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Analysis of Pulsatile Flow in Elastic Artery

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Abstract- In this paper, we have considered a Pulsatile flow in an elastic arterial tube and witnessed the efforts on the flow due to elasticity of the tube. The expression for "Volumetric flow rate" and "impedance of n^{th} harmonic" have been found. Conclusions have been drawn with the aid of graphs. MATLAB software has been used for sketching graphs.

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Analysis of Pulsatile Flow in Elastic Artery

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Abstract- In this paper, we have considered a Pulsatile flow in an elastic arterial tube and witnessed the efforts on the flow due to elasticity of the tube. The expression for “Volumetric flow rate” and “impedance of n^{th} harmonic” have been found. Conclusions have been drawn with the aid of graphs. MATLAB software has been used for sketching graphs.

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Nomenclature :-

Q_n	=	Volumetric flow rate
Z_n	=	Impedance
Q_r	=	Radial Component of Velocity
Q_θ	=	Transverse Component of Velocity
Q_z	=	Axial Component of Velocity
P	=	Pressure

I. Introduction

In a Pulsatile flow in an elastic arterial tube, the following effects on the flow due to the elasticity of the tube take place :-

i. As the wall of the tube is elastic, therefore due to the deformation of the wall, the flow will be radial together with axial.
ii. There is an axial variation of pressure and the shape of the curve between pressure and time will vary with z. Also, the pressure gradient will have a radial component.
iii. The boundary conditions for continuity of shear and radial stresses in the fluid and the elastic material at the common boundary give a coupling between fluid flow and elastic deformation.

Thurston [1] attempted to study all of the rheological properties of blood with a model including non-Newtonian viscosity, viscoelasticity, and thixotropy, Liepsch 'and Moravec [2] investigated the flow of a shear thinning blood, analog fluid in pulsatile flow through arterial branch model and observed large differences in velocity profiles relative to those measured with Newtonian fluids having the high shear rate viscosity of the analog fluid/ Rindt *et al.* [3] considered both experimentally and numerically the two-dimensional steady and pulsatile flow, Nazemi *et al.* [4] made important contributions to the identification of atherogenic sites. Rodkiewicz *et al.* [5] used several different non-Newtonian models for blood for simulation of blood flow in large arteries and they' observed that there is no effect of the yield stress of blood on either the

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1. G.8. Thurston, Rheological parameters for the viscosity, visco-elasticity and thixotropy of blood, *Biorheology* 16, 149-155, (1979).

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velocity profiles or the wall shear stress, Boesiger *et al,* [6] used magnetic resonance imaging (MRI) to study arterial homodynamics. Perktold *et al* [7] modeled the flow in stenotic vessels as that of an incompressible Newtonian fluid in the rigid vessels. Sharma and Kapur [8] made a mathematical analysis of blood flow through arteries using finite element method, Dutta and Tarbell [9] studied the two different rheological models of blood displaying shearing thinning viscosity and oscillatory flow visco-elasticity. Lee and Libby [10] made a study of vulnerable atherosclerotic plaque containing a large necrotic core, and covered by this fibrous cap,

Korenga *et al* [11] considered biochemical factors such as gene expression and albumin transport in atherogenessis and in plaque rupture, which were shown to .be activated by hemodynarnic factors in wall shear stress. Rachev *et al* [12] considered a model for geometric and mechanical adaptation of arteries. Rees and Thompson [13] studied a simple model derived from laminar boundary layer theory to investigate the flow of blood in arteria″ stenoses up to Reynolds numbers of 1000. Tang *et al* [14] analysed triggering events, which are believed to be primarily homo-dynamic including cap tension, bending of torsion of the artery. Zendehbudi and Moayery [15] made a comparison of physiological and simple pulsatile flows through stenosed arteries.

Berger- and Jou [16] measured wall shear stress downstream of axi-symmetric stenoses in the presence of hernodynamic forces acting on the plaque, which may be responsible for plaque rupture. Botnar *et al* [17] based on the correspondence between MRI velocity measurements and numerical simulations used two approaches to study in detail the role of different flow patterns for the initiation and amplification of atherosclerotic plaque sedimentation. Stroud *et al* [18] found differences in flow fields and in quantities such as wall shear stress among stenotic vessels with same degree of stenosis. Sharma *et al* [19] made a mathematical analysis of blood flow through arteries using finite element Galerkin approaches.

In the current study, we are interested in the analysis of blood flow in elastic arteries.

Basic equation are Mathematical Information :-

Let (q_r, q_θ, q_z) be the components of velocity in radial, transverse and axial directions respectively.

Due to the assumptions, the velocity profile is given by

$$q_r = q_r(r,z,t),\ q_\theta = 0,\ q_z = q_z(r,z,t)$$

and

$$p = p(r,z,t)$$

The equation of continuity gives

$$\frac{1}{r}\frac{\partial}{\partial r}(r.\, q_r) + \frac{\partial q_z}{\partial z} = 0 \qquad \text{.............(1)}$$

And the equations of motion on neglecting inertial term are given by

$$\rho\frac{\partial q_r}{\partial t} = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 q_r}{\partial r^2} + \frac{\partial^2 q_z}{\partial z^2} + \frac{1\partial\, q_r}{r\ \partial t} - \frac{q_r}{r^2}\right) \qquad \text{................(2)}$$

and

$$\rho\frac{\partial q_z}{\partial t} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 q_z}{\partial r^2} + \frac{\partial^2 q_z}{\partial r^2} + \frac{1\partial\, q_r}{r\ \partial r}\right) \qquad \text{................(3)}$$

Ref

6. P. Boesiger, S.E. Maier, L. Kecheng, M.B. Scheidegger and D. Meier, Visualisation and quantification of the human blood flow by magnetic resonance imaging, *J. of Biomechanics* 25, 55-67, (1992).

Notes

Let (u_r, u_θ, u_z) be the components of deformation vector of the material of the wall of the tube and $\tau_{rr}, \tau_{r\theta}, \tau_{rz}, \tau_{\theta\theta}, \tau_{\theta z}$ and τ_{zz} are the components of the symmetric stress tensor.

Here, $u_\theta = 0$ and ρ_ω is the density of the material of the wall, G the shear modulus and W the negative mean of normal stress. Then the equation of elasticity are

$$\rho_w \frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} \quad \text{.................................} (4)$$

$$\rho_w \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} \quad \text{.................................} (5)$$

$$\tau_{ij} = 2G \in_{ij} - \Omega\delta_{ij} \quad \text{.................................} (6)$$

$$\delta_{ij} = 1 \text{ If } i = j \text{ and } \delta_{ij} = 0 \; i \neq j \quad \text{.................................} (7)$$

$$\in_{rr} = \frac{\partial u_r}{\partial r}, \; \in_{r\theta} = 0 = \in_{\theta r}, \in_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \in_{zr} \quad \text{.................................} (8)$$

$$\in_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta} = 0, \; \in_{\theta r} = 0 = \in_{r\theta} \quad \text{.................................} (9)$$

$$\in_{zz} = \frac{\partial u_z}{\partial z}, \; \in_{z\theta} = 0 = \in_{\theta z} \quad \text{.................................} (10)$$

Above equations for $(u_r, 0, u_z)$ become :

$$\rho_w \frac{\partial^2 u_r}{\partial t^2} = G\left(\frac{\partial^2 u_r}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right) - \frac{\partial \Omega}{\partial r} \quad \text{.................................} (11)$$

$$\rho_w \frac{\partial^2 u_z}{\partial t^2} = G\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right) - \frac{\partial \Omega}{\partial z} \quad \text{.................................} (12)$$

And the equation of continuity becomes :

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \quad \text{.....................................} (13)$$

The partial differential equations for q_r, q_z and p are the same as those for $\frac{\partial u_r}{\partial t}, \frac{\partial u_z}{\partial t}$ and Ω and both sets are independent Due to the coupling between fluid flow and elastic deformation, we have following boundary conditions :

(i) From the symmetry of velocity field

$$q_r = 0, \frac{\partial q_z}{\partial r} = 0 \qquad \text{at } r = 0$$

(ii) From the continuity of motion at the interface of wall of the tube, we have

$$q_r = \frac{\partial u_r}{\partial t}, q_z = \frac{\partial u_z}{\partial t} \qquad \text{at } r = a \text{ (inner radius of tube)}$$

(iii) From the continuity of the shear stress and radial stress at the inner wall, we have

$$\mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right) = G\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \qquad \text{at } r = a$$

$$\text{and } -p + 2\mu\frac{\partial q_z}{\partial r} = -\Omega + 2G\frac{\partial u_r}{\partial r} \qquad \text{at } r = a$$

(iv) It is assumed that the outer wall is constrained radially and axially, then we have

$$G = \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0 \qquad \text{at } r = b \text{ (outer radius)}$$

If the inner wall is perturbed and the perturbations are small, then the boundary condition can be taken the same as that at the undisturbed inner wall. Since the outer wall is constrained radially and axially, therefore we can replace the boundary conditions by some others.

Suppose the solutions of equations (1), (2) and (3) are of the form

$$q_r = \mathrm{U}_1(r)e^{-iy_n z}e^{in\omega t} \qquad \text{.......... (14)}$$

$$q_z = \mathrm{U}_2(r)e^{-iy_n z}e^{in\omega t} \qquad \text{.......... (15)}$$

$$\text{and} \quad P = P(r)e^{-iy_n z}e^{in\omega t} \qquad \text{.......... (16)}$$

Using (14), (15) and (16), equations (1), (2) and (3) becomes :

$$\frac{d^2U_1}{dr^2} + \frac{1}{r}\frac{dU_1}{dr} - \frac{U_1}{r^2} - y_n^2 U_1 - \frac{\rho}{\mu} in\omega U_1 = \frac{1}{\mu}\frac{dp}{dr} \qquad \text{.......... (17)}$$

$$\frac{d^2U_2}{dr^2} + \frac{1}{r}\frac{dU_2}{dr} - y_n^2 U_2 - \frac{\rho}{\mu} in\omega U_2 = \frac{1}{\mu}(-iy_r)P \qquad \text{.......... (18)}$$

$$\text{and} \quad -iU_2 y_n + \frac{dU_1}{dr} + \frac{U_1}{r} = 0 \qquad \text{.......... (19)}$$

Let us take $\frac{in\omega}{v} + y_n^2 = K_n^2 (v = \frac{\mu}{\rho})$, So that the equations (17), (18) and (19) reduce to

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$$\frac{d^2U_1}{dr^2}+\frac{1}{r}\frac{dU_1}{dr}-\left(K_n^2+\frac{1}{r}\right)U_1=\frac{1}{\mu}\frac{dp}{dr} \quad \text{.................................... (20)}$$

$$\frac{d^2U_2}{dr^2}+\frac{1}{r}\frac{dU_2}{dr}-K_n^2U_2=-\frac{iy_n}{\mu}p \quad \text{.................................... (21)}$$

and $$\frac{d}{dr}(rU_1)=iy_nrU_2 \quad \text{.................................... (22)}$$

Since the expressions

$$\left.\begin{aligned} X &= A_1J_1\left(iy_nr\right)+A_2J_1\left(ik_nr\right)\\ \text{and}\quad Y &= B_1J_0\left(iy_nr\right)+B_2J_0\left(ik_nr\right)\end{aligned}\right] \quad \text{.................................... (23)}$$

are the solutions of the respective equations

$$\left.\begin{aligned} &\frac{d^2X}{dr^2}+\frac{1}{r}\frac{dX}{dr}-\left(K_n^2+\frac{1}{r^2}\right)X=-\frac{in\omega}{v}A_1J_1\left(iy_nr\right)\\ \text{and}\quad &\frac{d^2Y}{dr^2}+\frac{1}{r}\frac{dY}{dr}-K_n^2Y=-\frac{in\omega}{v}B_1J_1\left(iy_nr\right)\end{aligned}\right] \quad \text{........................ (24)}$$

with the help of the equations (17) – (24), we get

$$\left.\begin{aligned} U_1(r) &= -i\left[C_1Y_nJ_1\left(iy_nr\right)+C_2Y_nJ_1(ik_nr)\right]\\ U_2(r) &= -i\left[C_1Y_nJ_0\left(iy_nr\right)+C_2Y_nJ_0(ik_nr)\right]\\ \text{and}\qquad P(r) &= -C_1in\omega\rho J_0\left(iy_nr\right)\end{aligned}\right) \quad \text{.............................. (25)}$$

Where C_1 and C_2 are arbitrary constants

Putting these values in (14), (15) and (16), we get general solutions as

$$\left.\begin{aligned} q_r &= -\sum_n i[C_1Y_nJ_1(iy_nr)+C_2Y_nJ_1(ik_nr)]e^{in\omega t-iy_nz}\\ q_z &= -\sum_n i[C_1Y_nJ_0(iy_nr)+C_2Y_nJ_0(ik_nr)]e^{in\omega t-iy_nz}\\ \text{and}\ P &= -\sum_n C_1in\omega\rho\, J_0(iy_nr)e^{in\omega t-iy_nz}\end{aligned}\right] \quad \text{.......... (26)}$$

Let Q_n be the volumetric flow rate for the n-th harmonic, then

$$Q_n=\int_0^a 2\pi rq_z dr$$

$$= -2\pi a[C_1J_1(iy_na) + C_2J_2(ik_na)]\, e^{in\omega t - iy_nz} \quad \text{.......................................} (27)$$

$$[\because \int_0^a rJ_0(r)dr = aJ_1(a)]$$

If Z_n be the impedance of n-th harmonic, then

$$Z_n = \frac{-in\omega\rho\, [iy_naJ_n(iy_na)]C_1}{2\pi a^2[C_1J_1(iy_na)C_2J_1(ik_na)]} \quad \text{.......................................} (28)$$

The solution for $\frac{\partial u_r}{\partial t}, \frac{\partial u_z}{\partial t}$ and W are similar to (26), but these solutions will have four more arbitrary constants. The boundary conditions (i) is trivially satisfied by (26). The other six boundary conditions give six equations to determine the six constants. These six equations is equivalent to an equation to find y_n of the form

$$ay_n = f\left(\frac{a^2\omega\rho}{\mu}, \frac{b}{a}, \frac{\rho_w v^2}{Ga^2}, \frac{\rho}{\rho_w}\right)$$

Where, $\frac{a^2\omega\rho}{\mu}$, $\frac{b}{a}$, $\frac{\rho_\omega v^2}{Ga^2}$ and $\frac{\rho}{\rho_\omega}$ are all dimensionless parameters.

II. Numerical Results and Discussion

In order to see the effects of various parameters on volumetric flow rate, impedance etc., the following values of the parameters are taken:

a = 1.0, 0.2, 0.3, 0.4, 0.5 (in cm)

ρ = 1.05 gm/cm^3

μ = 0.04gm/cm^{-sec}

ω = 8 rad./sec.

1st set for J_1 (iy_na) and J_2 (ik_na) are respectively

−.7, .5, .4, .1, .2″, −.6, .6, .3, .2, .3″

IInd set of values for J_1 (iy_na) and J_2 (ik_na) are respectively

−.3, .1, .2, .3, .4″ −0, .6, .7, .5, .1″

IIIrd set of values for J_1 (iy_na) and J_2 (ik_na) are

J_1 (iy_na)	J_2 (ik_na)
i. .68, .48, .43, .15, .21	i. .59, .61, .32, .25, .33
ii. .61, .45, .42, .17, .25	ii. .58, .63, .31, .30, .35
iii. .7, .5, .4, .1, .2	iii. .6, .6, .3 .2, .3

It has been observed that:

On changing the set of values for J_1 (iy_na) and J_2 (ik_na) arbitrarily within the range -.7 to +.7, the graphs show maximum deflections in the value of $|Q_n|$ when artery radius

= .4 but if 'a' lies between 0.2cm to 0.3 cm, the value of $|Q_n|$ becomes constant for 1st set of values but for 2nd set of values, the value of $|Q_n|$ in creases uniformly upto a =0.3cm (fig, (i)).

In figure (ii), it is observed that for value of 'a' between $a = .3cm$ to $a = .5\ cm$, the value of $|Q_n|$ increase as the value of 'n' increases from $n = 3$ to $n = 4$ i.e. Z_4 has more value of impedance that Z_3.

In fig (iii), if the difference in the value of $J_1\ (iy_n a)$ or $J_2\ (ik_n a)$ for three sets of values are small, then it has been observed that $|Q_n|$ is directly proportional to the value of 'a'.

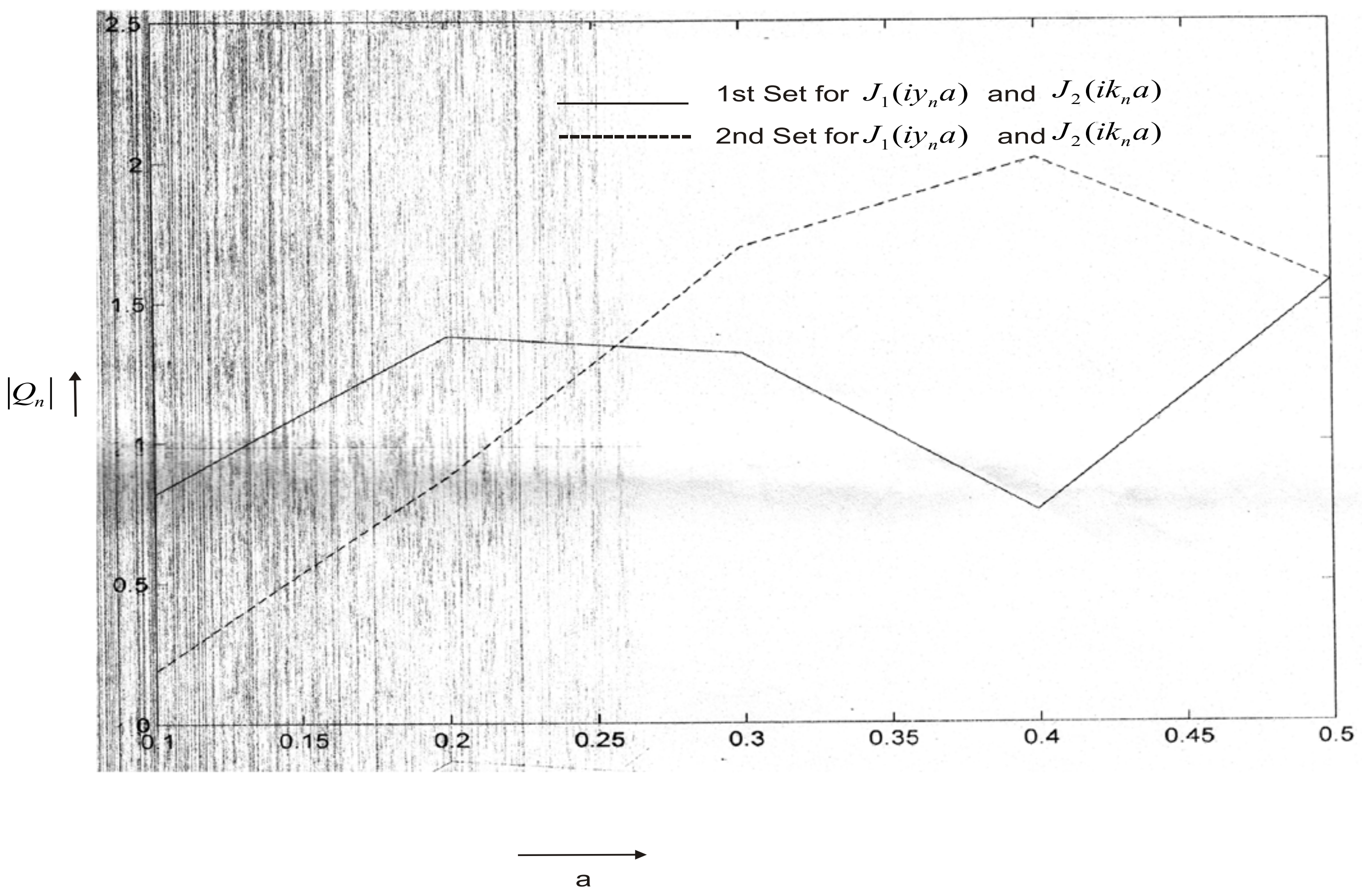

Figure : 1

Fig : Variation of $|Q_n|$ with 'a' for two different sets of values of $J_1(iy_n a)$ and $J_2(ik_n a)$

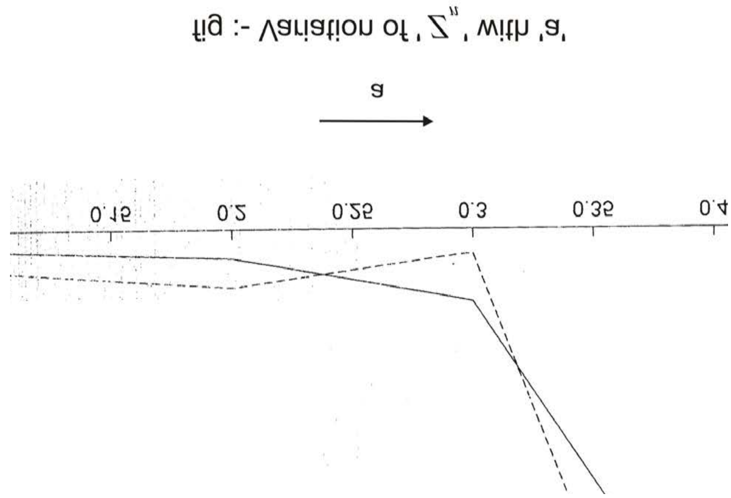

Figure : 2

Notes

$|Q_n|$ ↑

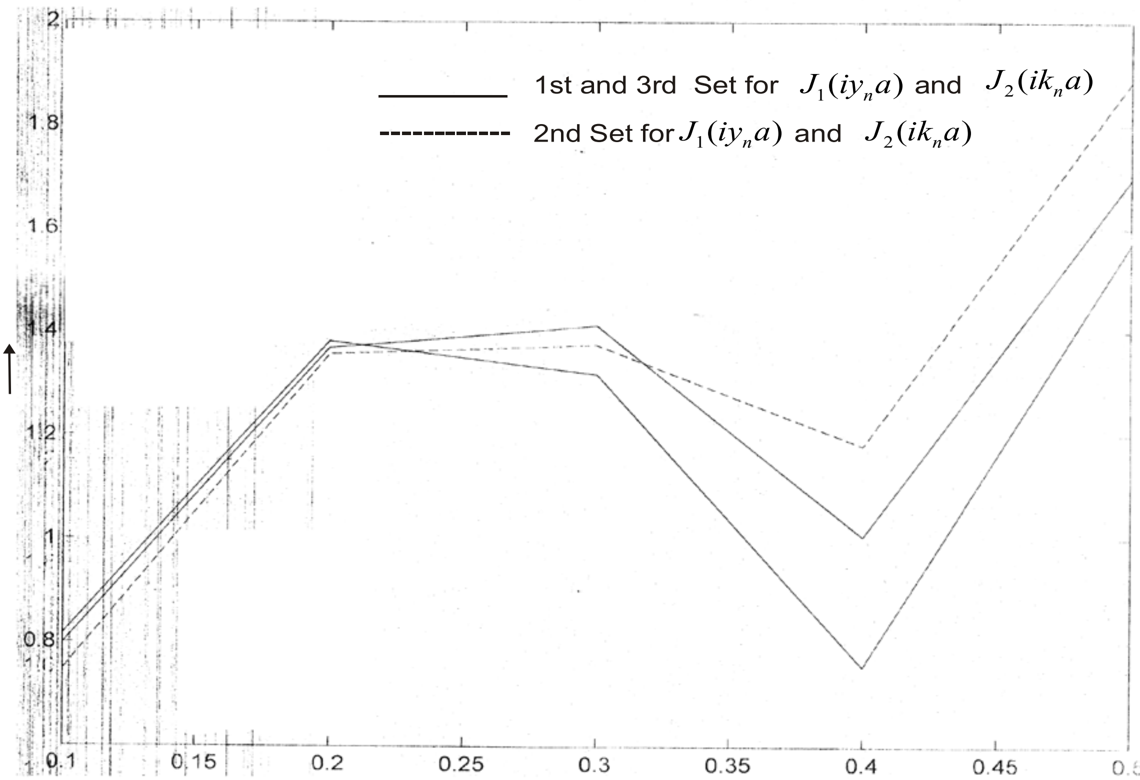

Figure : 3

⟶

Fig : Variation of $|Q_n|$ with 'a' for two different sets of values of $J_1(iy_na)$ and $J_2(ik_na)$

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