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Application of Hankel Transform of I-function of one Variable for Solving Axisymmetric Dirichlet Potential Problem

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Boundary conditions are

$$u(x, 0) = f(x),\ 0 < x < \infty$$

$$u(x, z) \to 0 \text{ as } \sqrt{(x^2 + z^2)} \to \infty,\ z > 0$$

Our main result is believed to be general and unified in nature. A number of known and new results can be obtained by specializing the coefficients and parameters involved in the kernel.

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Strictly as per the compliance and regulations of :

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Abstract- In the present paper we have solved the well known Axisymmetric Dirichlet problem for a half-space using the Hankel transform of I-function of one variable. Hankel transform is much effective tool for solving the boundary value problems involving cylindrical coordinates. Here we have considered the Axisymmetric Dirichlet problem for a half-space which is mathematically characterized by

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I. Introduction

Our aim is to find out the solution of boundary value problem involving cylindrical co-ordinates using Hankel transform of order zero.the most important partial differential equation in mathematicalphysics is the Laplace's equation or potential equation i.e.

$$\nabla^2 u = 0 \tag{1.1}$$

regardless of the co-ordinate system.

Laplace's equation arises in steady state heat conduction problems involving homogeneous solids. this same equation is satisfied by the gravitational potential in free space, the electrostatic potential in a uniform dielectric, the

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magnetic in the steady flow of currents in solid conductors, and the velocity potential of inviscid, irrotational fluids.The mathematical formulation of all potential problems is same despite the physical differences of the applications. Because of this, all solutions of the potential equations are collectively called potential functions, and the study of the many properties associated with these functions forms that branch of mathematics known as potentail theory. Here we have considered Axisymmetric Dirichlet problem for a half-space from the book by Andrews and Shivamoggi[1].

The Hankel transform arise naturally in solving boundary vaue problems formulated in cylindrical co-ordinates. They also occur in other applications such as determining the oscillations of a heavy chain suspended from one end, first treated by D. Bernoulli. This later problem is of some special historical significance since it was in this analysis of Bernoulli in 1703 that the Bessel function of order zero appeared for the first time.

The Hankel transform of order ν of a function $f(x)$ denoted by $H_\nu\{f(x);\rho\}$ is defined as

$$H_\nu\{f(x);\rho\} = \int_0^\infty xJ_\nu(\rho x)f(x)dx = g(\rho), \rho > 0, \nu > -\frac{1}{2} \tag{1.2}$$

The inverse Hankel transform is given by

$$f(x) = \int_0^\infty \rho J_\nu(x\rho)\, g(\rho)d\rho, \Re(\nu) > -1 \tag{1.3}$$

I-function of one variable introduced by Saxena V.P.[6] is defined as

$$I^{m,n}_{p_i,q_i:r} = I^{m,n}_{p_i,q_i:r}\left[x \left|^{\{(a_j,\alpha_j)_{1,n}\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\}}_{\{(b_j,\beta_j)_{1,m}\},\left\{(b_{ji},\beta_{ji})_{m+1,q_i}\right\}}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \theta(s)x^s ds \tag{1.4}$$

where $$\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j-\beta_j s)\prod_{j=1}^n \Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^r[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)]}$$

Here $\omega=\sqrt{-1}$, $p_i(i=1...r)$, $q_i(i=1...r)$, m , n are integers satisfying $0 \le n \le p_i$; $0 \le m \le q_i$; r is finite. α_j, β_j, α_{ji}, β_{ji} are real and positive; $a_j, , b_j, a_{ji}, b_{ji}$ are complex numbers. L is the Mellin-Barne's type of contour integral which runs from $-\omega\infty$ to $+\omega\infty$ with indentations.

II. Required Results

In this section we are mentioning the results required for the evaluation of the transform and the solution of the boundary value problem.

a) First Result

Hankel transform of x^μ given in Erdélyi[3] is

$$H_\nu\{x^\mu\} = \int_0^\infty (\rho x)^{\frac{1}{2}} J_\nu(\rho x)x^\mu dx$$

[1] Andrews Larry C,Shivamoggi Bhimsen K(2009)*Integral Transforms for Engineers*.PHI Learning Pvt Ltd,India,pp 274-285

$$= 2^{(\mu+\frac{1}{2})}\rho^{-\mu-1}\frac{\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{4})} \tag{2.1}$$

where $\rho > 0, -\Re(\nu) - \frac{3}{2} < \Re(\mu) < -\frac{1}{2}$

b) *Second Result*

The following result is taken from Erdélyi[3]

$$M\left\{e^{-ax}J_\nu(\beta x)\right\} = \int_0^\infty x^{s-1}e^{-ax}J_\nu(\beta x)dx$$

$$= \frac{\beta^\nu\Gamma(s+\nu)}{2^\nu a^{s+\nu}\Gamma(1+\nu)}\,{}_2F_1\left[\frac{s+\nu}{2},\frac{s+\nu+1}{2};\nu+1;\frac{-\beta^2}{a^2}\right] \tag{2.2}$$

where $\Re(\alpha) > |Im(\beta)|\,, \Re(s) > -\Re(\nu)$

c) *Third Result*

It is a well known result from Rainville[5]

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.3}$$

III. Hankel Transform of I-function of one Variable

For $\rho, \nu, \alpha \in C, \sigma > 0, \rho > 0, \alpha > 0$ satisfying the condition

$$\Re(\rho) + \Re(\nu) + \sigma_{1\le j\le m}^{min}\left[\frac{\Re(b_j)}{\beta_j}\right] > -\frac{3}{2}$$

and

$$\Re(\rho) + \sigma_{1\le j\le n}^{max}\left[\frac{1-\Re(a_j)}{\alpha_j}\right] < -\frac{1}{2}$$

Then there holds the formula

$$H_\nu\left\{\left(\frac{\rho}{x}\right)^{\frac{1}{2}} I_{p_i,q_i:r}^{m,n}\left[\alpha x^\sigma \left|^{\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}}_{\{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{,m+1,p_i}\}}\right.\right]\right\}$$

$$= \frac{2^{\frac{1}{2}}}{\rho} I_{p_i+2,q_i:r}^{m,n+1}\left[\alpha\left(\frac{2}{\rho}\right)^\sigma \left|^{(\frac{1}{4}-\frac{\nu}{2},\frac{\sigma}{2}),\{(a_j,\alpha_j)_{1,n}\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},(\frac{1}{4}+\frac{\nu}{2},\frac{\sigma}{2})}_{\{(b_j,\beta_j)_{1,m}\},\left\{(b_{ji},\beta_{ji})_{,m+1,p_i}\right\}}\right.\right] \tag{3.1}$$

Proof: Applying the definition of Hankel transform from eq.(1.2) to the left-hand side of eq.(3.1) and expressing the I-function in Mellin-Barne's type of contour integral, we get

$$= \int_0^\infty (\rho x)^{\frac{1}{2}} J_\nu(\rho x)\left\{\frac{1}{2\pi\omega}\int_L \theta(s)\alpha^s x^{\sigma s}ds\right\}dx$$

Changing the order of integration permissible under the conditions mentioned and using the result (2.1), we obtain

$$= \frac{1}{2\pi\omega}\int_L \theta(s)\alpha^s 2^{\sigma s+\frac{1}{2}}\rho^{-\sigma s-1}\frac{\Gamma(\frac{\sigma s}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{\nu}{2}-\frac{\sigma s}{2}+\frac{1}{4})}$$

Rearranging the terms and expressing the integral in I-function of one variable, we get the right-hand side of eq.(3.1).

Ref

[3] Erdelyi A, Magnus W, Oberhettinger F, Tricomi FG(1954) *Table of integral transforms Vol I,II*.Mc-Graw-Hill,Newyork

Substituting $\nu = 0$, we obtain the Hankel transform of order zero as

$$H_0\left\{\left(\frac{\rho}{x}\right)^{\frac{1}{2}} I^{m,n}_{p_i,q_i:r}\left[\alpha x^{\sigma}\,\Big|^{\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\}}_{\{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{,m+1,p_i}\}}\right]\right\}$$

$$= \frac{2^{\frac{1}{2}}}{\rho} I^{m,n+1}_{p_i+2,q_i:r}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\Big|^{(\frac{1}{4},\frac{\sigma}{2}),\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\},(\frac{1}{4},\frac{\sigma}{2})}_{\{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{,m+1,p_i}\}}\right] \tag{3.2}$$

where $\rho, \alpha \in C, \sigma > 0, \rho > 0, \alpha > 0$ satisfying the conditions

$$\Re(\rho) + \sigma^{min}_{1\le j\le m}\left[\frac{\Re(b_j)}{\beta_j}\right] > -\frac{3}{2}\text{and } \Re(\rho) + \sigma^{max}_{1\le j\le n}\left[\frac{1-\Re(a_j)}{\alpha_j}\right] < -\frac{1}{2}$$

IV. Axisymmetric Dirichlet Potential Problem

Consider the Axisymmetric Dirichlet problem for a half space which is mathematically characterized by

$$u_{xx} + \frac{1}{x}u_x + u_{zz} = 0, 0 < x < \infty, z > 0 \tag{4.1}$$

Boundary conditions are

$$u(x,0) = f(x), 0 < x < \infty$$

$$u(x,z) \to 0 \text{ as } \sqrt{(x^2+z^2)} \to \infty,\ z > 0$$

If we apply Hankel transform of order zero to the variable x in (4.1), we obtain the transformed problem as

$$U_{zz} - \rho^2 U = 0, z > 0 \tag{4.2}$$

Boundary conditions are

$$U(\rho,0) = F(\rho)$$
$$U(\rho,z) \to 0, \text{ as } z \to \infty$$

where

$$H_0\{u(x,z); x \to \rho\} = U(\rho,z) \tag{4.3}$$

$$H_0\{f(x);\rho\} = F(\rho) \tag{4.4}$$

The solution of (4.2) is

$$U(\rho,z) = F(\rho)e^{-\rho z} \tag{4.5}$$

Integrating eq.(4.5) by means of Hankel inversion formula,we have

$$u(x,z) = H_0^{-1}\left[F(\rho)e^{-\rho z};\rho\to x\right] = \int_0^{\infty}\rho F(\rho)e^{-\rho z}J_0(\rho x)d\rho \tag{4.6}$$

V. Solution of Axisymmetric Dirichlet Potential Problem

For $\rho, \alpha \in C, \sigma > 0, \rho > 0, \alpha > 0, \Re(z) > |Im(x)|$ satisfying the conditions

Notes

Notes

$$\sigma^{min}_{1\leq j\leq m}\left[\frac{\Re(b_j)}{\beta_j}\right] > 1 \qquad \text{and} \qquad \sigma^{max}_{1\leq j\leq n}\left[\frac{1-\Re(a_j)}{\alpha_j}\right] < -\frac{1}{2}$$

$$u(x,z) = \frac{2^{\frac{1}{2}}}{z}\sum_{k=0}^{\infty}\frac{1}{(n!)^2}\left(-\frac{x^2}{z^2}\right)^n$$

$$\times I^{m+3,n+1}_{p_i+4,q_i+3:r}\left[\alpha(2z)^{\sigma}\,\Big|^{(\frac{1}{4},\frac{\sigma}{2}),\{(a_j,\alpha_j)_{1,n}\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},(\frac{1}{4},\frac{\sigma}{2}),(\frac{1}{2},\frac{\sigma}{2}),(1,\frac{\sigma}{2})}_{(\frac{1}{2}+n,\frac{\sigma}{2}),(1+n,\frac{\sigma}{2}),(1,\sigma),\{(b_j,\beta_j)_{1,m}\},\left\{(b_{ji},\beta_{ji})_{,m+1,p_i}\right\}}\right] \tag{5.1}$$

Proof: Substitute

$$f(x) = \left(\frac{\rho}{x}\right)^{\frac{1}{2}} I^{m,n}_{p_i,q_i:r}\left[\alpha x^{\sigma}\,\Big|^{\{(a_j,\alpha_j)_{1,n}\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\}}_{\{(b_j,\beta_j)_{1,m}\},\left\{(b_{ji},\beta_{ji})_{,m+1,p_i}\right\}}\right] \text{ in eq. (3.1)}$$

Then the Hankel transform of order zero of $f(x)$ from eq. (3.2) is

$$F(\rho) = \frac{2^{\frac{1}{2}}}{\rho} I^{m,n+1}_{p_i+2,q_i:r}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\,\Big|^{(\frac{1}{4},\frac{\sigma}{2}),\{(a_j,\alpha_j)_{1,n}\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},(\frac{1}{4},\frac{\sigma}{2})}_{\{(b_j,\beta_j)_{1,m}\},\left\{(b_{ji},\beta_{ji})_{,m+1,p_i}\right\}}\right]$$

Substituting the value of $F(\rho)$ in eq. (4.4), we get

$$u(x,z) = \int_0^{\infty}\rho e^{-\rho z}J_0(\rho x)$$

$$\times\frac{2^{\frac{1}{2}}}{\rho} I^{m,n+1}_{p_i+2,q_i:r}\left[\alpha\left(\frac{2}{\rho}\right)^{\sigma}\,\Big|^{(\frac{1}{4},\frac{\sigma}{2}),\{(a_j,\alpha_j)_{1,n}\},\left\{(a_{ji},\alpha_{ji})_{n+1,p_i}\right\},(\frac{1}{4},\frac{\sigma}{2})}_{\{(b_j,\beta_j)_{1,m}\},\left\{(b_{ji},\beta_{ji})_{,m+1,p_i}\right\}}\right] d\rho$$

Expressing the I-function in Mellin-Barne's integral and changing the order of integration permissible under the specified conditions, we have

$$= 2^{\frac{1}{2}}\frac{1}{2\pi\omega}\int_L \alpha^s\theta(s)2^{\sigma s}\frac{\Gamma(\frac{3}{4}+\frac{\sigma s}{2})}{\Gamma(\frac{1}{4}-\frac{\sigma s}{2})}\left\{\int_0^{\infty}\rho^{-\sigma s}e^{-\rho z}J_0(\rho x)d\rho\right\}ds$$

Using the result(2.2), we obtain

$$= 2^{\frac{1}{2}}\frac{1}{2\pi\omega}\int_L \alpha^s\theta(s)2^{\sigma s}\frac{\Gamma(\frac{3}{4}+\frac{\sigma s}{2})}{\Gamma(\frac{1}{4}-\frac{\sigma s}{2})}\left\{\frac{\Gamma(1-\sigma s)}{z^{(1-\sigma s)}}{}_2F_1\left[\frac{(1-\sigma s)}{2},\frac{(2-\sigma s)}{2};1;-\frac{x^2}{z^2}\right]\right\}ds$$

Expanding ${}_2F_1$and using the result(2.3) then rearranging the terms, we obtain the right-hand side of eq.(5.1)

VI. Special Cases

Many special cases can be found by suitably specializing the parameters. One of the special case of our result is mentioned below.

Substituting $r=1$, I-function of one variable reduces to Fox's H-function of one variable assuming $a_{j1},\alpha_{j1},b_{j1},\beta_{j1}$ as a_j,α_j,b_j,β_j respectively.

For $\alpha\in C,\sigma>0,\alpha>0,\Re(z)>|Im(x)|$ satisfying the conditions

$$\sigma^{min}_{1\leq j\leq m}\left[\frac{\Re(b_j)}{\beta_j}\right] > 1 \qquad \text{and} \qquad \sigma^{max}_{1\leq j\leq n}\left[\frac{1-\Re(a_j)}{\alpha_j}\right] < -\frac{1}{2}$$

$$u(x,z) = \frac{2^{\frac{1}{2}}}{z}\sum_{k=0}^{\infty}\frac{1}{(n!)^2}\left(-\frac{x^2}{z^2}\right)^n$$

$$\times H^{m+3,n+1}_{p+4,q+3}\left[\alpha(2z)^{\sigma}\left|^{(\frac{1}{4},\frac{\sigma}{2}),\{(a_j,\alpha_j)_{1,p}\},(\frac{1}{4},\frac{\sigma}{2}),(\frac{1}{2},\frac{\sigma}{2}),(1,\frac{\sigma}{2})}_{(\frac{1}{2}+n,\frac{\sigma}{2}),(1+n,\frac{\sigma}{2}),(1,\sigma),\{(b_j,\beta_j)_{1,q}\}}\right.\right] \quad (6.1)$$

Notes

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