$. For each t ≥ t_0, $x_t \in$ D is defined by $x_t(s)$ = x (t + s), -τ ≤ s ≤ 0. For φ∈ PC ([-τ, 0], R^n), $|\varphi|_1$ is defined by $|\varphi|_1 = sup_{-\tau\leq s\leq 0}\|\varphi\|$, $|\varphi|_2$ is defined by $|\varphi|_2 = inf_{-\tau\leq s\leq 0}\|\varphi\|$. We can see that x (t) = 0 is a solution of (2.1) which we call the zero solution or trivial solution.

For a given σ ≥ 0 and φ∈ PC ([-τ, 0], R^n), the initial value problem of equation (2.1) is

$$\dot{x}(t)=f(t, x_t), \ \sigma\geq t_0, \ t\neq\tau_k \tag{2.2}$$

$$\Delta x(t) = x(\tau_k) - x(\tau_k^-)= I_k(x(\tau_k^-)), \ t = \tau_k, \ k = 1,2,... \text{ Where } x_\sigma = \varphi.$$

Throughout this paper we let the following hypothesis hold:

(H_1) For each function x(s): [σ − τ, ∞] →R^n, σ ≥ t_0, which continuous everywhere except a finite number of points τ_k at which $x(\tau_k^+)$ and $x(\tau_k^-)$ exist and $x(\tau_k^+) = x(\tau_k)$, where $f(t, x_t)$ is continuous for almost all t∈[σ,∞) and at the discontinuous f is the right continuous.

(H_2) f (t,φ) is lipschitzian in φ in each compact set in PC([-τ, 0], R^n).

(H_3) The functions $I_k: R^n\rightarrow R^n$, k = 1,2,3 ..... are such that for any H > 0, there exist a ρ > 0 such that if,

x∈ s(ρ) = {x ∈ R^n: ‖x‖ < ρ} implies that ‖x +I_k(x)‖ < H.

Under these hypothesis a unique solution of problem (2.2) exist throughout (σ, φ).

Let

K= {a ∈[R^+, R^+]: a(t) is monotone strictly increasing and a(0) = 0}

$$K_1 = \{w \in [R^+, R^+]: w(t) \in K \text{ and } 0 < w(s) < s,\ \ s > 0\}$$
$$PC_1(\rho) = \{\varphi \in PC([-\tau, 0], R^n): |\varphi|_1 < \rho\}$$
$$PC_2(\theta) = \{\varphi \in PC([-\tau, 0], R^n): |\varphi|_2 > \theta\}$$

Notes

We have the following definitions:

Definition: The trivial solution of (2.1) is said to be

(A_1) Strict practical stable, if for any $\sigma \geq t_0$ There exist (λ_1, A_1), $\lambda_1 \leq A_1$ such that $\varphi \in PC_1(\lambda_1)$ implies $\|x(t: \sigma, \varphi)\| < A_1$, $t \geq \sigma$, and for every $0 < \lambda_2 \leq \lambda_1$, there exist $0 < A_2 \leq \lambda_2$ such that $\varphi \in PC_2(\lambda_2)$ implies $\|x(t: \sigma, \varphi)\| > A_2$, $t \geq \sigma$.

(A_2) Strict Practically Uniformly Stable, if (A_1) holds for all $t \in R^+$.

Definition: The function V: $[t_0, \infty] \times s(\rho) \to R^+$ belongs to class v_0 if

I. The function V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times S(\rho)$ and for all, $t \geq t_0, V(t, 0) = 0$.

II. $V(t, x)$ is locally lipschitzian in $x \in S(\rho)$.

III. For each k = 1, 2,..... there exist finite limits.

$$\lim_{(t,y)\to(\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$$

$$\lim_{(t,y)\to(\tau_k^+, x)} V(t, y) = V(\tau_k^+, x)$$

With $V(\tau_k^+, x) = V(\tau_k, x)$ satisfied.

Definition: Let $V \in v_0$,for $(t, x) \in [\tau_{k-1}, \tau_k) \times S(\rho)$, D^+V is defined as:

$$D^+V(t,x(t)) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta} \{V(t+\delta, x(t+\delta)) - V(t, x(t))\}.$$

IV. Main Result

Now we consider the strict practical stability of the Impulsive functional differential equation (2.1) with following results:

Theorem: Assume that

(i) There exist (λ_1, A_1), $0 < \lambda_1 \leq A_1$ and $V_1 \in v_0$, such that $b_1(\|x(t)\|) \leq V_1(t,x(t)) \leq a_1(\|x(t)\|)$, $a_1, b_1 \in K$

(ii) For any solution x(t) of (2.1), $V_1(t+s, x(t+s)) \leq V_1(t,x(t))$ for $s \in [-\tau, 0]$, implies that

$$D^+V_1(t, x(t)) \leq 0.$$

Also for all $K \in Z^+$ and $x \in S(\rho)$

$$V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1+d_k) V_1(\tau_k, x(\tau_k^-)),\ \text{ where } d_k \geq 0 \text{ and } \textstyle\sum_{k=1}^{\infty} d_k < \infty$$

(iii) For any $0 < \lambda_2 \leq \lambda_1$ and $V_2 \in v_0$

$$b_2(\|x(t)\|) \leq V_2(t,x(t)) \leq a_2(\|x(t)\|),\ \ a_2, b_2 \in K$$

(iv) For any solution x(t) of (2.1), V_2(t + s), x(t+ s)) ≥ V_2(t,(x(t)) for s∈ [-τ,0], implies

$$D^+V_2(t, x(t)) \geq 0.$$

Also for all K∈ Z^+ and x∈ S(ρ)

$$V_2(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \geq (1 - c_k)\, V_2(\tau_k, x(\tau_k^-)),$$

Where 0≤ $c_k < 1$ and $\sum_{k=1}^{\infty} c_k < \infty$

Then the trivial solution of (2.1) is strict practical uniformly stable.

Proof: Since $\sum_{k=1}^{\infty} d_k < \infty$ and $\sum_{k=1}^{\infty} c_k < \infty$.

It follows that, $\prod_{k=1}^{\infty}(1+d_k)$ =Mand $\prod_{k=1}^{\infty}(1-c_k)$ =N, Obviously 1 ≤ M < ∞ , 0 < N ≤ 1

Let 0 < A_1 < ρand σ ≥ t_0 be given and σ∈ $[\tau_k,\ \tau_{k+1}]$ for some k ∈Z, Such that M$a_1(\lambda_1) < b_1 A_1$

Then we claim that φ ∈ $PC_1(\lambda_1)$ implies ‖x(t)‖ < A_1 , t ≥ σ

Obviously for any t∈ [τ,σ], there exist θ∈ [τ,0], such that

$$V_1(t, x(t)) = V_1(\sigma + \theta, x(\sigma + \theta)) \leq a_1(\, \| x(\sigma + \theta) \| \,) =$$

$$a_1 \| x_\sigma(\theta) | = a_1 \| \varphi(\theta) \| \leq a_1\ (\lambda_1)$$

Then, we claim that

$$V_1(t, x(t)) \leq a_1, \quad \sigma \leq t < \tau_k \tag{3.1}$$

If the inequality (3.1), does not hold, then there exist a $\hat{t}\in(\sigma\ ,\tau_k)$ such that

$$V_1(\hat{t}, x(\hat{t})) > a_1\ (\lambda_1) \geq V_1(\sigma, x(\sigma))$$

which implies that there exist a $\check{t}\in(\sigma, \hat{t}\,]$, such that

$$D^+V_1(\check{t}, x(\check{t})) > 0 \tag{3.2}$$

and

$$V_1(\check{t} + s, x(\check{t} + s)) \leq V_1(\hat{t}, (x(\hat{t}))\, , \text{where } s \in [-\tau, 0]$$

by condition (ii), which implies that $D^+V_1(\check{t}, x(\check{t}))$≤0. This contradicts inequality (3.2)

So inequality (3.1) holds.

From condition (ii), we have

$$V_1(\tau_k, x(\tau_k)) = V_1(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + d_k)\, V_1(\tau_k, x(\tau_k^-)) \leq (1 + dk)\, a_1(\lambda 1)$$

Next, we claim that

$$V_1(t, x(t)) \leq (1 + dk) a_1(\lambda 1),\ \tau_k \leq t \leq \tau_{k+1} \tag{3.3}$$

Notes

Notes

If inequality (3.3) does not hold then, there exist $\hat{s}\in(\tau_k,\ \tau_{k+1})$, such that

$$V_1(\hat{s},x(\hat{s}))\ \geq\ (1+d_k)\,a_1(\lambda_1)\ \geq\ V_1(\tau_k,x(\tau_k))$$

which implies that there exist an $\check{s}\in\ (\tau_k,\ \tau_{k+1})$, such that

$$D^+V_1(\check{s},x(\check{s}))>0 \tag{3.4}$$

And

$$V_1(\check{s}+\ s),x(\check{s}+\ s))\ \leq\ V_1(\hat{s},(x(\hat{s})),\text{where }\ s\in[-\tau,0]$$

by condition (ii), which implies that $D^+V_1(\hat{s},\ x(\hat{s}))\leq 0$, This Contradicts inequality (3.4) so inequality (3.3) holds.
And from condition (ii), we have

$$V_1(\tau_{k+1},x(\tau_{k+1})=V_1(\tau_{k+1},x(\tau_{k+1}{}^-)+I_k(x(\tau_{k+1}{}^-)))\leq(1+$$

$$d_k)\,V_1(\tau_{k+1},x(\tau_{k+1}{}^-))\leq(1+\ d_k)\ (1\ +\ d_k)\ a_1(\lambda 1).$$

By a simple induction, we can easily prove that in general form for, m = 0,1,2,3 ……

$$V_1(t,x(t))\ \leq\ (1\ +\ d_{k+m})\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\leq(1\ +\ d_k a_1(\lambda 1),\text{Where }\tau_{k+m}\leq t\leq\tau_{k+m+1}$$

Which together with inequality (3.1) and condition (i), we have

$$b_1(\|x(t)\|)\ \leq\ V_1(t,\ x(t))\ \leq\ M\,a_1((\lambda_1)\ <\ b_1(A_1),\ t\geq\sigma$$

Thus we have

$$\|x(t)\|\ <\ A_1,\ t\geq\sigma$$

Now, let $0\ <\ \lambda_2\leq\lambda_1$ and choose $0<A_2<\lambda_2,$ such that $a_2(A_2)\ <\ Nb_2(\lambda_2)$.
Next, we claim that $\varphi\in PC_2(\lambda_2)$ implies $\|\ x(t)\ \|>A_2$, $t\geq\sigma$
If it holds,then $\varphi\in\ PC_1(\lambda_1)\cap PC_2(\lambda_2)$ implies

$$A_2<\|x(t)\|\ <\ A_1,\ t\geq\sigma$$

Obviously for any, $t\in[\sigma-\tau,\ \sigma]$, there exist a $\theta\in[-\tau,\ 0]$, such that

$$V_2(t,\ x(t))=V_2(\sigma+\ \theta,\ x(\sigma+\theta))\geq b_2(\|x(\sigma+\theta)\|)=b_2\|x_\sigma(\theta)\|$$

$$=b_2\|\varphi(\theta)\|\ \geq\ b_2\ (\lambda_2)$$

Then, we claim that

$$V_2(t,\ x(t))\geq b_2(\lambda_2),\ \sigma\leq t\ <\ \tau_k \tag{3.5}$$

If inequality (3.5) does not hold, then there exist $\bar{t}\in(\sigma, \tau_k)$, such that

$$V_2(\bar{t}, x(\bar{t})) < b_2(\lambda_2) \leq V_2(\sigma, x(\sigma))$$

Which implies that there exist a $t_1\in(\sigma, \bar{t})$, such that

$$D^+V_2(t_1, x(t_1)) < 0 \tag{3.6}$$

And

$$V_2(t_1+s, x(t_1+s) \geq V_2(t_1, x(t_1)), \quad s\in[-\tau, 0]$$

By condition (iv), we have, $D^+V_2(t_1, x(t_1)) \geq 0$, This contradicts inequality (3.6) So, inequality (3.5) holds.
From condition (iv), we have

$$V_2(\tau_k, x(\tau_k) = V_2(\tau_k, x(\tau_k^-)+I_k(x(\tau_k^-)))\geq(1-c_k)V_2(\tau_k, x(\tau_k^-)) \geq (1-c_k)b_2(\lambda_2)$$

Next, we claim that

$$V_2(t, x(t))\geq(1-c_k)b_2(\lambda_2), \quad \tau_k\leq t<\tau_{k+1} \tag{3.7}$$

If the inequality (3.7) does not hold then there exist an $\bar{r}\in(\tau_k, \tau_{k+1})$, such that

$$V_2(\bar{r}, x(\bar{r})) < (1-c_k)\, b_2(\lambda_2) \leq V_2(\tau_k, x(\tau_k))$$

Which implies that there exist an $\check{r}\in(\tau_k, \bar{r})$, such that

$$D^+V_2(\check{r}, x(\check{r}))<0 \tag{3.8}$$

And

$$V_2(\check{r}+s, x(\check{r}+s) \geq V_2(\check{r}, x(\check{r})), \quad s\in[-\tau,0]$$

By condition (iv), we have, $D^+V_2(\check{r}, x(\check{r})) \geq 0$. Which contradicts inequality (3.8) So, inequality (3.7) holds
And from condition (iv), we have

$$V_2(\tau_{kk+1}, x(\tau_{kk+1})) = V_2(\tau_{k+1}, x(\tau_{k+1}^-)+I_k(x(\tau_{k+1}^-))) \geq (1-c_k)V_2(\tau_{k+1}, x(\tau_{k+1}^-))$$
$$\geq(1-c_{k+1})(1-c_k)\, b_2(\lambda_2).$$

And by a simple induction we can prove that, in general, for,m = 0,1,2,3….

$$V_2(t,x(t))\geq(1-c_{k+m})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\leq(1-c_k)b_2(\lambda_2), \quad \tau_{k+m}\leq t\leq\tau_{k+m+1}$$

Which together with inequality (3.5) and condition (iii), we have

$$a_2(\|x(t)\|) \leq V_2(t, x(t)) \leq Nb_2(\lambda_2) > a_2(A_2), \quad t\geq\sigma$$

Notes

Thus we have, $\|x(t)\| > A_2$, $t \geq \sigma$
Thus the zero solution of (2.1) is strict practical uniformly stable.
The proof of theorem is complete.

IV. Conclusion

Notes

In this paper, we investigated the strict practical stability criteria in the form of theorem for impulsive functional differential equations, which is more useful as compared to practical stability. It gives rate of decay of the solution, so it is finer concept which can give us more precise information. In future we can modify this theorem to get less restricted conditions to verify strict practical stability.

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Computation of a Summation Formula Clung To Recurrence Relation

By Salahuddin, M. P. Chaudhary & Upendra Kumar Pandit
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Abstract - In this paper we have established a summation formula clung to contiguous relation and recurrence relation.

Keywords : gaussian hypergeometric function, contiguous function, recurrence relation of gamma function, bailey summation theorem and legendre duplication formula.

GJSFR-F Classification : 2010 MSC No: 33C60 , 33C70 , 33D15 , 33D50 , 33D60.

COMPUTATION OF A SUMMATION FORMULA CLUNG TO RECURRENCE RELATION

Strictly as per the compliance and regulations of :

Computation of a Summation Formula Clung To Recurrence Relation

Salahuddin [α], M. P. Chaudhary [σ] & Upendra Kumar Pandit [ρ]

Abstract - In this paper we have established a summation formula clung to contiguous relation and recurrence relation.

Keywords : gaussian hypergeometric function, contiguous function, recurrence relation of gamma function, bailey summation theorem and legendre duplication formula.

I. Basic Introduction

The Pochhammer symbol or **generalized factorial function** or **shifted factorial** or **falling factorial** is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{bmatrix} 1 & ; \quad n=0 \\ a(a+1)(a+2).....(a+n-1); & n=1,2,3.... \end{bmatrix} \tag{1}$$

where $a \neq 0, -1, -2, ...$ and the notation Γ stands for Gamma function.

$$(b)_{-n} = \frac{\Gamma(b-n)}{\Gamma(b)} = \frac{(-1)^n}{(1-b)_n} \; ; \tag{2}$$

where $\quad b \neq .....-3,-2,-1,0,1,2,3..... \quad$ and $\quad n = 1,2,3,.....$

If $m = 1, 2, , 3, 4 \ldots$ and $\quad n = 0, 1, 2, 3, \ldots,$ then

$$(b)_{mn} = m^{mn} \left(\frac{b}{m}\right)_n \left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-2}{m}\right)_n \left(\frac{b+m-1}{m}\right)_n \tag{3}$$

Generalized Gaussian hypergeometric function of one variable is followed as

$$ {}_AF_B \left[\begin{array}{ccc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

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or

$$ {}_AF_B\left[\begin{array}{ccc}(a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ccc}(a_j)_{j=1}^A & ; & \\ & & z \\ (b_j)_{j=1}^B & ; & \end{array}\right] = \sum_{k=0}^{\infty}\frac{((a_A))_k z^k}{((b_B))_k k!} \tag{4} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$ (a-b)\ {}_2F_1\left[\begin{array}{ccc}a,\ b\ ; & \\ c\ \ ; & z\end{array}\right] = a\ {}_2F_1\left[\begin{array}{ccc}a+1,\ b\ ; & \\ c\ \ ; & z\end{array}\right] - b\ {}_2F_1\left[\begin{array}{ccc}a,\ b+1\ ; & \\ c\ \ ; & z\end{array}\right] \tag{5} $$

Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\ \Gamma(z) \tag{6} $$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$ \sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{7} $$

$$ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{8} $$

$$ = \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{9} $$

Bailey summation theorem [Prudnikov, p.491(7.3.7.8)]is defined as follows

$$ {}_2F_1\left[\begin{array}{cc}a, 1-a & ;\ \frac{1}{2} \\ c & ;\ 2\end{array}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} \tag{10} $$

II. Main Result of Summation Formula

$$ {}_2F_1\left[\begin{array}{ccc}a\ , & -a-51 & ;\ \frac{1}{2} \\ c & & ;\ 2\end{array}\right] $$

$$ = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+51}}\Bigg[\frac{1}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+51}{2})}\Big\{+392156293632163883140634416093593600000 $$

$$ -74021568132398532674161337237569536000000a $$

$$ +462959815063911749743748213599543296000 0a^2 $$

Notes

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$$-1320609504346225933240581617829869568000a^{3}$$
$$+181193044645895982439149503813419315200a^{4}$$
$$-93118568613791555759574842325948672000a^{5}$$
$$-319305701534575139018589169352966400a^{6}+43290665140850487112626960992505600a^{7}$$
$$+1459526532650544719250939889840000a^{8}-877191741085992023754067557552000a^{9}$$
$$-4575088744242843615633002444000a^{10}+1005808770049491450042585156000a^{11}$$
$$+150812277064668533760470050 0a^{12}-514201602572270839356312000a^{13}$$
$$-1509631951413201653983400a^{14}+43020161183318919600a^{15}+4606163067186269875 00a^{16}$$
$$+5854726103680396800a^{17}-6623190522634400a^{18}-744188213168400a^{19}$$
$$-6243725988500a^{20}-9352543200a^{21}+138868600a^{22}+795600a^{23}+1300a^{24}$$
$$+1016165681947732630569197025827487744 0000c$$
$$-14463672559592909749687559057939103744000ac$$
$$+7213434913697898840014034554316432998400a^{2}c$$
$$-1652471481542170230203835091179109662720a^{3}c$$
$$+174658211280962153516139481169788612608a^{4}c$$
$$-4969371876573656445853216824494834688a^{5}c$$
$$-475604102922385185557746261486438656a^{6}c$$
$$+26909425529563820334801481917460224a^{7}c+76684770531028704267263360787936 0a^{8}c$$
$$-4686074174974908708180859439020 8a^{9}c-1208658606098948851658543849776a^{10}c$$
$$+35787345557980426589741460624a^{11}c+1310981764885248713045228020a^{12}c$$
$$-41215984661472863960652 48a^{13}c-620856397972347884319336a^{14}c$$
$$-6671743900559708043216a^{15}c+6558127104990267950 0a^{16}c+189652043324273587 2a^{17}c$$
$$+115302113690146 24a^{18}c-53516453726736a^{19}c-1025323839540a^{20}c-4458181728a^{21}c$$
$$-1725256a^{22}c+31824a^{23}c+52a^{24}c+107661639700814075973953620933607424000 00c^{2}$$
$$-122206533915334111305556214698066575360 00ac^{2}$$
$$+4964847384594138558862158629137140940800a^{2}c^{2}$$
$$-917355347585720296300460971959282892800a^{3}c^{2}$$
$$+72848183281079430345996459810027110400a^{4}c^{2}$$
$$-461333397991092789366017263631462400a^{5}c^{2}$$
$$-234490573731887780307040658002739200a^{6}c^{2}$$
$$+5607883301081174731722705232665600a^{7}c^{2}+404532742605787477269983966899200a^{8}c^{2}$$
$$-7562908655286410131260619449600a^{9}c^{2}-482314643379144818090700710400a^{10}c^{2}$$

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$$+16497683403078933715422672000a^{11}c^2 + 31587544172278628267246240 0a^{12}c^2$$

$$+3057010359220073337681600a^{13}c^2 - 713427909344230690896000a^{14}c^2$$

$$-16190956361896224480 00a^{15}c^2 - 4986468790418544000a^{16}c^2 + 157146003545731200a^{17}c^2$$

$$+1822144519395200a^{18}c^2 + 4701184488000a^{19}c^2 - 34522488000a^{20}c^2 - 245044800a^{21}c^2$$

$$-436800a^{22}c^2 + 652901172598488055962549838438150963200 0c^3$$

$$-607527203112514220836692567188777730048 0ac^3$$

$$+203230437246199673444891125740266481254 4a^2c^3$$

$$-302067862057231092388694972300478971904a^3c^3$$

$$+171807205556893633035072183135331614 72a^4c^3$$

$$+30451627801462170139375295753024716 8a^5c^3$$

$$-59282000451431295121172336261369856a^6c^3$$

$$+1515324587191551134306980047022 08a^7c^3 + 93558710014022955396854597958656a^8c^3$$

$$+11273311081462894266871974067 2a^9c^3 - 82507658092986358200872009472a^{10}c^3$$

$$-87988149734201622303490310 4a^{11}c^3 + 31393626885755822215632832a^{12}c^3$$

$$+6860746929250194791690 88a^{13}c^3 - 66723688561216625452 8a^{14}c^3$$

$$-139936842051141632640a^{15}c^3 - 128549380235811392 0a^{16}c^3 + 1721586727276416a^{17}c^3$$

$$+89439005591936a^{18}c^3 + 471090459840a^{19}c^3 + 340500160a^{20}c^3 - 3267264a^{21}c^3$$

$$-5824a^{22}c^3 + 26128664157156468138072396472957009920 00c^4$$

$$-20230557750460236966755663345284546560 00ac^4$$

$$+55976066752616361846001242560072908800 0a^2c^4$$

$$-66329561671679848284049655989702656000a^3c^4$$

$$+24555120056093433441458482927779840 00a^4c^4$$

$$+12375358515135363621736882244736000 0a^5c^4$$

$$-88488263426316038368213504326400 00a^6c^4$$

$$-14240352781409956633110343795200 0a^7c^4 + 11807147375650944114408332352000a^8c^4$$

$$+19585353401745628653813384000 0a^9c^4 - 70445839577747270051578800 00a^{10}c^4$$

$$-17528211459227053351166400 0a^{11}c^4 + 9269200774452512894000 00a^{12}c^4$$

$$+6064650048474121104000 0a^{13}c^4 + 437436273068892240000a^{14}c^4$$

$$-4560723705938400000a^{15}c^4 - 81537114234576000a^{16}c^4 - 305264559600000a^{17}c^4$$

$$+1313672360000a^{18}c^4 + 12252240000a^{19}c^4 + 24024000a^{20}c^4$$

$$+746448719593841678194641789420029607936c^5$$

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$$
\begin{gathered}
-4848423422491983408074334140976573972 48ac^5 \\
+11098480168944757355191207022473799 2704a^2c^5 \\
-10317233687279462544440111356456501248a^3c^5 \\
+1950939528571816894404651335140638 72a^4c^5 \\
+236493214875780019782567068522188 80a^5c^5 \\
-7967507845695813932055290677811 20a^6c^5 - 3126389268620074980188047934 3616a^7c^5 \\
+823155784842685883358254178816a^8c^5 + 30059392071751438726127790720a^9c^5 \\
-2326353187477596442959830 40a^{10}c^5 - 1522791942814168592697 3312a^{11}c^5 \\
-81852811808947624244800a^{12}c^5 + 2562162110375475928320a^{13}c^5 \\
+364163020462906579 20a^{14}c^5 + 37364831997292800a^{15}c^5 - 2170921221076608a^{16}c^5 \\
-14204266876800a^{17}c^5 - 15116221120a^{18}c^5 + 98017920a^{19}c^5 + 192192a^{20}c^5 \\
+159936761724037288421717018180321280000c^6 \\
-87492127371110824883734156363903795200ac^6 \\
+16516009700292256474294598785512243200a^2c^6 \\
-1173671805520416747746112210311577600a^3c^6 \\
+177539130978892518650372653383 6800a^4c^6 \\
+28622640470259127964352987414528 00a^5c^6 - 363135500365849146023937572864 00a^6c^6 \\
-3479560389180212846538284236800a^7c^6 + 2098728159618006248183255 0400a^8c^6 \\
+2413106726268002114404454400a^9c^6 + 108192397469842939309056 00a^{10}c^6 \\
-715366249224232583577600a^{11}c^6 - 9335180202255799091200a^{12}c^6 \\
+347359470239029248 00a^{13}c^6 + 131913011269079 0400a^{14}c^6 + 6809187280896000a^{15}c^6 \\
-17303430144000a^{16}c^6 - 235243008000a^{17}c^6 - 512512000a^{18}c^6 \\
+265763840147261875822157681934729216 00c^7 \\
-122567724037584096490919339698358845 44ac^7 \\
+18958758184417158469812875050918871 04a^2c^7 \\
-9876339246552438748626259617723187 2a^3c^7 \\
-1653746068979970114211156315095 04a^4c^7 \\
+2383166266278887193826963246940 16a^5c^7 + 5908658433080967378318473297 92a^6c^7 \\
-24360365506821176917455165849 6a^7c^7 - 1564667020221574346476019712a^8c^7 \\
+115842060758279701503111168a^9c^7 + 1563137249722283122348032a^{10}c^7 \\
-16296935846993847631872a^{11}c^7 - 419485986370536177664a^{12}c^7
\end{gathered}
$$

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$$
\begin{gathered}
-1367035409521926144a^{13}c^{7}+22577009199833088a^{14}c^{7}+191257494405120a^{15}c^{7} \\
+274524856320a^{16}c^{7}-1344245760a^{17}c^{7}-2928640a^{18}c^{7} \\
+35070934807163820820574544351723520000c^{8} \\
-13619022737975545734554656579780608000ac^{8} \\
+17094042739830324058091479337533440000a^{2}c^{8} \\
-6067926395459215226369951249203200a^{3}c^{8} \\
-2382423862432601243329431773184000a^{4}c^{8}+1395817124252228631605978726400 0a^{5}c^{8} \\
+22391383557921068818730803200 0a^{6}c^{8}-11140323745178391030031564800a^{7}c^{8} \\
-192606890208950053905868800a^{8}c^{8}+3143176542139018125312000a^{9}c^{8} \\
+805635522203373987840000a^{10}c^{8}+45492253807679078400a^{11}c^{8} \\
-1004782518133954560 0a^{12}c^{8}-71472471662592000a^{13}c^{8}+91162266624000a^{14}c^{8} \\
+2285217792000a^{15}c^{8}+5601024000a^{16}c^{8}+373936129332143623419537730431877120c^{9} \\
-121881993596477660886548463168258048ac^{9} \\
+12244622122591681114343641738379264a^{2}c^{9} \\
-2546410607640370619633091721297 92a^{3}c^{9} \\
-19663670233357415566573248610304a^{4}c^{9}+5600847946702256866767627878 40a^{5}c^{9} \\
+18095746125004754228760657920a^{6}c^{9}-310605830872726263126540288a^{7}c^{9} \\
-102074766313533761172 70528a^{8}c^{9}+23223077801219253534720a^{9}c^{9} \\
+2355636749511444439040a^{10}c^{9}+1396666394732799590 4a^{11}c^{9} \\
-117516116735553536a^{12}c^{9}-1384090399211520a^{13}c^{9}-2582047170560a^{14}c^{9} \\
+10156523520a^{15}c^{9}+24893440a^{16}c^{9}+326244314975104400841524012646400 00c^{10} \\
-88813468513138902356037715623936 00ac^{10} \\
+7007702800107856248737834729472 00a^{2}c^{10}-5276222909482495066628397465600a^{3}c^{10} \\
-113163947678026437893915934720 0a^{4}c^{10}+1328631196452522577931796480 0a^{5}c^{10} \\
+881345196341559210265804800a^{6}c^{10}-30764588660387028615168 00a^{7}c^{10} \\
-331378512757105965465600a^{8}c^{10}-1597994314386127257600a^{9}c^{10} \\
+4000308020175667200 0a^{10}c^{10}+408320886900326400a^{11}c^{10}-797107863552 00a^{12}c^{10} \\
-12797219635200a^{13}c^{10}-35846553600a^{14}c^{10}+23505591020177341187259835088896 00c^{11} \\
-530849241329367710710766663368704ac^{11}+32040821902056030454476919799808a^{2}c^{11} \\
+1548691193163573591934406492 16a^{3}c^{11}-4794024376510400642713701580 8a^{4}c^{11} \\
+16508310593497412701519872a^{5}c^{11}+291856481624826313868574 72a^{6}c^{11} \\
+1394154197632331673108 48a^{7}c^{11}-68722462989704966307 84a^{8}c^{11}
\end{gathered}
$$

Notes

$$-6846181001001035366 4a^9c^{11} + 313126051801006080a^{10}c^{11} + 5905660917252096a^{11}c^{11}$$

$$+14157389955072a^{12}c^{11} - 46535344128a^{13}c^{11} - 130351104a^{14}c^{11}$$

$$+1407536728020310735218823659520 00c^{12} - 2614291973499429934468127784960 0ac^{12}$$

$$+11618463325817878010549239808 00a^2c^{12} + 1981617584456773656296816640 0a^3c^{12}$$

$$-151468081652607053892812800 0a^4c^{12} - 13007910157465601064960000a^5c^{12}$$

$$+667491473107756064768000a^6c^{12} + 7452079025079848140800a^7c^{12}$$

$$-838909492793229312 00a^8c^{12} - 1363318831300608000a^9c^{12} - 1212846415872000a^{10}c^{12}$$

$$+44208576921600a^{11}c^{12} + 144472473600a^{12}c^{12} + 7033255557317565417731034972160c^{13}$$

$$-1062714470690285719045451808768ac^{13} + 3277941453215632539707139686 4a^2c^{13}$$

$$+100226923740396503870944051 2a^3c^{13} - 35211188622667072624394240a^4c^{13}$$

$$-593591726750187523276800a^5c^{13} + 10030581135956355645440a^6c^{13}$$

$$+182420416817402609664a^7c^{13} - 3434758898785320 96a^8c^{13} - 15530359717232640a^9c^{13}$$

$$-481849039257 60a^{10}c^{13} + 136026390528a^{11}c^{13} + 444530688a^{12}c^{13}$$

$$+293814830383077351070105600000c^{14} - 356305824164232460857507840 00ac^{14}$$

$$+687943867344091157102592000a^2c^{14} + 337835095904808625766400 00a^3c^{14}$$

$$-571007455827728793600000a^4c^{14} - 15474236362054041600000a^5c^{14}$$

$$+791813202180833280 00a^6c^{14} + 2727329130086400000a^7c^{14} + 59363898163200 00a^8c^{14}$$

$$-97161707520000a^9c^{14} - 381026304000a^{10}c^{14} + 1025922942165343499620188160 0c^{15}$$

$$-9818387219877429453772554 24ac^{15} + 94378071966287098145669 12a^2c^{15}$$

$$+83110564558287723823104 0a^3c^{15} - 5375071988357110169600a^4c^{15}$$

$$-26982524008999157760 0a^5c^{15} - 2035673092765777 92a^6c^{15} + 2540972975063040 0a^7c^{15}$$

$$+104736510443520a^8c^{15} - 259097886720a^9c^{15} - 1016070144a^{10}c^{15}$$

$$+29869828988705671689011 2000c^{16} - 2208774390360694786621440 0ac^{16}$$

$$+347491652083243548672 00a^2c^{16} + 1525650459557057003520 0a^3c^{16}$$

$$+26018544311402496 00a^4c^{16} - 321049486924185600 0a^5c^{16} - 12890056359936000a^6c^{16}$$

$$+136026390528000a^7c^{16} + 666796032000a^8c^{16} + 72163098146267657037414 40c^{17}$$

$$-401332490274906285539328ac^{17} - 199469481915167381913 6a^2c^{17}$$

$$+20784195017805555302 4a^3c^{17} + 943705218464612352a^4c^{17} - 2515752081948672 0a^5c^{17}$$

$$-145384545976320a^6c^{17} + 320062095360a^7c^{17} + 1568931840a^8c^{17}$$

$$+14353770849318731776000 0c^{18} - 5795478578754119270400ac^{18}$$

$$-58868619943503462400a^2c^{18} + 20460253945528320 00a^3c^{18} + 15071438176256000a^4c^{18}$$

$$-117356101632000a^5c^{18} - 767033344000a^6c^{18} + 2323444065633933721600c^{19}$$

Notes

$$-649251921274223984 64ac^{19} - 916234973881565184a^2c^{19} + 13778265172869120a^3c^{19}$$

$$+124580748328960a^4c^{19} - 247065477120a^5c^{19} - 1614807040a^6c^{19}$$

$$+300929768772075520 00c^{20} - 5436453465096192 00ac^{20} - 9210673653350400a^2c^{20}$$

$$+56825059737600a^3c^{20} + 557108428800a^4c^{20} + 3042162628152524 80c^{21}$$

$$-3200279126212608ac^{21} - 59990496772096a^2c^{21} + 108238209024a^3c^{21} + 1061158912a^4c^{21}$$

$$+2311397048320000c^{22} - 11807804620800ac^{22} - 231525580800a^2c^{22} + 12408428953600c^{23}$$

$$-20535312384ac^{23} - 402653184a^2c^{23} + 41943040000c^{24} + 67108864c^{25}\Big\}+$$

$$+\frac{1}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+52}{2})}\Big\{-174640699428027153866146029067960320000 00a$$

$$+2068628735580311792995020774147538944 0000a^2$$

$$-893782773509521896640037682838123315 2000a^3$$

$$+177818752314291438536422827187968583 6800a^4$$

$$-152958349994552708911714321312984796160a^5$$

$$+84580791261276992924025160195502 3616a^6$$

$$+625073884704101388272682931412331456a^7-1635178039526152627678090954634 2272a^8$$

$$-1387715913059429859324257558607888a^9 + 294229809526677119082875916142 40a^{10}$$

$$+219727227149693714709941805 5796a^{11} - 915462138570260726731254 3112a^{12}$$

$$-200905589594665939444404 9063a^{13} - 22907465907142301442089065a^{14}$$

$$+678068948453250922512126a^{15} + 18820430104162303836518a^{16}$$

$$+699135138402554237 67a^{17} - 3106032871839381055a^{18} - 48230575964749464a^{19}$$

$$-173954956930612a^{20} + 2022903890007a^{21} + 24154252265a^{22} + 87862086a^{23} + 2678a^{24}$$

$$-663a^{25} - a^{26} + 17464069942802730897824646237782016000000c$$

$$-5400736761335170870603999636120535040 0000ac$$

$$+4176749357171813611694513140282515456 0000a^2c$$

$$-135732553774528999612009861420347248640 00a^3c$$

$$+205467171794561086065066525377099765 7600a^4c$$

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$$-318781053860485082203178447567944 3200a^6c$$

$$+537679455896348207294860188265593600a^7c+21561567877933197657113200287 8400a^8c$$

$$-110643444225065599477507298796 9600a^9c - 3900521188164301838303830970800a^{10}c$$

$$+129935275957691323318972230 1200a^{11}c + 182426037934496265901187 83300a^{12}c$$

$$-681341880584185413666746400a^{13}c - 19313434798930849743184200a^{14}c$$

Notes

$$
\begin{gathered}
+754754437503539804760 0a^{15}c+6009125692951825000700a^{16}c \\
+754286259085774272 00a^{17}c-939894569306264 00a^{18}c-9694744240381200a^{19}c \\
-81018694656900a^{20}c-120521200800a^{21}c+1807184600a^{22}c+10342800a^{23}c \\
+16900a^{24}c+3332108025754864996621860032093356032 0000c^2 \\
-61112319783077679498758852113706385408000ac^2 \\
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+17346834336799238866980948672915187 2a^7c^2 \\
+43889630745475193234758592147775 68a^8c^2-30487945465896598352737520311939 2a^9c^2 \\
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-350917550607624a^{19}c^2-6666815293138a^{20}c^2-28956944016a^{21}c^2-11176308a^{22}c^2 \\
+206856a^{23}c^2+338a^{24}c^2+28282653946454862828385645890424012800000c^3 \\
-37847109234486369007419794151897563136000ac^3 \\
+1713457937103544455695173560232449146 8800a^2c^3 \\
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+25431434993231939895777252470553600a^7c^3 \\
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-2028663240492155118701115926400a^{10}c^3+86464008832314800874790512 00a^{11}c^3 \\
+1360478905870410824014753600a^{12}c^3+127225083231978894162000 00a^{13}c^3 \\
-3129032111897029379184 00a^{14}c^3-6976348681997275152000a^{15}c^3 \\
-209019023090340320 00a^{16}c^3+683604505843459200a^{17}c^3+7884558877395200a^{18}c^3 \\
+20265613368000a^{19}c^3-149805656000a^{20}c^3-1061860800a^{21}c^3-1892800a^{22}c^3
\end{gathered}
$$

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$$+1440300849635277480118978884647308492800 0c^4$$

$$-150960224819365534135896082504816892313 60ac^4$$

$$+549850181306804818472775808469427211468 8a^2c^4$$

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$$-2655476442099055634432065592 64a^{10}c^4-269551643666341209691120948 8a^{11}c^4$$

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$$-2561830314196148179184a^{14}c^4-455189373853931951520a^{15}c^4$$

$$-4154321981773140320a^{16}c^4+574896506443459 2a^{17}c^4+290840371773952a^{18}c^4$$

$$+1529982133680a^{19}c^4+1104543440a^{20}c^4-10618608a^{21}c^4-18928a^{22}c^4$$

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$$+3422210792000a^{18}c^5+31855824000a^{19}c^5+62462400a^{20}c^5$$

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$$-87936530871567952790257707663829342617 6ac^6$$

$$+21322634206871890157691014184841917235 2a^2c^6$$

$$-20888741739044083190415260963003547648a^3c^6$$

$$+4453502394251834568997438150039306 24a^4c^6$$

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$$+476970442495828897042416866157066624a^5c^6$$
$$-17329923235451789527564097139758080a^6c^6$$
$$-6408355161658316801415678239078 4a^7c^6 + 18059446978369943024459816568 32a^8c^6$$
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$$+416416a^{20}c^6 + 24183042634414344581262003916505088 0000c^7$$
$$-140430911900394092446841334069303705600ac^7$$
$$+278342515450299241729726387709804544 00a^2c^7$$
$$-207550771145550821978660619233525760 0a^3c^7$$
$$+722454153797750835287655407288320 0a^4c^7$$
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$$+445617469440a^{16}c^8 - 2184399360a^{17}c^8 - 4759040a^{18}c^8$$
$$+4404068776039340967770266733117440000c^9$$
$$-178326131008147415788221034030694400 0ac^9$$
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$$+13209108250053065305305697484800 0c^{13} - 2508072701878821815695504834560 0ac^{13}$$

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$$+11410098189869793397361344512 00a^2c^{13} + 18804311236219616172716851200a^3c^{13}$$

$$-14973584596144425995534336 00a^4c^{13} - 1254124855209896209612 8000a^5c^{13}$$

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$$+66763806738731096840601 6a^3c^{16} - 440182277838848078643 2a^4c^{16}$$

$$-218306735322712178688a^5c^{16} - 16004581454492467 2a^6c^{16} + 2063361646854144 0a^7c^{16}$$

$$+85040625745920a^8c^{16} - 210517032960a^9c^{16} - 825556992a^{10}c^{16}$$

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$$+140283054011842560 0a^4c^{17} - 24505074238095360 00a^5c^{17} - 9827187621888000a^6c^{17}$$

$$+104020180992000a^7c^{17} + 509902848000a^8c^{17} + 5116743183106645604433920c^{18}$$

$$-287184942520039832027136ac^{18} - 14030155526802563399 68a^2c^{18}$$

$$+149541088213486534656a^3c^{18} + 676438446397063168a^4c^{18} - 1815914972971008 0a^5c^{18}$$

$$-104933473648640a^6c^{18} + 231155957760a^7c^{18} + 1133117440a^8c^{18}$$

$$+96911375027352371200000c^{19} - 3941797062701481984000ac^{19}$$

$$-398799410099650560 00a^2c^{19} + 1397449692807168000a^3c^{19} + 10287895281664000a^4c^{19}$$

$$-80296280064000a^5c^{19} - 524812288000a^6c^{19} + 149643012054701834240 0c^{20}$$

$$-42055817340773203968ac^{20} - 592824246953050112a^2c^{20} + 8950947523854336a^3c^{20}$$

$$+80929203683328a^4c^{20} - 160592560128a^5c^{20} - 1049624576a^6c^{20}$$

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$$+18518937847201792000c^{21} - 335979518676172800ac^{21} - 5690809542246400a^2c^{21}$$
$$+35177417932800a^3c^{21} + 344876646400a^4c^{21} + 1791460360218214400c^{22}$$
$$-18900506860584960ac^{22} - 354288643604480a^2c^{22} + 63958941696a^3c^{22} + 627048448a^4c^{22}$$
$$+1304260771840000c^{23} - 6673976524800ac^{23} - 130862284800a^2c^{23} + 6717597286400c^{24}$$
$$-11123294208ac^{24} - 218103808a^2c^{24} + 21810380800c^{25} + 33554432c^{26}\Big\}\Big] \qquad (11)$$

Notes

III. Derivation of Main Formula

Substituting $b = -a - 51, z = \frac{1}{2}$ in given result (5),we get

$$(2a+51)\ {}_2F_1\left[\begin{matrix} a\ , & -a-51 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]$$

$$= a\ {}_2F_1\left[\begin{matrix} a+1\ , & -a-51 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] + (a+51)\ {}_2F_1\left[\begin{matrix} a\ , & -a-50 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]$$

Now involving the result which is established in Ref[5], we can establish the main result.

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3. Submission of Manuscripts,

4. Manuscript's Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

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· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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