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On Some Properties of the Rising Sun Function

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ONSOMEPROPERTIESOFTHERISINGSUNFUNCTION

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On Some Properties of the Rising Sun Function

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I. Introduction

The rising sun function was used as a tool in the proof of the famous Lebesgue's theorem on the differentiability of a real valued monotone function without using the theory of integration [3]. In this paper, some properties of the rising sun function are presented and an operator on the space of all bounded real functions defined on a closed and bounded interval $[a,b]$ is introduced and its properties are investigated.

In what follows X, $\mathbb{R}$ and $\mathbb{N}$ stand for a topological space, the real line and the set of all positive integers respectively. Also $\mathscr{B}$ stands for the Banach space of all bounded real functions defined on a closed and bounded interval $[a,b]$ where $a,b \in \mathbb{R}$ and $a < b$ under the supremum norm.

II. Preliminaries

1.1 Definition [6]: The rising sun function of a function $f \in \mathscr{B}$ is defined by

$$f_{\odot}(x) = \sup\{f(y) / x \le y \le b\}.$$

1.2 Definition: For $f \in \mathscr{B}$ we define the following.

$$(i)\quad {}_{\odot}f(x) = \sup\{f(y) / a \le y \le x\}$$

$$(ii)\quad f^{\odot}(x) = \inf\{f(y) / x \le y \le b\}$$

$$(iii)\quad {}^{\odot}f(x) = \inf\{f(y) / a \le y \le x\}$$

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1.3 Definition [5]: A function $f: X \to \mathbb{R}$ is said to be semi-continuous at a point $p \in X$ if for every $\varepsilon > 0$ and every neighborhood U of p in X there exists a non-empty open set $W \subset U$ such that $|f(x) - f(p)| < \varepsilon \;\; \forall \; x \in W$. We say that a function f is semi-continuous on X if it is semi-continuous at every point of X.

1.4 Definition [6]: A function $f: X \to \mathbb{R}$ is said to be lower semicontinuous (lsc) at a point $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood U of x such that

$$f(y) - f(x) > -\varepsilon \;\; \forall \;\; y \in U.$$

We say that a function f is *lsc* on X if it is *lsc* at every point of X.

1.5 Definition [6]: A function $f: X \to \mathbb{R}$ is said to be upper semicontinuous (usc) at a point $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood G of x such that

$$f(y) - f(x) < \varepsilon \;\; \forall \;\; y \in G.$$

We say that a function f is *usc* on X if it is *usc* at every point of X.

1.6 Definition [4]: A function $f: X \to \mathbb{R}$ is said to be lower semi-quasicontinuous $(lsqc)$ at a point $x \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $f(y) - f(x) > -\varepsilon \;\; \forall \;\; y \in W$.

We say that a function f is *lsqc* on X if it is *lsqc* at every point of X.

1.7 Definition [4]: A function $f: X \to \mathbb{R}$ is said to be upper semi-quasicontinuous $(usqc)$ at a point $x \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $f(y) - f(x) < \varepsilon \;\; \forall \;\; y \in W$.

We say that a function f is *usqc* on X if it is *usqc* at every point of X.

1.8 Definition [5]: A function $f: X \to \mathbb{R}$ is said to be cliquish at a point $x \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $|f(y) - f(z)| < \varepsilon \;\; \forall \;\; y, z \in W$

We say that a function f is cliquish on X if it is cliquish at every point of X.

1.9 Definition: Let $f: [a,b] \to \mathbb{R}$. We define $f(a-) = f(a)$ and $f(b+) = f(b)$.

We say that $f(p+)$ exists at $p \in [a,b)$ and we write $f(p+) = L$, where $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon \;\; \forall \;\; x \in (p, p+\delta) \subset [a,b]$. Similarly for $p \in (a,b]$ we write

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$f(p-)=l\in\mathbb{R}$ if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x)-l|<\varepsilon \ \ \forall\, x\in(p-\delta,p)\subset[a,b].$$

1.10 Definition: A function $f:[a,b]\to\mathbb{R}$ is said to be quasicontinuous at a point $p\in[a,b]$ if $f(p+)$ and $f(p-)$ exist.

1.11 Definition: A function $f:[a,b]\to\mathbb{R}$ is said to be symmetrically continuous at a point $x\in[a,b]$ if $\lim_{h\to 0}\left[f(x+h)-f(x-h)\right]=0$.

1.12 Definition: A function $f:[a,b]\to\mathbb{R}$ is Darboux continuous if for all $p,q\in[a,b]$ and for each c between $f(p)$ and $f(q)$ there is an x between p and q such that $f(x)=c$.

1.13 Definition[2]: An operator P on a linear space L is said to be sublinear if (i) $P(x+y)\le P(x)+p(y)$ $\forall$ $x,y\in$ L and (ii) $P(\lambda x)=\lambda P(x)$ for any positive real number λ and every $x\in$ L.

III. Relations among $f_{\odot}, {}_{\odot}f, f^{\odot}$ and ${}^{\odot}f$

In this section the relations between the rising sun function and its analogues that are introduced are presented in the following propositions.

2.1 Proposition: For $f\in\mathcal{B}$, (a) $f^{\odot}=-(-f)_{\odot}$ and (b) ${}^{\odot}f=-{}_{\odot}(-f)$.

2.2 Proposition: For $f\in\mathcal{B}$, (a) $(f_{\odot})_{\odot}=f_{\odot}$ $\quad(b)$ ${}_{\odot}({}_{\odot}f)={}_{\odot}f$

$$(c)\ \left(f^{\odot}\right)^{\odot}=f^{\odot} \qquad (d)\ {}^{\odot}\left({}^{\odot}f\right)={}^{\odot}f$$

2.3 Proposition: For $f\in\mathcal{B}$ and $x\in[a,b]$,

(i) ${}_{\odot}\left(f_{\odot}\right)(x)=f_{\odot}(a)$ $\quad(ii)$ $\left(f_{\odot}\right)^{\odot}(x)=f_{\odot}(b)$ $\quad(iii)$ ${}^{\odot}\left(f_{\odot}\right)(x)=f_{\odot}(x)$

(iv) $\left({}_{\odot}f\right)_{\odot}(x)={}_{\odot}f(b)$ $\quad(v)$ $\left({}_{\odot}f\right)^{\odot}(x)={}_{\odot}f(x)$ $\quad(vi)$ ${}^{\odot}\left({}_{\odot}f\right)(x)={}_{\odot}f(a)$

(vii) $\left(f^{\odot}\right)_{\odot}(x)=f^{\odot}(b)$ $\quad(viii)$ ${}_{\odot}\left(f^{\odot}\right)(x)=f^{\odot}(x)$ $\quad(ix)$ ${}^{\odot}\left(f^{\odot}\right)(x)=f^{\odot}(a)$

(x) $\left({}^{\odot}f\right)_{\odot}(x)={}^{\odot}f(x)$ $\quad(xi)$ ${}_{\odot}\left({}^{\odot}f\right)(x)={}^{\odot}f(a)$ $\quad(xii)$ $\left({}^{\odot}f\right)^{\odot}(x)={}^{\odot}f(b)$

2.4 Remark: In view of the previous propositions it is enough to investigate the properties of the rising sun function and the properties of ${}_{\odot}f$, $f^{\odot}$ and ${}^{\odot}f$ follow analogously.

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IV. Characterisations of $f_{\odot}, {}_{\odot}f, f^{\odot}$ and ${}^{\odot}f$

3.1 proposition: For $f \in \mathscr{B}$, $f_{\odot}$ is the smallest decreasing function dominating f

More precisely,

(a) $f_{\odot}(x) \geq f(x)\ \ \forall\ x \in [a,b]$

(b) $f_{\odot}$ is decreasing on $[a,b]$

(c) If g satisfies (a) and (b) above, then $f_{\odot}(x) \leq g(x)\ \ \forall\ x \in [a,b]$.

3.2 Proposition: For $f \in \mathscr{B}$, ${}_{\odot}f$ is the smallest increasing function dominating f. More precisely, (a) ${}_{\odot}f(x) \geq f(x)\ \ \forall\ x \in [a,b]$

(b) ${}_{\odot}f$ is increasing on $[a,b]$

(c) If g satisfies (a) and (b) above, then ${}_{\odot}f(x) \leq g(x)\ \ \forall\ x \in [a,b]$.

3.3 Proposition: For $f \in \mathscr{B}$,

(a) $f^{\odot}$ is the largest increasing function such that $f^{\odot}(x) \leq f(x)\ \ \forall\ x \in [a,b]$

(b) ${}^{\odot}f$ is the largest decreasing function such that ${}^{\odot}f(x) \leq f(x)\ \ \forall\ x \in [a,b]$

V. The Rising Sun Operator

4.1 Definition: Define $T : \mathscr{B} \to \mathscr{B}$ by $Tf = f_{\odot}$. We call this operator T, the rising sun operator on $\mathscr{B}$.

4.2 Proposition: The rising sun operator T is sublinear on $\mathscr{B}$. More precisely,

(a) $T(f+g) \leq Tf + Tg\ \ \forall\ \ f,g \in \mathscr{B}$

(b) $T(\lambda f) = \lambda Tf$ for every real number $\lambda > 0$ and every $f \in \mathscr{B}$.

Proof: Let $f,g \in \mathscr{B}$ and $x \in [a,b]$.

(a) For $y \in [x,b]$, $(f+g)(y) = f(y) + g(y) \leq f_{\odot}(x) + g_{\odot}(x) = (f_{\odot} + g_{\odot})(x)$

$$\Rightarrow (f+g)(y) \leq (f_{\odot} + g_{\odot})(x)\ \ \forall\ \ y \in [x,b]$$
$$\Rightarrow (f+g)_{\odot}(x) \leq (f_{\odot} + g_{\odot})(x)\ \ \forall\ \ x \in [a,b]$$

Hence $T(f+g) \leq Tf + Tg\ \ \forall\ \ f,g \in \mathscr{B}$.

(b) Suppose that λ is a positive real number and $f \in \mathscr{B}$.

Then
$$(\lambda f)_{\odot}(x) = \sup\{(\lambda f)(y) / x \leq y \leq b\}$$
$$= \lambda \sup\{f(y) / x \leq y \leq b\}$$
$$= \lambda\, f_{\odot}(x)$$
$$T\ \lambda f\ = \lambda\, Tf.$$

Notes

4.3 Remark: From the following example, it is clear that $T(f+g) \neq Tf + Tg$.

4.4 Example: Define $f:[0,1] \to \mathbb{R}$ and $g:[0,1] \to \mathbb{R}$ by $f(x) = x$ and

$$g(x) = 1 - x \ \forall \ x \in [0,1] \quad \Rightarrow (f+g)(x) = 1. \forall \ x \in [0,1]$$

Then $(f+g)_{\odot}(x) = 1$ and $(f_{\odot} + g_{\odot})(x) = 2 - x \ \forall \ x \in [0,1]$.

Hence $T(f+g) \neq Tf + Tg$.

4.5 Proposition: For $f \in \mathcal{B}$ and $k \in \mathbb{R}$, $T(f+k) = Tf + k$.

4.6 Proposition: For $f, g \in \mathcal{B}$, (a) $f \le g \Rightarrow Tf \le Tg$ (b) $T(f \vee g) = Tf \vee tg$

where $(f \vee g)(x) = \max\{f(x), g(x)\}$.

4.7 Proposition: If $\{f_\alpha / \alpha \in \Delta\}$ is a collection of functions in $\mathcal{B}$ and if

$\bigvee_{\alpha \in \Delta} f_\alpha = \sup\{f_\alpha / \alpha \in \Delta\}$ exists in $\mathcal{B}$ then $T\left(\bigvee_{\alpha \in \Delta} f_\alpha\right) = \bigvee_{\alpha \in \Delta} T(f_\alpha)$.

4.8 Remark: From the following example it can be observed that

$T(f \wedge g) \neq Tf \wedge Tg$, where $(f \wedge g)(x) = \min\{f(x), g(x)\}$.

4.9 Example: Define $f:[0,1] \to \mathbb{R}$ and $g:[0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 - 2x & if \ \ 0 \le x \le \frac{1}{2} \\ 2x - 1 & if \ \ \frac{1}{2} \le x \le 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2x & if \ \ 0 \le x \le \frac{1}{2} \\ -2x + 2 & if \ \ \frac{1}{2} \le x \le 1 \end{cases}$$

Then $f_{\odot}(x) = 1 \ \forall \ x \in [0,1]$ and $g_{\odot}(x) = \begin{cases} 1 & if \ \ 0 \le x \le \frac{1}{2} \\ -2x + 2 & if \ \ \frac{1}{2} \le x \le 1 \end{cases}$

$$\Rightarrow (f_{\odot} \wedge g_{\odot})(x) = g_{\odot}(x) \ \ \forall \ x \in [0,1]$$

Also $(f \wedge g)(x) = \begin{cases} 2x & if \ \ 0 \le x \le \frac{1}{4} \\ -2x + 1 & if \ \ \frac{1}{4} \le x \le \frac{1}{4} \\ 2x - 1 & if \ \ \frac{1}{2} \le x \le \frac{3}{4} \\ -2x + 2 & if \ \ \frac{3}{4} \le x \le 1 \end{cases}$

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$$\Rightarrow (f \wedge g)_{\odot}(x) = \begin{cases} \frac{1}{2} & \text{if } \ 0 \le x \le \frac{3}{4} \\ -2x+2 & \text{if } \ \frac{3}{4} \le x \le 1 \end{cases}$$

Hence $(f \wedge g)_{\odot}(x) \ne (f_{\odot} \wedge g_{\odot})(x)$.

4.10 Proposition: T is continuous on $\mathcal{B}$.

Proof: Let $f_n \in \mathcal{B}$ for $n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathcal{B}$.

Let $\varepsilon > 0$ be given. Then there exists a positive integer N such that

$$n \ge N \Rightarrow |f_n(x) - f(x)| < \varepsilon \ \forall \ x \in [a,b]$$

$$\Rightarrow -\varepsilon < f_n(x) - f(x) < \varepsilon \ \ \forall \ x \in [a,b]$$

$$\Rightarrow f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \ \ \forall \ x \in [a,b].$$

Let $x \in [a,b]$ and choose $y \in [x,b]$. Then $y \in [a,b]$.

$$\Rightarrow f(y) - \varepsilon < f_n(y) < f(y) + \varepsilon \ \ \forall \ n \ge N$$

$$n \ge N \Rightarrow f_n(y) < f(y) + \varepsilon \le f_{\odot}(x) + \varepsilon$$

$$\Rightarrow f_n(y) < f_{\odot}(x) + \varepsilon \ \forall \ y \in [x,b]$$

$$\Rightarrow (f_n)_{\odot}(x) \le f_{\odot}(x) + \varepsilon.$$

Similarly $f_{\odot}(x) - \varepsilon < (f_n)_{\odot}(x)$.

Hence $|(f_n)_{\odot}(x) - f_{\odot}(x)| \le \varepsilon \ \ \forall \ n \ge N$ and for every $x \in [a,b]$.

Hence $(f_n)_{\odot} \to f_{\odot}$ uniformly on $[a,b]$.

Hence $f_n \to f$ in $\mathcal{B}$ $\Rightarrow Tf_n \to Tf$ in $\mathcal{B}$.

$\Rightarrow$ T is continuous on $\mathcal{B}$.

4.11Proposition: (a) $T^n f = f_{\odot} \ \ \forall \ f \in \mathcal{B}$ and for every $n \in \mathbb{N}$

(b) For $f \in \mathcal{B}$, the cycle of T is the set $\{f, f_{\odot}\}$.

4.12 Proposition: If $f \in \mathcal{B}$ and f is monotonically decreasing then f is a fixed point of T.

4.13 Remark: The set of all fixed points of T is the set of all monotonically decreasing functions on $[a,b]$. Let F denote the set of all fixed points of T

Then $F = \{f \in \mathcal{B} / Tf = f\} = \{f \in \mathcal{B} / f \text{ is decreasing}\}$.

Notes

4.14 Proposition: The set F of all fixed points of the rising sun operator T is closed in $\mathscr{B}$.

4.15 Proposition: Fix $f \in F$. Let $F^* = \{f \in \mathscr{B} / Tf = f\}$.

Then (a) F^* is closed in $\mathscr{B}$.

(b) $(F^*, \leq)$ is a $\vee -$semilattice under the relation $\leq$ defined on F^* by

$$f \leq g \Leftrightarrow f(x) \leq g(x) \ \forall \ x \in [a,b]$$

4.16 Proposition: The operator T is bounded. More precisely,

(i) $\|Tf\| \leq \|f\| \ \forall \ f \in \mathscr{B}$ and (ii) $\|T\| = 1$

VI. Invariant Properties

5.1 Proposition: Let $f \in \mathscr{B}$. If f is *usc* at a point $x \in [a,b]$ then so is $Tf = f_\odot$.

Proof: Let $\varepsilon > 0$ be given and $x \in [a,b]$. Since f is *usc* at $x \in [a,b]$, there exists a $\delta > 0$ such that $f(t) - f(x) < \varepsilon \ \forall t \in (x-\delta, x+\delta) \bigcap [a,b] = U$.

$\Rightarrow f_\odot(x) + \varepsilon \geq f(x) + \varepsilon > f(t) \ \forall t \in U$

$\Rightarrow f_\odot(x) + \varepsilon > f(t) \ \forall t \in U$

Suppose that $x < y$.

$$\Rightarrow f_\odot(x) + \varepsilon > f_\odot(x) \geq f_\odot(y) \geq f(y)$$

$$\Rightarrow f_\odot(x) + \varepsilon > f(y).$$

If $x < t$ and $y \in [t,b] \Rightarrow x < y$

$$\Rightarrow f_\odot(x) + \varepsilon > f(y)$$
$$\Rightarrow f_\odot(x) + \varepsilon \geq f_\odot(t)$$

Suppose that $t \leq x$. Then $y \in [t,b] \Rightarrow t \leq y \leq x$ or $x < y < b$.

If $t \leq y \leq x$ then $y \in U \Rightarrow f_\odot(x) + \varepsilon > f(y) \Rightarrow f_\odot(x) + \varepsilon > f_\odot(t)$.

If $x < y \leq b$ then $f_\odot(x) + \varepsilon > f_\odot(t)$. Hence $f_\odot(x) + \varepsilon > f_\odot(t) \ \forall t \in U$.

$$\Rightarrow f_\odot \text{ is } usc \text{ at } x.$$

5.2 Proposition: Let $f \in \mathscr{B}$. If f is *lsc* at a point $x \in [a,b]$ then so is $Tf = f_\odot$.

Proof: Let $\varepsilon > 0$ be given and $x \in [a,b]$. Since f is *lsc* at $x \in [a,b]$, there exists a $\delta_1 > 0$ such that $f(t) - f(x) > \frac{-\varepsilon}{2} \;\; \forall t \in (x-\delta_1, x+\delta_1) \bigcap [a,b] = U$

$$\Rightarrow f_{\odot}(t) + \frac{\varepsilon}{2} \geq f(t) + \frac{\varepsilon}{2} > f(x) \;\; \forall t \in U$$

$$\Rightarrow f_{\odot}(t) + \frac{\varepsilon}{2} > f(x) \;\; \forall t \in U$$

Since $f_{\odot}(x) - \frac{\varepsilon}{2}$ is not an upper bound of the set $\{f(y) / x \leq y \leq b\}$, there exists a point $y \in [x,b]$ such that $f(y) > f_{\odot}(x) - \frac{\varepsilon}{2}$.

If $y = x$ then $f_{\odot}(t) + \frac{\varepsilon}{2} > f(x) > f_{\odot}(x) - \frac{\varepsilon}{2} \;\; \forall t \in U$

$$\Rightarrow f_{\odot}(t) + \varepsilon > f_{\odot}(x) \;\; \forall t \in U .$$

$y \neq x$. If $x < z < y$ then $f_{\odot}(z) \geq f_{\odot}(y) \geq f(y) > f_{\odot}(x) - \frac{\varepsilon}{2}$

$$\Rightarrow f_{\odot}(z) > f_{\odot}(x) - \frac{\varepsilon}{2} > f_{\odot}(x) - \varepsilon .$$

If $z \leq x$ then $f_{\odot}(z) \geq f_{\odot}(x) > f_{\odot}(x) - \varepsilon$.

Choose $\delta > 0$ such that $a \leq x - \delta < x < x + \delta \leq y$.

Then $f_{\odot}(z) > f_{\odot}(x) - \varepsilon \;\; \forall z \in (x-\delta, x+\delta) \bigcap [a,b]$. Hence $f_{\odot}$ is *lsc* at x.

5.3 Corollary: Let $f \in \mathscr{B}$. If f is continuous at a point $x \in [a,b]$, then $Tf = f_{\odot}$ is continuous at x.

5.4 Proposition: For any $f \in \mathscr{B}$, $Tf = f_{\odot}$ is *lsqc* at every $x \in (a,b]$.

Proof: Let $f \in \mathscr{B}$ and $x \in (a,b]$. Let $\varepsilon > 0$ be given and let U be a neighborhood of x in $[a,b]$. Then there exists a $\delta > 0$ such that $(x-\delta, x+\delta) \bigcap [a,b] \subset U$. Choose $W = (x-\delta, x) \bigcap [a,b]$.

W is a non-empty open subset of U.

$$y \in W \Rightarrow x - \delta < y < x$$

$$\Rightarrow f_{\odot}(y) \geq f_{\odot}(x)$$

$$\Rightarrow f_{\odot}(y) + \varepsilon > f_{\odot}(y) \geq f_{\odot}(x)$$

Notes

Hence $f_{\odot}(y)+\varepsilon > f_{\odot}(x)\ \forall\ y \in W$.

Thus for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $W \subset U$ such that $f_{\odot}(y)+\varepsilon > f_{\odot}(x)\ \forall\ y \in W$.

5.5 Proposition: Let $f \in \mathscr{B}$. If $f:[a,b] \to \mathbb{R}$ is *lsqc* at the point a then $Tf = f_{\odot}$ is *lsqc* at a.

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Proof: Let $f \in \mathscr{B}$ and $\varepsilon > 0$ be given. Let U be a neighborhood of a in $[a,b]$. Since f is *lsqc* at a, there exists a non-empty open set $W \subset U$ such that $f(t) - f(x) > -\varepsilon\ \forall t \in W$.

$$\Rightarrow f_{\odot}(t) \geq f(t) > f(a) - \varepsilon\ \ \forall t \in W$$

$$\Rightarrow f_{\odot}(t) > f(a) - \varepsilon\ \ \forall t \in W.$$

Since $f_{\odot}(a) - \varepsilon$ is not an upper bound of $\{f(y) / a \leq y \leq b\}$, there exists a point $y \in [a,b]$ such that $f_{\odot}(a) - \varepsilon < f(y)$.

If $y = a$ then $f_{\odot}(t) + \varepsilon > f(a) > f_{\odot}(a) - \varepsilon\ \ \forall t \in W$.

Suppose that $y \neq a$.

Since $a \in U$ and U is open in $[a,b]$, there exists a $\delta_1 > 0$ such that $[a, a+\delta_1) \subset U$. Choose $\delta_2 > 0$ such that $a < a + \delta_2 < y$.

Put $\delta = \min\{\delta_1, \delta_2\}$ and $W_1 = (a, a+\delta)$. Clearly, W_1 is a non-empty open set such that $W_1 \subset U$.

Then $z \in W_1 \Rightarrow z < y$

$$\Rightarrow f_{\odot}(z) \geq f_{\odot}(y) \geq f(y) > f_{\odot}(a) - \varepsilon$$

Hence $f_{\odot}(z) + \varepsilon > f_{\odot}(a)\ \forall\ z \in W_1$

$$\Rightarrow\ f_{\odot} \text{ is } lsqc \text{ at } a.$$

5.6 Proposition: Let $f \in \mathscr{B}$. Then

(a) $f_{\odot}$ is *usqc* at every $x \in [a,b)$.

(b) If $f:[a,b] \to \mathbb{R}$ is *usqc* at b then so is $f_{\odot}$.

5.7 Proposition [5]: Let $f:[a,b] \to \mathbb{R}$ and $p \in [a,b]$. If $f(p+)$ exists then f is cliquish at p.

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5.8 Corollary: For any $f \in \mathcal{B}$, $Tf = f_{\odot}$ is cliquish on $[a,b]$.

5.9 Proposition: For any $f \in \mathcal{B}$, $Tf = f_{\odot}$ is quasicontinuous on $[a,b]$.

VII. Variant Properties

6.1 Symmetric continuity: It is not necessary that the rising sun function of a symmetrically continuous function is symmetrically continuous. For example, define $f:[-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \ if \ \ x \neq 0 \\ 0 \ \ if \ \ x = 0 \end{cases}$$

Then $$f_{\odot}(x) = \begin{cases} x^2 \ \ if \ \ -1 \leq x \leq 0 \\ 1 \ \ \ if \ \ \ 0 < x \leq 1 \end{cases}$$

Clearly f is symmetrically continuous on $[-1,1]$, but $f_{\odot}$ is not.

6.2 Semi-continuity: The semi-continuity of f need not imply the semi-continuity of $f_{\odot}$ as is evident from the following example.

Define $f:[-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & if \ \ x = 0 \\ (x+1)2^{\frac{-2}{x}} & if \ \ 0 < x \leq 1 \\ x+1 & if \ \ -1 \leq x < 0 \end{cases}$$

Then $$f_{\odot}(x) = \begin{cases} 1 & if \ \ -1 \leq x \leq 0 \\ \frac{1}{2} & if \ \ 0 < x \leq 1 \end{cases}$$

Clearly f is semi-continuous on $[-1,1]$. But $f_{\odot}$ is not semi-continuous at $x = 0$.

6.3 Darboux continuity: It is not necessary that the rising sun function of a Darboux continuous function is Darboux continuous. The function $f:[-1,1] \to \mathbb{R}$ defined in the above example is Dourbox continuous on $[-1,1]$ but its rising sun function is not Dourbox continuous.

6.4 Differentiability: The rising sun function of a differentiable function is not necessarily differentiable as can be observed from the following example.

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Notes

Define $f:[0,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases} +\sqrt{\frac{2x}{5}-x^2} & if\ \ 0\le x\le \frac{2}{5} \\ -\sqrt{\frac{6x}{5}-\frac{8}{25}-x^2} & if\ \ \frac{2}{5}\le x\le \frac{4}{5} \\ +\sqrt{\frac{9x}{5}-\frac{4}{5}-x^2} & if\ \ \frac{4}{5}\le x\le 1 \end{cases}$$

Then $$f_{\odot}(x)=\begin{cases} \frac{1}{5} & if\ \ 0\le x\le \frac{1}{5} \\ f(x) & if\ \ \frac{1}{5}\le x\le \frac{9}{10} \\ \frac{1}{10} & if\ \ \frac{2}{5}\le x\le \frac{9}{10} \\ f(x) & if\ \ \frac{9}{10}\le x\le 1 \end{cases}$$

Clearly f is differentiable at $a=0.3732$, but $f_{\odot}$ is not differentiable at this point.

6.5 Pointwise Convergence: If $\{f_n\}$ converges pointwise to f on $[a,b]$, it is not necessary that $\{Tf_n\}$ converges to Tf as can be seen from the following example. Define $f_n:[0,1]\to\mathbb{R}$ by

$$f_n(x)=\begin{cases} 1 & if\ \ x\in\left\{\frac{1}{n}/n\in\mathbb{N}\right\} \\ 0 & if\ \ x\notin\left\{\frac{1}{n}/n\in\mathbb{N}\right\} \end{cases}$$

Then $\{f_n\}$ converges pointwise to 0.

But $(f_n)_{\odot}(x)=\begin{cases} 1 & if\ \ x=0 \\ 0 & if\ \ 0<x\le 1 \end{cases}$ does not converge to 0.

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