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Ψ -Eventual Stability of Differential Systems with Impulses

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GJSFR-F Classification : MSC 2010: 34CXX; 34DXX; 34A37; 34K45



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1. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations of negligible duration in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is: in the form of impulses. It is known for example that many biological phenomenon involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Impulsive differential equations are adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jumps change of their states. For the description of the continuous change of such processes ordinary differential equations are used, while the moments and the magnitude of the change by jumps are given by the jump conditions. According to the way in which the moments of the change by jumps are determined, IDE are classified into two categories: Equations with fixed moments of impulsive effect and equations with unfixed (variable) moments of impulsive effect. The solutions of IDE with variable impulsive moments are piecewise continuous functions but unlike the solutions of the systems with fixed moments of impulse effect, different solutions of these IDE have different points of discontinuity. This leads to number of difficulties in the investigation of IDS with variable impulsive moments. That is why these systems have been an object of numerous investigations.

Moreover, when the trivial solution of the system does not exist, we may still have stability eventually, which generalizes Lyapunov Stability. For example, for the practical point of view, if a ship remains in an upright position, it is called stable. However, since the environmental forces acting on it as well as ship's disposition w.r.t. sea will change in time, the determination of a safe minimum amount of stability i.e.

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stability criterion becomes necessary. If one follows the intuitive concept of stability, it is natural to think that if the amplitude of the ship in every perceived combinations of ship- environment conditions, remain smaller than a pre-determined safe value, the ship should be considered stable, The mathematical counterparts of this definition are eventual stability and boundedness. For the problems arisen in this situations, to be solved a new notion of eventual stability is introduced by V. Lakshmikantham, S. Leela, A. A. Martynyuk.

The problem of Ψ - stability for the systems of ordinary differential equations has been studied by many authors e. g. J. Morchalo [4] introduced the notion of Ψ - stability, Ψ - uniform stability and Ψ - asymptotic stability of trivial solution of non linear system $x' = f(t, x)$. He has considered Ψ as scalar continuous function. Aurel Diamandescu [2] and [3] has introduced Ψ -stability by taking Ψ as diagonal matrix.

In [6], the criteria of eventual stability are established for impulsive differential systems with fixed moments of impulses by using piecewise Lyapunov functions by Zhang Yu. In [5], Soliman extended the notion of eventual stability to impulsive differential systems with variable moments.

In our paper we have established criterion of Ψ-Eventual Stability for impulsive differential systems with variable moments of impulses by using piecewise continuous auxiliary functions which are analogous to Lyapunov's functions. The paper is organized as follows: In section 2, some preliminary notes and definitions which will be used throughout the paper are introduced. In section 3, two theorems for Ψ - Eventual Stability and Ψ -Uniform Eventual Stability are proved. One example has been given in support of our theoretical results. Conclusion is given in Section 4.

II. PRELIMINARY NOTES AND DEFINITIONS

Let R^n denote n -dimensional Euclidean space with norm $\| \cdot \|$.

Let R_H^s be the s - dimensional Euclidean space with a suitable norm $\| \cdot \|$. Let $R^n = [0, \infty)$ and $R_H^s = \{x \in R^n : \|x\| < H\}$.

Consider the system of differential equations with impulses

$$\begin{aligned} \dot{x} &= f(t, x) + g(t, y), & t \neq \tau_i(x, y) & \quad \Delta x = A_t(x) + B_t(y), & t = \tau_i(x, y) \\ \dot{y} &= h(t, x, y), & t \neq \tau_i(x, y) & \quad \Delta y = C_t(x, y), & t = \tau_i(x, y) \end{aligned} \quad (2.1)$$

Where

$$\begin{aligned} x \in R^n, y \in R^m, f: R^+ \times R_H^n \rightarrow R^n, g: R^+ \times R_H^m \rightarrow R^m, h: R^+ \times R_H^n \times R_H^m \rightarrow R^m \\ A_t: R_H^n \rightarrow R^n, B_t: R_H^m \rightarrow R^m, C_t: R_H^n \times R_H^m \rightarrow R^m, \tau_i: R_H^n \times R_H^m \rightarrow R^1 \end{aligned}$$

$$\Delta x \Big|_{t=\tau(x, y)} = x(t+0) - x(t-0), \quad \Delta y \Big|_{t=\tau(x, y)} = y(t+0) - y(t-0).$$

Let $t_0 \in R^+$, $x_0 \in R_H^n$, $y_0 \in R_H^m$ Let $x(t, t_0, x_0, y_0)$, $y(t, t_0, x_0, y_0)$ be the solution of the system (2.1), satisfying the initial conditions $x(t_0 + 0, t_0, x_0, y_0) = x_0$, $y(t_0 + 0, t_0, x_0, y_0) = y_0$ The solution $(x(t), y(t))$ of the system (2.1) are piecewise continuous functions with points of discontinuities of the first type in which they are left continuous, i.e. at the moment t_i when the integral curve of the solution $(x(t), y(t))$ meets the hypersurface.

$$\sigma_t = \{(t, x, y) \in R^+ \times R_H^n \times R_H^m : t = \tau_i(x, y)\}$$

The following relations are satisfied:

$$x(t_i - 0) = x(t_i), \quad x(t_i + 0) = x(t_i) + A_i(x(t_i)) + B_i(y(t_i))$$

$$y(t_i - 0) = y(t_i), \quad y(t_i + 0) = y(t_i) + C_i(x(t_i), y(t_i))$$

Let $\tau_0(x, y) = 0$ for $(x, y) \in R_H^n \times R_H^m$.

Following [1], we define the sets $G_i = \{(t, x, y) \in R^+ \times R_H^n \times R_H^m : \tau_{i-1}(x, y) < t < \tau_i(x, y)\}$.

Let $\Psi : R^+(0, \infty)$ be a continuous finite function such that $\Psi(t_0) = \Psi_0$

Definition1. Let the sets K & K_1 be defined as

$$K = \{w \in C(R^+, R^+) : \text{strictly increasing and } w(0) = 0\},$$

$$K_1 = \{\phi \in C(R^+, R^+) : \text{increasing and } \phi(s) < s \text{ for } s > 0\}$$

We use the classes v_0 of piecewise continuous functions which are analogue to Lyapunov functions.

Definition2. We say that the function $V : R^+ \times R_H^n \times R_H^m \rightarrow R^+$ belongs to the class v_0 if the following conditions hold:

1. The function V is continuous in $\cup_{i=1}^{\infty} G_i$ and is locally Lipschitzian with respect to x and y in each of the sets G_i
2. $V(t, 0, 0) = 0$ for $t \in R^+$.
3. For each $i = 1, 2, 3, \dots$ and for any point $(t_0, x_0, y_0) \in \sigma_i$, there exist the finite limits

$$V(t_0 - 0, x_0, y_0) = \lim_{\substack{(t, x, y) \rightarrow (t_0, x_0, y_0) \\ (t, x, y) \in G_i}} V(t, x, y)$$

$$V(t_0 + 0, x_0, y_0) = \lim_{\substack{(t, x, y) \rightarrow (t_0, x_0, y_0) \\ (t, x, y) \in G_{i+1}}} V(t, x, y)$$

and the equality $V(t_0 - 0, x_0, y_0) = V(t_0, x_0, y_0)$ holds.

4. For any point $(t, x, y) \in \sigma_i$ the following inequality holds:

$$V(t + 0, x + A_t(x) + B_t(y), y + C_t(x, y)) \leq V(t, x, y). \quad (2.2)$$

Let $V \in v_0$. For $(t, x, y) \in \cup_{i=1}^{\infty} G_i$ Following [1], we define

$$V'_{(2.1)}(t, x, y) = \limsup_{s \rightarrow 0} \frac{1}{s} [V(t + s, x + s(f(t, x) + g(t, y)), y + s(h(t, x, y))) - V(t, x, y)]$$

Definition3 The set $\{(x, y) \in R_H^n \times R_H^m : x = 0 \text{ and } y = 0\}$ of the system (2.1) is said to be

1. Eventually stable if for all $\varepsilon > 0$ for all $t_0 \in R^+$ there exists $\tau = \tau(\varepsilon) > 0$ and $\delta = \delta(t_0, \varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|x_0\| + \|y_0\| < \delta$ implies $\|x(t, t_0, x_0, y_0)\| + \|y(t, t_0, x_0, y_0)\| < \varepsilon$, $t \geq t_0 \geq \tau(\varepsilon)$
2. Uniformly eventually stable if $\delta = \delta(\varepsilon)$ i.e. δ is independent of t_0 .
3. Ψ - eventually stable if for all $\varepsilon > 0$ for all $t_0 \in R^+$, there exists $\tau = \tau(\varepsilon) > 0$ and $\delta = \delta(t_0, \varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon$ for $\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta$ and $t \geq t_0 \geq \tau(\varepsilon)$.
4. Ψ - uniformly eventually stable if for all $\varepsilon > 0$, for all $t_0 \in R^+$, there exists $\tau = \tau(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon$ for $\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta$ and $t \geq t_0 \geq \tau(\varepsilon)$.

Definition 4. We say that conditions (A) hold if the following are satisfied:

(A1) The functions $f(t, x)$, $g(t, y)$ and $h(t, x, y)$ are continuous in their definition domains, $f(t, 0) = g(t, 0) = 0$ and $h(t, 0, 0) = 0$ for $t \in R^+$.

(A2) The functions A_t , B_t and C_t are continuous in their definition domains and $A_t(0) = B_t(0) = C_t(0, 0) = 0$

(A3) $x \in R_H^n$ and $y \in R_H^m$ then $\|xy + A_t(x) + B_t(y)\| \leq \|x\|$ and $\|y + C_t(x, y)\| \leq \|y\|$,

(A4) The functions $\tau_i(x, y)$ are continuous and for $(x, y) \in R_H^n \times R_H^m$, the following relations hold:

$$0 < \tau_1(x, y) < \tau_2(x, y) < \dots < \lim_{t \rightarrow \infty} \tau_i(x, y) = \infty \text{ uniformly in } R_H^n \times R_H^m \text{ and}$$

$$\inf_{R_H^n \times R_H^m} \tau_{i+1}(x, y) - \sup_{R_H^n \times R_H^m} \tau_i(x, y) \geq \theta > 0, \quad i=1, 2, \dots$$

(A5) For each point $(t_0, x_0, y_0) \in R^+ \times R_H^n \times R_H^m$ the solution $x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)$ is unique and defined in (t_0, ∞) .

(A6) The integral curve of each of the solutions of system (2.1) meets each of the hyper surfaces $\{\sigma_i\}$ at most once.

III. MAIN RESULTS

In this section, we extend the work of Kulev and Bainov [1] and A. A. Soliman [5] and established Ψ - eventually stability and Ψ - uniformly eventually stability for impulsive differential system with variable moments.

Theorem 1. Assume that

(H1) Conditions (A) holds.

(H2) There exists functions $V \in v_0, a \in K$ such that $a(\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\|) \leq V(t, x, y)$, $(t, x, y) \in R^+ \times R_H^n \times R_H^m$ where Ψ is a function defined in section 2.

(H3) $V_{(2.1)}(t, x, y) \leq p(t)w(v(t, x, y))$ for $(t, x, y) \in G_i, i=1, 2, 3, \dots$, the functions $p, w: R^+ \rightarrow R^+$ are locally integrable.

(H4) There exists a no. $L > 0$ such that $|w((t, x, y))| \leq L : (t, x, y) \in R^+ \times R_H^n \times R_H^m$ and $\int_{t_0}^{\infty} |p(s)| ds < \infty$. Then the set $\{(x, y) \in R_H^n \times R_H^m : x=0 \text{ and } y=0\}$ is Ψ - eventually stable set of the system (2.1).

Proof: Let $\varepsilon > 0$ be given and let the number $\tau = \tau(\varepsilon) > 0$ be chosen so that for $t \geq \tau(\varepsilon)$

$$\int_t^{\infty} |p(s)| ds < \frac{a(\varepsilon)}{2L} \quad (\text{This is possible because of condition (H4)}) \quad (3.1)$$

Let $t_0 \geq \tau(\varepsilon)$ From property 4 of definition 2, it follows that there exists a number $\delta(t_0, \varepsilon) > 0$ such that for

$$\|\Psi_0 x_0 + \Psi_0 y_0\| < \delta(t_0, \varepsilon), \quad V(t_0 + 0, x_0, y_0) \leq \frac{a(\varepsilon)}{2} \quad (3.2)$$

From (H3), (H4) and (3.2), we get

$$\begin{aligned} \int_{t_0}^t V_{(2.1)}(s, x(s), y(s)) ds &\leq \int_{t_0}^t p(s) w(v(s, x(s), y(s))) ds \leq \int_{t_0}^t |p(s)| |w(v(s, x(s), y(s)))| ds \\ &\leq L \int_{t_0}^t |p(s)| ds \leq L \int_{t_0}^{\infty} |p(s)| ds < L \frac{a(\varepsilon)}{2L} = \frac{a(\varepsilon)}{2} \text{ for } t \geq t_0 \end{aligned} \quad (3.3)$$

Without loss of generality, Let $\tau_{k+l} < t < \tau_{k+l+1}$

$$\begin{aligned} \text{Now } \int_{t_0}^t V_{(2.1)}(s, x(s), y(s)) ds &\leq \int_{t_0}^{\tau_1} V_{(2.1)}(s, x(s), y(s)) ds + \sum_{j=2}^{k+l} \int_{\tau_{j-1}}^{\tau_j} V_{(2.1)}(s, x(s), y(s)) ds + \int_{\tau_{k+l}}^t V_{(2.1)}(s, x(s), y(s)) ds \\ &= V(\tau_1, x(\tau_1), y(\tau_1)) - V(t_0 + 0, x(t_0), y(t_0)) + \sum_{j=2}^{k+l} \{V(\tau_j, x(\tau_j), y(\tau_j)) - V(\tau_{j-1} + 0, x(\tau_{j-1} + 0), y(\tau_{j-1} + 0))\} \end{aligned}$$

$$\begin{aligned}
 &+V(t, x(t), y(t))-V(\tau_{k+l}+0, x(\tau_{k+l}+0), y(\tau_{k+l}+0)) \\
 &=V(\tau_1, x(\tau_1), y(\tau_1))-V(t_0+0, x(t_0), y(t_0))+V(\tau_2, x(\tau_2), y(\tau_2))-V(\tau_1+0, x(\tau_1+0), y(\tau_1+0)) \\
 &+V(\tau_3, x(\tau_3), y(\tau_3))-V(\tau_2+0, x(\tau_2+0), y(\tau_2+0))+..... \\
 &+V(\tau_{k+l}, x(\tau_{k+l}), y(\tau_{k+l}))-V(\tau_{k+l-1}+0, x(\tau_{k+l-1}+0), y(\tau_{k+l-1}+0)) \\
 &+V(t, x(t), y(t))-V(\tau_{k+l}+0, x(\tau_{k+l}+0), y(\tau_{k+l}+0)) \\
 &\geq V(t, x(t), y(t))-V(t_0+0, x(t_0), y(t_0))
 \end{aligned}$$

From (H2), (3.1), (3.2) and (3.3)

$$\begin{aligned}
 a(\|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|) &\leq V(t, x, y) \leq V(t_0+0, x(t_0), y(t_0)) + \int_{t_0}^t V_{(2.1)}'(s, x(s), y(s)) ds < \\
 \frac{a(\varepsilon)}{2} + \frac{a(\varepsilon)}{2} &= a(\varepsilon) \text{ for } t \geq t_0 \geq \tau(\varepsilon)
 \end{aligned}$$

Thus for all $\varepsilon > 0$, for all $t_0 \in \mathbb{R}^+$, there exists $\tau = \tau(\varepsilon) > 0$ and $\delta = \delta(t_0, \varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon$ for $\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta$ and $t \geq t_0 \geq \tau(\varepsilon)$

Hence the set $\{(x, y) \in R_H^n \times R_H^m : x = 0 \text{ and } y = 0\}$ is Ψ - eventually stable set of the system (2.1).

Theorem 2. Assume that (H1) and (H3) of Theorem 1 holds. Moreover suppose that (H5) Let There exists functions $V \in \nu_0$, $a, b \in K$, $\phi \in K_1$ such that

$$a(\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\|) \leq V(t, x, y) \leq b(\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\|) : (t, x, y) \in \mathbb{R}^+ \times R_H^n \times R_H^m$$

(H6) For all $k \in \mathbb{N}$, $(x, y) \in R_H^n \times R_H^m$, $V(\tau_k, x(\tau_k^-) + A_t(x) + B_t(y), y(\tau_k^-) + C_t(x, y)) \leq \phi(V(\tau_k^- + 0, x(\tau_k^-), y(\tau_k^-)))$

(H7) There exists a constant $A > 0$ such that $\int_{\tau_{k-1}}^{\tau_k} |p(s)| ds < A$ and $\int_{\mu}^{\phi^{-1}(\mu)} \frac{ds}{w(s)} \geq A$

Then the set $\{(x, y) \in R_H^n \times R_H^m : x = 0 \text{ and } y = 0\}$ is Ψ - uniformly eventually stable set of the system (2.1).

Proof: Let $\varepsilon > 0$ and choose $\delta(\varepsilon) > 0$, $\tau = \tau(\varepsilon) > 0$ such that $\delta < b^{-1}(\phi(a(\varepsilon)))$ for $t_0 \geq \tau(\varepsilon)$.

In the following, we prove that for all $(x_0, y_0) \in R_H^n \times R_H^m$,

$$\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta \text{ implies } \|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon \text{ for } t \geq t_0 \geq \tau(\varepsilon).$$

Let $t_0 \in (\tau_{m-1}, \tau_m)$ i.e. G_m for some $m \in \mathbb{N}$

We first prove that $V(t, x, y) \leq \phi^{-1}(b(\delta))$ for $t_0 \leq t < \tau_m$ (3.4)

Clearly $V(t_0, x_0, y_0) \leq b(\|\Psi_0 x_0\| + \|\Psi_0 y_0\|) < b(\delta) < \phi^{-1}(b(\delta))$

Now for $t \in (t_0, \tau_m)$ if (3.1) does not hold, then there exists $\hat{t} \in (t_0, \tau_m)$ such that

$$V(\hat{t}, x(\hat{t}), y(\hat{t})) > \phi^{-1}(b(\delta)) > b(\delta) \geq V(t_0, x_0, y_0)$$

From the continuity of $V(t, x, y)$ in (τ_{m-1}, τ_m) and hence in (t_0, \hat{t}) there is an $s_1 \in (t_0, \hat{t})$ such that

$$V(s_1, x(s_1), y(s_1)) = \phi^{-1}(b(\delta)) \tag{3.5}$$

$$\begin{aligned}
 V(t, x(t), y(t)) &> \phi^{-1}(b(\delta)) : s_1 < t < \hat{t} \\
 V(t, x(t), y(t)) &\leq \phi^{-1}(b(\delta)) : t_0 \leq t \leq s_1
 \end{aligned}$$

and also there exists an $s_2 \in (t_0, s_1)$ such that

$$V(s_2, x(s_2), y(s_2)) = b(\delta) \tag{3.6}$$

$$V(t, x(t), y(t)) \geq b(\delta) : s_2 \leq t \leq s_1$$

Therefore we integrate (H3) between $[s_2, s_1]$

$$\begin{aligned} \int_{s_2}^{s_1} \frac{V'_{(2.1)}(t, x(t), y(t))}{w(V(t, x(t), y(t)))} dt &\leq \int_{s_2}^{s_1} p(t) dt \\ \frac{V(s_1, x(s_1), y(s_1))}{V(s_2, x(s_2), y(s_2))} \int_{s_2}^{s_1} \frac{du}{w(u)} &\leq \int_{s_2}^{s_1} p(t) dt \leq \int_{\tau_{m-1}}^{\tau_m} p(t) dt < A \end{aligned} \quad (3.7)$$

On the other hand from the inequalities (3.5), (3.6) and condition (H7)

$$\frac{V(s_1, x(s_1), y(s_1))}{V(s_2, x(s_2), y(s_2))} \int_{s_2}^{s_1} \frac{du}{w(u)} = \frac{\phi^{-1}(b(\delta))}{b(\delta)} \int_{b(\delta)}^{\phi^{-1}(b(\delta))} \frac{du}{w(u)} \geq A$$

which contradicts the inequality (3.7). Therefore our assumption was wrong and hence (3.4) holds.

From condition (H6)

$$\begin{aligned} V(\tau_m, x(\tau_m), y(\tau_m)) &= V(\tau_m, x(\tau_m^-) + A_t(x) + B_t(y), y(\tau_m^-) + C_t(x, y)) \leq \phi(V(\tau_m^-, x(\tau_m^-), y(\tau_m^-))) \\ &\leq \phi(\phi^{-1}(b(\delta))) < b(\delta) \end{aligned} \quad (3.8)$$

Next we prove that $V(t, x, y) \leq \phi^{-1}(b(\delta))$ for $\tau_m < t < \tau_{m+1}$

If inequality (3.9) does not hold good, there exists $\hat{r} \in (\tau_m, \tau_{m+1})$ such that

$$V(\hat{r}, x(\hat{r}), y(\hat{r})) > \phi^{-1}(b(\delta)) > b(\delta) \geq V(\tau_m, x(\tau_m), y(\tau_m)) \text{ [using (3.8)]}$$

From the continuity of $V(t, x, y)$ in (τ_m, τ_{m+1}) there is an $r_1 \in (\tau_m, \hat{r})$ such that

$$\begin{aligned} V(r_2, x(r_2), y(r_2)) &= \phi^{-1}(b(\delta)) \\ V(t, x(t), y(t)) &> \phi^{-1}(b(\delta)) : r_1 < t < \hat{r} \\ V(t, x(t), y(t)) &\leq \phi^{-1}(b(\delta)) : \tau_m \leq t \leq r_1 \end{aligned} \quad (3.10)$$

and also there exists an $r_2 \in (\tau_m, r_1)$ such that

$$\begin{aligned} V(r_2, x(r_2), y(r_2)) &= b(\delta) \\ V(t, x(t), y(t)) &\geq b(\delta) : r_2 \leq t \leq r_1 \end{aligned} \quad (3.11)$$

Then we integrate the inequality (H3) in $[r_2, r_1]$ as done before and get contradiction. Hence (3.9) holds.

Thus we see that $V(t, x, y) \leq \phi^{-1}(b(\delta))$ in $G_i : 1, 2, 3, \dots$ and hence in $\cup G_i$ (3.12)

Also as shown in (3.8), $V(\tau_m, x(\tau_m), y(\tau_m)) < b(\delta) : m = 1, 2, 3, \dots$ (3.13)

As $b(\delta) < \phi^{-1}(b(\delta))$, it follows by (3.4), (3.12) and (3.13) that

$$V(t, x(t), y(t)) < \phi^{-1}(b(\delta)) < a(\varepsilon)$$

therefore by condition (H5),

$$a(\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\|) \leq V(t, x, y) < a(\varepsilon) \text{ for all } t \geq t_0 \geq \tau(\varepsilon).$$

Thus we see that $\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon$ whenever $\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta$ for $t \geq t_0 \geq \tau(\varepsilon)$.

Hence the set $\{(x, y) \in R_H^n \times R_H^m : x = 0 \text{ and } y = 0\}$ is Ψ - uniformly eventually stable set of the system (2.1).

Example

Consider the system

$$\begin{aligned} \dot{x} &= A(t)x(t-r(t)) + B(t)y(t) & t \neq \tau_i & \quad x(\tau_k^+) = \alpha' x(\tau_k) + \beta' y(\tau_k) \\ \dot{y} &= D(t)y(t), & t \neq \tau_i & \quad y(\tau_k^+) = \gamma' y(\tau_k) \end{aligned} \quad (*)$$

Where $0 < r(t) < r'$, $x \in R$, $y \in R$, $A(t)$, $B(t)$, $D(t) \in C(R^+, R)$ and such that $|A(t)| < \alpha$, $|B(t)| < \beta$, $|D(t)| < \gamma$

Let us further assume that $x(s) \leq x(t)$ for $t - r' \leq s \leq t$.

Let (i) $\alpha' > 0, \beta' > 0, \gamma' > 0$ and $\alpha + T > \gamma$ (ii) $3\alpha'^2 + \beta'^2 < 2$ (iii) $\beta'^2 + \gamma'^2 < \alpha'^2$

$$(iv) \quad \tau_k - \tau_{k-1} < -\frac{\log(3\alpha'^2 + \beta'^2) + \log 2}{(2\alpha + \beta)}$$

Let us further define the following functions

$$\phi(s) = \frac{(3\alpha'^2 + \beta'^2)}{2} s, \quad w(s) = s, \quad p(t) = [2\alpha + \beta], \quad \Psi(t) = \max_{t \geq t_0} e^{-t}, \quad a, b \in K \quad \text{such that}$$

$$a(t) = \frac{t^2}{4e^{-t_0}}, \quad b(t) = \frac{t^2}{2e^{-t_0}} \quad \text{and} \quad V(t, x, y) = \frac{1}{2}(x^2 + y^2)e^{-t_0}$$

Therefore

$$\begin{aligned} V'(t, x, y) &= e^{-t_0} (xx' + yy') = e^{-t_0} \{A(t)x(t-r(t)) + B(t)y(t) + y(t)[D(t)y(t)]\} \\ &\leq e^{-t_0} \{x(t)[\alpha x(t) + \beta y(t)] + y(t)[\gamma y(t)]\} = e^{-t_0} [\alpha x^2(t) + \beta x(t)y(t) + \gamma y^2(t)] \\ &\leq e^{-t_0} [\alpha x^2(t) + \beta x(t)y(t) + \alpha y^2(t)] \leq e^{-t_0} \{\alpha[x^2(t) + y^2(t)] + \frac{\beta}{2}[x^2(t) + y^2(t)]\} \\ &= \frac{e^{-t_0}(2\alpha + \beta)}{2} [x^2(t) + y^2(t)] = p(t) \frac{[x^2(t) + y^2(t)]}{2} = p(t)w(V(t, x, y)) \end{aligned}$$

Thus (H3) holds. Let

$$\Psi(t) = \max_{t \geq t_0} e^{-t} = e^{-t_0}$$

$$\begin{aligned} a(|\Psi(t)x(t)| + |\Psi(t)y(t)|) &= a(e^{-t_0}|x(t)| + e^{-t_0}|y(t)|) = a(e^{-t_0}(|x(t)| + |y(t)|)) \\ &= \frac{e^{-2t_0}(|x(t)| + |y(t)|)^2}{4e^{-t_0}} = \frac{e^{-t_0}(|x(t)|^2 + |y(t)|^2 + 2|x(t)||y(t)|)}{4} \leq \\ &= \frac{e^{-t_0}(|x(t)|^2 + |y(t)|^2 + |x(t)|^2 + |y(t)|^2)}{4} = \frac{e^{-t_0}(|x(t)|^2 + |y(t)|^2)}{2} = V(t, x, y) \end{aligned}$$

$$\text{Also } b(|\Psi(t)x(t)| + |\Psi(t)y(t)|) = b(e^{-t_0}|x(t)| + e^{-t_0}|y(t)|) = b(e^{-t_0}(|x(t)| + |y(t)|))$$

$$\frac{e^{-2t_0}(|x(t)| + |y(t)|)^2}{2e^{-t_0}} \geq \frac{e^{-t_0}(|x(t)|^2 + |y(t)|^2)}{2e^{-t_0}} = V(t, x, y)$$

Thus (H5) holds. Now

$$\begin{aligned} V(\tau_k^+, x(\tau_k) + A_t(x) + B_t(y), y(\tau_k) + C_t(x, y)) &= V(\tau_k^+, x(\tau_k^+), y(\tau_k^+)) \\ &= V(\tau_k, \alpha' x(\tau_k) + \beta' y(\tau_k), \gamma' y(\tau_k)) = \frac{e^{-t_0}}{2} [(\alpha' x(\tau_k) + \beta' y(\tau_k))^2 + \gamma'^2 y^2(\tau_k)] \end{aligned}$$

$$\begin{aligned}
 &= \frac{e^{-t_0}}{2} [\alpha'^2 x^2(\tau_k) + \beta'^2 y^2(\tau_k) + 2\alpha' \beta' x(\tau_k)y(\tau_k) + \gamma'^2 y^2(\tau_k)] \\
 &\leq \frac{e^{-t_0}}{2} \{ \alpha'^2 x^2(\tau_k) + \alpha'^2 y^2(\tau_k) + \alpha' \beta' [x^2(\tau_k) + y^2(\tau_k)] \} \\
 &= \frac{e^{-t_0}}{2} (\alpha'^2 + \alpha' \beta') [x^2(\tau_k) + y^2(\tau_k)] \leq \frac{e^{-t_0}}{2} \left(\frac{3\alpha'^2 + \beta'^2}{2} \right) [x^2(\tau_k) + y^2(\tau_k)] \\
 &= \phi \left(\frac{e^{-t_0}}{2} [x^2(\tau_k) + y^2(\tau_k)] \right) = \phi(V(\tau_k, x(\tau_k), y(\tau_k)))
 \end{aligned}$$

Thus (H6) holds.

Again, if we choose $A = -\log \frac{3\alpha'^2 + \beta'^2}{2} > 0$

$$\begin{aligned}
 \int_{\tau_{k-1}}^{\tau_k} |p(s)| ds &= \int_{\tau_{k-1}}^{\tau_k} (2\alpha + \beta) ds = (2\alpha + \beta)(\tau_k - \tau_{k-1}) \\
 &< [-\log(3\alpha'^2 + \beta'^2) + \log 2] = -\log \frac{3\alpha'^2 + \beta'^2}{2} = A
 \end{aligned}$$

Also for any $\mu > 0$,

$$\begin{aligned}
 \phi^{-1}(\mu) \int_{\mu}^{\frac{2\mu}{(3\alpha'^2 + \beta'^2)}} \frac{ds}{w(s)} &= \int_{\mu}^{\frac{2\mu}{(3\alpha'^2 + \beta'^2)}} \frac{ds}{s} = [\log s]_{\mu}^{\frac{2\mu}{(3\alpha'^2 + \beta'^2)}} = \log \frac{2\mu}{(3\alpha'^2 + \beta'^2)} - \log \mu \\
 &= \log \frac{2}{(3\alpha'^2 + \beta'^2)} = -\log \frac{(3\alpha'^2 + \beta'^2)}{2} = A
 \end{aligned}$$

Thus all the conditions of Theorem 2 hold and hence the set $\{(x, y) \in R_H^n \times R_H^m : x=0 \text{ and } y=0\}$ is Ψ - uniformly eventually stable set of the system (*).

IV. CONCLUSION

In [5], no example has been given in support of the results and in example given in [6], zero solution being equilibrium, implies Lyapunov stability and thus difference between Lyapunov stability and Eventual stability has not been shown. But in the example given above, the zero solution is not stable in the sense of Lyapunov as it is not equilibrium but it is uniformly eventually stable. Moreover a weight function Ψ is also associated with state vectors.

Thus our result shows that the system may not be stable in the sense of Lyapunov even then it can be eventually stable.

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