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Highlights

Perturbed LamÈ System

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Sobolev Function Spaces

Nonlocal Boundary Conditions

Discovering Thoughts, Inventing Future

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Propagation of Stoneley Waves in Couple Stress Generalized Thermoelastic Media

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Abstract - The present paper is concerned with the propagation of Stoneley waves at the interface of two couple stress thermoelastic medium in context with Lord and Shulman (LS) and Green and Lindsay (GL) theories of thermoelasticity. It is observed that shear wave get decoupled from rest of the motion. After developing the formal solution, the secular equation for surface wave propagation is derived. The dispersion curve giving the phase velocity and attenuation coefficient related to wave number are plotted graphically to depict the effect of thermal relaxation times. The amplitude ratios of displacement components and temperature distribution are also computed numerically and shown graphically to depict the effect of thermal relaxation times. Some special cases are also deduced from the present investigation.

Keywords : *stoneley waves, couple stress, thermoelasticity, amplitude ratio, secular equation.*

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PROPAGATION OF STONELEY WAVES IN COUPLE STRESS GENERALIZED THERMOELASTIC MEDIA

Notes

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Rajneesh Kumar [α], Krishan Kumar [σ ρ] & Ravindra C. Nautiyal [σ]

Abstract - The present paper is concerned with the propagation of Stoneley waves at the interface of two couple stress thermoelastic medium in context with Lord and Shulman (LS) and Green and Lindsay (GL) theories of thermoelasticity. It is observed that shear wave get decoupled from rest of the motion. After developing the formal solution, the secular equation for surface wave propagation is derived. The dispersion curve giving the phase velocity and attenuation coefficient related to wave number are plotted graphically to depict the effect of thermal relaxation times. The amplitude ratios of displacement components and temperature distribution are also computed numerically and shown graphically to depict the effect of thermal relaxation times. Some special cases are also deduced from the present investigation.

Keywords : *stoneley waves, couple stress, thermoelasticity, amplitude ratio, secular equation.*

I. Introduction

The study of surface wave propagation along the free boundary of an elastic half-space or along the interface between two dissimilar elastic half-spaces has been a subject of continued interest for many years. These waves are well known in the study of seismic waves, geophysics and non- destructive evaluation, and there is a rich literature available in surface waves in terms of classical elasticity (see, e.g., Achenbach (1973); Brekhoviskikh (1960); Bullen and Bolt (1985); Ewing et al. (1957); Love (1911); Udias (1999)). Surface waves propagating along the free boundary of an elastic half-space, non-attenuated in their direction of propagation and damped normal to the boundary are known as Rayleigh waves in the literature, after their discoverer (Rayleigh (1885)). The phase velocity of Rayleigh wave is a single valued function of the parameters of an elastic half-space. These waves are nondispersive and their velocity is somewhat less than the velocity of shear waves in unbounded media.

Stoneley (1924) investigated the possible existence of waves similar to surface waves and propagating along the plane interface between two distinct uniform elastic solid half-spaces in perfect contact, and these are now universally known by his name. Stoneley waves can propagate at interfaces between either two elastic solid media or an elastic solid medium and a liquid medium. These waves are harmonic waves propagating along the interface between two elastic half-spaces, produce continuous traction and displacement across the interface and attenuate exponentially with distance normal to the

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interface in both the half-spaces, provided the range of the elastic constants of the two solids lie within some suitable limits (Scholte, (1947)). Stoneley (1924) obtained the frequency equation for propagation of Stoneley waves and showed that such interfacial waves can exist only if the velocity of distortional waves in the two half-spaces is approximately same.

Nayfeh and Abdelrahman (2000) found an approximate model for wave propagation in rectangular rods and their geometric limits. Tomar (2005) studied the wave propagation in elastic plates with voids. Some problems of wave propagation in micropolar elasticity medium voids were discussed by Kumar and Deswal (2006). Tomar and Singh (2006) discussed the propagation of Stoneley waves at an interface between two microstretch elastic half-spaces. Kumar et al. (2012) discussed the wave propagation in micropolar thermoelastic layer with two temperatures.

The ideas underlying the couple stress linear theory of elasticity were advanced by Voigt (1887) and the Cosserats (1909), but the subject was generalized and reached maturity only with the works of Toupin (1962), Green and Rivlin (1964) in addition, Kröner (1963) gave a physical aspects pertinent to crystal lattices and non local interpretation of the theory. It is noticed that the earlier application of couple stress elasticity theory, manly on quasi-static problems of stress concentration, met with success providing solution more adequate physically than classic solutions (Mindlin and Tiersten (1962), Weitsman (1965, 1967) Bogy and Sternberg (1967), Lakes (1982)).

Aggarwal and Alverson (1969) studied the effects of couple stresses on the diffraction of plane elastic waves by cylindrical discontinuities. Stefeniak (1969) presented the reflection of plane wave from a free surface in Cosserat medium. Sengupta and Ghosh (1974a, 1974b]) studied the effect of couple stress on surface waves in elastic media and propagation of waves in an elastic layer. Sengupta and Benergi (1978) investigated the effects of couple stresses on propagation of waves in an elastic layer immersed in an infinite liquid.

Anthonie (2000) studied the effects of couple stresses on the elastic bending of beams. Lubarda and Markenscoff (2000) investigated the conservation integrals in couple stress elasticity. Chen and Wang (2001) presented strain gradient theory with couple stress for crystalline solids. Circular inclusions in anti-plane strain couple stress elasticity have been investigated by Lubarda (2003). Selim (2006) studied the orthotropic elastic medium under the effect of initial and couple stresses. Shodja and Ghazisaeidi (2007) studied the effects of couple stresses in anti-plane problems of piezoelectric media with inhomogeneties. Radi (2007) find the effects of characteristic material lengths on mode III crack propagation in couple stress elasticity-plasticity materials by adopting an incremental version of the indeterminate theory couple stress plasticity displaying linear and isotropic strain hardening.

Gourgiotis and Gorgiadis (2008) investigated an approach based on distributed dislocations and dislocations for crack problems in couple stress theory of elasticity. Recently, the inplane orthotropic couple stress elasticity constant of elliptical call honey comb have been studied by Chung and Waas (2010). Kumar et al. (2012a, 2012b) studied the wave propagation in couple stress thermoelastic half space underlying an inviscid liquid layer and the propagation of SH-waves in couple stress elastic half space underlying an elastic layer.

The present study is concerned with the propagation of Stoneley waves in couple stress generalized thermoelastic medium in the context of Lord and Shulman (LS), Green and Lindsay (GL) theories of thermoelasticity. The dispersion curves giving the phase

Notes

velocity and amplitude ratios of displacement components and temperature distribution are obtained and depicted graphically. Some special cases of interest are also studied.

II. Basic Equations

Notes

Following Mindlin and Tiersten (1962), Lord and Shulman (1967), Green and Lindsay (1972) the equations governing the couple stress generalized thermoelastic medium in absence of body forces, the constitutive relations are

$$t_{ji,j} = \rho \ddot{u}_i , \tag{1}$$

$$m_{ji,j} + e_{ijk} t_{jk} = 0 , \tag{2}$$

$$K^* \nabla^2 T - \rho c_e \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) = T_o \beta \left(\frac{\partial u_{i,i}}{\partial t} + \tau_0 n_o \frac{\partial^2 u_{i,i}}{\partial t^2} \right), \tag{3}$$

$$t_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - \frac{1}{2} e_{ijk} m_{lk,l} - \beta \left(1 + \tau_1 \frac{\partial}{\partial t} \right) T \delta_{ij} , \tag{4}$$

$$m_{ij} = 4\alpha k_{ji} + 4\beta_1 k_{ij} , \tag{5}$$

$$k_{ij} = \phi_{j,i} , \tag{6}$$

$$\phi_i = \frac{1}{2} e_{ipq} u_{q,p} , \tag{7}$$

where

u_i are the displacement components, ϕ_i is rotational vector, ρ is density, t_{ij} are stress components, m_{ij} are couple stress components, ε_{ij} are strain components, e_{ijk} is alternate tensor, k_{ij} is curvature tensor, δ_{ij} is Kronecker's delta, ∇^2 is Laplacian operator, τ_0, τ_1 are thermal relaxation times with $\tau_1 \geq \tau_0 \geq 0$. Here $n_o = 1, \tau_1 = 0$, for LS theory and $n_o = 0, \tau_1 > 0$, for GL theory.

With the help of equations (2), (4)-(7) and without loss of generality assuming that $\beta' e_{lpq} e_{ijk} u_{q,pkl} = 0$, equation (1) takes the form

$$u_i = (\quad + \quad) u_{j,ij} + \mu \nabla^2 u_i + \alpha \left(e_{ijk} e_{kpq} u_{q,pi} \right)_{,ll} - \beta \left(1 + \tau_1 \frac{\partial}{\partial t} \right) T_{,i} \tag{8}$$

III. Formulation of the Problem

We consider two homogeneous isotropic couple stress generalized thermoelastic half spaces M_1 and M_2 connecting at the interface $x_3 = 0$. We consider a rectangular Cartesian coordinate system, (x_1, x_2, x_3) at any point on the plane horizontal surface and

x_1-axis in the direction of the wave propagation and x_3-axis taking vertically downward into the half-space so that all particles on a line parallel to x_2-axis are equally displaced, therefore, all the field quantities will be independent of x_2-coordinates. Medium M_2 occupies the region $-\infty < x_3 < 0$ and region $x_3 > 0$ is occupied by the half-space (medium M_1). The plane $x_3 = 0$ represents the interface between two media M_1 and M_2.

We take the displacements components as

$$u_i = (u_1, 0, u_3) \tag{9}$$

With the help of following Helmholtz's representation of displacement components u_1 and u_3 in terms of potentials ϕ and ψ

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3}, \quad u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} \tag{10}$$

and non dimensional quantities

$$\begin{gathered} x_1' = \frac{\omega^*}{c_1} x_1, x_3' = \frac{\omega^*}{c_1} x_3, t' = \omega^* t, t_{ij}' = \frac{t_{ij}}{\beta T_0}, m_{ij}' = \frac{\omega^* m_{ij}}{c_1 \beta T_0}, u_1' = \frac{\omega^*}{c_1} u_1, \\ u_3' = \frac{\omega^*}{c_1} u_3, T' = \frac{T}{T_0}, \tau_1' = \omega^* \tau_1, \tau_0' = \omega^* \tau_0, t_0' = \omega^* t_0, h' = \frac{c_1 h}{\omega^*}, \omega^{*2} = \frac{\lambda^2}{\rho \alpha}, c_1^2 = \frac{\lambda + 2\mu}{\rho}, \end{gathered} \tag{11}$$

where ω^* is the characteristic frequency,
equation (8) with the aid of equations (10) and (11) after suppressing the primes can be written as

$$\nabla^2 \phi - a_5 \left(1 + \tau_1 \frac{\partial}{\partial t}\right) T = \ddot{\phi}, \tag{12}$$

$$\nabla^2 \psi - a_6 \nabla^4 \psi = \frac{c_1^2}{c_2^2} \ddot{\psi}, \tag{13}$$

$$\nabla^2 T - a_7 \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) = a_8 \left(\frac{\partial u_{i,i}}{\partial t} + \tau_0 n_o \frac{\partial^2 u_{i,i}}{\partial t^2}\right), \tag{14}$$

where

$$\begin{gathered} a_1 = \frac{\mu}{\lambda + \mu}, a_2 = \frac{\alpha \omega^{*2}}{c_1^2 (\lambda + \mu)}, a_3 = \frac{\beta T_0}{\lambda + \mu}, a_4 = \frac{\rho c_1^2}{\lambda + \mu}, a_5 = \frac{a_3}{1 + a_1}, a_6 = \frac{a_2}{a_1}, \\ a_7 = \frac{\rho c_e c_1^2}{K^* \omega^*}, a_8 = \frac{c_1^2 \beta}{K^* \omega^*}, a_9 = \frac{\lambda}{\beta T_0}, a_{10} = \frac{\mu}{\beta T_0}, a_{11} = \frac{\alpha \omega^{*2}}{c_1^2 \beta T_0}, a_{12} = \frac{2 \alpha \omega^{*2}}{c_1^2 \beta T_0}, \\ a_{13} = \frac{2 \beta_1 \omega^{*2}}{c_1^2 \beta T_0} \end{gathered} \tag{15}$$

Notes

The equation (13) corresponds to purely transverse wave mode that decouples from rest of the motion and is not affected by the thermal effect.

IV. Solution of the Problem

We assume the solution of equations (12)-(14) of the form

$$(\phi,\psi,T)=(\phi_1(x_3),\psi_1(x_3),T_1(x_3))e^{i(kx_1-\omega t)} \tag{16}$$

where $c=\frac{\omega}{k}$ is the non-dimensional phase velocity, ω is the frequency and k is the wave number. Substituting the values ϕ,ψ,T from equation (16) in equations (12)-(14) and satisfying the radiation condition as $\phi,\ \psi,\ T\to 0$ as $x_3\to\infty$, we obtain the values of ϕ,ψ,T for medium M_1,

$$\phi=\left(A_1e^{-m_1x_3}+A_2e^{-m_2x_3}\right)e^{i(kx_1-\omega t)}, \tag{17}$$

$$\psi=\left(B_1e^{-m_3x_3}+B_2e^{-m_4x_3}\right)e^{i(kx_1-\omega t)}, \tag{18}$$

$$T=\left(b_1A_1e^{-m_1x_3}+b_2A_2e^{-m_2x_3}\right)e^{i(kx_1-\omega t)} \tag{19}$$

We attach bars for the medium M_2 and write the appropriate values of $\bar{\phi},\bar{\psi},\bar{T}$ for $M_2(x_3<0)$ satisfying the radiation conditions as:

$$\bar{\phi}=\left(\bar{A}_1e^{\bar{m}_1x_3}+\bar{A}_2e^{\bar{m}_2x_3}\right)e^{i(kx_1-\omega t)}, \tag{20}$$

$$\bar{\psi}=\left(\bar{B}_1e^{\bar{m}_3x_3}+\bar{B}_2{}^{\bar{m}_4x_3}\right)e^{i(kx_1-\omega t)},$$

$$\bar{T}=\left(\bar{b}_1\bar{A}_1e^{\bar{m}_1x_3}+\bar{b}_2\bar{A}_2e^{\bar{m}_2x_3}\right)e^{i(kx_1-\omega t)} \tag{22}$$

where m_1,m_2 are the roots of equation

$$\frac{d^4}{dx_3^4}+S\frac{d^2}{dx_3^2}+P=0 \tag{23}$$

and m_3,m_4 are the roots of equation

$$\frac{d^4}{dx_3^4}+S_1\frac{d^2}{dx_3^2}+P_1=0 \tag{24}$$

where

$$S=\left(A-2k^2\right),P=\left(k^4-k^2A+B\right),l_4=-\left(\frac{c_1^2\omega^2}{a_6c_2^2}\right),$$

$$S_1=-\left(\frac{1}{a_6}\right)-2k^2,P_1=k^4+\frac{k^2}{a_6}-l_4,b_i=-\left(\frac{a_8\omega hl_3}{m_i^2-k^2+a_7\omega^2l_2+ia_8\omega^3l_3a_5l_1}\right),i=1,2,$$

$$A=\left(\omega^2+l_2a_7\omega^2+a_5i\omega l_1a_8\omega^2l_3\right),\ B=\left(\omega^4l_2a_7\right),l_1=\tau_1+i\omega^{-1},l_2=\tau_0+i\omega^{-1},l_3=\tau_0n_0+i\omega^{-1}$$

Substituting the values of $\phi, \psi, T, \bar{\phi}, \bar{\psi}$ and $\bar{T}$ from equations (17)-(22) in equation (10), we obtained the displacement components

For medium M_1

$$u_1 = A_1 ike^{-m_1 x_3} + ikA_2 e^{-m_2 x_3} + m_3 B_1 e^{-m_3 x_3} + m_4 B_2 e^{-m_4 x_3}, \tag{25}$$

$$u_3 = -m_1 A_1 e^{-m_1 x_3} - m_2 A_2 e^{-m_2 x_3} + ikB_1 e^{-m_3 x_3} + ikB_2 e^{-m_4 x_3}, \tag{26}$$

For medium M_2

$$\bar{u}_1 = ik\bar{A}_1 e^{\bar{m}_1 x_3} + ik\bar{A}_2 e^{\bar{m}_2 x_3} - \bar{m}_3 \bar{B}_1 e^{\bar{m}_3 x_3} - \bar{m}_4 \bar{B}_2 e^{\bar{m}_4 x_3}, \tag{27}$$

$$\bar{u}_3 = \bar{m}_1 \bar{A}_1 e^{\bar{m}_1 x_3} \bar{m}_2 \bar{A}_2 e^{\bar{m}_2 x_3} + ik\bar{B}_1 e^{\bar{m}_3 x_3} + ik\bar{B}_2 e^{\bar{m}_4 x_3}. \tag{28}$$

V. BOUNDARY CONDITIONS

a) Mechanical Conditions

The boundary conditions are the continuity of normal stress, tangential stress, tangential couple stress, displacement components, temperature and rotations vector at the interface of elastic half spaces. Mathematically these can be written as

$$\left.\begin{aligned} t_{33} &= \bar{t}_{33} \\ t_{31} &= \bar{t}_{31} \\ m_{32} &= \bar{m}_{32} \\ u_1 &= \bar{u}_1 \\ u_3 &= \bar{u}_3 \\ \phi_2 &= \bar{\phi}_2 \end{aligned}\right\} \text{ at } x_3 = 0 \tag{29}$$

b) Thermal Condition

The thermal condition corresponding to insulated boundary is given by

$$\left.\begin{aligned} T &= \bar{T}, \\ K^* \frac{\partial T}{\partial x_3} &= \bar{K}^* \frac{\partial \bar{T}}{\partial x_3} \end{aligned}\right\} \text{ at } x_3 = 0. \tag{30}$$

Using equations (6) and (7) with aid of (11) in equations (4) and (5), we obtain the values of t_{ij} and m_{ij} as

$$t_{ij} = a_9 \delta_{ij} e + a_{10}\left(u_{i,j} + u_{j,i}\right) - a_{11} e_{ijk} e_{kpq} u_{q,pll} - \beta\left(1 + \tau_1 \frac{\partial}{\partial t}\right) T \delta_{ij}, \tag{31}$$

$$m_{ij} = a_{12} e_{ipq} u_{q,pj} + a_{13} e_{jpq} u_{q,pi}. \tag{32}$$

With the help of equations (31) and (32) with aid of (17)-(22) and (25)-(28) from equations (29) and (30), we obtain a system of eight homogeneous simultaneous linear equations and after some simplification, we obtain the secular equation as

Notes

$$c^2 = \frac{-\omega^2 a_9}{QQ}, \tag{33}$$

where

$$QQ = P_{12}Q_1 - P_{13}Q_2 + P_{13}Q_3 - P_{14}Q_4 + P_{15}Q_5 - P_{16}Q_6 + P_{17}Q_7 - P_{18}Q_8,$$

Notes

$$Q_1 = \frac{D2}{D1}, Q_2 = \frac{D3}{D1}, Q_3 = \frac{D4}{D1}, Q_4 = \frac{D5}{D1}, Q_6 = \frac{D6}{D1}, Q_7 = \frac{D7}{D1}, Q_8 = \frac{D8}{D1},$$

$$P_{11} = -a_9k^2 + (a_9 + a_{10})m_1^2 + b_1l_1, P_{12} = -a_9k^2 + (a_9 + a_{10})m_2^2 + b_2l_1, P_{13} = -ia_{10}km_3,$$

$$P_{14} = -ia_{10}km_4, P_{15} = -\bar{a}_9k^2 + (\bar{a}_9 + \bar{a}_{10})\bar{m}_1^2 + (b_1)^2 l_1, P_{16} = -\bar{a}_9k^2 + (\bar{a}_9 + \bar{a}_{10})\bar{m}_2^2 + (b_1)^2 l_1,$$

$$P_{17} = -i\bar{a}_{10}km_3, P_{18} = -i\bar{a}_{10}km_4, P_{21} = -2ika_{10}m_1 - a_{11}\left[(m_1)^4 + m_1ik^3\right],$$

$$P_{22} = -2ika_{10}m_2 - a_{11}\left[(m_2)^4 + m_2ik^3\right], P_{23} = -a_{10}\left[m_3^2 + k^2\right] - a_{11}\left[ikm_3^3 + 2m_3^2k^2 - m_3^4\right],$$

$$P_{24} = -a_{10}\left[m_4^2 + k^2\right] - a_{11}\left[ikm_4^3 + 2m_4^2k^2 - m_4^4\right], P_{25} = -2i\bar{a}_{10}k\bar{m}_1, P_{25} = -2i\bar{a}_{10}k\bar{m}_2,$$

$$P_{27} = \bar{a}_{10}\left[\bar{m}_3^2 + k^2\right] + \bar{a}_{11}\left[-k^4 + 2\bar{m}_3^2k^2 - \bar{m}_3^4\right], P_{28} = \bar{a}_{10}\left[\bar{m}_4^2 + k^2\right] + \bar{a}_{11}\left[-k^4 + 2\bar{m}_4^2k^2 - \bar{m}_4^4\right],$$

$$P_{33} = a_{12}\left((m_3)^3 + k^2m_3\right), P_{34} = a_{12}\left((m_4)^3 - k^2m_4\right), P_{37} = \bar{a}_{12}\left((\bar{m}_3)^3 + k^2\bar{m}_3\right),$$

$$P_{38} = \bar{a}_{12}\left((\bar{m}_4)^3 + k^2\bar{m}_4\right), P_{41} = -b_1km_1, P_{42} = -b_2km_2, P_{45} = -\bar{b}_1k\bar{m}_1, P_{46} = -\bar{b}_1k\bar{m}_2, P_{51} = ik,$$

$$P_{52} = ik, \; P_{53} = m_3, \; P_{54} = m_4, P_{55} = P_{56} = -ik, \; P_{57} = \bar{m}_3, P_{58} = \bar{m}_4, P_{61} = -m_1, P_{62} = -m_2,$$

$$P_{63} = P_{64} = ik, P_{65} = -\bar{m}_1, P_{66} = -\bar{m}_2, P_{67} = P_{68} = -ik, P_{71} = b_1, P_{72} = b_2, P_{75} = -\bar{b}_1, P_{76} = -\bar{b}_2,$$

$$P_{81} = -2ikm_1, \; P_{82} = -2ikm_2, P_{83} = -\left\{(m_2)^2 + k^2\right\}, P_{84} = -\left\{(m_4)^2 + k^2\right\},$$

$$P_{87} = \left\{(\bar{m}_3)^2 - k^2\right\}, P_{88} = \left\{(\bar{m}_4)^2 - k^2\right\}$$

and

$$D1 = \begin{bmatrix} P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{42} & 0 & 0 & P_{45} & P_{46} & 0 & 0 \\ P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & 0 & 0 \\ P_{82} & P_{83} & P_{84} & 0 & 0 & P_{87} & P_{88} \end{bmatrix}, \quad D2 = \begin{bmatrix} P_{21} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{41} & 0 & 0 & P_{45} & P_{46} & 0 & 0 \\ P_{51} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{73} & P_{74} & P_{75} & P_{76} & 0 & 0 \\ P_{81} & P_{83} & P_{84} & 0 & 0 & P_{87} & P_{88} \end{bmatrix},$$

$$D3=\begin{bmatrix} P_{21} & P_{22} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{41} & P_{42} & 0 & P_{45} & P_{46} & 0 & 0 \\ P_{51} & P_{52} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{74} & P_{75} & P_{76} & 0 & 0 \\ P_{81} & P_{82} & P_{84} & 0 & 0 & P_{87} & P_{88} \end{bmatrix}, \quad D4=\begin{bmatrix} P_{21} & P_{22} & P_{23} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{41} & P_{42} & 0 & P_{45} & P_{46} & 0 & 0 \\ P_{51} & P_{52} & P_{53} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{75} & P_{76} & 0 & 0 \\ P_{81} & P_{82} & P_{83} & 0 & 0 & P_{87} & P_{88} \end{bmatrix},$$

$$D5=\begin{bmatrix} P_{21} & P_{22} & P_{23} & P_{24} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{36} & P_{37} & P_{38} \\ P_{41} & P_{42} & 0 & 0 & P_{46} & 0 & 0 \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{76} & 0 & 0 \\ P_{81} & P_{82} & P_{83} & P_{84} & 0 & P_{87} & P_{88} \end{bmatrix}, \quad D6=\begin{bmatrix} P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{37} & P_{38} \\ P_{41} & P_{42} & 0 & 0 & P_{45} & 0 & 0 \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & 0 & 0 \\ P_{81} & P_{82} & P_{83} & P_{84} & 0 & P_{87} & P_{88} \end{bmatrix},$$

$$D7=\begin{bmatrix} P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{38} \\ P_{41} & P_{42} & 0 & 0 & P_{45} & P_{46} & 0 \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & 0 \\ P_{81} & P_{82} & P_{83} & P_{84} & 0 & P_{86} & P_{88} \end{bmatrix} \text{ and } D8=\begin{bmatrix} P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} \\ P_{41} & P_{42} & 0 & 0 & P_{45} & P_{46} & 0 \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & 0 \\ P_{81} & P_{82} & P_{83} & P_{84} & 0 & P_{86} & P_{87} \end{bmatrix}.$$

VI. Amplitude Ratios

The amplitude ratios of displacement components and temperature distribution at the interface $x_3 = 0$ for insulated boundary are given by

$$\frac{A_2}{A_1} = -\frac{P_{41}}{P_{42}}, \ \frac{B_1}{A_1} = L_1 L_2, \ \frac{B_2}{A_1} = L_3 - \frac{P_{41}}{P_{42}} L_1 L_2,$$

where

$$L_1 = -\frac{P_{21}}{P_{25}} + \frac{P_{22}P_{41}}{P_{42}P_{24}} + \frac{P_{11}P_{23}}{P_{24}P_{13}} - \frac{P_{23}P_{12}P_{24}}{P_{13}P_{42}P_{24}}, L_2 = \frac{P_{24}P_{13}}{(P_{24}P_{13} - P_{23}P_{14})}, L_3 = -\frac{P_{11}}{P_{13}} + \frac{P_{12}P_{14}}{P_{13}P_{24}}.$$

VII. Special Cases

For LS theory: Taking $\tau_1 = 0, n_0 = 1$ in equation (33), we obtain the secular equation for couple stress thermoelastic media with one relaxation time with the changed values of l_1, l_3 as

$$l_1 = i\omega^{-1}, l_3 = i\omega + \tau_0$$

For GL theory: In this case, taking $n_0 = 0$ in equation (33), we obtain the secular equation in couple stress thermoelastic medium with two relaxation times with the changed values of l_3 as

$$l_3 = i\omega$$

For C-T theory: Taking $\tau_1 = \tau_0 = 0$, in equation (33), yield the secular equation in couple stress thermoelastic medium with the changed values of l_1, l_2, l_3 as

$$l_1 = l_2 = l_3 = i\omega^{-1}$$

VIII. Numerical Discussion

With view of illustrating theoretical results derived in preceding sections, we now present some numerical results in this section. The physical data for medium M_1 and M_2 is given by

For medium M_1:

$$\lambda = 2.17 N/m^2, \mu = 3.28 N/m^2, \rho = 1.74 kg/m^3, c_e = 1.04 \times 10^3 Jkg^{-1} \deg^{-1},$$

$$K^* = 1.7 \times 10^2 Wm^{-1} \deg^{-1}, T_0 = .298^o K, \alpha = 2.05 N, \beta = 2.68 Nm^{-1} \deg^{-1}, \beta_1 = 1.5N$$

For Medium M_2:

$$\bar{\lambda} = 2.238 N/m^2, \bar{\mu} = 2.992 N/m^2, \rho = 2.65 kg/m^3, \bar{c}_e = 0.021 \times 10^3 Jkg^{-1} \deg^{-1},$$

$$\bar{K}^* = 2.4 \times 10^2 Wm^{-1} \deg^{-1}, \bar{T}_0 = .296^o K, \bar{\alpha} = 1.9 N, \bar{\beta} = 2.03 Nm^{-1} \deg^{-1}, \beta_1' = 2.4N$$

Using the above values of parameters, the dispersion curves of non-dimensional phase velocity, attenuation coefficient and amplitude ratios are shown graphically with respect to the non-dimensional wave number for insulated boundary. CGL region corresponds to the coupled theory of Green-Lindsay, region CLS corresponds to coupled theory of Lord-Shulman, region GL corresponds to Green-Lindsay and the region LS corresponds to Lord-Shulman theory.

From fig.1 it is noticed that the values of phase velocity increase for all the theories CLS, CGL, LS and GL for small values of wave number whereas the values of phase velocity decrease for large values of the wave number and the values for CGL remain higher than the other theories.

Notes

From fig.2 we observed that the values of attenuation coefficient decrease sharply for all the theories then increase monotonically for all the theories. The values for GL are higher for small values of wave number than CLS, CGL and LS for $0 \le k \le 1.15$ whereas the values for CGL remain higher than the other theories for intermediate values of wave number. But the values for LS are higher than the other as the values of the wave number increase.

Figs. 3, 4 and 5 are showing the behaviors of amplitude ratios of displacements and temperature with respect to wave number. The pattern of amplitude ratios in fig.3 is very similar. It decreases for all the four theories CLS, CGL, LS and GL. From fig.4 it is noticed that the values of amplitude ratio for CLS and CGL increase whereas it decrease for LS and GL theories. From fig. 5 it can be observed that the amplitude ratios increase for CLS and CGL theories.

Notes

IX. Conclusion

The propagation of Stoneley waves at the interface of two couple stress thermoelastic half spaces in the context of LS and GL theories of thermoelasticity have been studied. It has been observed that the transverse wave mode is not affected by thermal effect and get decoupled from rest of motion. The behavior and trends of variation of phase velocity is similar for CGL, CLS, LS and GL theories with difference in their magnitude values whereas the behavior of the attenuation coefficient is similar for all the theories. Appreciable effect of thermal relaxation times is obtained on phase velocity, attenuation coefficient and amplitude ratios of displacement components and thermal distribution.

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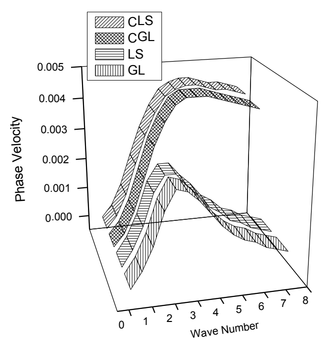

Figure 1 : variation of phase velocity w. r. t. wave number

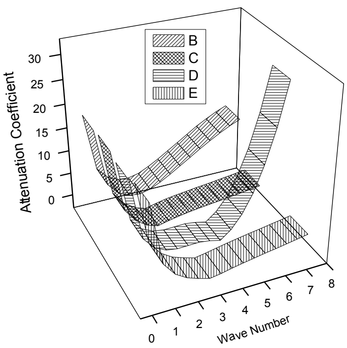

Figure 2 : variation of attenuation coefficient w. r. t. wave number

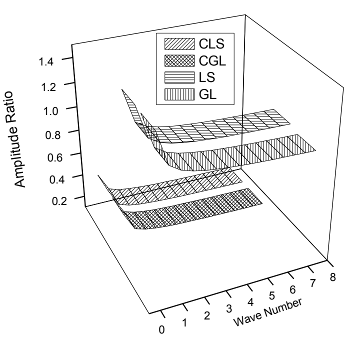

Figure 3 : variation of tangential amplitude verses wave number

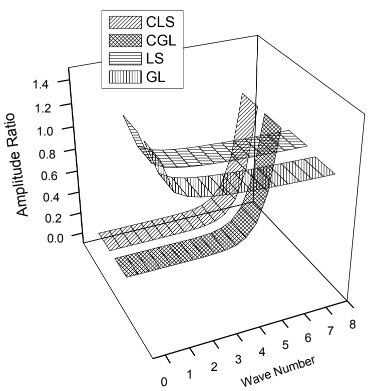

Figure 4 : variation of normal amplitude verses wave number

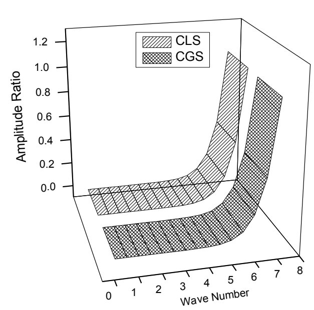

Figure 5 : variation of temperature distribution verses wave number

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Unsteady Couette Flow with Transpiration in a Rotating System

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Abstract - The unsteady Couette flow with transpiration of a viscous fluid in a rotating system has been considered. An exact solution of the governing equations has been obtained by using Laplace Transform Technique. Solutions for velocity distributions and the shear stresses have been obtained for small time $\boldsymbol{\tau} = \mathbf{0.05}$ as well as large time $\boldsymbol{\tau} = \mathbf{10.0}$. it is found that for small times the primary velocity profile increases with decrease in $\boldsymbol{K}^2$ with constant $\boldsymbol{Re}$ while the secondary velocity profile decreases with decrease in $\boldsymbol{K}^2$. It is also found that for large times, the primary flow increases with increase in $\boldsymbol{K}^2$, the secondary velocity behaves in an oscillatory manner near the moving plate and increases near the stationary plate. There exists a back flow in the region $\mathbf{0.0} \ll \boldsymbol{\varphi} \ll \mathbf{1.0}$. The shear stress due to primary flow decreases with increase in $\boldsymbol{K}^2$. On the other hand, it increases due to secondary flow with increase in rotation parameter with constant $\boldsymbol{Re}$ for small times. It is also observed that the shear stress for large time with constant $\boldsymbol{Re}$ shows layers of separation in both primary and secondary flow due to high rotation..

Keywords : *couette flow, transpiration, rotating system, shear stress.*

GJSFR-F Classification : *MSC 2010: 76U05*

UNSTEADY COUETTE FLOW WITH TRANSPIRATION IN A ROTATING SYSTEM

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Idowu, A.S [α], Joseph K. M [σ], Are, E.B [ρ] & Daniel K.S [ω]

Abstract - The unsteady Couette flow with transpiration of a viscous fluid in a rotating system has been considered. An exact solution of the governing equations has been obtained by using Laplace Transform Technique. Solutions for velocity distributions and the shear stresses have been obtained for small time $\boldsymbol{\tau} = \mathbf{0.05}$ as well as large time $\boldsymbol{\tau} = \mathbf{10.0}$. it is found that for small times the primary velocity profile increases with decrease in $\boldsymbol{K^2}$ with constant $\boldsymbol{Re}$ while the secondary velocity profile decreases with decrease in $\boldsymbol{K^2}$. It is also found that for large times, the primary flow increases with increase in $\boldsymbol{K^2}$, the secondary velocity behaves in an oscillatory manner near the moving plate and increases near the stationary plate. There exists a back flow in the region$\mathbf{0.0} \ll \boldsymbol{\varphi} \ll \mathbf{1.0}$. The shear stress due to primary flow decreases with increase in $\boldsymbol{K^2}$. On the other hand, it increases due to secondary flow with increase in rotation parameter with constant $\boldsymbol{Re}$ for small times. It is also observed that the shear stress for large time with constant $\boldsymbol{Re}$ shows layers of separation in both primary and secondary flow due to high rotation.

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Nomenclature

Re	Reynolds number
τ	Time
Ω	Uniform angular velocity
d	Distance between the two parallel plates
U	Uniform velocity
u	Velocity component along $x - direction$
v	Velocity component along $y - direction$
K^2	Rotation parameter which is the reciprocal of Ekman number
u_1	Velocity distribution due to primary flow
v_1	Velocity distribution due to secondary flow
$q(\varphi, \tau)$	Velocity distribution
τ_x	Non − dimensional shear stress due to primary flow for small time τ
τ_y	Non − dimensional shear stress due to secondary flow for small time τ
τ_{x_0}	Non − dimensional shear stress due to primary flow for large time τ
τ_{y_0}	Non − dimensional shear stress due to secondary flow for large time τ

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Author ω : Department of Mathematics, Kaduna State University.

$\nu = \frac{\mu}{\rho}$ Kinematic coefficient of viscosity

w_0 Transpiration parameter

φ Non−dimensional variable

I. Introduction

The study of the flow of a viscous incompressible fluid in rotating frame of reference has drawn considerable interest in recent years due to its wide applications in designing thermo siphon tubes, in cooling turbine blades, etc. Also Hydrodynamic coquette flow in a rotating system is a challenging approach to atmospheric Science that exerts its influence of rotation to help in understanding the behavior of Oceanic circulation and the formation of galaxies in taking into account the flow of electron is continuously liberated from the sun what is called Solar Wind. Several investigations have been carried out on various types of flow in a rotating frame of reference. Couette flow in a rotating systems leads to a start up process implying thereby a viscous layer boundary is suddenly set into motion and the rate of rotation becomes important in the application of geophysics and fluid engineering as already stated in such a way that the problem is to be analysed by the effect of rotation for small as well as large time τ. However, Vidyanidhi and Nigam (1967) expedite the situation of a secondary flow in a rotating channel by taking into account of the flow of a viscous fluid between two parallel plates in a rotating system with uniform angular velocity about an axis perpendicular to their planes under the influence of a constant pressure gradient. Barik, et al (2012) made an attempt to study the effects of heat and mass transfer on the flow over stretching sheet in the presence of a heat source. Seth et al (2011) studied MHD couette flow of class II in a rotating system. Jana and Datta (1977) have studied the steady Couette flow and heat transfer in a rotating system. Mazumder (1991) investigated an exact solution of an oscillatory Couette flow in a rotating system. Khaled (2012) also studied the transient magnetohydrodynamic in mixed double convection along a vertical plate embedded in a non − Darcy porous medium. Al − Odat (2012) studied transient non − Darcy mixed convection during a vertical surface in porous medium with such suction or injection. Seth et al. (1982) have studied unsteady Couette Flow in a rotating system. Our present problem is to study the unsteady Couette flow with transpiration confined between two plates, rotating with the same angular velocity Ω about an axis perpendicular to their planes and the flow is induced due to the motion of the upper plate. When the upper plate is not moving with uniform velocity, we have the problem considered by Berker (1979). It is important to note that when the upper plate is moving there is no reason to expect only the axially symmetric solution. In this context, it may be noted that Berker [4] established the existence of non − axisymmetric solution for the flow of an incompressible viscous fluid between two disk rotating about a common axis with d same angular velocity. In relating to the fact that Rajagopal [17] has studied, the now of viscoelastic fluid between two rotating disks is a decisive importance to an asymmetric solution of this problem. Singh, Gorla and Rajhans [20] investigated a periodic solution of oscillatory couette flow through a porous medium in a rotating system. Parter and Rajagopal [16] investigated the problem of flow of an incompressible viscous fluid between two parallel disks about a common axis with different angular speeds. They rigorously proved that the problem admits of solution that lacks axial symmetry. With regard to the existence theorem of Parter and Rajagopal it is pointed out that an extensive numerical computation were carried out by Lai et al. [13] with their study of non − axisymmetric

Ref.

4. Berker R. (1979), Arch. Mech. Stosow. 31, 265

flow between two parallel rotating disks. It becomes a significant approach to a viscous flow above a single rotating disk which gives a similar result as studied by Lai et al. [14]. Erdogan [6] has considered the unsteady flow induced by the rotation about non − coincident axes while both of the disks are initially rotating with the same angular velocity about a common axis. An extension of this problem of the unsteady flow produced by a sudden coincidence of the axes while two disks are initially rotating with the same angular velocity about non − coincident axes was studied by Ersoy [7]. Greenspan [9] devoted to the theory of unsteady hydrodynamic Couette flow in a rotating system.

The aim and objective of this our present work deals with the time development of a start up Couette flow with transpiration in a rotating frame of reference for small time τ and large time τ where the frictional layer of the upper plate is suddenly set into motion with uniform velocity U with the transparency of a rigid body rotation. A remarkable feature of this problem is to discuss the physical insight into the flow pattern for small time τ by applying Laplace transform technique. The velocity distributions and the shear stresses due to the primary and secondary flows are obtained for small as well as large time τ.

II. Formulation of the Problem

Consider the unsteady flow of a viscous incompressible fluid confined between parallel plates at a distance d apart, rotating with uniform angular velocity Ω about an axis normal to the plate with transpiration w_0. We choose the coordinate system in such a way that the $x-axis$ is along the plate (stationary plate) and $z-axis$ is perpendicular to it and $y-axis$ normal to the $xy-plane$. The flow is induced by motion of the upper plate while the lower plate is kept fixed. The upper upper plate moves with uniform velocity $U\ (t>0)$ in the $x-direction$. Since the plates are infinite along $x\ and\ y-directions$, all physical quantities will be the functions of $z\ and\ t$ only.

Denoting velocity $u\ and\ v$ along $x\ and\ y-directions$, respectively, the Navier Stoke equations in a rotating frame of reference are

$$\frac{\partial u}{\partial t}-w_0\frac{\partial u}{\partial z}=\nu\frac{\partial^2 u}{\partial z^2}+2\Omega v \tag{2.1}$$

$$\frac{\partial v}{\partial t}-w_0\frac{\partial v}{\partial z}=\nu\frac{\partial^2 v}{\partial z^2}-2\Omega u \tag{2.2}$$

Where $\nu=\frac{\mu}{\rho}$ is the kinematic coefficient of viscosity and w_0 is the transpiration parameter.

The initial and boundary conditions are

$$u=v=0\ \ for\ t\le 0, 0\le z\le d \tag{2.3}$$

$$u=v=0\ \ at\ \ z=0, t>0$$

$$u=U, v=0\ \ at\ z=0, t>0 \tag{2.4}$$

Let us define the dimensionless quantities as

$$u_1=\frac{u}{U}, v_1=\frac{v}{U}, \varphi=\frac{z}{d}, \tau=\frac{\nu t}{d^2}, w_0=\frac{Rev}{d} \tag{2.5}$$

Ref

6. Erdogan M. E. (1995), Unsteady Flow Induced by Rotation about Non − Coincident axes, Int. J. Non − Linear Mech. 30, 711.

Therefore, as a consequence of equation (2.5), equations (2.1) and (2.2) become

$$\frac{\partial u_1}{\partial \tau} - w_0 \frac{\partial u_1}{\partial \varphi} = \nu \frac{\partial^2 u_1}{\partial \varphi^2} + 2K^2 v_1 \tag{2.6}$$

$$\frac{\partial v_1}{\partial \tau} - w_0 \frac{\partial v_1}{\partial \varphi} = \nu \frac{\partial^2 v_1}{\partial \varphi^2} - 2K^2 u_1 \tag{2.7}$$

Where $K^2 = \frac{\Omega d^2}{\nu}$ is the rotating parameter which is the reciprocal of Eckman number.

Combining equations (2.6) and (2.7), we get

$$\frac{\partial q}{\partial \tau} - Re \frac{\partial q}{\partial \varphi} = \nu \frac{\partial^2 q}{\partial \varphi^2} - 2iK^2 q \tag{2.8}$$

Where $$q = u_1 + iv_1 \, , i = \sqrt{-1} \tag{2.9}$$

The initial and boundary conditions (2.3) and (2.4) become

$$q = 0 \;\; for\, \tau \leq 0, 0 \leq \varphi \leq 1 \tag{2.10}$$

$$q = 0 \;\; at\; \varphi = 0, \tau > 0$$

$$q = 1 \;\; at\; \varphi = 1, \tau > 0 \tag{2.11}$$

III. The Laplace Transform Technique Solution

In this section, the Laplace transforms technique which leads to the solution of equation (2.8) together with the initial and boundary condition (2.10) and (2.11) is described.

Substituting,

$$q(\varphi, \tau) = F(\varphi, \tau) e^{-2iK^2} \tag{3.1}$$

Equation (2.8) becomes

$$\frac{\partial F}{\partial \tau} - Re \frac{\partial F}{\partial \varphi} = \frac{\partial^2 F}{\partial \varphi^2} \tag{3.2}$$

The initial and boundary conditions (2.10) and (2.11) become

$$F(0,\tau) = 0,\;\; F(0,\tau) = 0 \;\; and\, F\,(1,\tau) = e^{2iK^2\tau} \tag{3.3}$$

Using the Laplace transform technique, equation (3.2) becomes

$$s\bar{F} = \frac{\partial^2 \bar{F}}{\partial \varphi^2} + Re \frac{\partial \bar{F}}{\partial \varphi} \tag{3.4}$$

Where

$$\bar{F} = \int_0^\infty F(\varphi, \tau) d\tau \tag{3.5}$$

Finally, the boundary conditions (3.3) take the form

$$\bar{F}(0) = 0 \; and \; \bar{F}(1) = \frac{1}{s - 2iK^2} \tag{3.6}$$

Notes

The solution of equation (3.4) together with conditions (3.6) is

$$\bar{F}(\varphi,s)=\frac{\sinh\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)\varphi}{s-2iK^2\sinh\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)} \tag{3.7}$$

Equation (3.7) is the solution of the problem. Since the flow is unsteady, we shall discuss the following cases:

Case 1: Solutions for sma ll time τ. In this case the method used by Ersoy [7] is used because it converges rapidly for small time $(\tau \ll 1)$ corresponding to large $s(\gg 1)$ and thus equation (3.7) can be written using Lorent's expansion as

$$\bar{F}(\varphi,s)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2iK^2)^2}{s^{n+1}}\left[e^{-(2m+1-\varphi)\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)}-e^{-(2m+1+\varphi)\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)}\right] \tag{3.8}$$

$$F(\varphi,\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2iK^2)^n(4\tau)^n e^{-\frac{Re}{2}}\left\{i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}}+2Re\sqrt{\tau}\right)+\exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}}-2Re\sqrt{\tau}\right)\right]-i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1+\varphi}{2}\right)\right)erfc\left(\frac{2m+1+\varphi}{8\sqrt{\tau}}+2Re\sqrt{\tau}\right)+\exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}}-2Re\sqrt{\tau}\right)\right]\right\} \tag{3.9}$$

Where

$$i^n erfc(x)=\int_x^{\infty} i^{n-1}erfc(\zeta)d\zeta$$

$$ierfc(x)=\int_x^{\infty} erfc(\zeta)d\zeta$$

$$i^0 erfc(x)=\operatorname{erf}(x) \tag{3.10}$$

The solution (3.9) can be written as

$$F(\varphi,\tau)=\sum_{n=0}^{\infty}(2K^2)^n i^n(4\tau)^n T_r \qquad r=0,2,4,6\ldots \tag{3.11}$$

Where

$$T_r=\sum_{m=0}^{\infty}e^{-\frac{Re}{2}}\left\{i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}}+2Re\sqrt{\tau}\right)+\exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}}-2Re\sqrt{\tau}\right)\right]-i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1+\varphi}{2}\right)\right)erfc\left(\frac{2m+1+\varphi}{8\sqrt{\tau}}+2Re\sqrt{\tau}\right)+\exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}}-2Re\sqrt{\tau}\right)\right]\right\}\quad r=0,2,4,6,8,\ldots \tag{3.12}$$

Ref

5. Dash S, Maji S, L, Guria M, Jana R. N.(2009), Unsteady Couette Flow in a Rotating System, Mathematical and Comp. Modeling, 50, 1211 – 1217.

On separating real and imaginary parts, we get the velocity distributions for primary and secondary flow respectively as follows

$$u_1 = e^{-\frac{Re}{2}}\{[T_0 - (2K^2)^2(4\tau)^2T_4 + (2K^2)^4(4\tau)^4T_8 + (2K^2)^6(4\tau)^6T_{12} \ldots]cos2K^2\tau + [(2K^2)(4\tau)T_2 - (2K^2)^3(4\tau)^3T_6 + (2K^2)^5(4\tau)^5T_{10} + (2K^2)^7(4\tau)^7T_{14} \ldots]sin2K^2\tau\} \quad (3.13)$$

$$v_1 = e^{-\frac{Re}{2}}\{[(2K^2)(4\tau)T_2 - (2K^2)^3(4\tau)^3T_6 + (2K^2)^5(4\tau)^5T_{10} + (2K^2)^7(4\tau)^7T_{14} \ldots]cos2K^2\tau - [T_0 - (2K^2)^2(4\tau)^2T_4 + (2K^2)^4(4\tau)^4T_8 + (2K^2)^6(4\tau)^6T_{12} \ldots]sin2K^2\tau\} \quad (3.14)$$

Differentiating equation (3.12) with respect to φ, we get

$$\frac{dT_r}{d\varphi} = e^{-\frac{Re}{2}}\left\{i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}} + 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right) + \exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}} - 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)\right] - i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1+\varphi}{2}\right)\right)erfc\left(\frac{2m+1+\varphi}{8\sqrt{\tau}} + 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right) + \exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}} - 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)\right]\right\} \quad (3.15)$$

This gives

$$\frac{dT_r}{d\varphi} = Y_{r-1}\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)e^{-\frac{Re}{2}}, r = 0,1,2,3, \ldots \quad (3.16)$$

Where

$$Y_{r-1} = \left\{i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}} + 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right) + \exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}} - 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)\right] - i^{2n}\frac{1}{2}\left[\exp\left(Re\left(\frac{2m+1+\varphi}{2}\right)\right)erfc\left(\frac{2m+1+\varphi}{8\sqrt{\tau}} + 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right) + \exp\left(-Re\left(\frac{2m+1-\varphi}{2}\right)\right)erfc\left(\frac{2m+1-\varphi}{8\sqrt{\tau}} - 2Re\sqrt{\tau}\right)\left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)\right]\right\}, r = 0,1,2.3, \ldots \quad (3.17)$$

The non – dimensional shear stress can be obtained as

$$\frac{dq}{d\varphi} = \left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)\sum_{n=0}^{\infty}(2K^2)^n i^n (4\tau)^n Y_{n-1}\, e^{-2iK^2 - \frac{Re}{2}} \quad (3.18)$$

The non – dimensional shear stresses due to primary flows and secondary flows at the plate $\varphi = 0$ can be respectively written as

$$\tau_x = \left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right)e^{-\frac{Re}{2}}\{[Y_{-1} - (2K^2)^2(4\tau)^2Y_3 + (2K^2)^4(4\tau)^4Y_7 + (2K^2)^6(4\tau)^6Y_{11} \ldots]cos2K^2\tau + [(2K^2)(4\tau)Y_1 - (2K^2)^3(4\tau)^3Y_5 + (2K^2)^5(4\tau)^5Y_9 + (2K^2)^7(4\tau)^7Y_{13} \ldots]sin2K^2\tau\}_{\varphi=0} \quad (3.19)$$

Notes

$$\tau_y = \left(\frac{1}{8\sqrt{\tau}} - \frac{Re}{2}\right) e^{-\frac{Re}{2}} \{[(2K^2)(4\tau)Y_1 - (2K^2)^3(4\tau)^3 Y_5 + (2K^2)^5(4\tau)^5 Y_9 + (2K^2)^7(4\tau)^7 Y_{13} \ldots] cos2 K^2\tau - [Y_{-1} - (2K^2)^2(4\tau)^2 Y_3 + (2K^2)^4(4\tau)^4 Y_7 + (2K^2)^6(4\tau)^6 Y_{11} \ldots] sin2K^2\tau\}_{\varphi=0} \quad (3.20)$$

Case 2: Solution for large time τ. For large time the method given in Batchelor[3] can be used. The solution of equation (2.8) subject to the conditions (2.10) and (2.11), can be written in the form

$$q(\varphi, s) = \frac{sin\,h\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)\varphi}{s-2iK^2 \sinh\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)} + F_1(\varphi,\tau) \quad (3.21)$$

Where the first term on the right hand side is the steady solution and $F_1(\varphi,\tau)$ shows the departure from steady state. $F_1(\varphi,\tau)$ Satisfies the differential equation

$$\frac{\partial F}{\partial \tau} + 2iK^2 F_1 = \frac{\partial^2 F}{\partial \varphi^2} + Re\frac{\partial F}{\partial \varphi} \quad (3.22)$$

With

$$F_1(0,\tau) = 0, F_1(1,\tau) = 0, F_1(0,\tau) = -\frac{sin\,h\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)\varphi}{\sinh\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)} \quad (3.23)$$

The solution of equation (3.22) may be written in the form

$$F_1(\varphi,\tau) = A_n sinn\pi\varphi e^{-\lambda_n^2\tau - \frac{Re}{2}} \quad (3.24)$$

Where

$$\lambda_n^2 = \frac{\pi^2 n^2 + 8iK^2}{4} \quad (3.25)$$

The coefficient A_n can be determined from the initial conditions which is

$$\sum_{n=0}^{\infty} A_n sinn\pi\varphi = -\frac{sin\,h\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)\varphi}{\sinh\left(\frac{\sqrt{Re^2+4s}-Re}{2}\right)} \quad (3.26)$$

The velocity distribution $q(\varphi, s)$ is given by

$$q(\varphi, s) = \frac{sin\,h(\sqrt{Re^2+8iK^2}-Re)\varphi}{sin\,h(\sqrt{Re^2+8iK^2}-Re)} + 4\sum_{n=1}^{\infty}\frac{n\pi(-1)^n e^{-\lambda_n^2\tau-\frac{Re}{2}}}{n^2\pi^2+8iK^2} sinn\pi\varphi \quad (3.27)$$

On separating the real and imaginary parts, we get

$$u_1 = \frac{S(4KRe\varphi)S(4KRe)+C(4KRe\varphi)C(4KRe)}{S^2(4KRe)+C^2(4KRe)} + 4\sum_{n=1}^{\infty}\frac{n\pi(-1)^n e^{-\lambda_n^2\tau-\frac{Re}{2}}}{(n^2\pi^2)^2+(8iK^2)^2} sinn\pi\varphi \times (n^2\pi^2 cos2K^2 - 8K^2 sin2K^2) \quad (3.28)$$

$$v_1 = \frac{C(4KRe\varphi)C(4KRe)-S(4KRe\varphi)S(4KRe)}{S^2(4KRe)+C^2(4KRe)} + 4\sum_{n=1}^{\infty}\frac{n\pi(-1)^n e^{-\lambda_n^2\tau-\frac{Re}{2}}}{(n^2\pi^2)^2+(8iK^2)^2} sinn\pi\varphi \times (8K^2 cos2K^2 + n^2\pi^2 sin2K^2) \quad (3.29)$$

Ref

3. Batchelor G. K. (1967), An introduction to fluid Dynamics, Cambridge press, Cambridge.

Where,

$$S(4KRe\varphi) = Sinh(4KRe\varphi)cos(4KRe)$$

$$C(4KRe\varphi) = cosh(4KRe\varphi)sin(4KRe)$$

$$S(4KRe) = Sinh(4KRe)cos(4KRe)$$

$$S(4KRe\varphi) = cosh(4KRe\varphi)sin(4KRe) \tag{3.30}$$

Notes

The second term of equations (3.28) and (3.29) represent the initial oscillations of the fluid velocity which decay experimentally with time and the frequency of the oscillation is $2K^2$.

The non − dimensional shear stresses due to primary and secondary flows at the stationary plate $\varphi = 0$ are given by

$$\left(\frac{\partial q}{\partial \varphi}\right)_{\varphi=0} = \frac{\sqrt{Re^2+8iK^2}-Re}{sinh\left(\sqrt{Re^2+8iK^2}-Re\right)} + 2\sum_{n=1}^{\infty}\frac{n^2\pi^2(-1)^n e^{-\left(\frac{n^2\pi^2+8iK^2}{4}\right)\tau}}{n^2\pi^2+8iK^2} \tag{3.31}$$

On separating the real and imaginary parts, we get the shear stresses components of due to the primary and the secondary flows respectively as

$$\tau_{x_0} = \frac{4KRe[S(4KRe)+C(4KRe)]}{S^2(4KRe)+C^2(4KRe)} + 2\sum_{n=1}^{\infty}\frac{n\pi(-1)^n e^{-\lambda_n^2\tau-\frac{Re}{2}}}{(n^2\pi^2)^2+(8iK^2)^2} sinn\pi\varphi \times (n^2\pi^2 cos2K^2 - 8K^2 sin2K^2) \tag{3.32}$$

$$\tau_{y_0} = \frac{4KRe[S(4KRe)+C(4KRe)]}{S^2(4KRe)+C^2(4KRe)} + 2\sum_{n=1}^{\infty}\frac{n\pi(-1)^n e^{-\lambda_n^2\tau-\frac{Re}{2}}}{(n^2\pi^2)^2+(8iK^2)^2} sinn\pi\varphi \times (+8K^2 sin2K^2 n^2\pi^2 cos2K^2) \tag{3.33}$$

Where

$S(4KRe)$ *and* $C(4KRe)$ are defined in (3.30).

IV. Discussion of Results

To study the effect of rotation with transpiration, the stream wise velocity profiles for primary and the secondary flows are plotted graphically against φ for different values of K^2 and constant Value for Re with small time ($\tau \ll 1$) and large time ($\tau \gg 1$). Taking $\tau = 0.05$ and $\tau = 10.0$

Figures 1 and 2 show the variation of primary and secondary velocities u_1 *and* v_1 respectively against φ for different values of rotation parameter K^2 with $= 0.05$. Here Re is assumed a value so that the transpiration of the flow in the system is laminar. It is evident from figure 1 that the primary velocity u_1 increases with decrease in K^2. Figure 2 shows that the secondary velocity v_1 increases with increase in K^2. It is also observed from figure 2 that the velocity profiles are skewed near the moving plate while the maximum peak occurs near the stationary plate this is due to the transpiration from the system.

Figures 3 and 4 are plotted such that they show the variation of primary and secondary velocities u_1 *and* v_1 respectively against φ for different values of rotation

Notes

parameter K^2 with $\tau = 10.0$. it is observed from figure3 that the primary velocity u_1 increases with increase in K^2. It is seen from figure 4 that v_1 behaves in an oscillatory manner near the moving plate due to the transpiration while it increases near the stationary plate with increase in K^2. It is stated that there exists a back flow which lies in the region $0.0 \ll \varphi \ll 1.0$. It comes to a conclusion that the start up flow will lead to an occurrence of a back flow for large time when the rate of rotation is high with constant transpiration.

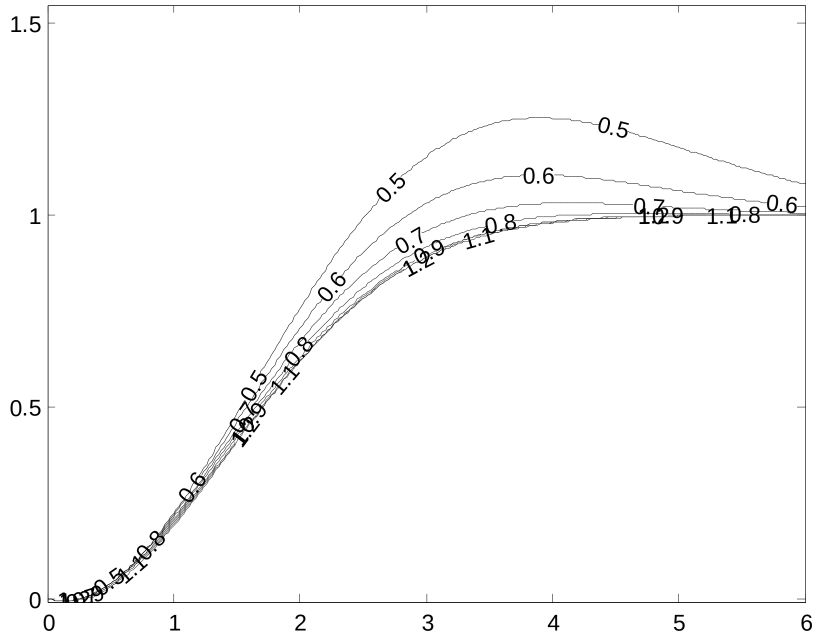

Figure 1 : Primary velocity profile u_1 against φ for diffent values of K^2 with $\tau = 0.05$

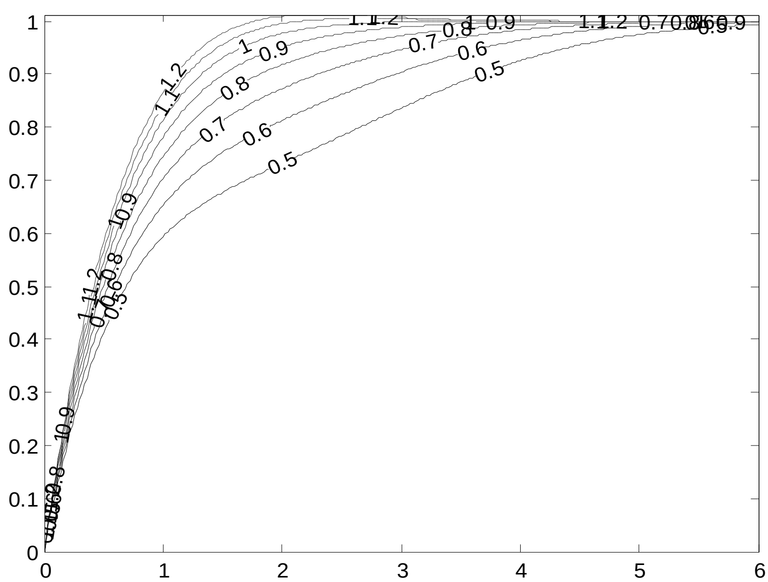

Figure 2 : Secondary velocity profile v_1 against φ for diffent values of K^2 with $\tau = 0.05$

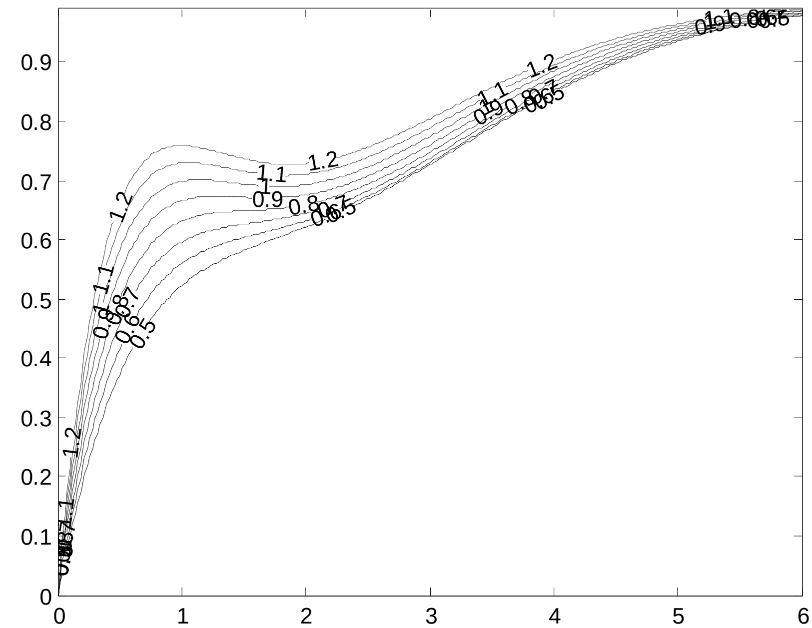

Figure 3 : Primary velocity profile v_1 against φ for diffent values of K^2 with $\tau = 10.0$

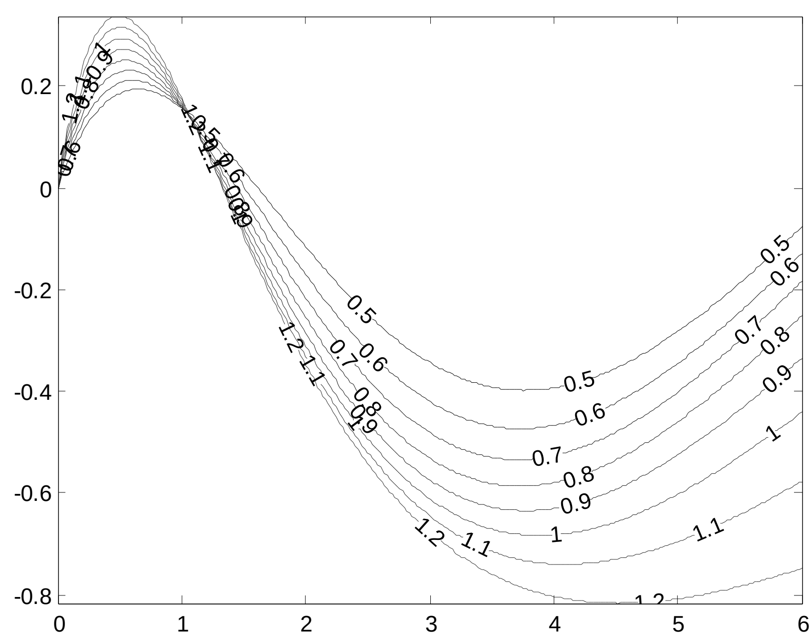

Figure 4 : Secondary velocity profile v_1 against φ for diffent values of K^2 with $\tau = 10.0$

Notes

Notes

Table 1 : Shear Stresses at the plate $\varphi = 0$ for small time $\tau = 0.05$ and $Re = 10.00$

K^2	Shear Stress due to Primary Flow (τ_x)	Shear Stress due to Secondary Flow (τ_y)
2.0	0.064991990	0.015195550
4.0	0.057874710	0.027697940
6.0	0.047290120	0.035139770
8.0	0.035079770	0.035767860
10.0	0.023563680	0.028661410
12.0	0.015325670	0.013854810

Table 2 : Shear Stresses at the plate $\varphi = 0$ for Large time $\tau = 10.0$ and $Re = 10.0$

K	Shear Stress due to Primary Flow (τ_{x_0})	Shear Stress due to Secondary Flow (τ_{y_0})
1.0	9.2544900	-3.1758700
1.5	6.7748620	-5.9575870
2.0	2.7346490	-7.0528330
2.5	0.79611640	-5.7605270
3.0	-2.5394700	-3.3887110
3.5	-2.7236030	-1.2372250
4.0	-2.0665060	0.15183220
4.5	-1.1879270	0.76656270
5.0	-0.4549140	0.83720510

Tables 1 and 2 presented the Shear stresses at the stationary plate $\varphi = 0$ due to the primary and secondary flows for small time $\tau = 0.05$ and large time $\tau = 10.0$ and $Re = 10.0$.

Table 1 shows that the shear stress due to primary flow (τ_x) decreases with increase in K^2 for small time τ. On the other hand, it increases due to secondary flow (τ_y) with increase in K^2. It is noticed from Table 2 that the shear stress due primary flow (τ_{x_0}) decreases from $K = 1.0\ to\ 2.5$ and increases from $K = 3.0\ to\ 5.0$. Similarly the shear stress due to secondary flow (τ_{y_0}) decreases from $K = 1.0\ to\ 2.0$ and increases from $K = 2.5\ to\ 5.0$.

V. Conclusion

An analysis and investigation has been made on unsteady Couette flow with transpiration in a rotating system for small time as well as large time τ. Laplace transform technique is applied for the solution of velocity distributions and shear stresses are analyzed for small time as well as large time τ.

Unsteady coquette flow with transpiration in a rotating system leads to a start up flow with occurrence of back flow for large time τ when the rate of rotation is high and the transpiration is constant. The significant of a study of the flow pattern is in such a way that the shear stresses due to primary and secondary flow for large time $\tau = 10.0$ show layer of separation when a back flow occurs.

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A Priori Estimate and Fourier's Method for Nonlocal Boundary Conditions of Mixed Problem for Singular Parabolic Equations in Sobolev Function Spaces

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GJSFR-F Classification : 35K20, 35B30, 35D05, 46E40, 46E99.

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A Priori Estimate and Fourier's Method for Nonlocal Boundary Conditions of Mixed Problem for Singular Parabolic Equations in Sobolev Function Spaces

Djibibe Moussa Zakari [α] & Tcharie Kokou [σ]

Abstract - The aims of this paper is to prove existence and uniqueness of following integral boundary conditions mixed problem for parabolic equation:

$$\begin{cases} \frac{\partial\theta}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) + b(t)\theta = \vartheta(x,t) \\ \theta(x,0) = \lambda(x),\ 0 \le x \le \ell \\ \int_0^\ell x\theta(x,t)\,dx = E(t),\ 0 \le t \le T \\ \int_0^\ell x^2\theta(x,t)\,dx = G(t),\ 0 \le t \le \ell, \end{cases} \tag{0.1}$$

The proofs are based on a priori estimates established in Sobolev function spaces and Fourier's method.

Keywords : *fourier's method, a priori estimate, nonlocal conditions, mixed problem, parabolic, sobolev espace.*

I. Introduction

This paper deals with existence and uniqueness of a class of parabolic equation with time and space-variable characteristics, with a nonlocal boundary condition. The precise statement of the problem is a follows: let $\ell > 0$, $T > 0$, and $\Omega = \{(x,t) \in \mathbb{R}^2 : 0 < x < \ell,\ 0 < t < T\}$. We shall determine a solution θ, in Ω of the differential equation

$$\frac{\partial\theta}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) + b(t)\theta = \vartheta(x,t), \quad (x,t) \in \Omega. \tag{1.1}$$

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satisfying the initial condition

$$\theta(x,0)=\lambda(x),\ 0\le x\le \ell. \tag{1.2}$$

and the integral conditions

$$\int_0^\ell x\theta(x,t)\,dx=E(t),\ 0\le t\le T \tag{1.3}$$

$$\int_0^\ell x^2\theta(x,t)\,dx=G(t),\ 0\le t\le T. \tag{1.4}$$

where $\lambda,\ E,\ G,\ a,b$ and ϑ are known functions

Assumption 1.1

For all $(x,t)\in\overline{\Omega}$, *we assume that*

$$a_0\le a(t)\le a_1,\quad a_2\le \frac{da(t)}{dt}\le a_3$$

$$b_0\le b(t)\le b_1,\quad b_2\le \frac{db(t)}{dt}\le b_3,$$

$$\frac{b(t)}{a(t)}=s.$$

where $a_0,\ a_1,\ a_2,\ a_3,\ b_0,\ b_1,\ b_2$ *are positive constants.*

The data satisfies the following compatibility conditions : For consistency, we have

$$\int_0^\ell x\lambda(x)=E(0),\quad \text{and } \int_0^\ell x^2\lambda(x)=G(0),$$

The importance of problems with integral conditions has been pointed out by Samarskii[11]. Mathematical modelling by evolution problems with a nonlocal constraint of the form $\frac{1}{1-\alpha}\int_\alpha^1 u(x,t)\,dx=\chi(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. Cannon [7] used the potentiel method, combining a Dirichlet and an intégral condition for a parabolic equation. L. A. Mouravey and V. Philinovoski [10] used the maximum principle, combining a Neumann and an integral condition for heat equation. Ionkin [8] and L. Bougoffa[4] used the Fourier method for same purpose.

Recently, mixed problems with integral conditions for generalization of equation (1.1) have been treated using the energy-integral method. See N. E. Benouar and N. I. Yurchuk [1], N. E. Benouar and A. Bouziani [2],[3], A. Bouziani[5], [6], M. Z. Djibibe el al. [12],[13], N. I. Yurchuk[14],[15], M. Mesloub, A. Bouziani and N. Kechkar[9]. Differently to these works, in the present paper we combine a priori estimate and Fourier's method to prove existence and uniqueness solution of the problem (1.1)- (1.4).

The result of the paper are new. It is interesting to note that the application of Fourier method to this nonlocal problem is made possible thanks, essentially, to the use of a Sobolev function space.

Ref.

[1] N. E. Benouar and N. I. Yurchuk; Mixed problem with an integral condition for parabolique equation with the Bessel operator, *Differentsial'nye Uravneniya*, 27 (1991), p. 2094 - 2098.

Notes

To this, we reduce the inhomogeneous boundary conditions (1.3) and (1.4) to homogeneous conditions, by introducing a new unknown function z by $z(x,t) = \theta(x,t) - \eta(x,t)$, where

$$\eta(x,t) = \frac{10l^4x^2 - 12(l^5+20)x + 3l(l^5+60)}{10\ell^3}E(t) + \frac{10\ell^5x^2 - 12(\ell^6-30)x + 3\ell(l^6-80)}{10\ell^4}G(t). \tag{1.5}$$

Then, problem (1.1), (1.2), (1.3) and (1.4) is transformed into the following homogeneous boundary value problem

$$\frac{\partial z}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) + b(t)z = \xi(x,t), \quad (x,t)\in\Omega, \tag{1.6}$$

$$z(x,0) = \Lambda(x),\ 0 \le x \le \ell, \tag{1.7}$$

$$\int_0^\ell xz(x,t)\,dx = 0,\ 0 \le t \le T, \tag{1.8}$$

$$\int_0^\ell x^2z(x,t)\,dx = 0,\ 0 \le t \le T, \tag{1.9}$$

where

$$\begin{aligned}\xi(x,t) =& \vartheta(x,t) - \frac{10l^4x^2 - 12(l^5+20)x + 3l(l^5+60)}{10\ell^3}E'(t)\\ &- \frac{10\ell^5x^2 - 12(\ell^6-30)x + 3\ell(l^6-80)}{10\ell^4}G'(t)\\ &+ a(t)\left(6\ell - \frac{6(\ell^5+20)}{5\ell^3x^2} + \right)E(t) + a(t)\left(4\ell - \frac{6(\ell^6-30)}{5\ell^4x^2}\right)G(t)\\ &- \frac{10l^4x^2 - 12(l^5+20)x + 3l(l^5+60)}{10\ell^3}b(t)E(t)\\ &- \frac{10\ell^5x^2 - 12(\ell^6-30)x + 3\ell(l^6-80)}{10\ell^4}b(t)G(t),\end{aligned} \tag{1.10}$$

$$\begin{aligned}\Lambda(x) =& \lambda(x) - \frac{10l^4x^2 - 12(l^5+20)x + 3l(l^5+60)}{10\ell^3}E(0)\\ &+ \frac{10\ell^5x^2 - 12(\ell^6-30)x + 3\ell(l^6-80)}{10\ell^4}G(0).\end{aligned} \tag{1.11}$$

Here, taking account assumption 1.1, we assume that the function Λ satisfy conditions of (1.8) and (1.9), that is

$$\int_0^\ell x\Lambda(x)\,dx = \int_0^\ell x^2\Lambda(x)\,dx = 0. \tag{1.12}$$

Instead of searching for the function θ, we search for the function z. So the solution of problem (1.1), (1.2), (1.3) and (1.4) will be given by $\theta(x,t) = z(x,t) + \eta(x,t)$.

The general difficult which arises to us is the presence of integral conditions which complicates the application of standard methods. It may, however, be worth while if this type of problem can be transformed into another equivalent problem which involves no integral conditions. For this, we convert problem (1.6), (1.7), (1.8) and (1.9) to the following classical problem.

Theorem 1.1

The problem (1.6), (1.7), (1.8) *and* (1.9) *is equivalent to the following classical problem :*

$$\frac{\partial z}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) + b(t)z = \xi(x,t), \quad (x,t) \in \Omega, \tag{1.13}$$

$$z(x,0) = \Lambda(x), \; 0 \le x \le \ell, \tag{1.14}$$

$$z(t,\ell) - u(0,t) = \frac{1}{\ell a(t)}\int_0^\ell (x^2 - \ell x)\xi(x,t)\,dx, \; 0 \le t \le T, \tag{1.15}$$

$$\frac{\partial z}{\partial x}(\ell,t) = -\frac{1}{\ell^2 a(t)}\int_0^\ell x^2\xi(x,t)\,dx, \; 0 \le t \le T. \tag{1.16}$$

Proof

Firstly, multiplying (1.6) with x, and integrating the obtained result with respect x over $(0,\ell)$, we obtain

$$\frac{\partial}{\partial t}\int_0^\ell xz\,dx - a(t)\int_0^\ell \frac{1}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right)\,dx + b\int_0^\ell xz\,dx = \int_0^\ell x\xi(x,t)\,dx. \tag{1.17}$$

Integrating by parts the integrals on the left-hand side of (1.17), and taking into account condition (1.8), we get

$$\ell\frac{\partial z}{\partial x}(\ell,t) + z(\ell,t) - z(0,t) = -\frac{1}{a(t)}\int_0^\ell x\xi(x,t)\,dx. \tag{1.18}$$

Secondly, multiplying (1.6) with x^2 and integrating the result obtained over $(0,\ell)$, he have

$$-a(t)\int_0^\ell \frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right)\,dx = \int_0^\ell x^2\xi(x,t)\,dx. \tag{1.19}$$

Integrating by parts the integrals on the left-hand side of (1.19).

$$\frac{\partial z}{\partial x}(\ell,t) = -\frac{1}{\ell^2 a(t)}\int_0^\ell x^2\xi(x,t)\,dx \tag{1.20}$$

Combining the equalities (1.19) and (1.20), we have

$$z(\ell,t) - z(0,t) = \frac{1}{\ell a(t)}\int_0^\ell (x^2 - \ell x)\xi(x,t)\,dx. \tag{1.21}$$

Finaly, it remains to prove that $\int_0^\ell xz(x,t)\,dx = 0$ and $\int_0^\ell x^2 z(x,t)\,dx = 0$.

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By using (1.6) and taking into account (1.18) and (1.20) we get

$$\frac{d}{dt}\int_0^\ell xz(x,t)\,dx + b\int_0^\ell xz(x,t)\,dx = 0,\ \ 0\le t\le T$$

$$\frac{d}{dt}\int_0^\ell x^2z(x,t)\,dx + b\int_0^\ell x^2z(x,t)\,dx = 0,\ \ 0\le t\le T$$

By virtue of the compatibility of the conditions, it follows that

$$\int_0^\ell xz(x,t)\,dx = \int_0^\ell x^2z(x,t)\,dx = 0.$$

This complete the proof of Theorem (1.1).

By introducing the new unknown function

$$u(x,t) = z(x,t) - \alpha(x,t)\int_0^\ell x^2\xi(x,t)\,dx - \beta(x,t)\int_0^\ell x\xi(x,t)\,dx,$$

where

$$\alpha(x,t) = \frac{-3x^3+3\ell x^2+\ell^5 a(t)}{\ell^4 a(t)}$$

and

$$\beta(x,t) = \frac{2x^3-3\ell x^2-\ell^3 a(t)}{\ell^3 a(t)},$$

the problem (1.13), (1.14), (1.15) and (1.16) is transformed into the following local boundary conditions problem,

$$\frac{\partial u}{\partial t} - \frac{a(t)}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) + bu = f(x,t), \quad (x,t)\in\Omega, \tag{1.22}$$

$$u(x,0) = \varphi(x),\ \ 0\le x\le \ell, \tag{1.23}$$

$$z(\ell,t) = u(0,t),\ \ 0\le t\le T, \tag{1.24}$$

$$\frac{\partial u}{\partial x}(\ell,t) = 0,\ \ 0\le t\le T, \tag{1.25}$$

where

$$f(x,t) = \xi(x,t) + \frac{3x^3-3\ell x^2-a'(t)\ell^5}{\ell^4 a'(t)}\int_0^\ell x^2\frac{\partial\xi}{\partial t}(x,t)\,dx - \frac{18(2x-\ell)}{\ell^4}\int_0^\ell x^2\xi(x,t)\,dx$$

$$+\frac{(3x^3-3\ell x^2-a(t)\ell^5)b(t)}{\ell^4 a(t)}\int_0^\ell x^2\xi(x,t)\,dx + \frac{6(4x-3\ell)}{\ell^3}\int_0^\ell x\xi(x,t)\,dx$$

Global Journal of Science Frontier Research (F) Volume XIII Issue V Version I Year 2013

$$-\frac{(2x^3-3\ell x^2-\ell^3 a(t))b(t)}{\ell^3 a(t)}\int_0^\ell x\xi(x,t)\,dx-\frac{2x^3-3\ell x^2-\ell^3 a'(t)}{\ell^3 a'(t)}\int_0^\ell x\frac{\partial\xi}{\partial t}(x,t)\,dx,$$

$$\varphi(x)=\Lambda(x)+\frac{3x^3-3\ell x^2-a(0)\ell^5}{\ell^4 a(0)}\int_0^\ell x^2\xi(x,0)\,dx-\frac{2x^3-3\ell x^2-\ell^3 a(0)}{\ell^3 a(0)}\int_0^\ell x\xi(x,o)\,dx.$$

Notes

II. Main Results

a) An Energy Inequality

The problem (1.17), (1.18), (1.19) and (1.20) can be considered as solving the following operator equation :

$$Lu=(\varphi,f)=\mathcal{F},$$

where L is an operator defined on E into F. E is the banach space of functions $u\in L^2(\Omega)$, satisfying conditions (1.19) and (1.20) with the norm

$$\|u\|_E^2=\int_{\Omega_\tau}\left(x^2u^2+\left|x\frac{\partial u}{\partial x}\right|^2+\left|x\frac{\partial u}{\partial t}\right|^2\right)dxdt+\sup_{0\le t\le T}\int_0^\ell\left(x^2u^2+\left|x\frac{\partial u}{\partial x}\right|^2\right)dx$$

and F is the Hilbert space $L^2(\Omega)\times L^2(0,\ell)$ which consists of elements $\mathcal{F}=(\varphi,f)$ with the norm

$$\|\mathcal{F}\|_F^2=\int_0^\ell\left(\varphi^2(x)+\left(\frac{d\varphi}{dx}\right)^2\right)dx+\int_{\Omega_\tau}f^2(x,t)\,dxdt.$$

Let $D(L)$ be the set of all function u, for which u, xu, $x^2\frac{\partial u}{\partial x}\in L^2(0,\ell)$ and u, $x\frac{\partial u}{\partial x}$, $x\frac{\partial u}{\partial t}$, $x^2\frac{\partial u}{\partial x}$, $\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\in L^2(\Omega)$.

Theorem 2.1

There exists a positive constant c, *such that for each function* $u\in D(L)$ *we have*

$$\|u\|_E\le c\|Lu\|_F. \tag{2.1}$$

Proof

We consider the scalar product in $L^2(\Omega_\tau)$ with $0\le t\le\tau$ of equation (1.22) and $xu(x,t)+x^2\frac{\partial u}{\partial t}$, yields

$$\int_{\Omega_\tau}xu\frac{\partial u}{\partial t}\,dxdt+\int_\Omega x^2\left(\frac{\partial u}{\partial t}\right)^2dxdt-\int_{\Omega_\tau}\frac{a(t)u}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)dxdt$$
$$-\int_{\Omega_\tau}a(t)\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,dxdt+\int_{\Omega_\tau}b(t)xu^2\,dxdt+\int_{\Omega_\tau}b(t)x^2u\frac{\partial u}{\partial t}\,dxdt$$
$$=\int_{\Omega_\tau}xuf(x,t)\,dxdt+\int_{\Omega_\tau}x^2f(x,t)\frac{\partial u}{\partial t}\,dxdt. \tag{2.2}$$

Global Journal of Science Frontier Research (F) Volume XIII Issue V Version I Year 2013

Notes

Integrating by parts certains integrals of left-hand side of (2.2), we get

$$\int_{\Omega_\tau} xu\frac{\partial u}{\partial t}\,dxdt = \int_0^\ell xu^2(x,\tau)\,dx - \frac{1}{2}\int_0^\ell x\varphi^2(x)\,dx, \tag{2.3}$$

$$-\int_{\Omega_\tau} \frac{a(t)u}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)dxdt = \int_{\Omega_\tau} a(t)\left(x\frac{\partial u}{\partial x}\right)^2 dxdt, \tag{2.4}$$

$$\begin{aligned}\int_{\Omega_\tau} b(t)x^2u\frac{\partial u}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell b(0)x^2u(x,\tau)\,dx - \frac{1}{2}\int_0^\ell b(0)x^2\varphi^2(x)\,dx\\ -\frac{1}{2}\int_{\Omega_\tau} b'(t)x^2u^2\,dxdt,\end{aligned} \tag{2.5}$$

$$-\int_{\Omega_\tau} a(t)\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,dxdt = \frac{1}{2}\int_0^\ell a(t)\left(x\frac{\partial u}{\partial x}\right)^2 dx - \frac{1}{2}\int_0^\ell a'(t)\left(x\frac{d\varphi}{dx}\right)^2 dx. \tag{2.6}$$

Substituting the equalities (2.3), (2.4), (2.5) and (2.6) into (2.2), it follows that

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\tau}(2b(t)x - b'(t)x^2)u^2\,dxdt + \int_{\Omega_\tau} a(t)\left(x\frac{\partial u}{\partial x}\right)^2 dxdt + \int_{\Omega}\left(x\frac{\partial u}{\partial t}\right)^2 dxdt\\ &+\int_0^\ell xu^2(x,\tau)\,dx + \frac{b(0)}{2}\int_0^\ell x^2u(x,\tau)\,dx + \frac{1}{2}\int_0^\ell a(t)\left(x\frac{\partial u}{\partial x}\right)^2 dx = \int_{\Omega_\tau} xuf(x,t)\,dxdt\\ &+\int_{\Omega_\tau} x^2f(x,t)\frac{\partial u}{\partial t}\,dxdt + \frac{1}{2}\int_0^\ell x\varphi^2(x)\,dx + \frac{b(0)}{2}\int_0^\ell x^2\varphi^2(x)\,dx + \frac{1}{2}\int_0^\ell a'(t)\left(x\frac{d\varphi}{dx}\right)^2 dx.\end{aligned} \tag{2.7}$$

Estimating the first and the two first integrals of the right-hand side of (2.7), by applying elementary inequalities, we get

$$\int_{\Omega_\tau} xuf(x,t)\,dxdt \le \frac{\varepsilon_1}{2}\int_{\Omega_\tau} xu^2\,dxdt + \frac{1}{2\varepsilon_1}\int_{\Omega_\tau} xf(x,t)\,dxdt, \tag{2.8}$$

$$\int_{\Omega_\tau} x^2f(x,t)\frac{\partial u}{\partial t}\,dxdt \le \frac{1}{2}\int_{\Omega_\tau}\left(x\frac{\partial u}{\partial t}\right)^2 dxdt + \frac{1}{2}\int_{\Omega_\tau} x^2f^2(x,t)\,dxdt. \tag{2.9}$$

Therefore, by formulas (2.7), (2.8) and (2.9), we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\tau}(2b(t) - \varepsilon_1 - b'(t)x)xu^2\,dxdt + \frac{1}{2}\int_{\Omega}\left(x\frac{\partial u}{\partial t}\right)^2 dxdt + \int_0^\ell xu^2(x,\tau)\,dx\\ &+\frac{b(0)}{2}\int_0^\ell x^2u(x,\tau)\,dx + \frac{1}{2}\int_0^\ell a(t)\left(x\frac{\partial u}{\partial x}\right)^2 dx \le \frac{1+\ell^2}{2\varepsilon_1}\int_{\Omega_\tau} f^2(x,t)\,dxdt\\ &+\frac{1}{2}\int_0^\ell x\varphi^2(x)\,dx + \frac{b(0)}{2}\int_0^\ell x^2\varphi^2(x)\,dx + \frac{1}{2}\int_0^\ell a'(t)\left(x\frac{d\varphi}{dx}\right)^2 dx.\end{aligned} \tag{2.10}$$

Hence, if $\varepsilon_1 > 0$ satisfies $\varepsilon_1 \leq \min_{\Omega_\tau} \left\{ 2b(t) - x\frac{db(t)}{dt} \right\}$, and choosing $\varepsilon_2 = 2b_0 - \varepsilon_1 - \ell b_3$, then inequality (2.10) implies

$$\frac{\varepsilon_2}{2}\int_{\Omega_\tau} xu^2\,dxdt + a_0\int_{\Omega_\tau}\left(x\frac{\partial u}{\partial x}\right)^2 dxdt + \frac{1}{2}\int_{\Omega}\left(x\frac{\partial u}{\partial t}\right)^2 dxdt$$
$$+\frac{2+b_0\ell}{2\ell}\int_0^\ell x^2u^2(x,\tau)\,dx + \frac{a_0}{2}\int_0^\ell\left(x\frac{\partial u}{\partial x}\right)^2 dx \leq \frac{1+\ell^2}{2\varepsilon_1}\int_{\Omega_\tau} f^2(x,t)\,dxdt$$
$$+\frac{1+b_1\ell}{2}\int_0^\ell x\varphi^2(x)\,dx + \frac{a_3}{2}\int_0^\ell\left(x\frac{d\varphi}{dx}\right)^2 dx. \tag{2.11}$$

Therefore, by formulas (2.11) and of assumption (1.1), we obtain

$$\int_{\Omega_\tau} xu^2\,dxdt + \int_{\Omega_\tau}\left(x\frac{\partial u}{\partial x}\right)^2 dxdt + \int_{\Omega}\left(x\frac{\partial u}{\partial t}\right)^2 dxdt + \int_0^\ell x^2u^2(x,\tau)\,dx +$$
$$+\int_0^\ell\left(x\frac{\partial u}{\partial x}\right)^2 dx \leq \frac{A}{B}\left[\int_{\Omega_\tau} f^2(x,t)\,dxdt + \int_0^\ell x\varphi^2(x)\,dx + \int_0^\ell\left(x\frac{d\varphi}{dx}\right)^2 dx\right], \tag{2.12}$$

where

$$A = \min\left(\frac{2+b_0\ell}{2\ell}, \frac{a_0}{2}, \frac{1}{2}, \frac{\varepsilon_2}{2}\right) \quad B = \max\left(\frac{1+\ell^2}{2\varepsilon_1}, \frac{1+b_1\ell}{2}, \frac{a_3}{2}\right),$$

The right-hand side of (2.12) is independent of τ, replacing the left-hand side by the upper with respect to τ. Thus inequality (2.1) holds, where

$$c = \sqrt{\frac{\min\left(\frac{2+b_0\ell}{2\ell}, \frac{a_0}{2}, \frac{1}{2}, \frac{\varepsilon_2}{2}\right)}{\max\left(\frac{1+\ell^2}{2\varepsilon_1}, \frac{1+b_1\ell}{2}, \frac{a_3}{2}\right)}}$$

This completes the proof of Theorem (1.1).

b) Solvability of the Problem

Now we shall start to prove the existence of the boundary value problem (1.13), (1.14), (1.15) and (1.16). We use the Fourier's method.

Consider the fouction $u_n(x,t) = v_n(x)w_n(t)$, where $v_n(t)$ is a eigenfunction of the following boundary value problem

$$\begin{cases} -\dfrac{1}{x^2}\dfrac{d}{dx}\left(x^2\dfrac{dv_n(x)}{dx}\right) + sv_n(x) = \alpha_n v_n(x) \\ v_n(0) = v_n(\ell) \\ \dfrac{dv_n(\ell)}{dx} = 0 \end{cases}$$

Notes

Notes

where α_n is the eigenvalue corresponding to the eigenfunction $\nu_n(x)$, and $w_n(t)$ satisfying the initial problem

$$\begin{cases} \dfrac{dw_n(t)}{a(t)dt} - \alpha_n w_n(t) = f_n(t) \\ \\ w_n(0) = \varphi_n. \end{cases}$$

Here

$$\varphi(x) = \sum_{n=1}^{+\infty} \varphi_n \nu_n(x)$$
$$\varphi'(x) = \sum_{n=0}^{+\infty} \rho_n \nu_n(x)$$
$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t)\nu_n(x).$$

Lemma 2.1
Using the Parseval-Steklov *equality, we have*

$$\|(f,\varphi)\|_F = \sum_{n=1}^{+\infty} \left(\int_0^T f_n^2(t)\,dt + \varphi_n^2 + \rho_n^2 \right).$$

The direct computation, the solution of the initial problem is giving by

$$w_n(t) = \varphi_n e^{\alpha_n t} + \int_0^t f_n(t) e^{\alpha_n (t-\tau)}\,dt.$$

By virtue principle of superposition, the solution of the boundary value problem (1.13), (1.14), (1.15) *and* (1.16) *is giving by the series*

$$u(x,t) = \sum_{1}^{+\infty} \nu_n(x) w_n(t). \tag{2.13}$$

Theorem 2.2
Let assumption 1.1 be fulfilled. Then for any $f \in L^2(\Omega)$ *and* $\varphi \in L_2(0,\ell)$ *which* $\dfrac{d\varphi}{dx} \in L^2(0,\ell)$, *problem*(1.13), (1.14), (1.15) *and* (1.16) *admits a unique solution and its represented by series* (2.13) *which converge in* E.

Proof

Consider the partial sum $S_n(x,t) = \sum_{k=1}^{n} \nu_k(x) w_k(t)$ of the series (2.13).

By applaying the Theorem 1.1, then it follows that

$$\left\| \sum_{i=1}^{n} \nu_k(x) w_k(t) \right\| \leq C \sum_{n=1}^{+\infty} \left(\int_0^T f_n^2(t)\,dt + \varphi_n^2 + \rho_n^2 \right) \tag{2.14}$$

The series $\sum_{n=1}^{+\infty} \int_0^T f_n^2(t)\, dt = \int_\Omega f^2(x,t)\, dxdt$, $\sum_{n=1}^{+\infty} \varphi_n^2$ and $\sum_{n=1}^{+\infty} \rho_n^2$ converge.

Therefore, from (2.14) it follows that the series (2.13) converge in E.

This completes the proof of the Theorem 2.2.

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Generalized and Perturbed Lamé System

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Abstract - In this work, we study the existence, the uniqueness and the regularity of the solution for some boundary value problems gouverned by perturbed and generalized dynamical Lamé system operator.

Keywords : *lamé system (elasticity), perturbed, existence, unique-ness, regularity.*

GJSFR-F Classification : *MSC 2010: 35B40, 35B65, 35C20*

GENERALIZED AND PERTURBED LAM SYSTEM

Notes

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B. Merouani [α], A. Boulaouad [σ] & M. Me.ah [ρ]

Abstract - In this work, we study the existence, the uniqueness and the regularity of the solution for some boundary value problems gouverned by perturbed and generalized dynamical Lamé system operator.

Keywords : *lamé system (elasticity), perturbed, existence, unique-ness, regularity.*

I. Introduction

1. Notations.- Ω is a bounded and connected open set of $\mathbb{R}^n$ $(n = 2,3)$ with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, a lipschitzian manifold of dimension $n-1$, were $\Gamma_i \subset \Gamma$, $i = 1,2$, with $\text{mes}(\Gamma_1) > 0$ and $\Gamma_1 \cap \Gamma_2 = \phi$.

2. Position of the Problem.- We consider firstly the mathematical model of the perturbed Lamé system :

$$-L_p u + F(u),$$

were $F(u)$ is the perturbation and

$$L_p u = \mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + (\lambda + \mu) \nabla (div(u)),$$

p, q are two real numbers such that $p \in]1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$,

λ and μ are the Lamé coefficients subjected to the constraint $\lambda + \mu \geq 0$ and $\lambda > 0$,

ν denotes the outgoing normal vector to Γ_2.

For $p = 2$, we recover the classical dynamical Lamé system.

Given f and $\varphi = (\varphi_{i,j})_{1<i,j<n}$,such that $\varphi_{i,j} = \varphi_{j,i} \in C^{0,1}(\overline{\Omega})$ and $\varphi_{i,j}(x) > 0, \forall\ x \in \Gamma_2$.We study the existence, the uniqueness and the regularity of the complex-valued solution $u = u(x)$, $x \in \Omega$, for the following problem :

$$(P) \begin{cases} -L_p u + F(u) = f, \ \text{ in } Q & (2.1) \\ u = 0, \ \text{ on } \Gamma_1 & (2.2) \\ \sigma(u).\nu + \varphi(x)\ u = 0, \ \text{ on } \Gamma_2 & (2.3) \end{cases}$$

Here $\sigma(u) = (\sigma_{ij}(u))_{1<i,j<n}$ is the matrix of the constraints tensor $\sigma_{ij}(u) = \lambda div(u)\delta_{ij} + 2\mu\ \varepsilon_{ij}(u)$, were $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), 1 \leq i,j \leq n$, are the components of the deformation tensor.

In this work, we consider the cases $F(u) = 0$, $F(u) = |u|^{\rho}\, u$ with $\rho = p-2 > 0$, $F(u) = u^3$ and $F(u) \equiv a(x,t)$.

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We distinguish the cases:

when $\Gamma_2 = \phi$, (P) becomes a Dirichlet problem,

when $\Gamma_1 = \phi$, and $\varphi \equiv 0$ on Γ_2, (P) becomes a Neumann problem,

when , $\Gamma_2 \neq \phi$ and $\varphi \equiv 0$ on Γ_2, (P) becomes a mixed problem : Dirichlet-Neumann,

when Γ_1 , $\Gamma_2 \neq \phi$ and $\varphi(x) \neq 0$ on Γ_2, (P) becomes a mixed problem : Dirichlet-(2.3).

Of course, when it is question of a Neumann problem ($\Gamma_1 = \phi$ and $\varphi \equiv 0$ on Γ_2), we suppose verified the necessary condition of existence that is, the data are orthogonal to the rigid displacements :

$$\int_\Omega f.vdx = \int_\Gamma 0.vds = 0,$$

for any v of the form

$$v(x,y) = \left\{ \begin{array}{c} a+cy \\ b-cx \end{array} \right\},$$

with a, b, c arbitrary real numbers.

In the remaining part of this paper we study with details the last cas with $F(u) = |u|^\rho u$.

The main result is

Theorem 2.1.- We suppose that

$$f \in (W^{-1,q}(\Omega))^n).$$

Then, there exist a function $u = u(x)$ solution of the problem (P) with :

$$u \in (W^{1,p}(\Omega))^n),$$

Before giving the proof, we make the following remarks :

Remark 2.1.- The space $V = (H_0^1(\Omega))^n \cap (L^p(\Omega))^n$, were $p = \rho + 2$, is separable (i.e. admits a countable dense subset).

In fact, V is identified, by the application $v \to \left\{ v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_n} \right\}$, to a closed subspace of

$(L^p(\Omega))^n \times (L^2(\Omega))^n \times ... \times (L^2(\Omega))^n$,separable and uniformly convex, in such way that it possible to project a countable dense set on this subspace.

Remarque 2.3.- The application defined on $(L^p(\Omega))^n$ by $u \longrightarrow |u|^{p-2} u$, is $(L^q(\Omega))^n$-valued, moreover it is continuous. To see that, if

$u \in (L^p(\Omega))^n$, $|u|^{p-2} u$ est mesurable and

$$\int_\Omega \left| |u|^{p-2} u \right|^q dx = \int_\Omega |u|^p dx < \infty \Longrightarrow u \in (L^q(\Omega))^n.$$

We deduce that $\forall u \in (W^{1,p}(\Omega))^n$, $\forall\, i$, $1 \le i \le n$,

$$\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \in (L^q(\Omega))^n.$$

So, it is possible to define the real-valued application :

$$\left((W^{1,p}(\Omega))^n\right)^2 \longrightarrow \mathbb{R},\ (u,v) \longmapsto a_p(u,v).$$

$$a_p(u,v) = \ \mu \sum_{i,j=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\ dx + (\lambda+\mu) \int_\Omega div(u)\ div(v)\ dx.$$

For any u in $(W^{1,p}(\Omega))^n$, the application $(W^{1,p}(\Omega))^n \longrightarrow \mathbb{R}$, $v \longrightarrow a_p(u,v)$, is a continuous linear form. then $c.f.$ [5] there exist a unique element $A(u)$ of $(W^{-1,q}(\Omega))^n$, such that

$$a(u,v)_p = \langle A(u), v\rangle\,, \forall v \in (W_0^{1,p}(\Omega))^n.$$

The application $(W^{1,p}(\Omega))^n \longrightarrow (W^{-1,q}(\Omega))^n$, $u \longrightarrow A(u)$, is noted :

$$-L_p u = \; -\mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - (\lambda+\mu)\nabla(div(u)),$$

and is called a **p-Lamé application.**

The following proposition gives some properties of $-L_p$:

Proposition 2.1.- The operator $-L_p : (W^{1,p}(\Omega))^n \rightarrow (W^{-1,q}(\Omega))^n$ is bounded, hemicontinuous, monotone and coercitive.

Demonstration: Using the expression of the norm in dual space espace dual and Lebesgue's dominated convergence theorem, we prove that $-L_p$ is bounded and hemicontinuous. From the convexity of the real application $t \longrightarrow |t|^p$, we deduce the monotonicity of $-L_p$.

Proposition 2.2.- The problem (P) and the variational problem $(P.V)$:

$$a_p(u,v) + (|u|^{p-2}\, u, v) = (f,v) + (-\varphi(x)\ (u,v), \forall v \in (W^{1,p}(\Omega))^n,$$

are equivalent.

Demonstration: Indeed, it suffices to observe that $u = 0$ on $\Gamma_1 = 0 \Leftrightarrow\in (W_0^{1,p}(\Omega))^n$,and the variationnal equality is then equivalent to

$$-L_p u + |u|^{p-2}\, u = f \;\; \text{in } \Omega,$$

because $(D(\Omega))^n$ is dense in $(W_0^{1,p}(\Omega))^n$.

Let us return to the demonstration of **Theorème2.1.**

(i) Construction of approximated solutions :

We look for $u_m = \sum_{i=1}^{n} \lambda_i v_i$ solution of the following problem (P_m) :

$\forall j,\ 1 \leq j \leq m$:

$$a_p(u_m, v_j) + (|u_m|^{p-2}\, u_m, v_j) = (f, v_j) + (-\varphi(x)\ (u_m, v_j)$$

We obtain a second order nonlinear differential system. Let be the function

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

$$F(\lambda_1,...,\lambda_m) = \left(\left\langle A(\sum_{i=1}^{n} \lambda_i v_i), v_j \right\rangle - ((f,v_j) + (-\varphi(x)(u_m,v_j)).\right)_{1\leq j\leq m}$$

(ii) Establishment of priori estimates.-

- Of the coercivité of to one deducts that $||u_m||$ is a bounded;
- The operator has a bounded $\Longrightarrow (A(u_m))\ _{m\in\mathbb{N}}$ is a bounded in $V\prime$;
- $\exists u \in V, \exists \chi \in V' \Longrightarrow \begin{cases} u_p \rightharpoonup u,\ \sigma(V,V'), \\ A(u_m) \rightharpoonup \chi,\ \sigma(V',V). \end{cases}$

(iii) Passage to the limit via compactness.

- The monotony and the hemicontinuous $\Longrightarrow \chi = A(u)$.

What finishes the demonstration of the **Theorem 2.1.**

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Mathematical Modeling of Gonorrheal Disease a Case Study with Reference to Anantapur District-Andhrapradesh-India

By Dr. G.Prabhakararao
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Abstract - In this paper we have analyzed the Mathematical modeling of Gonorrheal disease, the spread of a contagious disease involves interactions of two populations: the susceptible and the infectives. In some diseases these two populations are from different species. For example, malaria is not passed directly between animals but by the anopheline mosquitoes, and schistosomiasis is passed from animal to animal only through contact with water in which live snails that can incubate the disease-causing helminthes. In other diseases, the infection can be passed direct from invectives to susceptible: Viral diseases like chicken-pox, measles, and influenza, and bacterial diseases like tuberculosis can pass through a population much like flame through fuel.

Keywords : modeling, infectious diseases, epidemics, stratified populations, susceptible, chain-branched reactions.

GJSFR-F Classification : MSC 2010: 00A71 , 93A30

MATHEMATICAL MODELING OF GONORRHEAL DISEASE A CASE STUDY WITH REFERENCE TO ANANTAPUR DISTRICT-ANDHRAPRADESH-INDIA

Strictly as per the compliance and regulations of :

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I. Introduction

The spread of a contagious disease involves interactions of two populations: the susceptible and the infectives. In some diseases these two populations are from different species. For example, malaria is not passed directly between animals but by the anopheline mosquitoes, and schistosomiasis is passed from animal to animal only through contact with water in which live snails that can incubate the disease-causing helminthes. In other diseases, the infection can be passed direct from invectives to susceptible: Viral diseases like chicken-pox, measles, and influenza, and bacterial diseases like tuberculosis can pass through a population much like flame through fuel.

There are useful analogies between epidemics and chemical reactions. A theory of epidemics was derived by W.O.Kermack, a chemist, and A.G.Mc Kendrick, a physician, who worked at the Royal college of Surgeons in Edinburgh between 1900 and 1930.They introduced and used many novel mathematical ideas in studies of populations [4].One important result of theirs is that any infection determines a threshold size for the

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4. W.O.Kermark and A.G.Mc.Kendrick,A contribution to the theory of epidemics Proc.Roy.Soc. (A) 115 (1923), 700-721,139(1932), 55

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susceptible population, above which an epidemic will propagate. Their theoretical epidemic threshold is observed in practice, and it measures to what extent a real population is vulnerable to spread of an epidemic. At roughly the same time V.I.Semenov [6] derived a theory of combustion that identified explosion meets beyond which combinations of pressure and temperature cause chemicals to begin explosive chain-branched reactions

These models can serve as building blocks to study other diseases, for example ones having intermediate hosts, and diseases in stratified populations, for example where there are mixing groups that have various contact probabilities, like families, preschools, and social groups. Some of these extensions are described in the exercises

II. Spread of an Epidemic and Deterministic Models

Consider the situation where a small group of people having an infectious disease mixes with a sizable population susceptible to carry the infection. The equations a use how many people will ultimately catch the disease? Will the infection spread rapidly or die out gradually? How does the epidemic evolve in time? In order to discuss this situation, we consider a fixed population and assume that three kinds of individuals in it exist viz, susceptibles, infectives and removals. Susceptibles can acquire the infection upon effective contact with an infective. Infectives have the disease and are capable of transmitting it. Removals are those who have passed through the disease process but are no longer susceptible or infective. We assume that a person who as permanent immunity and also the disease has a negligibly short incubation period. This implies that an individual who contracts the disease becomes infective immediately after words. The flow of the disease is describe as follows

$$S \rightarrow I \rightarrow R$$

Indicating that susceptible might become infectives, and invectives might be removed, but there is no supply of new susceptible to the processes.

We now begin with the Reed-Frost model which describes the spread of infection in population due to random sampling.

In a given population let S_n, I_n and R_n be the number of susceptible infected persons and removals at the nth sampling time at the end of nth sampling interval, the corresponding numbers are S_{n+1}, I_{n+1} and R_{n+1}.

The probability that a susceptible avoids contact with all of the infectives during the sampling interval is

$$q_n = (1-p)^{I_n}$$

Ref.

6. Martin Braun The threshold theorem of epidemiology, Differentail equations and their applications Springer-Verlag Publishers.

Therefore, the probability that

$$S_{n+1} = k \text{ is}$$

$$\binom{S_n}{k} q_n^{\ k} (1-p)^{S_n - k}$$

which is a simple binomial distribution. That is, the probability that k survive the sampling interval as susceptible is the number of ways k can be selected from among candidates times the probability that k avoid effective contact and the probability that have effective contact.

If

$$S_{n+1} = k \text{ then } I_{n+1} = S_{n+1} - k \text{ and } R_{n+1} = R_n + I_n$$

This calculation is summarized by the formula

$$\Pr[S_{n+1} = \frac{k}{S_n} \, and \, I_n] = \binom{S_n}{k} q_n^{\ k} (1-p)^{S_n - k}$$

and this called the Reed-Frost model [1].

The Kermack-Mc Kendrick model [2] is a nonrandom model that describes the proportion in large populations.

Let x_n denote the number of susceptible, y_n the number of infectives and the number of removals. It might be expected that the number of susceptible in the next sampling interval would be

$$x_n = (1-p)^{y_n} x_n$$

since the factor $(1-p)^{y_n}$ is the proportion of susceptible who avoid effective contact with all infectives.

Let $a = -\log(1-p)$ so that e^{-a} is the probability that a given susceptible will successfully avoid contact with each infective during the sampling interval. Assume that the sampling interval is the same as the interval of infectiousness. Supposing that those leaving the susceptible class enter directly into the infectious class, and that a proportion, say b, of the infectives remain infective at the end of each sampling interval. Then

$$x_{n+1} = e^{-a y_n} x_n \text{ and}$$

Ref

1. N.T.S.Bailey, The theory of infectives disease and its application, Charle Griffin, London, 1975.

$$y_{n+1} = (1-e^{-ay_n})x_n + by_n$$

determine their numbers. This model is the Kermack-Mc Kendrick model.

We now consider simple deterministic models without removal [3].In a given population at time *t*, let s be the number of susceptible *S (t)*, *I (t)* be the number of infected persons. Let *n* be the initial number of susceptible in the population and let one person is infected initially. Assuming that the rate of decrease of *S (t)* or the rate of increase of *I (t)*, is proportional to the product of the number of susceptible and the number of infected. We obtain a model

$$\frac{dS}{dt} = -\beta SI \,, \frac{dI}{dt} = -\beta SI$$

Since *S (0)* is *n* and *I (0)* is one and *S (t) +I (t) =N+1*.we obtain

$$\frac{dS}{dt} = -\beta S(n+1-S)$$

Integrating using the initial conditions

$$S(t) = \frac{n(n+1)}{n+e^{(n+1)\beta t}} \,,\; I(t) = \frac{(n+1)e^{(n+1)\beta t}}{n+e^{(n+1)\beta t}}$$

So that

$$\underset{n\to\infty}{Lt} S(t) = 0 \,,\; \underset{n\to\infty}{Lt} I(t) = n+1$$

and, ultimately, all persons will be infected.

A modification of above model may be obtain by assuming that a susceptible person can become infected at a rate proportional to *SI* and an infected person can recover and become susceptible again at a rate γI so that we get the model

$$\frac{dS}{dt} = -\beta SI + \gamma I \quad \frac{dI}{dt} = \beta SI - \gamma I \qquad (2.1)$$

which gives

$$S(t) + I(t) = N = S(0) + I(0) = S_0 + I_0 \,(I_0 \neq 0) \qquad (2.2)$$

from (2.1) and (2.2)

$$\frac{dI}{dt} = (\beta N - \gamma)I - \beta I^2 = kI - \beta I^2$$

Ref.

3. J.N.Kapur Mathematical models in Biology and Medicine,East-west press private limited 2000.

Ref

5. Classic Papers in the theory of epidemics.See also A.G.Mc.Kendrick, Mathematics applied to medical problems, Roy.Soc.Edinb.,1926.

Integrating, we obtain

$$I(t)=\begin{bmatrix} \dfrac{e^{kt}}{\dfrac{\beta[e^{kt}-1]}{k}+I_0^{-1}} & (k\neq 0) \\ \dfrac{1}{\beta t+I_0^{-1}} & (k=0) \end{bmatrix}$$

As $t\rightarrow\infty$,

$$I(t)=\begin{bmatrix} N-\rho \quad if \quad N\succ\rho=\dfrac{\gamma}{\beta} \\ 0 \qquad if \quad N\leq\rho=\dfrac{\gamma}{\beta} \end{bmatrix}$$

We now discuss the deterministic model taking into account the number of persons removed from the population by recovery, immunization, death, hospitalization or by any other means. We make use of the following assumptions [5].

I. The population remains at affixed level N in the time interval under consideration. This means, of course, that we neglect births, deaths from causes unrelated to the disease under consideration, immigration and emigration.

II.The rate of change of the susceptible population is proportional to the product of the number of *(S)* and the number of members of *(I)*.

III. Individuals are removed from the infectious class *(I)* at the proportional to the size of *(I)*.

Let *S(t), I(t),* and *R(t)* denote the number of individuals in classes *(S),(I),*and *(R)*,respectively, at time t. It follows immediately form rules (I-III) that *S (t), I (t),* and *R (t)* satisfies the system of differential equations.

$$\frac{dS}{dt}=-rSI$$

$$\frac{dI}{dt}=rSI-\gamma I \qquad (2.3)$$

$$\frac{dR}{dt}=\gamma I$$

For some positive constants r and γ.The proportionality constant r is called the infection rate, and the proportionality constant γ is called removal rate.

The first two equations of (2.1, 2.2) do not depend on R.Thus,we need only consider the system of equations.

$$\frac{dS}{dt} = -rSI, \quad \frac{dI}{dt} = rSI - \gamma I \qquad (2.4)$$

For the two unknown functions *S (t)* and *I (t)*.Once *S (t)* and *I (t)* are known, we can solve for *R (t)* form the third equation of (2.3). Alternately, observe that $\frac{d(S+I+R)}{dt}$.

Thus,

$$S(t) + I(t) + R(t) = constant = N$$

so that $R(t)=N-S(t)-I(t)$,

the orbits of (2.2) are the solution curves of the first-order equation

$$\frac{dI}{dt} = \frac{rSI - \gamma I}{-rSI} = -1 + \frac{\gamma}{rS} \qquad (2.5)$$

Integrating this differential equation gives

$$I(S) = I_0 + S_0 - S + \rho \log \frac{S}{S_0} \qquad (2.6)$$

where S_0 and I_0 are the number of susceptible and infectives at the initial time $t = t_0$ and $\rho = \frac{\gamma}{r}$. To analzethe behavior of the curves(2.6),we compute $I'(S) = -1 + \frac{\rho}{S}$. The quantity $-1 + \frac{\rho}{S}$ is negative for $S \succ \rho$,and positive for $S \prec \rho$.Hence, *I(S)* is an increasing function of S for $S \prec \rho$ and a decreasing function of S for $S \succ \rho$.Next, observe that $I(0) = -\infty$ and $I(S_0) = I_0 \succ 0$.Consequently, there exists a unique point S_∞ with $0 \prec S_\infty \prec S_0$, such that $I(S_\infty) = 0$, and $I(S) \succ 0$ for $S_\infty \prec S \leq S_0$. The point $(S_\infty, 0)$ is an equilibrium point of (2.2) since both $\frac{dS}{dt}$ and $\frac{dI}{dt}$ vanish when $I=0$.

Let us see what all this implies about the spread of the disease within the population. As t runs from t_0 to ∞, the point *(S(t) ,I(t))*travels along the curve (2.4), and it moves along the curve in the direction of decreasing *S*, since *S(t)* decreases monotonically with time. Consequently, if S_0 is less than, then I(t) decreases

Notes

Global Journal of Science Frontier Research (F) Volume XIII Issue V Version I Year 2013

monotonically to zero, and $S(t)$ decreases monotonically to S_∞.Thus, if a small group of infectives I_0 is inserted into a group of susceptible S_0, with $S_0 \prec \rho$,then the disease will die out rapidly. On the other hand, if S_0 is greater than, then $I(t)$ increases as $S(t)$ decreases to ρ, and it achieves a maximum value when $S=\rho$. It only starts decreasing when number of susceptible falls below the threshold value ρ. From these results we may draw the following conclusions.

Conclusions I. An epidemic will occur if the number of susceptible in a population exceeds the threshold value $\rho = \frac{\gamma}{r}$.

II. The spread of the disease does not stop for lack of a susceptible population; it stops only for lack of infectives. In particular, some individuals will escape the disease altogether.

a) Spread of Infection within a Family

What happens when an infectious diseases is introduced into a small family? What is the likelihood of spread within the family? We can answer these questions by carefully counting the possibilities.

Consider a fixed population and assume that three kinds of individuals in it are defined by a disease: susceptibles,infectives, and removals.Susceptibles can acquire the infection upon effective contact with an infective,infectives have the disease and are capable of transmitting it, and removals are those who have passed through the disease process but are no longer susceptible or infective.

$$S \to I \to R$$

indicating that susceptible might become infectives,and invectives might be removed, but there is no supply of new susceptible to the processes.

There are problems with taking such a simple view of a disease. For example, there are great variations in the level of susceptibility, infectiousness, and immunity among individuals in populations. Also, these definitions may depend on stratifying attributes such as age groups, genetic type or mixing groups, etc, there are many diseases whose transmission mechanisms are not known, nor how long are latency periods between becoming infective and the appearance of symptoms.

III. Calculation of the Severity of an Epidemic

Suppose that there are smooth functions $S(t), I(t), and\ R(t)$ and a small interval h such that $x_n = S(nh), y_n = I(nh),\ and\ z_n = R(nh)$. Since the sampling interval h is small, we must rescale a and b

Let $a = rh\ and\ b = 1 - h\sigma$ .Then setting $t = nh$, we have

$$S(t+h) = e^{-rhI(t)}S(t)$$

$$I(t+h) = (1-h\sigma)I(t) + (1-e^{-rhI(t)})S(t)$$

$$R(t+h) = R(t) + (1-(1-h\sigma))I(t)$$

It follows that

$$S(t+h) - S(t) = (e^{-rhI(t)} - 1)S(t) \sim -rhI(t)S(t)$$

$$I(t+h) - I(t) = -h\sigma I(t) + (1-e^{-rhI(t)})S(t)$$

$$\sim -h\sigma I(t) + rhI(t)S(t)$$

$$R(t+h) - R(t) = h\sigma I(t)$$

Dividing these equations by h and passing to the limit $h = 0$ gives three differential equations for approximations to $S(t), I(t), and\ R(t)$; which we write as

$$\frac{ds}{dt} = -rIS$$

$$\frac{dI}{dt} = rIS - \sigma I$$

$$\frac{dR}{dt} = \sigma I$$

This system of equations is the continuous-time version of Kermark and Mc Kendrick's model.Incidentally,the calculation just completed gives a neat derivation of the law of mass action in chemistry in which the rate at which to chemical species ,say having concentrations S and , I interact is proportional to product $S\,I$. Thus, the Law of mass action follows from the binomial distribution of random interactions since the expected number of interactions occurring in a specified (short) time interval is

$$(1-q_n)S_n = (1-e^{-rhI_n})S_n \sim rhI_nS_n$$

Notes

Obviously, this law and the two model derived for epidemics depend on the assumption that the populations are thoroughly mixing as the process continues

We can solve the differential equations and so determine the severity of an epidemic.Taking the ratio of the first two equations gives

$$\frac{dS}{dI} = -\frac{rIS}{rIS - \sigma I} = \frac{-rS}{rS - \sigma}$$

Therefore

$$dI = (\frac{\sigma}{rS} - 1)\ dS$$

Integrating this equation gives

$$\mathrm{I} = (\frac{\sigma}{\mathrm{r}})\log \mathrm{S} \text{-} \mathrm{S} + C$$

where c is a constant of integration that is determined by the initial conditions:

$$C = I_0 - \log S_0 + S_0$$

Typical trajectories

In the infinitesimal sampling process, the threshold level of S* becomes

$$S* = \frac{(1-b)}{(1-e^a)} \sim (\frac{\sigma}{r})$$

S^* is the value of S for which $\frac{DI}{Dt} = 0$.We see that trajectories starting near but about this value describe epidemics that end at a comparable distance below this value.Trajectories that start well about S^* end up near $S = 0$.However, in each case the final size of the susceptible population, S (infinity), is where the trajectory meets the $I = 0$ axis Therefore ,solving the equation

$$\mathrm{S} \text{-} (\frac{\sigma}{r}) \log \mathrm{S} = \mathrm{C}$$

For its smaller of two roots gives the final size .this is not possible to do in a convenient form; however, it is easy to do using a computer. In this way ,we can estimate an epidemic's severity once we have estimated the infectiousness (a or r) and the removal rate (b or σ)

Recurrent diseases. Finally, there are diseases in which removals can eventually become susceptible again .This is the case for a variety of sexually transmitted diseases, for example gonorrhea. The flow of such a disease is depicted by the graph

$$S \to I \to S\,.$$

Without further discussion, we can write down a model of such a disease:

$$\frac{dS}{dt} = -rSI + \sigma I$$

$$\frac{dI}{dt} = rSI - \sigma I$$

Since $I + S$ is constant (its derivative is zero), we can reduce these equations to a single equation.

$$\frac{dS}{dt} = (\sigma - rS)(I_0 + S_0 - S)\,.$$

We see that if $S^* = \dfrac{\sigma}{r} \prec I_0 + S_0$, *then* $S \to S^*$ in this case! In the other case $S \to I_0 + S_0$, and the infection dies out of the population. When $S \to S^*$, the disease is epidemic. Stratification of the population, latency periods of the disease, hidden carriers, seasonal cycling of contact rates, and many other factors confound the study of epidemics, but the simple modals derived here provide useful and interesting methods.

IV. Spread of Gonorrhea in Anantapur District (a Case Study)

Gonorrhea ranks first today among reportable communicable diseases in the United States. There are more reported cases of gonorrhea every year than the combined totals for syphilis, measles, mumps, and infectious hepatitis. This painful and dangerous disease, which is caused by the gonococcus germ, is spread from person to person by sexual contact. A few days after the infection there is usually itching and burning of the genital area, particularly while urinating. About the same time a discharge develops which males will notice, but which females may not notice. Infected women may have no easily recognizable symptoms, even while the disease does substantial internal damage. Gonorrhea can only be cured by antibiotics (usually penicillin). However, treatment must be given early if the disease is to be stopped from doing serious damage to the body. If untreated, gonorrhea can result in blindness, sterility, arthritis, heart failure, and ultimately, death.

In this section we construct a mathematical model of the spread of gonorrhea. Our work is greatly simplified by the fact that the incubation period of gonorrhea is very short (3-7 days) compared to the often quite long period of active infectiousness. Thus, we will

Notes

assume in our model that an individual becomes infective immediately after contracting gonorrhea. In addition, gonorrhea does not confer even partial immunity to those individuals who have recovered from it. Immediately after recovery, an individual is again susceptible. Thus, we can split the sexually active and promiscuous portion of the population into two groups, susceptibles and infectives. Let $c_1(t)$ be the total number of promiscuous males, $c_2(t)$ be the total number of promiscuous females, $x(t)$ the total number of infective males, and $y(t)$ the total number of infective females, at time t. Then, the total numbers of susceptible males and susceptible females are $c_1(t)-x(t)$ and $c_2(t)-y(t)$ respectively. The spread of gonorrhea is presumed to be governed by the following rules:

I. Males infectives are cured at a rate a_1 proportional to their total number, and female infectives are cured at a rate a_2 proportional to their total number. The constant a_1 is larger than a_2 since infective males quickly develop painful symptoms and therefore seek prompt medical attention. Female infectives, on the other hand, are usually asymptomatic, and therefore are infectious for much longer periods.

II. New infectives are added to the male population at a rate b_1 proportional to the total number of male susceptibles and female infectives. Similarly, new infectives are added to the female population at a rate b_2 proportional to the total number of female susceptibles and male infectives.

III. The total numbers of promiscuous males and promiscuous females remain at constant levels c_1 and c_2, respectively.

It follows immediately from rules I-III that

$$\begin{aligned}\frac{dx}{dt} &= -a_1 x + b_1(c_1 - x)y \\ \frac{dy}{dt} &= -a_2 y + b_2(c_2 - y)x\end{aligned} \qquad (4.1)$$

If $x(t_0)$ and $y(t_0)$ are positive, then $x(t)$ and $y(t)$ are positive for all $t \ge t_0$

If $x(t_0)$ is less than c_1 and $y(t_0)$ is less than c_2 then $x(t)$ is less than c_1 *and* $y(t)$ is less than c_2 for all $t \ge t_0$

We can show that equation (4.1)

(a) Suppose that $a_1 a_2$ *is less than* $b_1 b_2 c_1 \ c_2$ .Then, every solution $x(t)$, $y(t)$ of (4.1) with $0 \prec x(t_0) \prec c_1$ *and* $0 \prec y(t_0) \prec c_2$, approaches the equilibrium solution.

$$x = \frac{b_1 b_2 c_1 c_2 - a_1 a_2}{a_1 b_2 + b_1 b_2 c_2}, \ y = \frac{b_1 b_2 c_1 c_2 - a_1 a_2}{a_2 b_1 + b_1 b_2 c_1}$$

as t approaches infinity. In other words, the total number of infective males and infective females will ultimately level off.

Proof: The result can be established by splitting the rectangle $0 \prec x \prec c_1$ *and* $0 \prec y \prec c_2$ into regions in which both $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$ have fixed signs. This is a accomplished in the following manner. Setting $\frac{dx}{dt}=0$ in equation (4.1), and solving for y as a function of x gives.

$$-a_1 x + b_1 (c_1 - x) y = 0$$

$$b_1 y (c_1 - x) = a_1 x$$

$$y = \frac{a_1 x}{b_1 (c_1 - x)} = \phi_1 x$$

Similarly, setting $\frac{dy}{dt} = 0$ in (4.1)

$$-a_2 y + b_2 x (c_1 - y) = 0$$

$$b_2 x (c_2 - y) = a_2 y$$

$$x = \frac{a_2 y}{b_2 (c_2 - y)}$$

$$x b_2 (c_2 - y) = a_2 y$$

$$x b_2 c_2 - x b_2 y = a_2 y$$

$$x b_2 c_2 = a_2 y + x b_2 y$$

$$x b_2 c_2 = y (a_2 + x b_2)$$

$$\frac{x b_2 c_2}{a_2 + x b_2} = y$$

$$y = \frac{x b_2 c_2}{a_2 + x b_2} = \phi_2 x$$

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Observe that $\phi_1 x$ and $\phi_2 x$ are monotonic increasing functions of *x;* $\phi_1 x$ approaches infinity as x approaches c_1 and $\phi_2 x$ approaches c_2 as x approaches infinity. Second, observe that the curves $y=\phi_1 x$, $y=\phi_2 x$ intersect at $(0,0)$ and at (x_0, y_0) where

$$x_0 = \frac{b_1 b_2 c_1 c_2 - a_1 a_2}{a_1 b_2 + b_1 b_2 c_2}, y_0 = \frac{b_1 b_2 c_1 c_2 - a_1 a_2}{a_2 b_1 + b_1 b_2 c_1}$$

Third, observe that $\phi_2 x$ is increasing faster than $\phi_1 x$ at $x=0$, since

$$\phi_2^{'}(0) = \frac{b_2 c_2}{a_2} \succ \frac{a_1}{b_1 c_1}$$

Hence, $\phi_2 x$ lies above for $0 \prec x \prec x_0$ and lies below $\phi_1 x$ for $x_0 \prec x \prec c_1$. The point (x_0, y_0) is an equilibrium point of (4.1) since both $\frac{dx}{dt} and \frac{dy}{dt}$ are zero when $x=x_0$ and $y=y_0$.

Finally, observe that $\frac{dx}{dt}$ is positive at any point (x,y) above the curve $y=\phi_1 x$,and negative at any point (x,y) below this curve. Similarly, $\frac{dy}{dt}$ is positive at any point (x,y) below curve $y=\phi_2 x$, and negative point (x,y) above this curve. Thus, the curves $y=\phi_1 x$,and $y=\phi_2 x$ split the rectangle $0 \prec x \prec c_1$ *and* $0 \prec y \prec c_2$ into four regions in which $\frac{dx}{dt} and \frac{dy}{dt}$ haved fixed sings.

It can be established that any solution *x (t), y (t)* of (4.1) which starts in region I at time $t=t_0$, will remain in this region for all future time $t \ge t_0$ that and approach the equilibrium solution $x=x_{0,} y=y_0$ as *t* approaches infinity.

Any solution *x (t), y (t)* of (4.1) which starts in region II at time $t=t_0$, will remain in this region II for all future time , must approach the equilibrium solution $x=x_{0,} y=y_0$ as *t* approaches infinity.

Any solution *x (t), y (t)* of (4.1) which starts in region IV at time $t=t_0$, will remain in this region IV for all future time , must approach the equilibrium solution $x=x_{0,} y=y_0$ as *t* approaches infinity.

We make use of the above mentioned deterministic model to study of severity of Gonorrhea diseased in Anantapur district during the period of 1995-2003 based on the data collection from the Head Quarters of Hospital Anantapur during this period.

Year wise male population in Anantapur district and case study of gonorrheal disease form the recorded data of Government Head Quarters Hospital, Anantapur, Andhra Pradesh

Table 1

Years	Total Male population	Total number of promiscuous Males	Total number of Infective Males	Total number of Males Cured
1995	1686038	16860.38	7250	6975
1996	1698463	16984.63	5178	4846
1997	1723455	17234.55	6147	5872
1998	1748654	17486.54	3065	2969
1999	1798563	17985.63	6598	5985
2000	1810943	18109.43	3657	3465
2001	1859588	18595.88	2713	2285
2002	1910464	19104.64	2577	2299
2003	1945084	19450.84	1954	1835

Year wise Female population in Anantapur district and case study of gonorrheal disease form the recorded data of Government Head Quarters Hospital, Anantapur, Andhra Pradesh

Table 2

Years	Total Female population	Total number of promiscuous Females	Total number of Infective Females	Total number of Females Cured
1995	1516549	15165.49	3549	3020
1996	1576910	15769.10	1988	1866
1997	1624704	16247.04	1769	1665
1998	1672291	16722.91	1935	1798
1999	1695168	16951.68	1701	1595
2000	1755574	17555.74	1278	1152
2001	1780890	17808.90	1356	1265
2002	1801626	18016.26	1093	943
2003	1839792	18397.92	899	844

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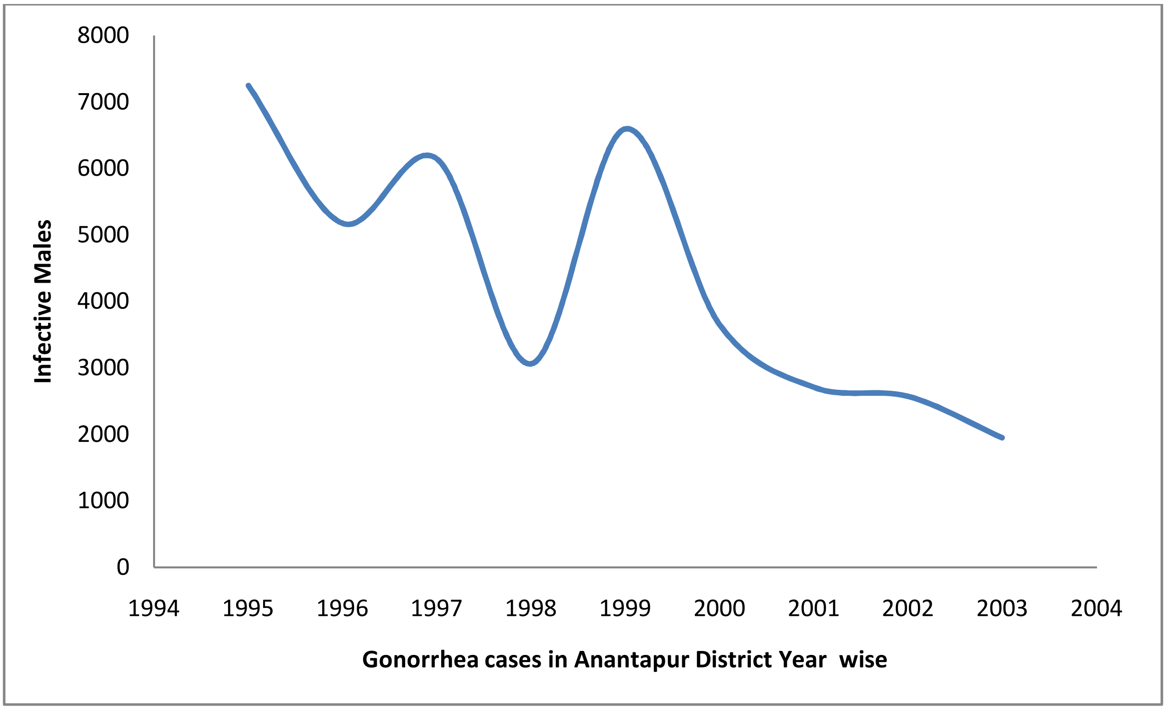

Figure 1 : **Profile of Infective Males with Gonorrheal Disease during the period of 1995-2003**

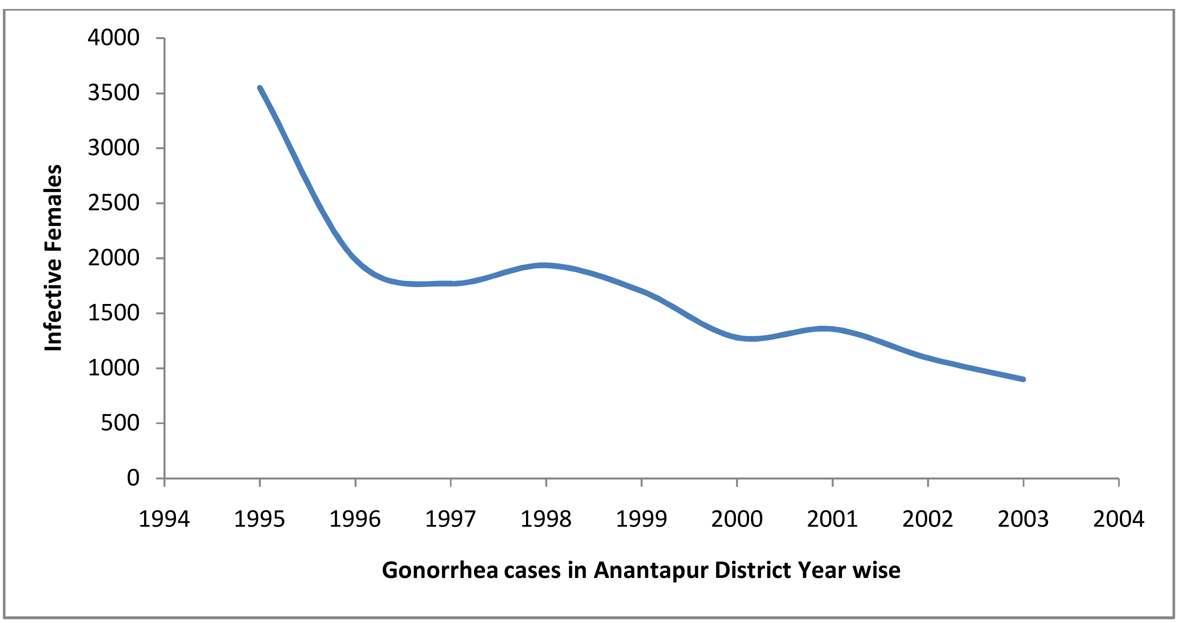

Figure 2 : **Profile of Infective Females with Gonorrheal Disease during the period of 1995-2003**

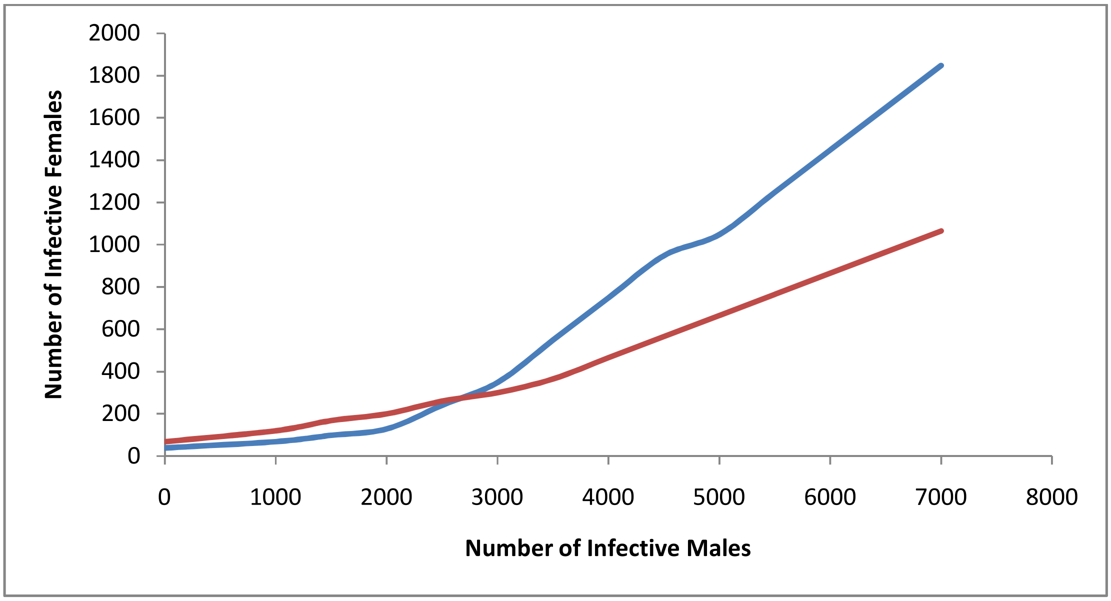

Figure 3 : **Profile of Number of Infective Males Vs Number of Infective Females during the period of 1995-2003**

V. Discussion

The Study of epidemic is quite interesting and important. When a small group of people having infected with an epidemic disease is inserted into a large population which is capable of catching the disease, the question arises what happens as time evolves. Will the disease die out rapidly or its spreads? How many people will catches is disease? To answer these questions, we have chosen a mathematical modeling consisting of a system of differential equations which govern the spread of infected disease within the population and analyse the behavior of its solution. This approach will lead to the famous threshold theorem of Epidemiology which states that an epidemic will occur only if the number of people who are susceptible to the disease exceeds a certain threshold value. In this paper, we discussed the spread of Gonorrheal disease among males and females in Anantapur District during the period 1995-2003, based on the recorded data available in Government Head Quarters Hospital Anantapur. As already stated, we assume in our model that an individual become infective immediately after contacting the Gonorrhea. The proportional rate a_1, a_2 and b_1 *and* b_2 are quite difficult to evaluate. However, we have made the crude estimate of these proportional constants based on available data. It is interesting note that condition $a_1 a_2 \prec b_1 b_2 c_1 c_2$ is satisfied by the said constants. This

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condition is equivalent to $1 \prec \left(\frac{b_1c_1}{a_2}\right)\left(\frac{b_2c_2}{a_1}\right)$. The expression $\left(\frac{b_1c_1}{a_2}\right)$ can be interpreted as the average number of males that one female infective contact during her infectious period, if every male is susceptible. Similarly the expression $\left(\frac{b_2c_2}{a_1}\right)$ can be interpreted as the average number of females that one male infective contact during his infectious period, if every female is susceptible. These quantity $\left(\frac{b_1c_1}{a_2}\right)$ and $\left(\frac{b_2c_2}{a_1}\right)$ are called the maximal female and male contacts rates. In view of the fact that this product of maximal male and female contact rates is greater than one, we may conclude that the solution of the mathematical model approaches the equilibrium solution and the Gonorrheal disease will approach a non zero steady state in course of time. This equilibrium solution also implies that the total number of infective males and infective females will ultimately level off, and from Fig (3) the point of the equilibrium approximately gives x_0 (Infective males) =260, y_0 (Infective females) =266. We may conclude that this Epidemic disease does not die out but ultimately approach a steady state with reference to its severity among the population of Anantapur District.

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Formation of a Summation Formula in Connection with Hypergeometric and Gamma Function

By Salahuddin & Intazar Husain
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Abstract - The main objective of the present paper is to establish a summation formula in connection with Hypergeometric and Gamma function.

Keywords : *contiguous relation, gauss second summation theorem, recurrence relation.*

GJSFR-F Classification : *MSC 2010: 11S80, 40A25*

FORMATION OF A SUMMATION FORMULA IN CONNECTION WITH HYPERGEOMETRIC AND GAMMA FUNCTION

Strictly as per the compliance and regulations of :

Formation of a Summation Formula in Connection with Hypergeometric and Gamma Function

Salahuddin [α] & Intazar Husain [σ]

Abstract - The main objective of the present paper is to establish a summation formula in connection with Hypergeometric and Gamma function.

Keywords and Phrases : *contiguous relation, gauss second summation theorem, recurrence relation .*

I. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lll} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{lll} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lll} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers and $\mid z \mid = 1$.

Contiguous Relation is defined by

[Andrews p.363(9.16)]

$$ (a-b)\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ c\ \ ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ll} a+1,\ b\ \ ; & \\ c\ \ ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ll} a,\ b+1\ \ ; & \\ c\ ; & z \end{array}\right] \tag{2} $$

Gauss second summation theorem is defined by [Prudnikov, 491(7.3.7.8)]

$$ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ \frac{a+b+1}{2}\ ; & \frac{1}{2} \end{array}\right] = \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \tag{3} $$

$$ = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})} \tag{4} $$

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Recurrence relation is defined by

$$\Gamma(z+1) = z\,\Gamma(z) \tag{5}$$

II. Main Summation Formula

$$ {}_2F_1\left[\begin{array}{ll} a, \quad b \;; & \frac{1}{2} \\ \frac{a+b+38}{2} \;; & \end{array}\right] = \frac{2^b\,\Gamma(\frac{a+b+38}{2})}{(a-b)\,\Gamma(b)}\times $$

$$ \times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{262144(-466206625753989120000a+801771087841984512000a^2-596902689054602035200a^3)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+\right.\right. $$

$$ +\frac{262144(259410161783191633920a^4-74489848286749655040a^5+15114758112640368640a^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(-2257478987452416000a^7+254687695850864640a^8-22057499145876480a^9)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(1479487730772480a^{10}-77059448294400a^{11}+3106062568960a^{12}-95888257920a^{13})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(2223453120a^{14}-37454400a^{15}+432480a^{16}-3060a^{17}+a^{18}+466206625753989120000b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(227646051134629478400ab-845615187148721356800a^2b+204544425045918744576a^3b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(-414766676276709949440a^4b+229435681205960048640a^5b-26709695220289536000a^6b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(6536552959206359040a^7b-463676439510789120a^8b+60870335749893120a^9b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(-2674527779404800a^{10}b+203102891427840a^{11}b-5385879757440a^{12}b+239858989440a^{13}b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$ +\frac{262144(-3503822400a^{14}b+87279360a^{15}b-535500a^{16}b+6300a^{17}b+801771087841984512000b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+ $$

$$+\frac{262144(845615187148721356 80ab^2+3767653738922744217 60a^2b^2-371061079468442910 72a^3b^2)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(946619218456657920 00a^4b^2-825005824582754304 0a^5b^2+483448493742391296 0a^6b^2)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(-286088788492683264a^7b^2+73488037086731520a^8b^2-2830362252157440a^9b^2)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(385171067635200a^{10}b^2-9214702632576a^{11}b^2+714494944800a^{12}b^2-9600101280a^{13}b^2)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(427616640a^{14}b^2-2450448a^{15}b^2+58905a^{16}b^2+596902689054602035 20b^3)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(204544425045918744576ab^3+3710610794684429107 2a^2b^3+1481523391702916136 96a^3b^3)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(-595460874960838656 0a^4b^3+1479843527378554060 8a^5b^3-655755966671984640a^6b^3)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(38066811299364 2496a^7b^3-1241163223255296 0a^8b^3+3161240677647360a^9b^3)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(-67636609729920a^{10}b^3+9071028833664a^{11}b^3-112495612320a^{12}b^3+8492373120a^{13}b^3)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(-45912240a^{14}b^3+1947792a^{15}b^3+25941016178319163392b^4+414766676276709949 44ab^4)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(946619218456657920 00a^2b^4+595460874960838656 0a^3b^4+2124840505791725568 0a^4b^4)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(-4173332340975114 24a^5b^4+97920219203364096 0a^6b^4-23674554904366080a^7b^4)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(1319008023476352 0a^8b^4-239192223842880a^9b^4+5864768599536 0a^{10}b^4)}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{262144(-654569667360a^{11}b^{4}+84112741440a^{12}b^{4}-423644760a^{13}b^{4}+30260340a^{14}b^{4})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(7448984828674965504b^{5}+22943568120596004864ab^{5}+8250058245827543040a^{2}b^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(1479843527378554060 8a^{3}b^{5}+417333234097511424a^{4}b^{5}+1333407400586376192a^{5}b^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(-13876520995814400a^{6}b^{5}+30294926136059904a^{7}b^{5}-403001238644928a^{8}b^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(211792515536448a^{9}b^{5}-1997908688160a^{10}b^{5}+463267099008a^{11}b^{5}-2102818536a^{12}b^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(254186856a^{13}b^{5}+1511475811264036864b^{6}+2670969522028953600ab^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(4834484937423912960a^{2}b^{6}+655755966671984640a^{3}b^{6}+979202192033640960a^{4}b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(13876520995814400a^{5}b^{6}+39814926411617280a^{6}b^{6}-223040891969280a^{7}b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(449675387995680a^{8}b^{6}-3081641824800a^{9}b^{6}+1511564503680a^{10}b^{6}-5776974000a^{11}b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(1251677700a^{1}2b^{6}+2257478987452416000b^{7}+653655295920635904ab^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(2860887884926832 64a^{2}b^{7}+380668112993642496a^{3}b^{7}+236745549043660 80a^{4}b^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(30294926136059904a^{5}b^{7}+223040891969280a^{6}b^{7}+576376172064000a^{7}b^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(-1635980781600a^{8}b^{7}+3032869276800a^{9}b^{7}-8351853840a^{10}b^{7}+3796297200a^{11}b^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(2546876958508646 4b^{8}+46367643951078912ab^{8}+73488037086731520a^{2}b^{8})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(12411632232552960a^{3}b^{8}+13190080234763520a^{4}b^{8}+403001238644928a^{5}b^{8})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(449675387995680a^{6}b^{8}+1635980781600a^{7}b^{8}+3817288490400a^{8}b^{8}-4298748300a^{9}b^{8})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(7307872110a^{10}b^{8}+2205749914587648b^{9}+60870335749893l2ab^{9}+2830362252157440a^{2}b^{9})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(3161240677647360a^{3}b^{9}+239192223842880a^{4}b^{9}+2117925155336448a^{5}b^{9}+3081641824800a^{6}b^{9})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(3032869276800a^{7}b^{9}+4298748300a^{8}b^{9}+9075135300a^{9}b^{9}+1479487730077248b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(267452777940480ab^{10}+385171067635200a^{2}b^{10}+67636609729920a^{3}b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(58647685995360a^{4}b^{10}+1997908688160a^{5}b^{10}+1511564503680a^{6}b^{10}+8351853840a^{7}b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(7307872110a^{8}b^{10}+7705944829440b^{11}+20310289142784ab^{11}+9214702632576a^{2}b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(9071028833664a^{3}b^{11}+654569667360a^{4}b^{11}+463267099008a^{5}b^{11}+5776974000a^{6}b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(3796297200a^{7}b^{11}+310606256896b^{12}+538587975744ab^{12}+714494944800a^{2}b^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(112495612320a^{3}b^{12}+84112741440a^{4}b^{12}+2102818536a^{5}b^{12}+1251677700a^{6}b^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(9588825792b^{13}+23985898944ab^{13}+9600101280a^{2}b^{13}+8492373120a^{3}b^{13}+423644760a^{4}b^{13})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(254186856a^5b^{13}+222345312b^{14}+350382240ab^{14}+427616640a^2b^{14}+45912240a^3b^{14})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(30260340a^4b^{14}+3745440b^{15}+8727936ab^{15}+2450448a^2b^{15}+1947792a^3b^{15}+43248b^{16})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{262144(53550ab^{16}+58905a^2b^{16}+306b^{17}+630ab^{17}+b^{18})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576b(4850003358787829760 0a+11328021094402621440a^2+21140011005367025664a^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(27950231060679229 44a^4+15740255487918735 36a^5+1287581877967257 60a^6)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(33175766006734848a^7+1737630410723328a^8+240371614509312a^9+8050302965760a^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(638746221312a^{11}+13154598912a^{12}+603267168a^{13}+6887040a^{14}+172176a^{15}+816a^{16})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(9a^{17}+48500033587878297600b+83577021037226754048a^2b+80340853662689525 76a^3b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(131057667389561896 96a^4b+918253385961963520a^5b+48322335834936 9344a^6b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(23020491316207616a^7b+5654596258046208a^8b+177925617745920a^9b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(23543562530048a^{10}b+465775063040a^{11}b+35113632736a^{12}b+392680960a^{13}b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(16797360a^{14}b+79968a^{15}b+1785a^{16}b-11328021094402621440b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(83577021037226754048ab^2+33618491552919846912a^3b^2+16105805776879616 00a^4b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(2326378986350354432a^5b^2+929606378513612 80a^6b^2+452185556039895 04a^7b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(128500024857600 0a^8b^2+296035410612480a^9b^2+5498064732160a^{10}b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(683281861696a^{11}b^2+7361244800a^{12}b^2+516583760a^{13}b^2+2423520a^{14}b^2+94248a^{15}b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(2114001100536702566 4b^3-8034085366268952576ab^3+3361849155291984691 2a^2b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(4891283590711992320a^4b^3+1283876372804157 44a^5b^3+165510896176890880a^6b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(3878969853214720a^7b^3+1734099087578880a^8b^3+28871685427200a^9b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(6174643478080a^{10}b^3+62291568640a^{11}b^3+7178203760a^{12}b^3+32463200a^{13}b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(2086920a^{14}b^3-2795023106067922944b^4+13105766738956189696ab^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(-16105805776879616 00a^2b^4+4891283590711992320a^3b^4+3099498365353815 04a^5b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(4639359573094400a^6b^4+5366111189204480a^7b^4+72693320663040a^8b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(29762423312160a^9b^4+267366915200a^{10}b^4+52694561360a^{11}b^4+222520480a^{12}b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(23535820a^{13}b^4+157402554879187353 6b^5-918253385961963520ab^5)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(2326378986350354432a^2b^5-1283876372804157 44a^3b^5+3099498365353815 04a^4b^5)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(9319447558181376a^{6}b^{5}+78989552345088a^{7}b^{5}+82270476664992a^{8}b^{5}+594896924160a^{9}b^{5})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(222498227952a^{10}b^{5}+831884256a^{11}b^{5}+150201324a^{12}b^{5}-1287581877967257600b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(483223358349369344ab^{6}-92960637851361280a^{2}b^{6}+16551089617689088 0a^{3}b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(-4639359573094400a^{4}b^{6}+9319447558181376a^{5}b^{6}+135601809678720a^{7}b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(604067587200a^{8}b^{6}+567933181200a^{9}b^{6}+1694579040a^{10}b^{6}+577697400a^{11}b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(331757660067348848b^{7}-230204913162076 16ab^{7}+4521855560398950 4a^{2}b^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(-3878969853214720a^{3}b^{7}+5366111189204480a^{4}b^{7}-78989552345088a^{5}b^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(135601809678720a^{6}b^{7}+901508929200a^{8}b^{7}+1637618400a^{9}b^{7}+1391975640a^{10}b^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(-173763041072332 8b^{8}+5654596258046208ab^{8}-1285000248576000a^{2}b^{8})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(1734099087578880a^{3}b^{8}-72693320663040a^{4}b^{8}+82270476664992a^{5}b^{8}-604067587200a^{6}b^{8})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(901508929200a^{7}b^{8}+2149374150a^{9}b^{8}+240371614509312b^{9}-177925617745920ab^{9})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(296035410612480a^{2}b^{9}-28871685427200a^{3}b^{9}+29762423312160a^{4}b^{9}-594896924160a^{5}b^{9})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(567933181200a^{6}b^{9}-1637618400a^{7}b^{9}+2149374150a^{8}b^{9}-8050302965760b^{10})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{1048576b(23543562530048ab^{10}-5498064732160a^{2}b^{10}+6174643478080a^{3}b^{10}-267366915200a^{4}b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(222498227952a^{5}b^{10}-1694579040a^{6}b^{10}+1391975640a^{7}b^{10}+638746221312b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(-465775063040ab^{11}+683281861696a^{2}b^{11}-62291568640a^{3}b^{11}+52694561360a^{4}b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(-831884256a^{5}b^{11}+577697400a^{6}b^{11}-13154598912b^{12}+35113632736ab^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(-7361244800a^{2}b^{12}+7178203760a^{3}b^{12}-222520480a^{4}b^{12}+150201324a^{5}b^{12}+603267168b^{13})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(-392680960ab^{13}+516583760a^{2}b^{13}-32463200a^{3}b^{13}+23535820a^{4}b^{13}-6887040b^{14})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(16797360ab^{14}-2423520a^{2}b^{14}+2086920a^{3}b^{14}+172176b^{15}-79968ab^{15}+94248a^{2}b^{15})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576b(-816b^{16}+1785ab^{16}+9b^{17})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}\Bigg\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{1048576a(48500033587878297600a-113280210944026214440a^{2}+21140011005367025664a^{3})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(-27950231060679229444a^{4}+1574025548791873536a^{5}-128758187796725760a^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(3317576600673484 8a^{7}-1737630410723328a^{8}+240371614509312a^{9}-8050302965760a^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(638746221312a^{11}-13154598912a^{12}+603267168a^{13}-6887040a^{14}+172176a^{15}-816a^{16})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(9a^{17}+48500033587878297600b+83577021037226754048a^{2}b-80340853662689525760a^{3}b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(131057667389561896 96a^4b-918253385961963520a^5b+4832233583493693 44a^6b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(-2302049131620761 6a^7b+565459625804620 8a^8b-177925617745920a^9b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(23543562530048a^{10}b-465775063040a^{11}b+35113632736a^{12}b-392680960a^{13}b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(16797360a^{14}b-79968a^{15}b+1785a^{16}b+1132802109440262144 0b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(8357702103722675404 8ab^2+3361849155291984691 2a^3b^2-1610580577687961600a^4b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(232637898635035443 2a^5b^2-92960637851361280a^6b^2+4521855560398950 4a^7b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(-1285000248576000a^8b^2+296035410612480a^9b^2-5498064732160a^{10}b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(683281861696a^{11}b^2-7361244800a^{12}b^2+516583760a^{13}b^2-2423520a^{14}b^2+94248a^{15}b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(2114001100536702566 4b^3+803408536626895257 6ab^3+3361849155291984691 2a^2b^3)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(4891283590711992320a^4b^3-1283876372804157 44a^5b^3+16551089617689088 0a^6b^3)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(-3878969853214720a^7b^3+173409908757888 0a^8b^3-28871685427200a^9b^3)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(6174643478080a^{10}b^3-62291568640a^{11}b^3+7178203760a^{12}b^3-32463200a^{13}b^3)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(2086920a^{14}b^3+2795023106067922944b^4+131057667389561896 96ab^4)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(1610580577687961600a^2b^4+4891283590711992320a^3b^4+3099498365353381504a^5b^4)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(-4639359573094400a^6b^4+5366111189204480a^7b^4-72693320663040a^8b^4)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(29762423312160a^9b^4-267366915200a^{10}b^4+52694561360a^{11}b^4-222520480a^{12}b^4)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(23535820a^13b^4+1574025548791873536b^5+918253385961963520ab^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(2326378986350354432a^2b^5+1283876372804157744a^3b^5+3099498365353381504a^4b^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(9319447558181376a^6b^5-78989552345088a^7b^5+82270476664992a^8b^5-594896924160a^9b^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(222498227952a^{10}b^5-831884256a^{11}b^5+150201324a^{12}b^5+1287581877967257560b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(483223358349369344ab^6+929606378513612800a^2b^6+165510896176890880a^3b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(4639359573094400a^4b^6+9319447558181376a^5b^6+135601809678720a^7b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(-604067587200a^8b^6+567933181200a^9b^6-1694579040a^{10}b^6+577697400a^{11}b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(33175766006734848b^7+23020491316207616ab^7+45218555603989504a^2b^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(3878969853214720a^3b^7+5366111189204480a^4b^7+78989552345088a^5b^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(135601809678720a^6b^7+901508929200a^8b^7-1637618400a^9b^7+1391975640a^{10}b^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(1737630410723328b^8+5654596258046208ab^8+1285000248576000a^2b^8)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(1734099087578880a^3b^8+72693320663040a^4b^8+82270476664992a^5b^8+604067587200a^6b^8)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(901508929200a^7b^8+2149374150a^9b^8+240371614509312b^9+177925617745920ab^9)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(296035410612480a^2b^9+28871685427200a^3b^9+29762423312160a^4b^9+594896924160a^5b^9)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(567933181200a^6b^9+1637618400a^7b^9+2149374150a^8b^9+8050302965760b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(23543562530048ab^{10}+5498064732160a^2b^{10}+6174643478080a^3b^{10}+267366915200a^4b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(222498227952a^5b^{10}+1694579040a^6b^{10}+1391975640a^7b^{10}+638746221312b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(465775063040ab^{11}+683281861696a^2b^{11}+62291568640a^3b^{11}+52694561360a^4b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(831884256a^5b^{11}+577697400a^6b^{11}+13154598912b^{12}+35113632736ab^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(7361244800a^2b^{12}+7178203760a^3b^{12}+222520480a^4b^{12}+150201324a^5b^{12}+603267168b^{13})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(392680960ab^{13}+516583760a^2b^{13}+32463200a^3b^{13}+23535820a^4b^{13}+6887040b^{14})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(16797360ab^{14}+2423520a^2b^{14}+2086920a^3b^{14}+172176b^{15}+79968ab^{15}+94248a^2b^{15})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576a(816b^{16}+1785ab^{16}+9b^{17})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

Global Journal of Science Frontier Research (F) Volume XIII Issue V Version I Year 2013

$$+\frac{262144(46620662575398912000a+80177108784198451200a^2+59690268905460203520a^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(25941016178319163392a^4+7448984828674965504a^5+1511475811264036864a^6)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(225747898745241600a^7+25468769585086464a^8+2205749914587648a^9)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(147948773077248a^{10}+7705944829440a^{11}+310606256896a^{12}+9588825792a^{13})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(222345312a^{14}+3745440a^{15}+43248a^{16}+306a^{17}+a^{18}-46620662575398912000b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(227646051134629478400ab+84561518714872135680a^2b+204544425045918744576a^3b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(41476667627670994944a^4b+22943568120596004864a^5b+2670969522028953600a^6b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(653655295920635904a^7b+46367643951078912a^8b+6087033574989312a^9b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(267452777940480a^{10}b+20310289142784a^{11}b+538587975744a^{12}b+23985898944a^{13}b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(350382240a^{14}b+8727936a^{15}b+53550a^{16}b+630a^{17}b+80177108784198451200b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(-84561518714872135680ab^2+376765373892274421760a^2b^2+37106107946844291072a^3b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(94661921845665792000a^4b^2+8250058245827543040a^5b^2+4834484937423912960a^6b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(286088788492683264a^7b^2+73488037086731520a^8b^2+2830362252157440a^9b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(385171067635200a^{10}b^2+9214702632576a^{11}b^2+714494944800a^{12}b^2+9600101280a^{13}b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(427616640a^{14}b^2+2450448a^{15}b^2+58905a^{16}b^2-596902689054602035520b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(2045444250459187445 76ab^3-371061079468442910 72a^2b^3+14815233917029161369 6a^3b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(595460874960838656 0a^4b^3+147984352737855406 08a^5b^3+655755966671984640a^6b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(380668112993642496a^7b^3+12411632232552960a^8b^3+3161240677647360a^9b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(67636609729920a^{10}b^3+9071028833664a^{11}b^3+112495612320a^{12}b^3+8492373120a^{13}b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(45912240a^{14}b^3+1947792a^{15}b^3+259410161783191633392b^4-414766676276709949 44ab^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(94661921845665792000a^2b^4-5954608749608386560a^3b^4+2124840505791725568 0a^4b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(417333234097511424a^5b^4+979202192033640960a^6b^4+23674554904366080a^7b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(13190080234763520a^8b^4+239192223842880a^9b^4+586476859953 60a^{10}b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(654569667360a^{11}b^4+84112741440a^{12}b^4+423644760a^{13}b^4+30260340a^{14}b^4)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(-744898482867496555 04b^5+22943568120596004864ab^5-8250058245827543040a^2b^5)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(147984352737855406 08a^3b^5-417333234097511424a^4b^5+1333407400586376192a^5b^5)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(13876520995814400a^{6}b^{5}+30294926136059904a^{7}b^{5}+403001238644928a^{8}b^{5})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(211792515536448a^{9}b^{5}+1997908688160a^{10}b^{5}+463267099008a^{11}b^{5}+2102818536a^{12}b^{5})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(254186856a^{13}b^{5}+1511475811264036864b^{6}-2670969522028953600ab^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(4834484937423912960a^{2}b^{6}-655755966671984640a^{3}b^{6}+979202192033640960a^{4}b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(-13876520995814400a^{5}b^{6}+39814926411617280a^{6}b^{6}+223040891969280a^{7}b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(449675387995680a^{8}b^{6}+3081641824800a^{9}b^{6}+1511564503680a^{10}b^{6}+5776974000a^{11}b^{6})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(1251677700a^{12}b^{6}-2257478987452416000b^{7}+6536552959206359040ab^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(-2860887884926832640a^{2}b^{7}+3806681129936424960a^{3}b^{7}-23674554904366080a^{4}b^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(30294926136059904a^{5}b^{7}-223040891969280a^{6}b^{7}+576376172064000a^{7}b^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(1635980781600a^{8}b^{7}+3032869276800a^{9}b^{7}+8351853840a^{10}b^{7}+3796297200a^{11}b^{7})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(254687695850864640b^{8}-463676439510789120ab^{8}+734880370867315200a^{2}b^{8})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(-124116322325529600a^{3}b^{8}+131900802347635200a^{4}b^{8}-403001238644928a^{5}b^{8})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(449675387995680a^{6}b^{8}-1635980781600a^{7}b^{8}+3817288490400a^{8}b^{8}+4298748300a^{9}b^{8})}{\left[\prod_{\sigma=0}^{18}\left\{a-b-2\sigma\right\}\right]\left[\prod_{\varsigma=1}^{17}\left\{a-b+2\varsigma\right\}\right]}+$$

$$+\frac{262144(7307872110a^{10}b^{8}-2205749914587648b^{9}+6087033574989312ab^{9}-2830362252157440a^{2}b^{9})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(3161240677647360a^{3}b^{9}-239192223842880a^{4}b^{9}+211792515536448a^{5}b^{9}-3081641824800a^{6}b^{9})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(3032869276800a^{7}b^{9}-4298748300a^{8}b^{9}+9075135300a^{9}b^{9}+147948773077248b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(-267452777940480ab^{10}+385171067635200a^{2}b^{10}-67636609729920a^{3}b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(58647685995360a^{4}b^{10}-1997908688160a^{5}b^{10}+1511564503680a^{6}b^{10}-8351853840a^{7}b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(7307872110a^{8}b^{10}-77059448294440b^{11}+20310289142784ab^{11}-9214702632576a^{2}b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(9071028833664a^{3}b^{11}-654569667360a^{4}b^{11}+463267099008a^{5}b^{11}-5776974000a^{6}b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(3796297200a^{7}b^{11}+310606256896b^{12}-538587975744ab^{12}+714494944800a^{2}b^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(-112495612320a^{3}b^{12}+84112741440a^{4}b^{12}-2102818536a^{5}b^{12}+1251677700a^{6}b^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(-9588825792b^{13}+23985898944ab^{13}-9600101280a^{2}b^{13}+8492373120a^{3}b^{13}-423644760a^{4}b^{13})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(254186856a^{5}b^{13}+222345312b^{14}-350382240ab^{14}+427616640a^{2}b^{14}-45912240a^{3}b^{14})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{262144(30260340a^{4}b^{14}-3745440b^{15}+8727936ab^{15}-2450448a^{2}b^{15}+1947792a^{3}b^{15}+43248b^{16})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\left.\left.\frac{262144(-53550ab^{16}+58905a^{2}b^{16}-306b^{17}+630ab^{17}+b^{18})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}\right\}\right] \tag{6}$$

III. Derivation of the Summation Formula

Replacing c with $\frac{a+b+38}{2}$ and z with $\frac{1}{2}$ in equation (2), and involving derived from reference(6), we get

$$L.H.S = a\,\frac{2^b\;\Gamma(\frac{a+b+38}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+2}{2})}\left\{\frac{131072(-466206625753989120000a+801771087841984512000a^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+\right.\right.$$

$$+\frac{131072(-59690268905460203520a^3+25941016178319163392a^4-7448984828674965504a^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(1511475811264036864a^6-225747898745241600a^7+25468769585086464a^8)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-2205749914587648a^9+147948773077248a^{10}-7705944829440a^{11}+310606256896a^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-9588825792a^{13}+222345312a^{14}-3745440a^{15}+43248a^{16}-306a^{17}+a^{18})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(466206625753989120000b+227646051134629478400ab-84561518714872135680a^2b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(204544425045918744576a^3b-41476667627670994944a^4b+2294356812059600486 4a^5b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-2670969522028953600a^6b+653655295920635904a^7b-46367643951078912a^8b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(6087033574989312a^9b-267452777940480a^{10}b+20310289142784a^{11}b-538587975744a^{12}b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(23985898944a^{13}b-350382240a^{14}b+8727936a^{15}b-53550a^{16}b+630a^{17}b)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(801771087841984512000b^2+84561518714872135680ab^2+376765373892274421760a^2b^2)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-37106107946844291072a^{3}b^{2}+94661921845665792000a^{4}b^{2}-82500582458275430040a^{5}b^{2})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(48344849374239129600a^{6}b^{2}-28608878884926832640a^{7}b^{2}+73488037086731520a^{8}b^{2})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(-2830362252157440a^{9}b^{2}+385171067635200a^{10}b^{2}-9214702632576a^{11}b^{2})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(714494944800a^{12}b^{2}-9600101280a^{13}b^{2}+427616640a^{14}b^{2}-2450448a^{15}b^{2}+58905a^{16}b^{2})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(5969026890546020352 0b^{3}+20454442504591874457 6ab^{3}+37106107946844291072a^{2}b^{3})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(14815233917029161369 6a^{3}b^{3}-59546087496083865600a^{4}b^{3}+14798435273785540608a^{5}b^{3})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(-6557559666719846400a^{6}b^{3}+3806681129936424960a^{7}b^{3}-1241163223255296 0a^{8}b^{3})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(3161240677647360a^{9}b^{3}-67636609729920a^{10}b^{3}+9071028833664a^{11}b^{3}-112495612320a^{12}b^{3})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(8492373120a^{13}b^{3}-45912240a^{14}b^{3}+1947792a^{15}b^{3}+25941016178319163392b^{4})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(41476667627670994944ab^{4}+94661921845665792000a^{2}b^{4}+59546087496083865600a^{3}b^{4})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(21248405057917255680a^{4}b^{4}-41733323409751142 4a^{5}b^{4}+9792021920336409600a^{6}b^{4})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(-2367455490436608 0a^{7}b^{4}+1319008023476352 0a^{8}b^{4}-239192223842880a^{9}b^{4})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(58647685995360a^{10}b^{4}-654569667360a^{11}b^{4}+84112741440a^{12}b^{4}-423644760a^{13}b^{4})}{\left[\prod_{\delta=0}^{17}\left\{a-b-2\delta\right\}\right]\left[\prod_{\eta=1}^{18}\left\{a-b+2\eta\right\}\right]}+$$

$$+\frac{131072(30260340a^{14}b^4+74489848286749655504b^5+2294356812059600486 4ab^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(8250058245827543040a^2b^5+14798435273785540608a^3b^5+417333234097511424a^4b^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(1333407400586376192a^5b^5-13876520995814400a^6b^5+30294926136059904a^7b^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-403001238644928a^8b^5+211792515536448a^9b^5-1997908688160a^{10}b^5+463267099008a^{11}b^5)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-2102818536a^{12}b^5+254186856a^{13}b^5+1511475811264036864b^6+2670969522028953600ab^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(4834484937423912960a^2b^6+655755966671984640a^3b^6+979202192033640960a^4b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(13876520995814400a^5b^6+39814926411617280a^6b^6-223040891969280a^7b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(449675387995680a^8b^6-3081641824800a^9b^6+1511564503680a^{10}b^6-5776974000a^{11}b^6)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(1251677700a^12b^6+225747898745241600b^7+653655295920635904ab^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(286088788492683264a^2b^7+380668112993642496a^3b^7+23674554904366080a^4b^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(30294926136059904a^5b^7+223040891969280a^6b^7+576376172064000a^7b^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(-1635980781600a^8b^7+3032869276800a^9b^7-8351853840a^{10}b^7+3796297200a^{11}b^7)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(25468769585086464b^8+46367643951078912ab^8+73488037086731520a^2b^8)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(1241163223255296 0a^3b^8+13190080234763520a^4b^8+4030012386449 28a^5b^8)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(449675387995680a^6b^8+1635980781600a^7b^8+3817288490400a^8b^8-4298748300a^9b^8)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(7307872110a^{10}b^8+220574991458 76648b^9+6087033574989312ab^9+2830362252157440a^2b^9)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(3161240677647360a^3b^9+239192223842880a^4b^9+211792515536448a^5b^9+3081641824800a^6b^9)}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(3032869276800a^7b^9+4298748300a^8b^9+9075135300a^9b^9+1479487730772 48b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(267452777940480ab^{10}+385171067635200a^2b^{10}+67636609729920a^3b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(58647685995360a^4b^{10}+1997908688160a^5b^{10}+1511564503680a^6b^{10}+8351853840a^7b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(7307872110a^8b^{10}+77059448294 40b^{11}+20310289142784ab^{11}+9214702632576a^2b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(9071028833664a^3b^{11}+654569667360a^4b^{11}+463267099008a^5b^{11}+5776974000a^6b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(3796297200a^7b^{11}+310606256896b^{12}+538587975744ab^{12}+714494944800a^2b^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(112495612320a^3b^{12}+84112741440a^4b^{12}+2102818536a^5b^{12}+1251677700a^6b^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(9588825792b^{13}+23985898944ab^{13}+9600101280a^2b^{13}+8492373120a^3b^{13}+423644760a^4b^{13})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(254186856a^5b^{13}+222345312b^{14}+350382240ab^{14}+427616640a^2b^{14}+45912240a^3b^{14})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(30260340a^4b^{14}+3745440b^{15}+8727936ab^{15}+2450448a^2b^{15}+1947792a^3b^{15}+43248b^{16})}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{131072(53550ab^{16}+58905a^2b^{16}+306b^{17}+630ab^{17}+b^{18})}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}\Bigg\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{1048576(48500033587878297600a-113280210944026214440a^2+211400110053670256664a^3)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}$$

$$+\frac{1048576(-27950231060679229444a^4+15740255487918735336a^5-1287581877967257600a^6)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(33175766006734848a^7-1737630410723328a^8+240371614509312a^9-8050302965760a^{10})}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(638746221312a^{11}-13154598912a^{12}+603267168a^{13}-6887040a^{14}+172176a^{15}-816a^{16})}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(9a^{17}+48500033587878297600b+83577021037226754048a^2b-80340853662689525576a^3b)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(13105766738956189696a^4b-918253385961963520a^5b+48322335834936934 4a^6b)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(-23020491316207616a^7b+5654596258046208a^8b-177925617745920a^9b)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(23543562530048a^{10}b-465775063040a^{11}b+35113632736a^{12}b-392680960a^{13}b)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(16797360a^{14}b-79968a^{15}b+1785a^{16}b+113280210944026214440b^2)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(83577021037226754048ab^2+33618491552919846912a^3b^2-16105805776879616 00a^4b^2)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(2326378986350354432a^5b^2-92960637851361280a^6b^2+45218555603989504a^7b^2)}{\left[\prod\limits_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod\limits_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(-1285000248576000a^{8}b^{2}+296035410612480a^{9}b^{2}-5498064732160a^{10}b^{2})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(683281861696a^{11}b^{2}-7361244800a^{12}b^{2}+516583760a^{13}b^{2}-2423520a^{14}b^{2}+94248a^{15}b^{2})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(21140011005367025664b^{3}+80340853662689525 76ab^{3}+336184915529198469 12a^{2}b^{3})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(4891283590711992320a^{4}b^{3}-1283876372804157 44a^{5}b^{3}+1655108961768908 80a^{6}b^{3})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(-3878969853214720a^{7}b^{3}+1734099087578880a^{8}b^{3}-28871685427200a^{9}b^{3})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(6174643478080a^{10}b^{3}-62291568640a^{11}b^{3}+7178203760a^{12}b^{3}-32463200a^{13}b^{3})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(2086920a^{14}b^{3}+27950231060679229 44b^{4}+13105766738956189696ab^{4})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(16105805776879616 00a^{2}b^{4}+4891283590711992320a^{3}b^{4}+309949836535381504a^{5}b^{4})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(-4639359573094400a^{6}b^{4}+536611118920448 0a^{7}b^{4}-72693320663040a^{8}b^{4})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(29762423312160a^{9}b^{4}-267366915200a^{10}b^{4}+52694561360a^{11}b^{4}-222520480a^{12}b^{4})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(23535820a^{13}b^{4}+1574025548791873536b^{5}+9182533859619635 20ab^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(2326378986350354432a^{2}b^{5}+128387637280415744a^{3}b^{5}+309949836535381504a^{4}b^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(9319447558181376a^{6}b^{5}-78989552345088a^{7}b^{5}+82270476664992a^{8}b^{5}-594896924160a^{9}b^{5})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(222498227952a^{10}b^{5}-831884256a^{11}b^{5}+150201324a^{12}b^{5}+1287581877967257760b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(4832233583493693440ab^{6}+9296063785136128 0a^{2}b^{6}+1655108961768908 80a^{3}b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(4639359573094400a^{4}b^{6}+9319447558181376a^{5}b^{6}+135601809678720a^{7}b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(-604067587200a^{8}b^{6}+567933181200a^{9}b^{6}-1694579040a^{10}b^{6}+577697400a^{11}b^{6})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(3317576600673484 8b^{7}+23020491316207616ab^{7}+45218555603989504a^{2}b^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(3878969853214720a^{3}b^{7}+5366111189204480a^{4}b^{7}+78989552345088a^{5}b^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(135601809678720a^{6}b^{7}+901508929200a^{8}b^{7}-1637618400a^{9}b^{7}+1391975640a^{10}b^{7})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(1737630410723328b^{8}+5654596258046208ab^{8}+1285000248576000a^{2}b^{8})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(1734099087578880a^{3}b^{8}+72693320663040a^{4}b^{8}+82270476664992a^{5}b^{8}+604067587200a^{6}b^{8})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(901508929200a^{7}b^{8}+2149374150a^{9}b^{8}+240371614509312b^{9}+177925617745920ab^{9})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(296035410612480a^{2}b^{9}+28871685427200a^{3}b^{9}+29762423312160a^{4}b^{9}+594896924160a^{5}b^{9})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(567933181200a^{6}b^{9}+1637618400a^{7}b^{9}+2149374150a^{8}b^{9}+8050302965760b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(23543562530048ab^{10}+5498064732160a^{2}b^{10}+6174643478080a^{3}b^{10}+267366915200a^{4}b^{10})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(222498227952a^5b^{10}+1694579040a^6b^{10}+1391975640a^7b^{10}+638746221312b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(465775063040ab^{11}+683281861696a^2b^{11}+62291568640a^3b^{11}+52694561360a^4b^{11})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(831884256a^5b^{11}+577697400a^6b^{11}+13154598912b^{12}+35113632736ab^{12}+7361244800a^2b^{12})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(7178203760a^3b^{12}+222520480a^4b^{12}+150201324a^5b^{12}+603267168b^{13}+392680960ab^{13})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(516583760a^2b^{13}+32463200a^3b^{13}+23535820a^4b^{13}+6887040b^{14}+16797360ab^{14})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\frac{1048576(2423520a^2b^{14}+2086920a^3b^{14}+172176b^{15}+79968ab^{15}+94248a^2b^{15}+816b^{16})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}+$$

$$+\left.\left.\frac{1048576(1785ab^{16}+9b^{17})}{\left[\prod_{\delta=0}^{17}\{a-b-2\delta\}\right]\left[\prod_{\eta=1}^{18}\{a-b+2\eta\}\right]}\right\}\right]-$$

$$-\frac{2^{b+1}\,\Gamma(\frac{a+b+38}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{131072(466206625753989120000a+801771087841984512000a^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+\right.\right.$$

$$+\frac{131072(596902689054602035200a^3+259410161783191633920a^4+74489848286749655040a^5)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(15114758112640368640a^6+2257478987452416000a^7+254687695850864640a^8)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(22057499145876480a^9+1479487730772480a^{10}+77059448294400a^{11}+3106062568960a^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(95888257920a^{13}+2223453120a^{14}+37454400a^{15}+432480a^{16}+3060a^{17}+a^{18})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-466206625753989120000b+227646051134629478400ab+845615187148721356800a^2b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(2045444250459187744576a^3b+4147666762767099494 4a^4b+229435681205960048 64a^5b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(267096952202895360 0a^6b+653655295920635904a^7b+46367643951078912a^8b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(6087033574989312a^9b+267452777940480a^{10}b+20310289142784a^{11}b+538587975744a^{12}b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(23985898944a^{13}b+350382240a^{14}b+8727936a^{15}b+53550a^{16}b+630a^{17}b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(801771087841984512 00b^2-845615187148721356 80ab^2+376765373892274421760a^2b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(371061079468442910 72a^3b^2+946619218456657920 00a^4b^2+825005824582754304 0a^5b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(4834484937423912960a^6b^2+2860887884926832 64a^7b^2+73488037086731520a^8b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(2830362252157440a^9b^2+385171067635200a^{10}b^2+9214702632576a^{11}b^2+714494944800a^{12}b^2)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(9600101280a^{13}b^2+427616640a^{14}b^2+2450448a^{15}b^2+58905a^{16}b^2-596902689054602035 20b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(2045444250459187744576ab^3-37106107946844291072a^2b^3+148152339170291613696a^3b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(59546087496083865 60a^4b^3+14798435273785540608a^5b^3+655755966671984640a^6b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(380668112993642496a^7b^3+12411632232552960a^8b^3+316124067764 7360a^9b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(67636609729920a^{10}b^3+9071028833664a^{11}b^3+112495612320a^{12}b^3+8492373120a^{13}b^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(45912240a^{14}b^{3}+1947792a^{15}b^{3}+25941016178319163392b^{4}-41476667627670994944ab^{4})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(94661921845665792000a^{2}b^{4}-59546087496083865600a^{3}b^{4}+21248405057917255680a^{4}b^{4})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(4173332340975114240a^{5}b^{4}+979202192033640960a^{6}b^{4}+23674554904366080a^{7}b^{4})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(13190080234763520a^{8}b^{4}+239192223842880a^{9}b^{4}+58647685995360a^{10}b^{4})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(654569667360a^{11}b^{4}+84112741440a^{12}b^{4}+423644760a^{13}b^{4}+30260340a^{14}b^{4})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-7448984828674965504b^{5}+22943568120596004864ab^{5}-82500582458275430400a^{2}b^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(14798435273785540608a^{3}b^{5}-4173332340975114240a^{4}b^{5}+1333407400586376192a^{5}b^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(138765209958144000a^{6}b^{5}+30294926136059904a^{7}b^{5}+403001238644928a^{8}b^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(211792515536448a^{9}b^{5}+1997908688160a^{10}b^{5}+463267099008a^{11}b^{5}+2102818536a^{12}b^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(254186856a^{13}b^{5}+1511475811264036864b^{6}-2670969522028953600ab^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(4834484937423912960a^{2}b^{6}-6557559666719846400a^{3}b^{6}+979202192033640960a^{4}b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-138765209958144000a^{5}b^{6}+39814926411617280a^{6}b^{6}+223040891969280a^{7}b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(449675387995680a^{8}b^{6}+3081641824800a^{9}b^{6}+1511564503680a^{10}b^{6}+5776974000a^{11}b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(1251677700a^{12}b^{6}-2257478987452416 00b^{7}+653655295920635904ab^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-2860887884926832 64a^{2}b^{7}+380668112993642496a^{3}b^{7}-236745549043660 80a^{4}b^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(30294926136059904a^{5}b^{7}-223040891969280a^{6}b^{7}+576376172064000a^{7}b^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(1635980781600a^{8}b^{7}+3032869276800a^{9}b^{7}+8351853840a^{10}b^{7}+3796297200a^{11}b^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(254687695850864 64b^{8}-46367643951078912ab^{8}+73488037086731520a^{2}b^{8})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-12411632232552960a^{3}b^{8}+1319008023476352 0a^{4}b^{8}-403001238644928a^{5}b^{8})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(449675387995680a^{6}b^{8}-1635980781600a^{7}b^{8}+3817288490400a^{8}b^{8}+4298748300a^{9}b^{8})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(7307872110a^{10}b^{8}-2205749914587648b^{9}+6087033574989312ab^{9}-2830362252157440a^{2}b^{9})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(3161240677647360a^{3}b^{9}-239192223842880a^{4}b^{9}+211792515536448a^{5}b^{9}-3081641824800a^{6}b^{9})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(3032869276800a^{7}b^{9}-4298748300a^{8}b^{9}+9075135300a^{9}b^{9}+147948773077248b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-267452777940480ab^{10}+385171067635200a^{2}b^{10}-67636609729920a^{3}b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(58647685995360a^{4}b^{10}-1997908688160a^{5}b^{10}+1511564503680a^{6}b^{10}-8351853840a^{7}b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(7307872110a^{8}b^{10}-7705944829440b^{11}+20310289142784ab^{11}-9214702632576a^{2}b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(9071028833664a^3b^{11}-654569667360a^4b^{11}+463267099008a^5b^{11}-5776974000a^6b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(3796297200a^7b^{11}+310606256896b^{12}-538587975744ab^{12}+714494944800a^2b^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-112495612320a^3b^{12}+84112741440a^4b^{12}-2102818536a^5b^{12}+1251677700a^6b^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-9588825792b^{13}+23985898944ab^{13}-9600101280a^2b^{13}+8492373120a^3b^{13}-423644760a^4b^{13})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(254186856a^5b^{13}+222345312b^{14}-350382240ab^{14}+427616640a^2b^{14}-45912240a^3b^{14})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(30260340a^4b^{14}-3745440b^{15}+8727936ab^{15}-2450448a^2b^{15}+1947792a^3b^{15}+43248b^{16})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{131072(-53550ab^{16}+58905a^2b^{16}-306b^{17}+630ab^{17}+b^{18})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}\Bigg\}-$$

$$-\frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a}{2})}\Bigg\{\frac{1048576(485000335878782976000a+113280210944026214440a^2+211400110053670256664a^3)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}$$

$$+\frac{1048576(27950231060679229440a^4+1574025548791873536a^5+128758187796725760a^6)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(3317576600673484 8a^7+1737630410723328a^8+240371614509312a^9+8050302965760a^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(638746221312a^{11}+13154598912a^{12}+603267168a^{13}+6887040a^{14}+172176a^{15}+816a^{16})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(9a^{17}+485000335878782976000b+835770210372267540448a^2b+803408536626895257 6a^3b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(131057667389561896 96a^4b+918253385961963520a^5b+48322335834936934 4a^6b)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(23020491316207616a^{7}b+5654596258046208a^{8}b+177925617745920a^{9}b)}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(23543562530048a^{10}b+465775063040a^{11}b+35113632736a^{12}b+392680960a^{13}b)}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(16797360a^{14}b+79968a^{15}b+1785a^{16}b-1132802109440262144 0b^{2})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(8357702103722675404 8ab^{2}+33618491552919846912a^{3}b^{2}+16105805776879616 00a^{4}b^{2})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(23263789863503544 32a^{5}b^{2}+929606378513612 80a^{6}b^{2}+45218555603989504a^{7}b^{2})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(1285000248576000a^{8}b^{2}+296035410612480a^{9}b^{2}+5498064732160a^{10}b^{2}+683281861696a^{11}b^{2})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(7361244800a^{12}b^{2}+516583760a^{13}b^{2}+2423520a^{14}b^{2}+94248a^{15}b^{2}+21140011005367025664b^{3})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-80340853662689525 76ab^{3}+33618491552919846912a^{2}b^{3}+4891283590711992320a^{4}b^{3})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(1283876372804157 44a^{5}b^{3}+165510896176890880a^{6}b^{3}+3878969853214720a^{7}b^{3})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(173409908757888 0a^{8}b^{3}+28871685427200a^{9}b^{3}+6174643478080a^{10}b^{3}+62291568640a^{11}b^{3})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(7178203760a^{12}b^{3}+32463200a^{13}b^{3}+2086920a^{14}b^{3}-2795023106067922944b^{4})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(13105766738956189696ab^{4}-1610580577687961600a^{2}b^{4}+4891283590711992320a^{3}b^{4})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(309949836535381504a^{5}b^{4}+4639359573094400a^{6}b^{4}+536611189204480a^{7}b^{4})}{\left[\prod\limits_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod\limits_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(72693320663040a^{8}b^{4}+29762423312160a^{9}b^{4}+267366915200a^{10}b^{4}+52694561360a^{11}b^{4})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(222520480a^{12}b^{4}+23535820a^{13}b^{4}+157402554879187353 6b^{5}-918253385961963520ab^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(2326378986350354432a^{2}b^{5}-1283876372804157 44a^{3}b^{5}+309949836535381504a^{4}b^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(9319447558181376a^{6}b^{5}+789895523450 88a^{7}b^{5}+82270476664992a^{8}b^{5}+594896924160a^{9}b^{5})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(222498227952a^{10}b^{5}+831884256a^{11}b^{5}+150201324a^{12}b^{5}-128758187796725760b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(483223358349369344ab^{6}-929606378513612 80a^{2}b^{6}+1655108961768908 80a^{3}b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-4639359573094400a^{4}b^{6}+9319447558181376a^{5}b^{6}+135601809678720a^{7}b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(604067587200a^{8}b^{6}+567933181200a^{9}b^{6}+1694579040a^{10}b^{6}+577697400a^{11}b^{6})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(331757660067348 48b^{7}-2302049131620761 6ab^{7}+4521855560398950 4a^{2}b^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-3878969853214720a^{3}b^{7}+5366111189204480a^{4}b^{7}-78989552345088a^{5}b^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(135601809678720a^{6}b^{7}+901508929200a^{8}b^{7}+1637618400a^{9}b^{7}+1391975640a^{10}b^{7})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-1737630410723328b^{8}+5654596258046208ab^{8}-1285000248576000a^{2}b^{8})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(1734099087578880a^{3}b^{8}-72693320663040a^{4}b^{8}+82270476664992a^{5}b^{8}-604067587200a^{6}b^{8})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(901508929200a^7b^8+2149374150a^9b^8+240371614509312b^9-177925617745920ab^9)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(296035410612480a^2b^9-28871685427200a^3b^9+29762423312160a^4b^9-594896924160a^5b^9)}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(567933181200a^6b^9-1637618400a^7b^9+2149374150a^8b^9-8050302965760b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(23543562530048ab^{10}-5498064732160a^2b^{10}+6174643478080a^3b^{10}-267366915200a^4b^{10})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(222498227952a^5b^{10}-1694579040a^6b^{10}+1391975640a^7b^{10}+638746221312b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-465775063040ab^{11}+683281861696a^2b^{11}-62291568640a^3b^{11}+52694561360a^4b^{11})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-831884256a^5b^{11}+577697400a^6b^{11}-13154598912b^{12}+35113632736ab^{12})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-7361244800a^2b^{12}+7178203760a^3b^{12}-222520480a^4b^{12}+150201324a^5b^12+603267168b^{13})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(-392680960ab^{13}+516583760a^2b^{13}-32463200a^3b^{13}+23535820a^4b^{13}-6887040b^{14})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$+\frac{1048576(16797360ab^{14}-2423520a^2b^{14}+2086920a^3b^{14}+172176b^{15}-79968ab^{15}+94248a^2b^{15})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}+$$

$$\left.\left.+\frac{1048576(-816b^{16}+1785ab^{16}+9b^{17})}{\left[\prod_{\sigma=0}^{18}\{a-b-2\sigma\}\right]\left[\prod_{\varsigma=1}^{17}\{a-b+2\varsigma\}\right]}\right\}\right]$$

On simplification the result (8) is established.

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New Results and Applications on Robust Stability and Tracking of Pseudo-Quadratic Uncertain MIMO Discrete-Time Systems

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Abstract - In this paper new fundamental theorems are stated, which allow to easily determine a first-order majorant system of a pseudo-quadratic uncertain MIMO discrete-time system.

For the above class of systems, some results and systematic methods are provided, which allow to solve, "via majorant system", several analysis problems of robust stability, stabilization and tracking of a generic reference signal with bounded variation in presence of a generic disturbance with bounded variation.

The utility and the efficiency of the main results proposed in this paper are illustrated with three significant examples.

Keywords : pseudo-quadratic mimo uncertain discrete-time systems; majorant system; robust stability; stabilization and tracking..

GJSFR-F Classification : MSC 2010: 11K45, 11R11

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Strictly as per the compliance and regulations of :

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I. Introduction

There exist many discrete-time systems linear but with uncertain parameters, uncertain pseudo-linear and with bounded coefficients, uncertain pseudo-quadratic and with bounded coefficients, having, in several cases, a bounded nonlinear additional term or that are solicited with non standard inputs, for whose not always an equilibrium state or the steady-state response exists.

Regarding this, consider: the demographic, economic, resource and traffic management, environmental, agricultural, biological, medical, sampled systems, etc.

For a given system of the above mentioned significant class, it is important to obtain an estimate of its evolution in a finite or infinite time, for all the initial conditions belonging to a prefixed compact set and for all the values of uncertainties, or to design a controller in order to practically stabilize it or, finally, to design a controller to force this system to track a sufficiently regular prefixed trajectory with a bounded error.

Despite the numerous scientific papers available in literature (e.g. [1]-[16]), some of which also very recent (e.g. [19], [23]-[25]), the following practical limitations remain: *1.* the considered classes of systems are often with little relevant interest to engineers; *2.* the considered signals (references, disturbances, etc.) are almost always standard waveforms (polynomial and/or sinusoidal ones); *3.* the controllers are often not very robust and/or do not allow satisfying more than a single specification.

In this paper a systematic method, in a more general framework with respect to the ones proposed in literature (see e.g. [1]-[16],[19], [23]-[25]), for the analysis, for the practical stabilization and the tracking of a significant class of pseudo-linear and pseudo-quadratic uncertain MIMO systems, with additional bounded nonlinearities and/or bounded disturbances, is considered. Some of these results are an extension of analogous results for the continuous-time systems provided in [18],[21],[22],[26],[27].

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In detail, in Section II the considered class of pseudo-quadratic uncertain MIMO systems is presented, the definition of majorant system is given and, finally, several analysis and synthesis problems are formulated. In Section III some basic theorems are stated, which allow to determine, by calculating the eigenvalues of appropriate matrices only in correspondence of the extreme values of the uncertain parameters and of the vertices of suitable polytopes of the state space, if the system is pseudo-quadratic, a majorant system of a pseudo-quadratic uncertain MIMO system. In Section IV some methods are proposed to analyze the robust practical stability, via majorant system, of a pseudo-quadratic uncertain MIMO system.

In Section V some theorems are provided, which allow to determine a state feedback control law to stabilize a pseudo-quadratic uncertain MIMO system and to design an integral controller with state reaction, that forces a LTI uncertain MIMO system to track a generic reference signal with bounded variation in presence of a generic disturbance with bounded variation too.

In Section VI the main results proposed in this paper are illustrated with three significant examples.

II. Problem formulation

Consider the following class of nonlinear discrete-time dynamic systems

$$\begin{aligned} x_{k+1} &= a(x_k, p, k) + A_0(x_k, w_k, p, k)x_k + \left(\sum_{i=1}^{n} A_i(x_k, w_k, p, k)x_{ik} \right) x_k + B(x_k, p, k)u_k \\ y_k &= C(x_k, p, k)x_k, \end{aligned} \tag{1}$$

where: $k \in Z$ is the time; $x \in R^n$ is the state; $u \in U \subset R^h$, with U compact set, is the control and/or disturbance input; $w \in W \subset R^l$, with W compact set, is a possible parametric input; $y \in R^m$ is the output; $p \in [p^-, p^+] \subset R^\nu$ is the vector of uncertain parameters; $A_i \in R^{n\times n}, i = 0,1,...,n$, $B \in R^{n\times h}$ and $C \in R^{m\times n}$ are bounded matrices continuous with respect to their arguments, with C of full rank, and multilinear with respect to p and w; $a \in R^n$ is a bounded continuous vector multilinear with respect to p, which models particular nonlinearities of the system.

The following preliminary notations and definitions are introduced:

$$\|x\|_P = \sqrt{x^T P x},\ \ \|x\| = \|x\|_I = \sqrt{x^T x},\ \ S_{P,\rho} = \left\{ x : \|x\|_P \le \rho \right\},\ \ C_{P,\rho} = \left\{ x : \|x\|_P = \rho \right\},\ \ \hat{C}_{P,\rho} \supseteq C_{P,\rho}, \tag{2}$$

where $P \in R^{n\times n}$ is a symmetric and positive definite (*p.d.*) matrix, x^T is the transpose of $x \in R^n$ and $\hat{C}_{P,\rho}$ is a compact set; $\lambda_{\max}(Q) = \max_{i=1,2,...,n} \|\lambda_i(Q)\|$, where $Q \in R^{n\times n}$;

$$\tau_{\max}(A) = -\frac{1}{\ln\left(\lambda_{\max}(A)\right)}, \quad \text{where } A \in R^{n\times n}. \tag{3}$$

Definition 1. Give the system (1), an hyper-interval $W_I = [w^-, w^+] \subseteq W$, a $\delta \ge 0$ and a *p.d.* symmetric matrix $P \in R^{n\times n}$. A positive first-order system $\rho_{k+1} = f(\rho_k, \delta), \rho_0 = \|x_0\|_P$, $\upsilon_k = \eta(\rho_k)$, where $\rho_k = \|x_k\|_P$, such that $\|y_k\| \le \upsilon_k$, $\forall k \ge 0, \forall x_0 \in R^n$, $\forall w_k \in W_I$, $\forall u_k \in U : \|u_k\| \le \delta$ and $\forall p \in [p^-, p^+]$, is said to be majorant system of the system (1).

Since a majorant system of a system belonging to the class of uncertain nonlinear systems (1), that obviously includes also the linear uncertain system, is a first-order time-invariant system, this system can be used to easily solve numerous analysis and synthesis problems. E.g. the analysis of practical stability, the analysis of the performances decreasing of a control system, or its impossibility to guarantee given performances in the hypothesis of deterioration and/or faults of its components (fault tolerance), the robust stabilization, the robust tracking.

The aim of the present paper is to establish new fundamental results which easily allow to determine a majorant system of the system (1) and to provide systematic methods, via majorant system, to solve, for brevity, only some main problems between the numerous analysis and synthesis above mentioned ones.

III. Basic Theorems

To easily determine and analyze a majorant system of the system (1) the following basic theorems are necessary.

Theorem 1. Let $P \in R^{n\times n}$ be a symmetric *p.d.* matrix, $Q(x,p) \in R^{n\times n}$ be a symmetric *s.p.d.* matrix, $g(x,p) \in R^n$ be a vector, continuous with respect to $x \in R^n$ and $p \in \wp \subset R^\nu$, with $\wp$ compact set; then $\forall \rho \geq 0$ it is:

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \leq \max_{x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2 \leq \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2 \quad (4)$$

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T g(x,p) \leq \max_{x\in C_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho \leq \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho. \quad (5)$$

Moreover, if $Q(x,p)$ is linear with respect to x it is

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \leq \max_{x\in C_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^3 \leq \max_{x\in \hat{C}_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^3, \quad (6)$$

while if $Q(x,p)$ is quadratic with respect to x it turns out to be

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \leq \max_{x\in C_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^4 \leq \max_{x\in \hat{C}_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^4. \quad (7)$$

More in general, if $Q(x,p)$ is continuous, pseudo-linear with respect to x and with bounded coefficients, i.e. if $Q(x,p) = \sum_{i=1}^{n} Q_i(x,p)x_i$, with bounded $Q_i(x,p)$, then

$$\begin{aligned} \max_{x\in C_{P,\rho},\, p\in\wp} x^T \left(\sum_{i=1}^{n} Q_i(x,p)x_i\right)x &\leq \max_{\substack{x\in C_{P,1},\, p\in\wp \\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n} Q_i(z,p)x_i P^{-1}\right)\rho^3 \leq \\ &\leq \max_{\substack{x\in \hat{C}_{P,1},\, p\in\wp \\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n} Q_i(z,p)x_i P^{-1}\right)\rho^3, \end{aligned} \quad (8)$$

whereas if $Q(x,p)$ is continuous, pseudo-quadratic with respect to x and with bounded coefficients, i.e. if $Q(x,p) = \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(x,p)x_i x_j$, with bounded $Q_{ij}(x,p)$, then

$$\begin{aligned} \max_{x\in C_{P,\rho},\, p\in\wp} x^T \left(\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(x,p)x_i x_j\right)x &\leq \max_{\substack{x\in C_{P,1},\, p\in\wp \\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(z,p)x_i x_j P^{-1}\right)\rho^4 \leq \\ &\leq \max_{\substack{x\in \hat{C}_{P,1},\, p\in\wp \\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(z,p)x_i x_j P^{-1}\right)\rho^4. \end{aligned} \quad (9)$$

Proof. Note that, if $f(x)\in R$ is a continuous function with respect to $x\in R^n$ and X_1, X_2, $X_1\subset X_2$, are compact subsets of R^n, it is $\max_{x\in X_1} f(x)\le \max_{x\in X_2} f(x)$. Moreover, since P is *p.d.*, there exists a symmetric nonsingular matrix S such that $P=S^2$. Hence, by posing $z=Sy$, it is

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \le \max_{y\in C_{P,\rho}, x\in C_{P,\rho},\, p\in\wp} y^T Q(x,p)y = \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} z^T S^{-1}Q(x,p)S^{-1}z = \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(S^{-1}Q(x,p)S^{-1})z^T z =$$
$$= \max_{x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(SS^{-1}Q(x,p)S^{-1}S^{-1})\rho^2 = \max_{x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2 \le \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2, \quad (10)$$

and so (4). Similarly

$$\begin{aligned}&\max_{x\in C_{P,\rho},\, p\in\wp} x^T g(x,p) \le \max_{y\in C_{P,\rho}, x\in C_{P,\rho},\, p\in\wp} y^T g(x,p) = \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} z^T S^{-1} g(x,p) \le \\ &\le \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} \|z\|\|S^{-1}g(x,p)\| = \max_{x\in C_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho \le \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho,\end{aligned} \quad (11)$$

and hence (5).
Inequalities (6) easily follow from the fact that, if $Q(x,p)$ is linear with respect to x, $Q(x,p)|_{x\in C_{P,\rho}} = Q(x,p)|_{x\in C_{P,1}}\,\rho$. Inequalities (7), (8), (9) analogously follow.

Remark 1. Theorem 1 can be easily generalized to the case in which $Q(x,p)$ is a homogeneous function of degree ν with respect to x.

Remark 2. Clearly, if $Q(x,p)$ and $g(x,p)$ are independent of x, inequalities (4) and (5) hold with the equal sign. Moreover, if $Q(x,p)$ depends on x, it is quite difficult to compute $\max_{x\in C_{P,\rho}} x^T Q(x,p)x$ because $x^T Q(x,p)x|_{x\in C_{P,\rho}}$ has, in general, different points of relative maximum, of relative minimum and of "inflection"; the second and third members of (4), ((6),(8)), ((7),(9)) allow an easier computation of an upper bound on $x^T Q(x,p)x|_{x\in C_{P,\rho}}$ proportional to ρ^2, ρ^3, ρ^4 , respectively, as it will be shown later on. A similar talking is valid if $g(x,p)$ depends on x.

Theorem 2. Let $P\in R^{n\times n}$ be a *p.d.* symmetrix matrix, $C\in R^{m\times n}$ be a matrix with rank m and $B\in R^{n\times h}$ be a matrix with rank h. Then

$$\upsilon = \|Cx\| \le \sqrt{\lambda_{\max}(CP^{-1}C^T)}\,\rho, \quad \forall x\in C_{P,\rho} \quad (12)$$

$$\|Bu\|_P \le \sqrt{\lambda_{\max}(B^T PB)}\,\|u\|, \quad \forall u\in R^h. \quad (13)$$

Proof. By taking into account that, if F is a real matrix $m\times n$ with rank m, the matrix $F^T F$ has $n-m$ null eigenvalues and m positive eigenvalues equal to the ones of FF^T and, by posing $z=Sx$, where S is a symmetric nonsingular matrix such that $P=S^2$, it is

$$\begin{aligned}&\upsilon = \|Cx\| = \sqrt{x^T C^T Cx} = \sqrt{z^T S^{-1}C^T CS^{-1}z} \le \sqrt{\lambda_{\max}\left(S^{-1}C^T CS^{-1}\right)}\,\|z\| = \\ &= \sqrt{\lambda_{\max}\left(CS^{-1}S^{-1}C^T\right)}\,\|Sx\| = \sqrt{\lambda_{\max}\left(CP^{-1}C^T\right)}\,\rho\end{aligned} \quad (14)$$

and so (12). The proof of (13) easily follows.

Theorem 3. Let $A = \sum_{i_1, i_2, ..., i_\mu \in \{0,1\}} A_{i_1 i_2 ... i_l} \pi_1^{i_1} \pi_2^{i_2} ... \pi_\mu^{i_\mu} \in R^{nxn}$ be a matrix multilinearly depending on the parameters $[\pi_1 \;\; \pi_2 \;\; ... \;\; \pi_\mu]^T = \pi \in \prod = \{\pi \in R^\mu : \pi^- \le \pi \le \pi^+\}$ and let $P \in R^{n \times n}$ be a symmetric *p.d.* matrix. Then the maximum of $\lambda_{\max\limits_{\pi \in \prod}}(QP^{-1})$, where $Q = A^T PA$, is attained in one of the 2^μ vertices of $\prod$.

Proof. Note that for a constant $\pi_j, \, j \ne i$, it is $A = A_0 + \pi_i A_1, \; \pi_i \in [\pi_i^-, \; \pi_i^+]$. Moreover, since for Theorem 1 and Remark 2 it is $\lambda_{\max}(QP^{-1}) = \max\limits_{x \in C_{P,1}} x^T Qx$, it turns out to be

$$\begin{aligned} &\max_{\pi_i \in [\pi_i^-, \, \pi_i^+]} \lambda_{\max}\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)P^{-1}\right) = \max_{\pi_i \in [\pi_i^-, \, \pi_i^+], \, x \in C_{P,1}} x^T\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)\right)x = \\ &\max_{\pi_i \in [\pi_i^-, \, \pi_i^+], \, x \in C_{P,1}} x^T\left(A_1^T PA_1\pi_i^2 + \left(A_0^T PA_1 + A_1^T PA_0\right)\pi_i + A_0^T PA_0\right)x. \end{aligned} \tag{15}$$

Therefore, said $\hat{\pi}_i$, $\hat{x}$ the point of maximum of $f(\pi_i, x) = x^T\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)\right)x\Big|_{x \in C_{P,1}, \pi_i \in [\pi_i^-, \pi_i^+]}$, it is

$$\begin{aligned} &\max_{\pi_i \in [\pi_i^-, \, \pi_i^+]} \lambda_{\max}\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)P^{-1}\right) = \max_{\pi_i \in [\pi_i^-, \, \pi_i^+], \, x \in C_{P,1}} x^T\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)\right)x = \\ &\max_{\pi_i \in [\pi_i^-, \, \pi_i^+]} \left(\hat{x}^T A_1^T PA_1\hat{x}\pi_i^2 + 2\hat{x}^T A_0^T PA_1\hat{x}\pi_i + \hat{x}^T A_0^T PA_0\hat{x}\right) = \max_{\pi_i \in [\pi_i^-, \, \pi_i^+]} \left(\hat{c}_2\pi_i^2 + \hat{c}_1\pi_i + \hat{c}_0\right). \end{aligned} \tag{16}$$

Since $\hat{c}_2 = \hat{x}^T A_1^T PA_1\hat{x} \ge 0$, it is

$$\max_{\pi_i \in [\pi_i^-, \, \pi_i^+]} \left(\hat{c}_2\pi_i^2 + \hat{c}_1\pi_i + \hat{c}_0\right) = \max\left\{\hat{c}_2(\pi_i^-)^2 + \hat{c}_1\pi_i^- + \hat{c}_0, \quad \hat{c}_2(\pi_i^+)^2 + \hat{c}_1\pi_i^+ + \hat{c}_0\right\}. \tag{17}$$

The proof easily follows from (16) and (17).

From Theorem 3 the next Corollary easily follows.

Corollary 1. Let $A = \sum_{i_1, i_2, ..., i_\mu \in \{0,1,2\}} A_{i_1 i_2 ... i_l} \pi_1^{i_1} \pi_2^{i_2} ... \pi_\mu^{i_\mu} \in R^{nxn}$ be a matrix multiquadratic depending on the parameters $[\pi_1 \;\; \pi_2 \;\; ... \;\; \pi_\mu]^T = \pi \in \prod = \{\pi \in R^\mu : \pi^- \le \pi \le \pi^+\}$ and $P \in R^{n \times n}$ a symmetric *p.d.* matrix. Then an upper bound of the maximum of $\lambda_{\max\limits_{\pi \in \prod}}(QP^{-1})$, where $Q = A^T PA$, is equal to the maximum of $\lambda_{\max}(A_e^T PA_e P^{-1})$, attained in one of the $2^{2\mu}$ vertices of $\prod \times \prod$, where A_e is obtained from matrix A by substituting the product $\pi_i \pi_{\mu+i}$, $i = 1, 2, ..., \mu$, to π_i^2.

Theorem 3 can be generalized as follows.

Theorem 4. Let A be the matrix

$$A = \sum_{i_1, i_2, ..., i_\mu \in \{0,1\}} A_{i_1 i_2 ... i_\mu} g_1(\pi_1)^{i_1} g_2(\pi_2)^{i_2} ... g_\mu(\pi_l)^{i_\mu} \in R^{nxn}, \tag{18}$$

in which $[\pi_1 \;\; \pi_2 \;\; ... \;\; \pi_\mu]^T = \pi \in \prod = \{\pi \in R^\mu : \pi^- \le \pi \le \pi^+\}$ and each function $g_i, i = 1, 2, .., \mu,$ is continuous with respect to π_i, and let $P \in R^{n \times n}$ be a *p.d.* symmetric matrix. Then the maximum of $\lambda_{\max}(QP^{-1})$, where $Q = A^T PA$, is attained in one of 2^μ vertices of Γ, where

$$\Gamma = \left\{ \gamma \in R^{\mu} : \min[g_1 ... g_{\mu}] \le \gamma \le \max[g_1 ... g_{\mu}] \right\}. \quad (19)$$

Proof. The proof follows from Theorem 3, by making the change of variable $\gamma = g(\pi) = \left[g_1(\pi_1)\; g_2(\pi_2) ... g_{\mu}(\pi_{\mu}) \right]$ and by noting that $\max_{\pi \in \Pi} \lambda_{\max}\left(Q(g(\pi)) P^{-1} \right) = \max_{\gamma \in \Gamma} \lambda_{\max}\left(Q(\gamma) P^{-1} \right)$.

Remark 3. For Theorem 3 and Corollary 1 the computation of $\max_{x \in \hat{C}_{P,\rho}} \lambda_{\max}(Q(x,p)P^{-1})$, if $Q(x,p)$ is linear or quadratic with respect to x, is very easy if $\hat{C}_{P,1} \supseteq C_{P,1}$ is an hyper-rectangle (or a polytope decomposable into hyper-rectangles). To this aim, to compute the vertices of $\hat{C}_{P,1}$, it is easy to prove that the point of contact of the hyper-line orthogonal to the versor $e_i, i = 1,2,...,n,$ of R^n and tangent to the hyper-ellipse $C_{P,1}$ (see Fig. 1) is

$$p_i = \frac{P^{-1} e_i}{\sqrt{e_i^T P^{-1} e_i}}. \quad (20)$$

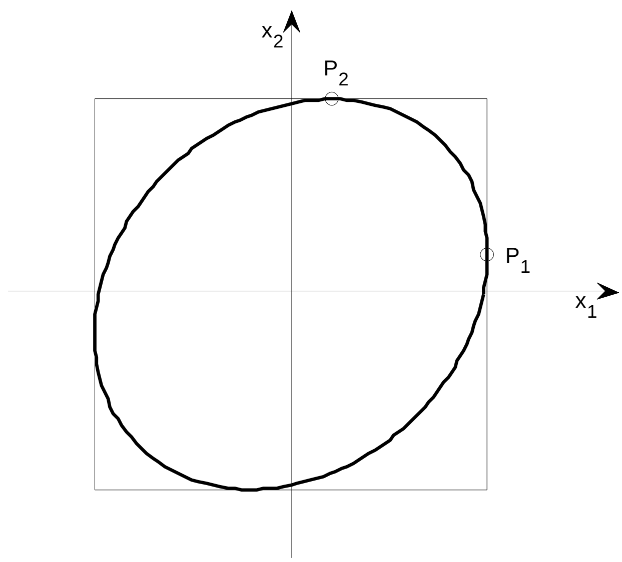

Figure 1 : Points of contact of the hyper-rectangle circumscribed to $C_{P,1}$.

Theorem 5. Consider the quadratic discrete-time system

$$\rho_{k+1} = a_2 \rho_k^2 + a_1 \rho_k + a_0, \quad a_0 \ge 0,\; a_1 \ge 0,\; a_2 \ge 0, \quad \rho_0 \ge 0. \quad (21)$$

If $a_1 > 1$ the system (21) is unstable. If $0 < a_1 < 1$ and $a_2 = 0, \forall \rho_0 \ge 0$ the system evolves toward the equilibrium point $\rho_e = \frac{a_0}{1 - a_1}$ with time constant $\tau = -\frac{1}{\ln a_1}$. Finally, if $a_2 > 0,\; 0 < a_1 < 1$ and $(1 - a_1)^2 - 4 a_2 a_0 > 0$, said ρ_1, ρ_2, $\rho_1 < \rho_2$, the roots of the algebraic equation $a_2 \rho^2 + (a_1 - 1)\rho + a_0 = 0, \;\; \forall \rho_0 \in [0, \rho_2)$, the system evolves toward the equilibrium point $\rho_e = \rho_1$ with time constant of the linearized system equal to $\tau_l = -\frac{1}{\ln \alpha_1}$, where $\alpha_1 = a_1 + 2 a_2 \rho_e$.

Proof. If $a_1 > 1$ it is always $\rho_{k+1} > \rho_k$ and, hence, the system is unstable. The remaining part of the proof easily follows by making standard manipulations and from Fig. 2.

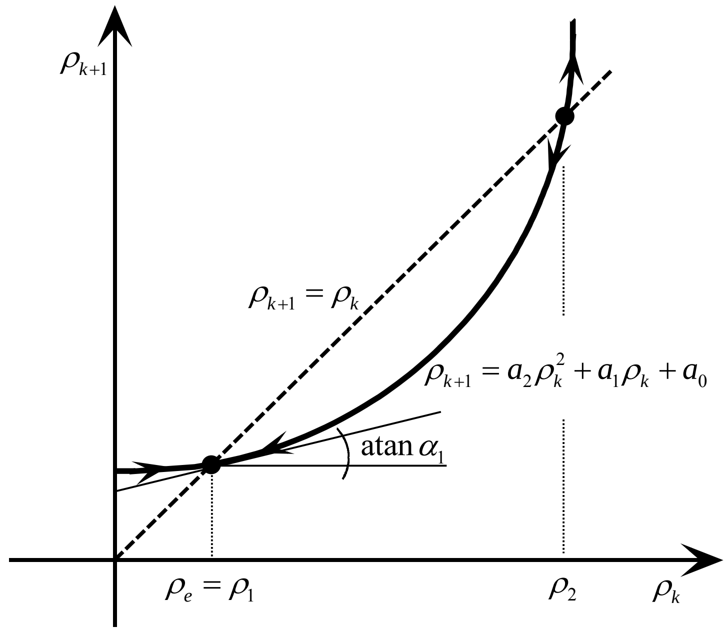

Figure 2 : Graphical illustration of Theorem 5.

Theorem 6. Let $A \in R^{n\times n}$ be a matrix with ν real distinct eigenvalues λ_i, $i = 1, \ldots, \nu$, and with $\mu = \frac{n-\nu}{2}$ distinct pair of complex conjugate eigenvalues $\lambda_{h\pm} = \alpha_h \pm j\omega_h$, $h = 1, \ldots, \mu$; moreover, let u_i, $i = 1, \ldots, \nu$, and $u_{h\pm} = u_{ah} \pm ju_{bh}$, $h = 1, \ldots, \mu$, be the associated eigenvectors. Then, by denoting with Z^* the conjugate transpose of the matrix of the eigenvectors $Z = \left[u_1 \ldots u_\nu \; u_{a1} + ju_{b1} \; u_{a1} - ju_{b1} \ldots u_{a\mu} + ju_{b\mu} \; u_{a\mu} - ju_{b\mu}\right]$ and with $\Lambda = diag\left(\lambda_1, \ldots, \lambda_\nu, \alpha_1 + j\omega_1, \alpha_1 - j\omega_1, \ldots, \alpha_\mu + j\omega_\mu, \alpha_\mu - j\omega_\mu\right)$ the diagonal matrix of the eigenvalues, the matrix

$$P = \left(ZZ^*\right)^{-1} = \left[\sum_{i=1}^{\nu} u_i u_i^T + 2\sum_{h=1}^{\mu}\left(u_{ah}u_{ah}^T + u_{bh}u_{bh}^T\right)\right]^{-1} \quad (22)$$

turns out to be always *p.d.*; moreover it is

$$\lambda_{\max}\left(QP^{-1}\right) = \lambda_{\max}^2(A) \Rightarrow \tau_{\max}(A) = -\frac{2}{\ln\left(\lambda_{\max}\left(QP^{-1}\right)\right)}, \quad (23)$$

where $Q = A^T PA$.

Proof. As, for hypothesis, the eigenvalues of A are distinct, the matrix of the eigenvectors Z is nonsingular. Hence the matrix ZZ^* is *p.d.* and, therefore, also its inverse P is *p.d.* . Moreover, since $A = Z\Lambda Z^{-1}$, it is

$$QP^{-1} = A^T PAP^{-1} = A^* PAP^{-1} = \left(Z^*\right)^{-1}\Lambda^* Z^*\left(Z^*\right)^{-1} Z^{-1}Z\Lambda Z^{-1}ZZ^* = \left(Z^*\right)^{-1} Z^{-1}Z\Lambda Z^{-1} = \left(Z^*\right)^{-1}\left(\Lambda^*\Lambda\right)Z^*. \quad (24)$$

Hence the eigenvalues of QP^{-1} are $\lambda_j^*\lambda_j = \left\|\lambda_j\right\|^2$, $j = 1, \ldots, n$, from which the proof follows.

IV. Stability Analysis

First, consider the linear and time-invariant uncertain MIMO system

$$x_{k+1} = A(p)x_k + B(p)u_k, \quad y_k = C(p)x_k, \quad \text{with}$$
$$A(p) = \sum_{i_1,i_2,\dots,i_\nu \in\{0,1\}} A_{i_1 i_2 \dots i_l} p_1^{i_1} p_2^{i_2} \dots p_\nu^{i_\nu} \in R^{nxn}, \; B(p) = \sum_{i_1,i_2,\dots,i_\nu \in\{0,1\}} B_{i_1 i_2 \dots i_l} p_1^{i_1} p_2^{i_2} \dots p_\nu^{i_\nu} \in R^{nxn}, \tag{25}$$
$$C(p) = \sum_{i_1,i_2,\dots,i_\nu \in\{0,1\}} C_{i_1 i_2 \dots i_l} p_1^{i_1} p_2^{i_2} \dots p_\nu^{i_\nu} \in R^{nxn}, \; p \in [p^-, p^+] \subset R^\nu.$$

It is well-known that this system is asymptotically stable if $a = \max_{p\in[p^-,p^+]} \lambda_{\max}(A) < 1$. If the goal is only to study the asymptotic stability without calculating a, then the Jury criterion to the characteristic polynomial $d(\lambda, p) = \det(\lambda I - A(p))$ can be applied or it is possible to use one of the several methods to establish the definite positivity of the matrix $P(p)$, which is solution of the Lyapunov equation $A^T(p)P(p)A(p) - P(p) = -Q$, with Q $p.d.$. As it can be easily realized, both the methods are very onerous because of the strong nonlinearity with respect to the parameters p both of $d(\lambda, p)$ and $P(p)$. Clearly the computation of a is even more onerous.

Let P be a $p.d.$ symmetric matrix and fixed a $\rho_k \ge 0, \; \forall x_k \in C_{P,\rho_k}$ from the first of (25) for $u_k = 0$ and from Theorem 1 easily follows that

$$\rho_{k+1} = \|x_{k+1}\|_P = \sqrt{x_k^T A^T(p) P A(p) x_k}\Big|_{x_k \in C_{P,\rho_k}} \le a\rho_k \Rightarrow \rho_k \le a^k \rho_0, \tag{26}$$

where

$$a = \sqrt{\max_{p\in[p^-,p^+]} \lambda_{\max}(Q(p)P^{-1})} = \sqrt{\max_{p\in V_p} \lambda_{\max}(Q(p)P^{-1})}, \; Q(p) = A^T(p)PA(p), \tag{27}$$

in which V_p is the set of the 2^ν vertices of the hyper-rectangle $[p^-, p^+]$.

It is interesting to note that the last of (26) provides an upper bound of the free evolution of the system (25) $\forall p \in [p^-, p^+]$. Clearly the goodness of this bound depends on P; a not appropriate matrix P could provide a value of a greater than 1 also when $p^- = p^+$ and the system is asymptotic stable.

If for a given $\hat{p} \in [p^-, p^+]$ the matrix $A(\hat{p})$ has distinct eigenvalues, condition almost always verified, the relative matrix P given by (22) is always $p.d$ and for (23) it is always that $a = \sqrt{\lambda_{\max}(Q(\hat{p})P^{-1})} = \lambda_{\max}(A(\hat{p}))$, also when $A(\hat{p})$ has not all the eigenvalues with magnitude less than 1.

From this reasoning, from the theorems stated in Section III and from the fact that $\|x_1 + x_2\|_P \le \|x_1\|_P + \|x_2\|_P, \; \forall x_1, x_2 \in R^n$, the following theorems easily derive.

Theorem 7. Give a matrix $A(p) = \sum_{i_1,i_2,\dots,i_\nu \in\{0,1\}} A_{i_1 i_2 \dots i_l} p_1^{i_1} p_2^{i_2} \dots p_\nu^{i_\nu} \in R^{nxn}$, $p \in \wp \subset R^\nu$, with $\wp$ a compact set. An estimate of the $\max_{p\in\wp} \lambda_{\max}(A(p))$ can be obtained by covering the set $\wp$ with N hyper-rectangles $[p_i^-, p_i^+]$ and by computing the maximum of $\left\{a_i : a_i = \sqrt{\max_{p\in V_{ip}} \lambda_{\max}\left(Q(p_i)P_i^{-1}\right)}, Q(p_i) = A^T(p_i)P_i A(p_i), i = 1,2,..,N\right\}$, where $p_i = \dfrac{p_i^- + p_i^+}{2}$ or it is a near value, $P_i = \left(Z_i Z_i^*\right)^{-1}$, where Z_i is the eigenvectors matrix of $A(p_i)$ and V_{ip} is the set of vertices of $[p_i^-, p_i^+]$.

Theorem 8. Suppose there exist a $\hat{p} \in [p^-, p^+]$ such that the dynamic matrix $A(\hat{p})$ of the multivariable uncertain system (25) has distinct eigenvalues; said $P = \left(ZZ^*\right)^{-1}$, where Z is the eigenvectors matrix of $A(\hat{p})$, a majorant system of the system (25) is

$$\rho_{k+1} = a\rho_k + b\delta, \quad \upsilon_k = c\rho_k, \tag{28}$$

in which: $\rho_k = \|x_k\|_P$, $\delta \geq \|u_k\|$, $\upsilon_k = \|y_k\|$, $a = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(Q(p)P^{-1}\right)}$, $Q(p) = A^T(p)PA(p)$, $b = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(B^T(p)PB(p)\right)}$, $c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(C(p)P^{-1}C^T(p)\right)}$, where V_p is the set of the 2^ν vertices of the hyper-rectangle $[p^-, p^+]$.

Remark 4. By using the theorems stated in Section III, Theorem 8 can be easily extended also to the case in which the system (25) is pseudo-linear, i.e. to the case in which the matrices A, B, C depend also on x_k and k and they are bounded.

Now consider the quadratic uncertain MIMO system

$$\begin{aligned} x_{k+1} &= A_0(p)x_k + \left(\sum_{i=1}^{n} A_i(p)x_{ik}\right)x_k + B(p)u_k \\ y_k &= C(p)x_k, \end{aligned} \tag{29}$$

where $A_i(p), i = 0,1,...,n$, $B(p)$, $C(p)$ are multilinear functions of $p \in [p^-, p^+]$. The following theorem holds.

Theorem 9. Suppose that there exists a $\hat{p} \in [p^-, p^+]$ such that the matrix $A_0(\hat{p})$ in (29) has distinct eigenvalues; by choosing $P = \left(ZZ^*\right)^{-1}$, where Z is the eigenvectors matrix of $A_0(\hat{p})$, a majorant system of the system (29) turns out to be

$$\rho_{k+1} = a_1\rho_k + a_2\rho_k^2 + b\delta, \quad \upsilon_k = c\rho_k, \tag{30}$$

where: $\rho_k = \|x_k\|_P$, $\delta \geq \|u_k\|$, $\upsilon_k = \|y_k\|$, $a_1 = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(Q_1(p)P^{-1}\right)}$, $Q_1(p) = A_0^T(p)PA_0(p)$, $a_2 = \sqrt{\max_{p \in V_p, x \in V_x} \lambda_{\max}\left(Q_2(p,x)P^{-1}\right)}$, $Q_2(p,x) = \left(\sum_{i=1}^{n} A_i^T(p)x_i\right)P\left(\sum_{i=1}^{n} A_i(p)x_i\right)$, $b = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(B^T(p)PB(p)\right)}$, $c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(C(p)P^{-1}C^T(p)\right)}$, V_p is the set of the 2^ν vertices of the hyper-rectangle $[p^-, p^+]$ and V_x is the set of the 2^n vertices of the hyper-rectangle $[x^-, x^+]$ circumscribed to the hyper-circle $C_{P,1}$.

Proof. The proof easily follows from Theorems 1, 2, 3.

Remark 5. By using the results stated in Section III, Theorems 9 can be easily extended also to the case in which the system (29) is pseudo-quadratic, i.e. to the case in which the matrices A_i, B, C depends also on x_k and k and they are bounded.

Remark 6. Give the system $x_{k+1} = f(x_k, p)$, with $f(0,p) = 0$, being

$$\begin{aligned} x_{k+1} &= \partial f / \partial x_k\big|_{x_k=0} x_k + 1/2\left[x_k^T \left\{\partial^2 f_1 / \partial x_{ik} \partial x_{jk}\right\}\big|_{x_k=\xi} x_k \; \ldots \; x_k^T \left\{\partial^2 f_n / \partial x_{ik} \partial x_{jk}\right\}\big|_{x_k=w_k} x_k\right]^T = \\ &= A_0(p)x_k + \left(\sum_{i=1}^{n} A_i(w_k, p)x_{ik}\right)x_k, \; 0 \leq w_k \leq x_k, \end{aligned} \tag{31}$$

if the second order partial derivatives of f are bounded then, by using the stated theorems, it is easy and systematic to estimate a region of asymptotic stability (RAS) of the origin and, moreover, to estimate the degree of linearity of the system by comparing a_1 to a_2 of a related majorant system.

V. Robust Stabilization and Tracking of a Pseudo-Quadratic Uncertain Mimo System

The problem of stabilization and tracking of an uncertain discrete-time system is very complex and, in some cases, it is impossible to solve, above all when the interval of uncertainties is very wide, unless an identification method of the parameters is used. It is sufficient to consider, e.g., the system $x_{k+1} = ax_k + u_k$, with a uncertain parameter belonging to an interval of amplitude greater than 2.
By considering bounded enough uncertainties, in many cases, the following results are very useful.

Theorem 10. Give the system (29), suppose that the state is measurable and that there exists a $\hat{p} \in [p^-, p^+]$ such that the couple $(A_0(\hat{p}),\ B(\hat{p}))$ is reachable. Said $\bar{\Lambda} = \{\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n\}$ a symmetric set of n distinct complex numbers with $\lambda_{\max}(\lambda_i) \le 1$, a possible control law to stabilize or to increase the RAS of the system (29) is the following

$$u_k = -K_0 x_k - \left(\sum_{i=1}^{n} K_i x_{ik} \right) x_k, \tag{32}$$

where K_0 is such that $eig\left(A_0(\hat{p}) - B(\hat{p})K_0\right) = r\bar{\Lambda},\ r \in [0,1)$ and K_i, with $i = 1,2,...,n,$ is such that to minimize $\|A_i(\hat{p}) - B(\hat{p})K_i\|_P$, where $P = \left(ZZ^*\right)^{-1}$, in which Z is the eigenvectors matrix of $A_0(\hat{p}) - B(\hat{p})K_0$.

Remark 7. If $rank\left(B(\hat{p})\right) > 1$, by posing $K0 = K_0$, $Ac = A_0(\hat{p})$, $Bc = B(\hat{p})$, $L = r\bar{\Lambda}$, the matrix K_0 can be computed by using the Matlab command *K0=place(Ac,Bc,L)*, based on the algorithm in [3], that uses the extra degrees of freedom to find a solution that minimizes the sensitivity of the closed-loop poles to perturbation in $A_0(\hat{p})$ or $B(\hat{p})$.
Instead, by posing $Ai = A_i(\hat{p})$, $Ki = K_i$, the matrices $K_i, i = 1,2,...,n,$ can be computed with the Matlab commands: *S=P^.5; Ki=pinv(S*Bc,S*Ai)* .
Finally the value of r can be used to reduce both a_1 and a_2, or to maximize $(1-a_1)/a_2$.

Consider now the LTI uncertain MIMO plant described by

$$x_{k+1} = A(p)x_k + B(p)u_k + E(p)d_k, \quad y_k = C(p)x_k + D(p)d_k, \tag{33}$$

where: $x_k \in R^n$ is the state, $u_k \in R^r$ is the control signal, $d_k \in R^l$ is the disturbance, $y_k \in R^m$ is the output, $A(p), B(p), E(p), C(p), D(p)$ are matrices of suitable dimensions, which are multilinear functions of the parameter vector $p \in \wp$. Suppose that $\wp = \left\{p : p \in \left[p^-, p^+\right]\right\} \subset R^\nu$ is an hyper-rectangle and that the following reachable condition

$$rank\left[B(\hat{p}) \quad A(\hat{p})B(\hat{p}) \quad ... \quad A^{n-1}(\hat{p})B(\hat{p})\right] = n \tag{34}$$

is satisfied for at least a $\hat{p} \in \wp$.
In the following, for simplicity of notations, the dependency of $A(p), B(p), E(p), C(p), D(p)$ on p will be omitted.

A main goal is :

1) to state new results to estimate, $\forall p \in \wp$ and for an assigned controller with integral (I) action of the system (33), the maximum time constant and the maximum tracking error of a generic reference signal r_k, with bounded variation $\delta r_k = r_k - r_{k-1}$, in presence of a generic disturbance d_k with bounded variation $\delta d_k = d_k - d_{k-1}$ too;

2) to design, if possible, a LTI controller such that to force the uncertain MIMO system (33) to track, with prefixed maximum time constant and maximum error, a generic reference signal r_k, with bounded variation, in presence of a generic disturbance d_k with bounded variation, $\forall p \in \wp$.

Regarding this, it is important to note that a relevant class of reference signals r_k with generic behavior but with bounded variation is very recurring in the practice and easily feasible by using digital technologies (see Fig. 3).

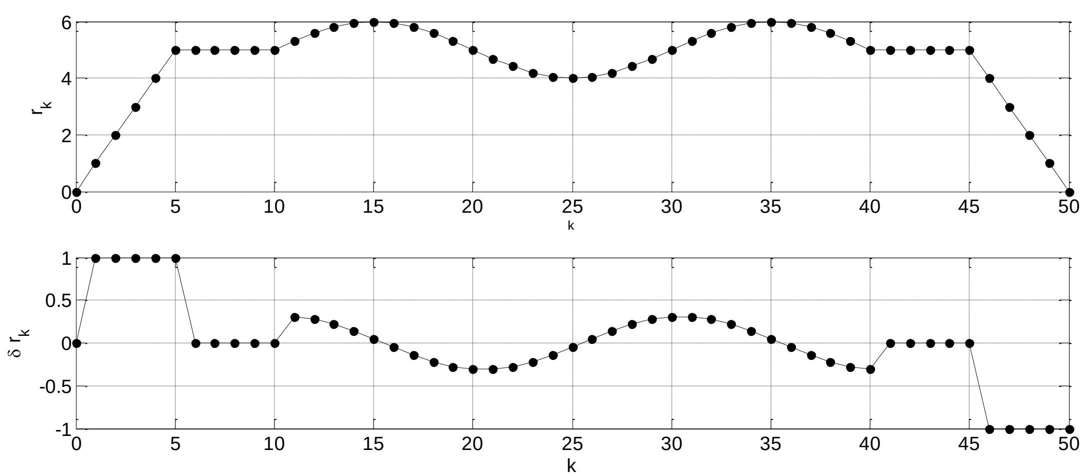

Figure 3 : A possible reference signal r_k with its bounded variation δr_k.

Remark 8. Note that, nowadays, the reference signal of a control system (e.g. manufacturing systems, …) is in general a non standard (not polynomial and/or cisoidal) signal whose variation is the desired "working velocity" (clearly finite, even if the planners and the builders make a great effort to make it as higher as it is possible).

The problems 1) and 2), not suitably solved in literature ([1]-[16],[19], [23]-[25]), are very important from a theoretical and practical point of view.

Among the several controllers available in literature, for brevity, in the following it will be considered only the well-known state feedback control scheme with an I action of Fig. 4.

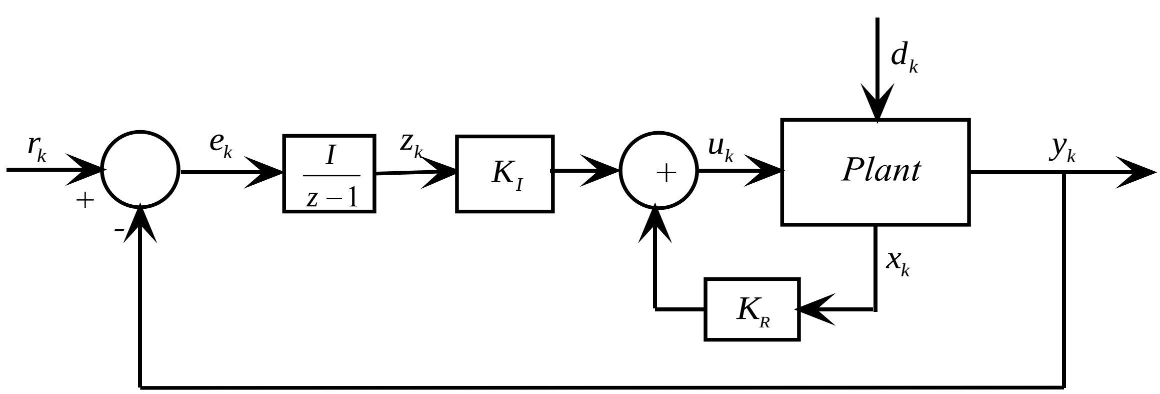

Figure 4 : State feedback control scheme with an I control action.

As regards suppose that there exists at least a $\hat{p} \in \wp$ such that, in addition to (34), also the following condition is satisfied

$$rank\begin{bmatrix} I-A & B \\ C & 0 \end{bmatrix} = n+m. \tag{35}$$

From the control scheme of Fig. 4 it easily derives that:

$$\begin{aligned} &x_{k+1} = Ax_k + Bu_k + Ed_k, \quad u = K_I z_k + K_R x_k, \quad e_k = r_k - Cx_k - Dd_k \\ &z_{k+1} = z_k + r_k - y_k = z_k + r_k - Cx_k - Dd_k , \end{aligned} \tag{36}$$

from which:

$$\xi_{k+1} = A_c \xi_k + B_c r_k + E_c d_k, \quad e_k = C_c \xi_k + r_k - Dd_k , \tag{37}$$

where:

$$\begin{aligned} &A_c = \begin{bmatrix} A+BK_R & BK_I \\ -C & I \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad E_c = \begin{bmatrix} E \\ -D \end{bmatrix}, \\ &C_c = \begin{bmatrix} -C & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} x \\ z \end{bmatrix}. \end{aligned} \tag{38}$$

In order to solve the problems 1) and 2) the following preliminary important result is necessary.

Theorem 11. The control system (37) can be described also by:

$$\zeta_{k+1} = A_c \zeta_k + B_c \delta r_{k+1} + E_c \delta d_{k+1}, \quad e_k = H_c \zeta_k, \quad H_c = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{39}$$

or equivalently by

$$C_c (zI - A_c)^{-1} [B_c \; E_c] + [I \; -D] = H_c (zI - A_c)^{-1} [B_c \; E_c](z-1), \tag{40}$$

where $\delta r_{k+1} = r_{k+1} - r_k$, $\delta d_{k+1} = d_{k+1} - d_k$, $e_k = r_k - y_k$ is the tracking error and I is the identity matrix of appropriate order.

Proof. Posed

$$(zI - A_c)^{-1} = \begin{bmatrix} F_1 & F_2 \\ C & (z-1)I \end{bmatrix}^{-1} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{41}$$

by using a known formula of the inverse of a partitioned matrix (see e.g. [17]), it is easy to prove that

$$\begin{aligned} &G_1 = \left(F_1 - \frac{F_2 C}{z-1}\right)^{-1}, \quad G_2 = \left(F_1 - \frac{F_2 C}{z-1}\right)^{-1} \frac{F_2}{z-1}, \\ &G_3 = -\frac{C}{z-1}\left(F_1 - \frac{F_2 C}{z-1}\right)^{-1}, \quad G_4 = \frac{1}{z-1}\left(I - C\left(F_1 - \frac{F_2 C}{z-1}\right)^{-1} \frac{F_2}{z-1}\right). \end{aligned} \tag{42}$$

From (38), from the last of (39) and by using (42), after tedious steps, (40) follows and hence the proof.

Remark 9. Note that

$$A_c = \begin{bmatrix} A+BK_R & BK_I \\ -C & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} -K_R & -K_I \end{bmatrix} = A_0 - B_0 K. \tag{43}$$

Therefore, if there exists a $\hat{p} \in \wp$ such that (34) and (35) are satisfied, the eigenvalues of A_c can be arbitrarily assigned.

Remark 10. Theorem 11 is very significant because it allows to evaluate or estimate, via majorant system, the tracking error e_k.

Now the following main result, concerning the robust tracking, can be stated.

Theorem 12. Let K_I, K_R be two matrices such that the matrix A_c, for $p = \hat{p}$, has distinct eigenvalues with magnitudes less than one. Then a majorant system of the system (39) with respect to the norm $\|\zeta\|_P$ with $P = \left(ZZ^*\right)^{-1}$, where Z is the eigenvectors matrix of A_c for $p = \hat{p}$, is

$$\rho_{k+1} = a_c \rho_k + b_c \max\|\delta r_{k+1}\| + e_c \max\|\delta d_{k+1}\|, \quad \|e_k\| = h_c \rho_k, \tag{44}$$

in which: $a_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(Q(p) P^{-1}\right)}$, $Q(p) = A_c^T(p) P A^T(p)$, $b_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(B_c^T(p) P B(p)\right)}$, $e_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(E_c^T(p) P E_c(p)\right)}$, $h_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(H_c(p) P^{-1} H_c^T(p)\right)}$, being V_p the set of the 2^ν vertices of the hyper-rectangle $[p_-, p_+]$.

Proof. The proof is standard.

Remark 11. Note that, if $\wp = \hat{p}$, then the time constant of the majorant system $\tau = -1/\ln a_c$ is positive and coincides, for Theorem 6, with the maximum time constant of the control system. Moreover, "at steady-state", the tracking error satisfies relation

$$\|e_k\| \le \frac{h_c b_c}{1-a_c} \max\|\delta r_{k+1}\| + \frac{h_c e_c}{1-a_c} \max\|\delta d_{k+1}\|. \tag{45}$$

Remark 12. Clearly, if the initial state of the control system is not null and/or r_k and/or d_k are "discontinuous" in zero, the tracking error e_k has an additional term, whose practical duration depends on the time constant τ of the majorant system.

VI. Examples

The following examples show the great utility and efficiency of the results stated in the previous sections.

Example 1. Consider the system

$$x_{k+1} = \left(A_0 + A_1 p_1 + A_2 p_2 + A_{12} p_1 p_2 + A_{11} p_1^2\right) x_k + \left(B_0 + p_1 \sin x_{1k} B_1\right) u_k, \quad y_k = C x_k, \text{ where}$$

$$A_0 = \begin{bmatrix} 0.6677 & 0.3864 \\ -0.2338 & 0.6265 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1259 & -0.1492 \\ 0.1623 & 0.1654 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0529 & -0.0886 \\ -0.1610 & 0.0188 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.0915 & -0.0685 \\ 0.0930 & 0.0941 \end{bmatrix}, \tag{46}$$

Global Journal of Science Frontier Research (F) Volume XIII Issue V Version I Year 2013

$$A_{11}=\begin{bmatrix}0.0433 & 0.1000\\ -0.0406 & -0.1035\end{bmatrix},\ B_0=\begin{bmatrix}0.0187\\ 0.1801\end{bmatrix},\ B_1=\begin{bmatrix}-0.0112\\ 0.1080\end{bmatrix},\ C=\begin{bmatrix}1 & 0\end{bmatrix},\ p_1\in[0.9\quad 1.1],\ p_2\in[0.9\quad 1.1].$$

By posing $\hat{p}_1=1,\ \hat{p}_2=1$ and by using Theorem 6 with $A=A(\hat{p})=A_0+A_1+A_2+A_{12}+A_{11}$, it is

$$P=\begin{bmatrix}1.3373 & 0.7606\\ 0.7606 & 1.4895\end{bmatrix}. \tag{47}$$

By using this P, Theorem 8 and Remark 4, a majorant system of the system (46) turns out to be:

$$\rho_{k+1}=a\rho_k+b\delta,\quad \upsilon_k=c\rho_k\ , \tag{48}$$

where, for Theorem 3, Corollary 1 and Theorem 4, it is:

$$\begin{aligned} a &= \sqrt{\max_{\substack{\pi_1=0.9,1.1\\ \pi_2=0.9,1.1\\ \pi_3=0.9,1.1}} \lambda_{\max}\left(A^T(\pi)PA(\pi)P^{-1}\right)},\quad A(\pi)=A_0+A_1\pi_1+A_2\pi_2+A_{12}\pi_1\pi_2+A_{11}\pi_1\pi_3\\ b &= \sqrt{\max_{\substack{\gamma_1=0.9,1.1\\ \gamma_2=-1,1}} B^T(\gamma)B(\gamma)},\quad B=B_0+\gamma_1\gamma_2 B_1\\ c &= \sqrt{CP^{-1}C}; \end{aligned} \tag{49}$$

hence $\rho_{k+1}=0.9574\rho_k+0.3688\delta,\quad \upsilon_k=1.0266\rho_k$.

From (48) it follows that $|y_k|\le \dfrac{cb}{1-a}\delta+ca^k\left(\rho_o-\dfrac{b}{1-a}\delta\right),\quad \forall x_0:\|x_0\|_P\le\rho_0$ and $\forall u_k:|u_k|\le\delta$; hence the system (46) is externally asymptotically stable and for $u_k=0$ also internally asymptotically stable.

Note that if $u_k=0$ the system (46) is linear and time invariant. The characteristic polynomial of $A(p)=A_0+A_1p_1+A_2p_2+A_{12}p_1p_2+A_{11}p_1^2$ is of the type

$$\begin{aligned} d(\lambda,p)&=\det\left(\lambda I-A(p)\right)=\lambda^2+\left(\alpha_0+\alpha_1p_1+\alpha_2p_2+\alpha_3p_1p_2+\alpha_4p_1^2\right)\lambda+\\ &+\left(\beta_0+\beta_1p_1+\beta_2p_2+\beta_3p_1p_2+\beta_4p_1^2+\beta_5p_2^2+\beta_6p_1^2p_2+\beta_7p_1p_2^2+\beta_8p_1^3+\beta_9p_1^2p_2^2+\beta_{10}p_1^3p_2+\beta_{11}p_1^4\right); \end{aligned}$$

therefore, it is very onerous both to establish the asymptotic stability of $x_{k+1}=A(p)x_k$ and to determine $a=\max_{p\in[p^-,\,p^+]}\lambda_{\max}(A(p))$. Numerically it is $a=0.9315$, while by using the proposed method, very efficient (see Theorem 3 and Corollary 1), an upper bound of a turns out to be $a=0.9574$. By using Corollary 1 and Theorem 7 with four rectangles, an upper bound of a turns out to be $a=0.9458$.

Example 2. Consider the system

$$\begin{aligned} y_{k+1}&=\left(p_1a_{11}+p_2a_{12}\right)y_k+\left(p_1a_{21}+p_2a_{22}\right)y_{k-1}+g_{11}(y_k,y_{k-1},p_1,p_2,k)y_k^2+g_{12}(y_k,y_{k-1},p_1,p_2,k)y_ky_{k-1}+\\ &+g_{22}(y_k,y_{k-1},p_1,p_2,k)y_{k-1}^2+\left(b_0+p_1b_1\right)u_k, \end{aligned} \tag{50}$$

where: $a_{11}=-0.730,\ a_{12}=-0.490,\ a_{21}=1.310,\ a_{22}=0.870,\ b_0=0.200,\ b_1=0.050;$
$p_1\in[0.9,\quad 1.1],\ p_2\in[0.8,\quad 1.2],\ g_{11}\in[0.08,\quad 0.12],\ g_{12}\in[0.07,\quad 0.13],\ g_{22}\in[-0.12,\quad -0.08].$

By posing $x_{1k}=y_k,\ x_{2k}=y_{k-1},\ \gamma_1=g_{11},\ \gamma_2=g_{12},\ \gamma_3=g_{22}$, system (50) can be put in the form

$$x_{k+1}=\begin{bmatrix}0 & 1\\ p_1a_{21}+p_2a_{22} & p_1a_{11}+p_2a_{12}\end{bmatrix}x_k+\left(\begin{bmatrix}0 & 1\\ \gamma_1 & \gamma_2/2\end{bmatrix}x_{1k}+\begin{bmatrix}0 & 1\\ \gamma_2/2 & \gamma_3\end{bmatrix}x_{2k}\right)x_k+\begin{bmatrix}0\\ b_0+b_1p_1\end{bmatrix}u_k \quad (51)$$
$$y_k=\begin{bmatrix}1 & 0\end{bmatrix}x_k.$$

By applying Theorem 9 with $\hat{p}_1=1,\ \hat{p}_2=1,$ a majorant system of the system (50) turns out to be

$$\rho_{k+1}=1.749\rho_k+1.413\rho_k^2+1.504\delta,\quad \upsilon_k=1.0484\rho_k. \quad (52)$$

This system is clearly unstable $\forall\delta\ge 0$, since $a_1=1.749>1$ (see Theorem 5).

By considering the closed-loop system of (51) with the control law

$$u_k=-\begin{bmatrix}-3.9075 & 5.7213\end{bmatrix}x_k-\left(\begin{bmatrix}0.400 & 0.200\end{bmatrix}x_{1k}+\begin{bmatrix}0.200 & -0.400\end{bmatrix}x_{2k}\right)x_k+v_k, \quad (53)$$

obtained by applying Theorem 10 with $\hat{p}_1=1,\ \hat{p}_2=1,$ $\hat{\gamma}_1=0.1,\ \hat{\gamma}_2=0.1,\ \hat{\gamma}_3=-0.1,\ r\bar{\Lambda}=\{0.3749+j0.3203,\ \ 0.3749-j0.3203\}$, a majorant system of the closed-loop system is

$$\rho_{k+1}=0.8046\rho_k+0.1488\rho_k^2+0.6276\delta,\quad \upsilon_k=0.6254\rho_k, \quad (54)$$

where

$$\rho=\|x\|_P,\ P=\begin{bmatrix}1.4728 & -2.2708\\ -2.2708 & 6.0579\end{bmatrix}. \quad (55)$$

From the first of (54) and from Theorem 5 it can be deducted that, whatever the values of uncertainties are, if $v_k=0\Rightarrow\delta=0,\ \forall x_0:\|x_0\|_P<1.3135$, it is $\|x_k\|_P<\|x_0\|_P,\forall k>0,$ and $\lim_{k\to\infty}x_k=0$ (see Fig 5a). Moreover the time constant of the linearized of the majorant system (54) turns out to be $\tau=4.5995$.
Always from the first of (54) and from Theorem 5 it can be deducted that, for any values of uncertainties, $\forall v_k:|v_k|\le 0.09$ and $\forall x_0:\|x_0\|_P\le 0.4296$, x_k remains always in the ellipse $C_{P,\,0.4296}$, while $\forall v_k:|v_k|\le 0.09$ and $\forall x_0:\|x_0\|_P\in(0.4296,\ 0.8839)$, it is $\|x_k\|_P<\|x_0\|_P,\forall k>0$ and for k big enough x_k goes in the ellipse $C_{P,\,0.4296}$ (see Fig 5b).

Moreover the time constant of the linearized of the majorant system (54) turns out to be $\tau=14.2978$.

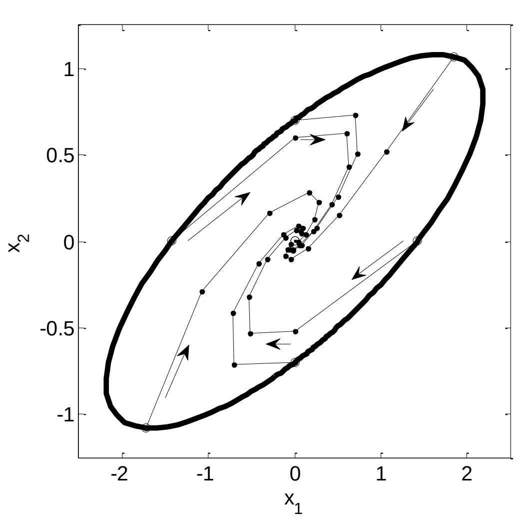

Figure 5a : RAS for $v_k=0$.

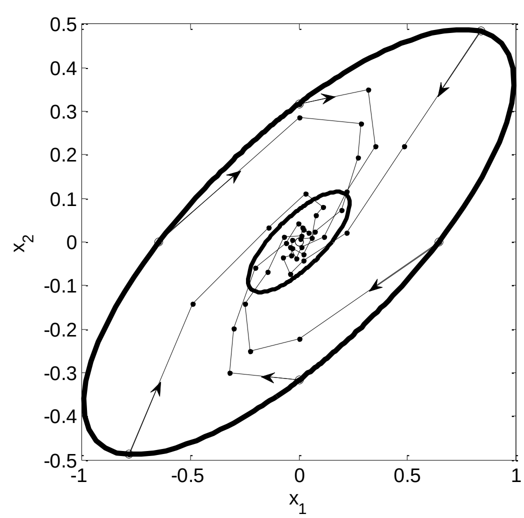

Figure 5b : Practical RAS for $v_k=0$.

Example 3. Consider the control system of the traffic of the road network in Fig. 6. By denoting with x_{ik} the distance of the vehicle i $(i = 1, 2)$ from the next one $i+1$ at the instant kT, where T is the sampling time,

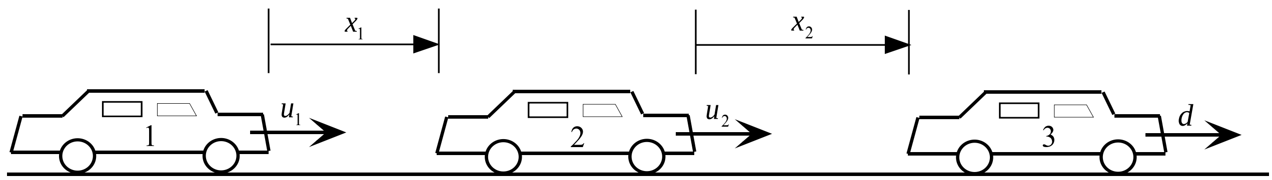

Figure 6 : Road network.

with u_{ik} the stretch of road that the vehicle i $(i=1, 2)$ must cover in the interval $[kT, (k+1)T]$ and with d_k the stretch of road that the head vehicle (number 3 in Fig. 6) will run in the interval $[kT, (k+1)T]$, it is:

$$x_{1k+1} = x_{1k} - (1+p_1)u_{1k} + (1+p_2)u_{2k}, \quad x_{2k+1} = x_{2k} - (1+p_2)u_{2k} + d_k, \tag{56}$$

where p_1, p_2 are the relative errors of actuation. From (56) it is:

$$x_{k+1} = Ax_k + B(p)u_k + Ed_k, \quad y_k = C_k x_k, \tag{57}$$

where:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(p) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} p_1 + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} p_2, \quad E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{58}$$

An easier model is the one in which for the vehicle number 1 the signal u_{2k} is a disturbance, i.e. in the hypothesis of a decentralized control. In this case it is:

$$x_{ik+1} = x_{ik} - (1+p_i)u_{ik} + d_{ik}, \quad i = 1, 2, \quad d_{1k} = u_{2k}, \quad d_{2k} = d_k, \tag{59}$$

By using the decentralized control law

$$z_{ik+1} = z_{ik} + (r_{ik} - y_{ik}), \quad u_{ik} = -0.5429 z_{ik} + 1.2929 y_{ik}, \quad i = 1, 2, \tag{60}$$

obtained such that the eigenvalues of A_{ic} for $\hat{p}_1 = \hat{p}_2 = 0$ are $0.5e^{\pm j\frac{\pi}{4}}$, in the hypothesis that $p_1 \in [-0.05, 0.05]$, $p_2 \in [-0.05, 0.05]$, upper bounds of the tracking errors can be determined by using the majorant systems

$$\rho_{ik+1} = 0.5760\rho_{ik} + 2.6085 \max|\delta r_{ik+1}| + 2.4843|\delta d_{ik+1}|, \quad |e_{ik}| = 1.1385\rho_{ik}, \quad i = 1, 2, \tag{61}$$

computed with Theorem 12.

In Fig. 7 the simulation results of the controlled road network, in the hypothesis of $p_1 = 0.05$, $p_2 = -0.05$, $r_{1k} = 10 \cdot 1_k$, $r_{2k} = 20 \cdot 1_k$, $d_k = \left(20 + 2\sin\frac{k}{10} - 2\cos\frac{k}{20}\right) \cdot 1_k$, $\zeta_0 = 0$, are shown. After the transient phase, due to the initial "discontinuous" of r_{1k}, r_{2k} and d_k, it is $|e_{2k}| \le 0.5827$, while from the majorant system it turns out to be that $|e_{2k}| \le 1.9993$.

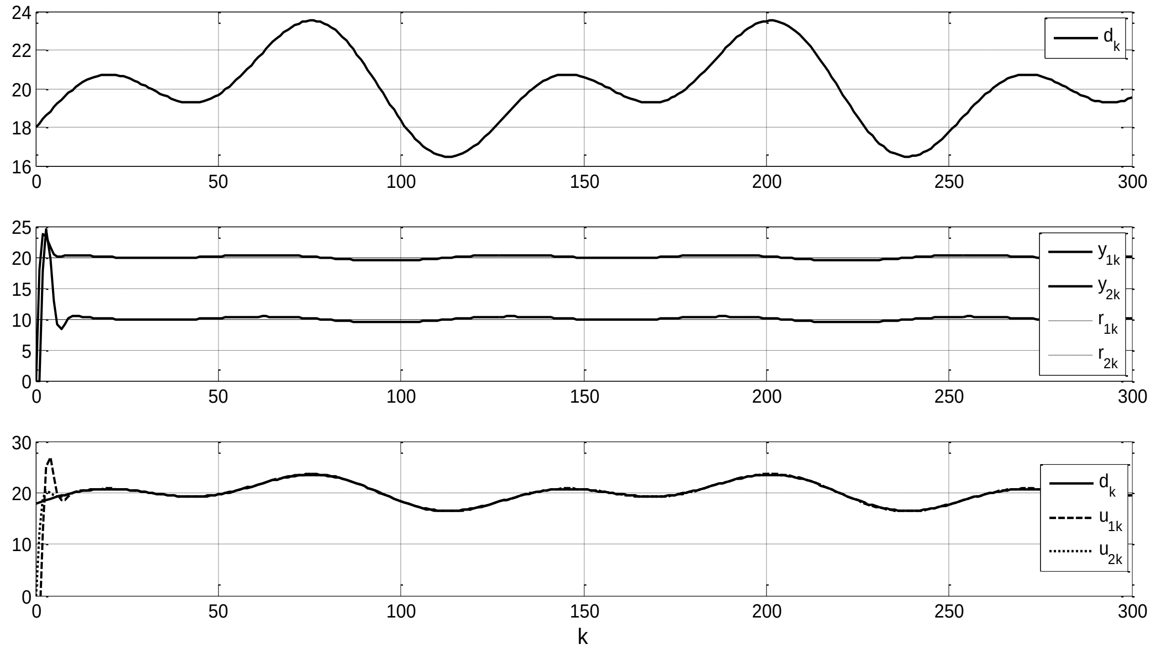

Figure 7 : Errors and control signals of the controlled road network.

VII. Conclusions and Future Developments

In this paper several basic theorems have been stated and proved. They allow to determine, by calculating the eigenvalues of suitable matrices only in correspondence of the vertices of appropriate polytopes, a majorant system of a pseudo-quadratic uncertain MIMO system.

By using the provided results, systematic methods have been derived, which allow to solve, via majorant system, several analysis and synthesis problems about the robust stability, robust stabilization and tracking of a generic reference signal with bounded variation, in presence of a generic disturbance with bounded too. The presented examples have shown the utility and the efficiency of the main proposed results.

Future developments are going on in the direction of the fault tolerance and of the robust tracking in the hypothesis of a non measurable state.

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A Summation Formula in the Light of Recurrence Relation

By Salahuddin & M. P. Chaudhary
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GJSFR-F Classification : MSC 2010: 65Q30, 40A25

Strictly as per the compliance and regulations of :

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Abstract - The main aim of the present paper is to compute a summation formula in the light of recurrence relation and contiguous relation.

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I. Introduction

The **Pochhammer symbol** or **generalized factorial function** or **shifted factorial** or **falling factorial** is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)\cdots(a+n-1) \ ; & \text{if } \ n=1,2,3,\ldots \\ 1 \ ; & \text{if } \ n=0 \\ n! \ ; & \text{if } \ a=1, \ n=1,2,3,\ldots \end{cases} \quad (1)$$

where $a \neq 0,-1,-2,\ldots$ and the notation Γ stands for Gamma function. Note that $(0)_0 = 1$.

If $m = 1,2,3,4,\ldots$ and $n = 0,1,2,3,4,\ldots$ then

$$(b)_{mn} = m^{mn}\left(\frac{b}{m}\right)_n \left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-2}{m}\right)_n \left(\frac{b+m-1}{m}\right)_n \quad (2)$$

$$(\alpha)_{p+q} = (\alpha)_p(\alpha+p)_q = (\alpha)_q(\alpha+q)_p \quad (3)$$

$$\text{If } 0 \leq k \leq \left[\frac{n}{m}\right], \text{ then } (b)_{n-mk} = \frac{(\frac{-1}{m})^{mk}(b)_n}{\prod_{j=1}^{m}(\frac{j-b-n}{m})_k} \quad (4)$$

$$(n-mk)! = \frac{(\frac{-1}{m})^{mk}\, n!}{\prod_{j=1}^{m-1}(\frac{j-n}{m})_k} \quad (5)$$

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where $[x]$ returns the largest integer less than or equal to x. In Slater's book[90], $(a)_n$ is denoted by (a, n).

$$(b)_{-m} = \frac{\Gamma(b-m)}{\Gamma(b)} = \frac{(-1)^m}{(1-b)_m} \tag{6}$$

where, $b \neq 0, \pm1, \pm2, \pm3, \pm4, \ldots$ and $m = 1, 2, 3, 4, \ldots$

Some properties of the Pochhammer's symbol are given by

$$(b)_{m-n} = (b)_m (b+m)_{-n} = (b)_{-n}(b-n)_m \tag{7}$$

$$(a_1)_{M+N}(a_2)_{M+N}\cdots(a_A)_{M+N} = [(a_A)]_{M+N} = [(a_A)]_M[(a_A)+M]_N \tag{8}$$

Generalized Gaussian hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_AF_B\left[\begin{matrix} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{matrix}\right] \equiv {}_AF_B\left[\begin{matrix} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{9}$$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$(a-b)\ {}_2F_1\left[\begin{matrix} a, & b\ ; & \\ & c\ \ ; & z \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1, & b & ; & \\ & c & ; & z \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a, & b+1\ ; & \\ & c\ \ ; & z \end{matrix}\right] \tag{10}$$

Recurrence relation of gamma function is defined as follows

$$\Gamma(z+1) = z\,\Gamma(z) \tag{11}$$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$\sqrt{\pi}\,\Gamma(2z) = 2^{(2z-1)}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \tag{12}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{13}$$

$$= \frac{2^{(a-1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{14}$$

Notes

Bailey summation theorem [Prud, p.491(7.3.7.8)]is defined as follows

$$ {}_2F_1\left[\begin{matrix} a, 1-a & ; & \frac{1}{2} \\ c & ; & 2 \end{matrix}\right] = \frac{\Gamma(\frac{c}{2})\,\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-1}\,\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c+1-a}{2})} \quad (15) $$

Notes

II. Main Result of Summation Formula

$$ {}_2F_1\left[\begin{matrix} a, & -a-52 & ; & \frac{1}{2} \\ c & & ; & 2 \end{matrix}\right] $$

$$ = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c+52}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a+52}{2})}\Big\{-34928139885605446284439249144578048000000a $$
$$ +42071734873936699642597555068454010880000a^2 $$
$$ -18702590782686940348571526156051922944000a^3 $$
$$ +3913402400021039256540281604096498585600a^4 $$
$$ -377223664134626507934927914708226309120a^5 $$
$$ +7989571832148498655051443520007784192a^6 $$
$$ +1174255602469543244429680529972051136a^7 $$
$$ -54242832591167079430159654587460224a^8 - 20994865799681897133280511135134 24a^9 $$
$$ +10032380413427373779303642294 9216a^{10} + 331639953946463027454780740 2916a^{11} $$
$$ -776280552699841842393452723 44a^{12} - 3592660349233917957600756619a^{13} $$
$$ +2960249949153674689741751a^{14} + 1779080539870652651300486a^{15} $$
$$ +230165663910377761839 26a^{16} - 1900074312182535693 49a^{17} - 706093008273058591 9a^{18} $$
$$ -5164052717078226 4a^{19} + 222305063914076a^{20} + 5674507689851a^{21} + 31183770761a^{22} $$
$$ +19511726a^{23} - 411034a^24 - 1339a^{25} - a^{26} + 34928139885605461795649292475564032000000c $$
$$ -10871389538903394038490594400911753216 0000ac $$
$$ +8566729039706254572244169588755709952 0000a^2c $$
$$ -2878240113798257309044526309784818073 6000a^3c $$
$$ +46379742513741206675808873792462397 44000a^4c $$
$$ -316847044725090296206110319628987166720a^5c $$
$$ -15638571309771173039141682196151961 60a^6c $$
$$ +115658155457130878755804139562561753 6a^7c $$
$$ -17988204353358933679295505292192256a^8c - 21313587773814526887122140075220 16a^9c $$
$$ +253300184233660405450472588361 60a^{10}c + 259286412148579320814176186101 6a^{11}c $$
$$ +1993426292336480580601171904a^{12}c - 1727286054597891762330490106a^{13}c $$
$$ -225675670931622348349203 20a^{14}c + 385153303273201498890036a^{15}c $$

$$+11442926416738757850624a^{16}c+5367614954693216911 4a^{17}c-1197921355762331200a^{18}c$$

$$-17815000979897264a^{19}c - 63344641295936a^{20}c + 419074409754a^{21}c + 4354571520a^{22}c$$

$$+12064676a^{23}c + 1664a^{24}c - 26a^{25}c + 66642160515097299932437200641867120640000c^2$$

$$-123530007507285230550469913161191653376000ac^2$$

$$+72836076162669006670353194997063853670400a^2c^2$$

$$-19198125365308987399707077724866986229760a^3c^2$$

$$+2390293356263580742022446233482892201984a^4c^2$$

$$-1066398696057937441132947595060752353 28a^5c^2$$

$$-403900983117771128834413551664063564 8a^6c^2$$

$$+42934512092237001359035974532934579 2a^7c^2$$

$$+2924092456618390432624901233201472a^8c^2-7175088666429903553069378738254 72a^9c^2$$

$$-5693513145785706123832854666512a^{10}c^2 + 6430569113889278661621722525 28a^{11}c^2$$

$$+104386455218626083797707309 24a^{12}c^2 - 2363829003612547284544902 08a^{13}c^2$$

$$-716509331695217118223016 8a^{14}c^2 - 1246234309741487150884 8a^{15}c^2$$

$$+1493203719892310086804a^{16}c^2 + 18100386873203990784a^{17}c^2 + 192199960148 3104a^{18}c^2$$

$$-1421335537237616a^{19}c^2 - 10307915165548a^{20}c^2 - 14758519776a^{21}c^2$$

$$-1421335537237616a^{19}c^2 - 10307915165548a^{20}c^2 - 14758519776a^{21}c^2 + 114938824a^{22}c^2$$

$$+449904a^{23}c^2 + 364a^{24}c^2 + 5656530789290972565677129178084802560000 0c^3$$

$$-76773094417412920099297189451582668800000ac^3$$

$$+35687278744536038663635683055562391552000a^2c^3$$

$$-7501438390916162127538789656958390763520a^3c^3$$

$$+713943658948690921885604796252682321920a^4c^3$$

$$-16705606842656372545944276887236804608a^5c^3$$

$$-1728757601740457366021472945000374272a^6c^3$$

$$+79518446332910960936680940949617664a^7c^3$$

$$+2438341023774939799001976789424640a^8c^3-1119385852150763027657384621547 52a^9c^3$$

$$-3040398019108634505172957171968a^{10}c^3 + 639122105137603626631278583 68a^{11}c^3$$

$$+2441885550484590174390709920a^{12}c^3 - 156528945944395783839516 8a^{13}c^3$$

$$-825533474094234334489632a^{14}c^3 - 874616845512945982761 6a^{15}c^3$$

$$+53252570814885682240a^{16}c^3 + 1580615102252764864a^{17}c^3 + 8756784251190976a^{18}c^3$$

$$-23906001927808a^{19}c^3 - 427022916320a^{20}c^3 - 1408543136a^{21}c^3 - 486304a^{22}c^3$$

Notes

$$+2912a^{23}c^3 + 288060169927055496023795776929461698560000c^4$$
$$-3072455316811095347799143802824212611072 0ac^4$$
$$+115434730559587543305808186543873507983336a^2c^4$$
$$-19452388870478036922897072233263047966 72a^3c^4$$
$$+1378195516217163254114637444153243156 48a^4c^4$$
$$-393039136430981961339529809072310272a^5c^4$$
$$-382841780809199389444887709129580032a^6c^4$$
$$+70828501211164745481601307963138 56a^7c^4 + 5498864290152829664289447730524 16a^8c^4$$
$$-74125692474125862133740531824 64a^9c^4 - 5152279918269030350559028286 40a^{10}c^4$$
$$+42281070951479729529909403 2a^{11}c^4 + 25129299946082451422095083 2a^{12}c^4$$
$$+2498262451265766805892272a^{13}c^4 - 38787567775641350473808a^{14}c^4$$
$$-854071404221074696800a^{15}c^4 - 2838712550035533600a^{16}c^4 + 52246466127055200a^{17}c^4$$
$$+528037255665920a^{18}c^4 + 1175371397200a^{19}c^4 - 5047362320a^{20}c^4 - 24744720a^{21}c^4$$
$$-21840a^{22}c^4 + 99746492016444789285621930706813845504 00c^5$$
$$-864058555400412687504926344581919172198 4ac^5$$
$$+26565178238662084701173154109072893542 40a^2c^5$$
$$-35802878977498866880526452090346648371 2a^3c^5$$
$$+1793528617698974707683992064441832243 2a^4c^5$$
$$+355442453909769139900305632836153344a^5c^5$$
$$-52527348703619549507837894069672960a^6c^5$$
$$-1250528252728629048238460931942 4a^7c^5 + 674540372525562889953117494307 84a^8c^5$$
$$+21939781818828958209849295411 2a^9c^5 - 4601677896397690716230863968 0a^{10}c^5$$
$$-5127512824163995221087593 28a^{11}c^5 + 12640580059018263309525568a^{12}c^5$$
$$+267811234488316292053280a^{13}c^5 + 7139890514453007168 0a^{14}c^5$$
$$-3648364550419329312 0a^{15}c^5 - 299274518303080320a^{16}c^5 + 153769503311424a^{17}c^5$$
$$+11859338370880a^{18}c^5 + 47353386080a^{19}c^5 + 25945920a^{20}c^5 - 96096a^{21}c^5$$
$$+2517115005590931497891492998623145230336c^6$$
$$-1801853977321067526697455847041071579136ac^6$$
$$+4557260477838915614446956375399192330 24a^2c^6$$
$$-486745130392021580622202015465338306 56a^3c^6$$
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$$+80230918540267183957108427091656704a^5c^6$$
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$$+850937419857440140152544 08192a^9c^6-22492849994165525417366676 48a^{10}c^6$$
$$-547855531547000933109934 08a^{11}c^6+168828804235914347452928a^{12}c^6$$
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$$-26912297408777605707360182675 0464a^6c^7$$
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$$-2891381923725459611166720a^{11}c^7-157544094713224693760 00a^{12}c^7$$
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$$-364748524722448701768140846210115174 40ac^8$$
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$$-393300928078289324915015388951871488a^3c^8$$
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$$+382403192371200a^{15}c^8-582581176320a^{16}c^8-5769054720a^{17}c^8-6223360a^{18}c^8$$
$$+880813755207868193554053346623488 0000c^9$$
$$-36965672922163116955642233124510760 96ac^9$$

Notes

$$+514645272515113686120708497604608000a^2c^9$$
$$-235717688865057770397098738259066688a^3c^9$$
$$-41066519525689133592496126735155 2a^4c^9+466540137786023665827145484369 92a^5c^9$$
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$$+16572689923220964515840 0a^{10}c^9-123413330656982036070 4a^{11}c^9$$
$$-29257328226543312896a^{12}c^9-86465927583107072a^{13}c^9+92291114426368 0a^{14}c^9$$
$$+5963024404480a^{15}c^9+5924638720a^{16}c^9-12446720a^{17}c^9$$
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$$-3045409581927875282238079912546140 16ac^{10}$$
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$$-40452740862753606890529175142400a^4c^{10}+19575351485021315872736842874 88a^5c^{10}$$
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$$+80040811817187427090064048390144a^2c^{12}-445178481287516116952381652992a^3c^{12}$$
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$$+61469093240338376246198272a^6c^{12}-1345274452880242716016 64a^7c^{12}$$
$$-16428668892122988560384a^8c^{12}-72880540472360484864a^9c^{12}$$
$$+12749996485759303 68a^{10}c^{12}+10554547802652672a^{11}c^{12}+299058020352a^{12}c^{12}$$

$$
\begin{gathered}
-148806647808a^{13}c^{12} - 206389248a^{14}c^{12} + 264182165001061306106113949696000c^{13} \\
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-20229621696039813 12a^{9}c^{13} + 6320432896081920a^{10}c^{13} + 94010016694272a^{11}c^{13} \\
+151584964608a^{12}c^{13} - 222265344a^{13}c^{13} + 12409623398297151947688965570560c^{14} \\
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+488921602091245066382213120000c^{15} - 64844547829545383354062340096ac^{15} \\
+1708414248369719055006826496a^{2}c^{15} + 48031603574786883105849344a^{3}c^{15} \\
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-686799912960a^{9}c^{16} - 1333592064a^{10}c^{16} + 445740409937736947990528000c^{17} \\
-36713644648914482370707456ac^{17} + 282998508385046888448000a^{2}c^{17} \\
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-209294333050880a^{5}c^{19} - 800136888320a^{6}c^{19} + 807403520a^{7}c^{19}
\end{gathered}
$$

Notes

$$+2992860241094036684800c^{20} - 98819016954688307200ac^{20} - 842719940499734528a^{2}c^{20}$$

$$+21682030516371456a^{3}c^{20} + 119353052758016a^{4}c^{20} - 573821681664a^{5}c^{20}$$

$$-1857028096a^{6}c^{20} + 37037875694403584000c^{21} - 821933144230854656ac^{21}$$

$$-9220521208053760a^{2}c^{21} + 97319414398976a^{3}c^{21} + 580453924864a^{4}c^{21} - 530579456a^{5}c^{21}$$

$$+358292072043642880c^{22} - 4926934588850176ac^{22} - 61881771360256a^{2}c^{22}$$

$$+238471348224a^{3}c^{22} + 1157627904a^{4}c^{22} + 2608521543680000c^{23} - 19531632214016ac^{23}$$

$$-240987930624a^{2}c^{23} + 201326592a^{3}c^{23} + 13435194572800c^{24} - 43201331200ac^{24}$$

$$-419430400a^{2}c^{24} + 43620761600c^{25} - 33554432ac^{25} + 67108864c^{26}$$

$$+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a+53}{2})}\Big\{1999997097524035804017235522077327360000000$$

$$-37759184926395849969376799217510383616000 0a$$

$$+2361784996108884300696721851980720209920 00a^{2}$$

$$-672902239587113360490886710250579832832 00a^{3}$$

$$+917456596097224322922963231788944736256 0a^{4}$$

$$-454011662043974405665799107718370637824a^{5}$$

$$-190073418163495502980463365657033958 40a^{6}$$

$$+23063338786656851070046181641510254 72a^{7} + 1559856529798205134430567133651321 6a^{8}$$

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$$-3263674121769242212100525677a^{13} - 110771538150352696821002641a^{14}$$

$$-1180020540747977790987 58a^{15} + 370061099915364634764 26a^{16}$$

$$+537950317559878419053a^{17} - 3590070561647193 91a^{18} - 80542804789151848a^{19}$$

$$-793098224405344a^{20} - 1487999580067a^{21} + 24813226529a^{22} + 188133842a^{2}3 + 450866a^{24}$$

$$-13a^{25} - a^{26} + 522166060729665280421696827332954685440000c$$

$$-745176201564121598450537124560894951424000ac$$

$$-745176201564121598450537124560894951424000ac$$

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$$+895859801250586838978867144779165588684 8a^{4}c$$

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$$-270777139616783687328852442045955712 00a^{6}c$$

$$+143215610142725917538701660412147289 6a^{7}c$$

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$$-4696511285748857960037016 0a^{14}c - 5807099735111127140280 84a^{15}c$$

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$$-6301160011713424a^{19}c - 150822856992808a^{20}c - 812684971098a^{21}c - 424572720a^{22}c$$

$$+11103196a^{23}c + 35464a^{24}c + 26a^{25}c + 55923601929362911377285543701967273984 0000c^2$$

$$-6377241965272594830269823451921207787520 00ac^2$$

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$$+24948277948518264814402797406247744a^8c^2$$

$$-41597828345098614930863890374624 0a^9c^2 - 3187963834679364341471059924424 0a^{10}c^2$$

$$+61694532595820165339877883344a^{11}c^2 + 2257246793664495150437056682 8a^{12}c^2$$

$$+25565589653600512276837148 8a^{13}c^2 - 5573490785959241092209496a^{14}c^2$$

$$-146528598112733273847696a^{15}c^2 - 5591317402847999983 16a^{16}c^2$$

$$+16855156515546615552a^{17}c^2 + 230600014236825376a^{18}c^2 + 723575163399088a^{19}c^2$$

$$-6151367250028a^{20}c^2 - 57945215328a^{21}c^2 - 151491704a^{22}c^2 + 4368a^{23}c^2 + 364a^{24}c^2$$

$$+343745761995310316138295779696817733632000c^3$$

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Notes

$$-1993824734611046049398 4a^{14}c^3 - 1362610394357550271190 4a^{15}c^3$$

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$$+85791249351808a^{19}c^3 + 82212210080a^{20}c^3 - 1139922784a^{21}c^3 - 3969056a^{22}c^3$$

$$-2912a^{23}c^3 + 1397851989274828680637947203964622602240 00c^4$$

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$$+233013036576320a^{18}c^4 + 2897260366000a^{19}c^4 + 8332083760a^{20}c^4 - 240240a^{21}c^4$$

$$-21840a^{22}c^4 + 4068175111500157240173397090771721099 6736c^5$$

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$$+96096a^{21}c^5 + 890322356751974338770220971661641488 7936c^6$$

$$-4936219104808499733342488455607167746048ac^6$$

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$$+10271583171931361 28a^{15}c^6-3042388046810112a^{16}c^6-55627391339520a^{17}c^6$$

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$$+3298243020196540416a^{14}c^7+34781366180081664a^{15}c^7+65605466234880a^{16}c^7$$

$$-468992409600a^{17}c^7-1992939520a^{18}c^7-1464320a^{19}c^7$$

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$$-1073451921137448652 8a^{13}c^8+1564899342413721 6a^{14}c^8+540380298332160a^{15}c^8$$

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$$-647212691205599241699328ac^{18} - 7677439917877812527104a^2c^{18}$$

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$$+304500878910237442048a^3c^{18}+2766935258250870784a^4c^{18}-27750913550581760a^5c^{18}$$

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$$-419430400a^2c^{24}+45365592064c^{25}+33554432ac^{25}+67108864c^{26}\Big\}\Big] \qquad (16)$$

Derivation of main result

Substituting $b=-a-52, z=\frac{1}{2}$ in given result (10),we get

$$(2a+52)\ {}_2F_1\left[\begin{matrix} a\ ,\ -a-52 & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right]$$

$$=a\ {}_2F_1\left[\begin{matrix} a+1\ ,\ -a-52 & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right]+(a+52)\ {}_2F_1\left[\begin{matrix} a\ ,\ -a-51 & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right]$$

Now involving the derived result of Ref[6], we can prove the main result.

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Title: The title page must carry an instructive title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) wherever the work was carried out. The full postal address in addition with the e-mail address of related author must be given. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining and indexing.

Abstract, used in Original Papers and Reviews:

Optimizing Abstract for Search Engines

Many researchers searching for information online will use search engines such as Google, Yahoo or similar. By optimizing your paper for search engines, you will amplify the chance of someone finding it. This in turn will make it more likely to be viewed and/or cited in a further work. Global Journals Inc. (US) have compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Key Words

A major linchpin in research work for the writing research paper is the keyword search, which one will employ to find both library and Internet resources.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy and planning a list of possible keywords and phrases to try.

Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

Acknowledgements: Please make these as concise as possible.

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.

Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

Color Charges: It is the rule of the Global Journals Inc. (US) for authors to pay the full cost for the reproduction of their color artwork. Hence, please note that, if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a color work agreement form before your paper can be published.

Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.

6. AFTER ACCEPTANCE

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

6.1 Proof Corrections

The corresponding author will receive an e-mail alert containing a link to a website or will be attached. A working e-mail address must therefore be provided for the related author.

Acrobat Reader will be required in order to read this file. This software can be downloaded

(Free of charge) from the following website:

www.adobe.com/products/acrobat/readstep2.html. This will facilitate the file to be opened, read on screen, and printed out in order for any corrections to be added. Further instructions will be sent with the proof.

Proofs must be returned to the dean at dean@globaljournals.org within three days of receipt.

As changes to proofs are costly, we inquire that you only correct typesetting errors. All illustrations are retained by the publisher. Please note that the authors are responsible for all statements made in their work, including changes made by the copy editor.

6.2 Early View of Global Journals Inc. (US) (Publication Prior to Print)

The Global Journals Inc. (US) are enclosed by our publishing's Early View service. Early View articles are complete full-text articles sent in advance of their publication. Early View articles are absolute and final. They have been completely reviewed, revised and edited for publication, and the authors' final corrections have been incorporated. Because they are in final form, no changes can be made after sending them. The nature of Early View articles means that they do not yet have volume, issue or page numbers, so Early View articles cannot be cited in the conventional way.

6.3 Author Services

Online production tracking is available for your article through Author Services. Author Services enables authors to track their article - once it has been accepted - through the production process to publication online and in print. Authors can check the status of their articles online and choose to receive automated e-mails at key stages of production. The authors will receive an e-mail with a unique link that enables them to register and have their article automatically added to the system. Please ensure that a complete e-mail address is provided when submitting the manuscript.

6.4 Author Material Archive Policy

Please note that if not specifically requested, publisher will dispose off hardcopy & electronic information submitted, after the two months of publication. If you require the return of any information submitted, please inform the Editorial Board or dean as soon as possible.

6.5 Offprint and Extra Copies

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Before start writing a good quality Computer Science Research Paper, let us first understand what is Computer Science Research Paper? So, Computer Science Research Paper is the paper which is written by professionals or scientists who are associated to Computer Science and Information Technology, or doing research study in these areas. If you are novel to this field then you can consult about this field from your supervisor or guide.

TECHNIQUES FOR WRITING A GOOD QUALITY RESEARCH PAPER:

1. Choosing the topic: In most cases, the topic is searched by the interest of author but it can be also suggested by the guides. You can have several topics and then you can judge that in which topic or subject you are finding yourself most comfortable. This can be done by asking several questions to yourself, like Will I be able to carry our search in this area? Will I find all necessary recourses to accomplish the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Administration Rules Listed Before Submitting Your Research Paper to Global Journals Inc. (US)

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.
- Do not give permission to anyone else to "PROOFREAD" your manuscript.
- Methods to avoid Plagiarism is applied by us on every paper, if found guilty, you will be blacklisted by all of our collaborated research groups, your institution will be informed for this and strict legal actions will be taken immediately.)
- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

Criterion for Grading a Research Paper (Compilation) by Global Journals Inc. (US)

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals Inc. (US).

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

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