Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 14 Issue 3 Version 1.0 Year 2014
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Fractional Calculus Pertaining to Generalized *H*-Functions

By Dr. Dinesh Kumar & Jitendra Daiya
Pratap University, India

Abstract- This paper is devoted to study of a pair of unified and extended fractional integral operators involving the multivariable *H*-Function, *I*-Function and general class of polynomials. Mellin transforms of these operators are investigated. Further, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of fractional integral operators involving simpler functions can be obtained as special cases of our main results.

Keywords and Phrases: fractional calculus, multivariable H-function, I-function, general class of polynomials, mellin transform.

GJSFR-F Classification : *MSC 2010: 26A33, 33C05, 33C40, 33C45, 33C60*

FRACTIONALCALCULUSPERTAININGTOGENERALIZED HFUNCTIONS

Strictly as per the compliance and regulations of :

© 2014. Dr. Dinesh Kumar & Jitendra Daiya. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Fractional Calculus Pertaining to Generalized H -Functions

Dr. Dinesh Kumar [α] & Dr. Jitendra Daiya [σ]

Abstract- This paper is devoted to study of a pair of unified and extended fractional integral operators involving the multivariable *H*-Function, *I*-Function and general class of polynomials. Mellin transforms of these operators are investigated. Further, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of fractional integral operators involving simpler functions can be obtained as special cases of our main results.

Keywords and Phrases: fractional calculus, multivariable H-function, I-function, general class of polynomials, mellin transform.

I. Introduction and Preliminaries

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using mathematical tools by models using mathematical tools from fractional calculus. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. Fractional derivatives are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering.

The multivariable H -function has been studied extensively by H.M. Srivastava and R. Panda in their two basic papers on the subject (see [18, pp.119-137] and [19, pp.265-274]). It is defined and represented in the following manner:

$$H\left[x_1,\dots,x_r\right]=H^{0,n:\{m_r,n_r\}}_{p,q:\{p_r,q_r\}}\left[\begin{matrix}x_1\\ \vdots\\ x_r\end{matrix}\middle|\begin{matrix}\left(a_j;\alpha'_j,\dots,\alpha^{(r)}_j\right)_{1,p}:\left\{\left(c^{(r)}_j,\gamma^{(r)}_j\right)_{1,p_r}\right\}\\ \\ \left(b_j;\beta'_j,\dots,\beta^{(r)}_j\right)_{1,q}:\left\{\left(d^{(r)}_j,\delta^{(r)}_j\right)_{1,q_r}\right\}\end{matrix}\right]$$

Author α : Department of Applied Sciences, Pratap University, Chandwaji, N.H. 8, Jaipur - 303104, India. e-mail: dinesh_dino03@yahoo.com
Author σ : Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur, India. e-mail: jitendra.daiya25@gmail.com

© 2014 Global Journals Inc. (US)

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi(\xi_1,\ldots,\xi_r) \prod_{i=1}^{r} \left(\theta_i(\xi_i)\; x_i^{\xi_i}\, d\xi_i \right), \tag{1.1}$$

where $$\phi(\xi_1,\ldots\xi_r) = \frac{\prod_{j=1}^{n} \Gamma\left(1-a_j+\sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j-\sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right) \prod_{j=1}^{q} \Gamma\left(1-b_j+\sum_{i=1}^{r} \beta_j^{(i)} \xi_i\right)}, \tag{1.2}$$

$$\theta_i(\xi_i) = \frac{\prod_{j=1}^{n_i} \Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\right) \prod_{j=1}^{m_i} \Gamma\left(d_j^{(i)}-\delta_j^{(i)}\xi_i\right)}{\prod_{j=n_i+1}^{p_i} \Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right) \prod_{j=m_i+1}^{q_i} \Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\right)}, \quad \forall i \in \{1,2,\ldots,r\} \tag{1.3}$$

Here, $\{m_r, n_r\}$ stands for $m_1, n_1; \ldots, m_r, n_r$ and $\left\{\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}\right\}$ stand for the sequence of r ordered pairs $\left(c_j', \gamma_j'\right)_{1,p_1}; \ldots; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}$.

In case $r=2$, it reduces to the H-function of two variables.

Recently, Ram and Kumar [9] have obtained the images of the product of two H-functions involving Saigo-Maeda operators; Kumar and Daiya [5] obtained the generalized fractional differentiation of the $\overline{H}$-function involving general class of polynomials; Saxena, Ram and Kumar [14] have obtained the generalized fractional integral formulae of the product of Bessel functions of the first kind involving Saigo-Maeda fractional integral operators.

Also, $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [16]:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k, \quad n = 0,1,2,\ldots \tag{1.4}$$

where m is an arbitrary positive integer and the coefficient $A_{n,k}\,(n,k \geq 0)$ are arbitrary constants, real or complex. On suitably specialize the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases [see Srivastava and Singh [20], pp. 158-161].

The I-function is generalization of Fox's H-function [8], defined and represented as [15]:

$$I_{p_i,q_i;r}^{m,n}[z] = I_{p_i,q_i;r}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi i} \int_L \phi(\xi) z^{\xi} d\xi, \tag{1.5}$$

Notes

© 2014 Global Journals Inc. (US)

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j\xi\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\alpha_{ji}\xi\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}\xi\right)\right\}}\,. \tag{1.6}$$

Notes

The integration path $L=L_{i\gamma\infty},\gamma\in\Re$ extends from $\gamma-i\infty$ *to* $\gamma+i\infty$, and is such that the poles of $\Gamma\left(1-a_j+\alpha_j\xi\right), j=\overline{1,n}$ (the symbol $\left(\overline{1,n}\right)$ is used for 1,2,…,n) do not coincide with the poles of $\Gamma\left(b_j-\beta_j\xi\right), j=\overline{1,m}$. The parameters p_i,q_i are non-negative integers satisfying the condition $0\le n\le p_i,\ 1\le m\le q_i$ for $i=\overline{1,r}$. The parameters $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}>0$ *and* $a_j,b_j,a_{ji},b_{ji}\in C$.

The Mellin transform of $f(x)$ will be denoted by $M\left[f(x)\right]$ or $F(s)$. If p and y are real, we write $s=p^{-1}+iy$. If $p\ge1,\ f(x)\in L_p(0,\infty)$, then for $p=1$ we have

$$M\left[f(x)\right]=F(s)=\int_0^\infty x^{s-1}f(x)\,dx, \tag{1.7}$$

and

$$f(x)=\frac{1}{2\pi i}\int_L F(s)\,x^{-s}\,ds. \tag{1.8}$$

For p > 1,

$$M\left[f(x)\right]=F(s)=\ell.i.m.\int_{1/x}^{x}x^{s-1}f(x)\,dx, \tag{1.9}$$

where $\ell.i.m.$ denotes the usual limit in the mean for L_p-spaces.

II. Definitions

The pair of new extended fractional integral operators are defined by the following equations:

$$D_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=t\,x^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)\times H\begin{bmatrix}\gamma_1\upsilon\\ \vdots\\ \gamma_n\upsilon\end{bmatrix}$$

$$\times\prod_{j=1}^{k}I_{p_{i'j},q_{i'j};r}^{m_j,n_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\left|\begin{matrix}\left(e_{j'j},E_{j'j}\right)_{1,n_j};\left(e_{j'i'j},E_{j'i'j}\right)_{n_j+1,p_{i'j}}\\ \left(f_{j'j},F_{j'j}\right)_{1,m_j};\left(f_{j'i'j},F_{j'i'j}\right)_{m_j+1,q_{i'j}}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{r}S_{N_i}^{M_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)\,dy\ , \tag{2.1}$$

© 2014 Global Journals Inc. (US)

and

$$R_{\gamma_n}^{\delta,\beta}\left[f(x)\right]=t\,x^{\delta}\int_x^{\infty}y^{-\delta-t\beta-1}\left(y^t-x^t\right)^{\beta}\times H\begin{bmatrix}\gamma_1\mu\\ \vdots\\ \gamma_n\mu\end{bmatrix}$$

$$\times\prod_{j=1}^{k}I_{p_{i'j},q_{i'j};r}^{m_j,n_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\left|\begin{matrix}\left(e_{j'j},E_{j'j}\right)_{1,n_j};\left(e_{j'i'j},E_{j'i'j}\right)_{n_j+1,p_{i'j}}\\ \left(f_{j'j},F_{j'j}\right)_{1,m_j};\left(f_{j'i'j},F_{j'i'j}\right)_{m_j+1,q_{i'j}}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{r}S_{N_i}^{M_i}\left[z_i\left(\frac{x^t}{y^t}\right)^{g_i}\left(1-\frac{x^t}{y^t}\right)^{h_i}\right]\psi\left(\frac{x^t}{y^t}\right)f(y)\,dy\ , \qquad (2.2)$$

where $\upsilon=\left(\frac{y^t}{x^t}\right)^{u_i}\left(1-\frac{y^t}{x^t}\right)^{v_i}$, $\mu=\left(\frac{x^t}{y^t}\right)^{u_i}\left(1-\frac{x^t}{y^t}\right)^{v_i}$; t,u_i and v_i are positive numbers. The kernels $\psi\left(\frac{y^t}{x^t}\right)$ and $\psi\left(\frac{x^t}{y^t}\right)$ appearing in (2.1) and (2.2) respectively, are assumed to be continuous functions such that the integrals make sense for wide classes of functions $f(x)$.

The conditions for the existence of these operators are as follows:

(i) $f(x)\in L_p(0,\infty)$, (ii) $1\le p,\ q<\infty,\ p^{-1}+q^{-1}=1$,

(iii) $\Re\left(\alpha+t\,a_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-q^{-1}$, (iv) $\Re\left(\beta+t\,b_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-q^{-1}$,

(v) $\Re\left(\delta+t\,a_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-p^{-1}$; where $j=\left(\overline{1,u_r}\right)$, $i'=\left(\overline{1,r}\right)$.

Condition (i) ensures that both operators defined in (2.1) and (2.2) exist and belong to $L_p(0,\infty)$.

These operators are extensions of fractional integral operators defined and studied by several authors like Erdélyi [2], Kober [4], Love [7], Saigo et al. [11], Saxena and Kiryakova [12], Saxena et al. [13], and many more.

III. Main Results

Theorem 3.1. If $f(x)\in L_p(0,\infty)$, $1\le p\le 2$; or $f(x)\in M_p(0,\infty)$, $p>2$, also following conditions satisfies:

$$p^{-1}+q^{-1}=1,\ \Re\left(\alpha+t\,a_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-q^{-1},\ \Re\left(\beta+t\,b_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-q^{-1},$$

Notes

© 2014 Global Journals Inc. (US)

for $j=\left(\overline{1,u_r}\right)$, then we obtain the following relation:

$$M\left\{D_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\}=M\left\{f(x)\right\}R_{\lambda_n}^{\alpha-s+1,\beta}\left[1\right]. \tag{3.1}$$

where $M_p(0,\infty)$ stands for the class of all functions $f(x)$ of $L_p(0,\infty)$ with $p>2$, which are inverse Mellin-transforms of the functions of $L_p(-\infty,\infty)$.

Notes

Proof. By taking Mellin transform of (2.1), we get

$$M\left\{D_{\lambda_n}^{\alpha,\beta}\left[f(x)\right]\right\}=\int_0^\infty x^{s-1}\left\{t\,x^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta}H\begin{bmatrix}\gamma_1\upsilon\\ \vdots\\ \gamma_n\upsilon\end{bmatrix}\right.$$

$$\times\prod_{j=1}^{k}I_{p_{i'j},q_{i'j};r}^{m_j,n_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\left|\begin{matrix}\left(e_{j'j},E_{j'j}\right)_{1,n_j};\left(e_{j'i'j},E_{j'i'j}\right)_{n_j+1,p_{i'j}}\\ \left(f_{j'j},F_{j'j}\right)_{1,m_j};\left(f_{j'i'j},F_{j'i'j}\right)_{m_j+1,q_{i'j}}\end{matrix}\right.\right]$$

$$\left.\times\prod_{i=1}^{r}S_{N_i}^{M_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)\,dy\right\}dx. \tag{3.2}$$

On interchanging the order of integration, which is permissible under the conditions, the result (3.1) follows easily in view of (2.2).

Theorem 3.2. If $f(x)\in L_p(0,\infty)$, $1\le p\le 2$; or $f(x)\in M_p(0,\infty)$, $p>2$, and satisfies the following conditions:

$$p^{-1}+q^{-1}=1,\ \Re\left(\delta+t\,a_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-p^{-1},\ \Re\left(\beta+t\,b_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-q^{-1},$$

for $j=\left(\overline{1,u_r}\right)$, then we obtain the following relation:

$$M\left\{R_{\gamma_n}^{\delta,\beta}\left[f(x)\right]\right\}=M\left\{f(x)\right\}D_{\lambda_n}^{\delta+s-1,\beta}\left[1\right]. \tag{3.3}$$

Proof. By taking Mellin transform of (2.2), we get

$$M\left\{R_{\lambda_n}^{\delta,\beta}\left[f(x)\right]\right\}=\int_0^\infty x^{s-1}\left\{t\,x^{\delta}\int_x^\infty y^{-\delta-t\beta-1}\left(y^t-x^t\right)^{\beta}H\begin{bmatrix}\gamma_1\mu\\ \vdots\\ \gamma_n\mu\end{bmatrix}\right.$$

$$\times\prod_{j=1}^{k}I_{p_{i'j},q_{i'j};r}^{m_j,n_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\left|\begin{matrix}\left(e_{j'j},E_{j'j}\right)_{1,n_j};\left(e_{j'i'j},E_{j'i'j}\right)_{n_j+1,p_{i'j}}\\ \left(f_{j'j},F_{j'j}\right)_{1,m_j};\left(f_{j'i'j},F_{j'i'j}\right)_{m_j+1,q_{i'j}}\end{matrix}\right.\right]$$

© 2014 Global Journals Inc. (US)

$$\times \prod_{i=1}^{r} S_{N_i}^{M_i}\left[z_i \left(\frac{x^t}{y^t}\right)^{g_i} \left(1-\frac{x^t}{y^t}\right)^{h_i} \right] \psi\left(\frac{x^t}{y^t}\right) f(y)\, dy \Bigg\} dx. \tag{3.4}$$

On interchanging the order of integration, the result (3.3) can easily be obtained with the help of (2.1)

Theorem 3.3. If $f(x) \in L_p(0,\infty)$, $v(x) \in L_p(0,\infty)$ $1 \le p \le 2$; or $f(x) \in M_p(0,\infty)$, $p > 2$, and satisfies the following conditions:

$$p^{-1}+q^{-1}=1,\ \Re\left(\alpha + t\, a_j \frac{f_{j'i'j}}{F_{j'i'j}} + t\sum_{i=1}^{n} u^{(i)} \frac{c_j^{(i)}}{\psi_j^{(i)}}\right) > -q^{-1},\ \Re\left(\beta + t\, b_j \frac{f_{j'i'j}}{F_{j'i'j}} + t\sum_{i=1}^{n} v^{(i)} \frac{c_j^{(i)}}{\psi_j^{(i)}}\right) > -q^{-1},$$

for $j = \left(\overline{1,u_r}\right)$, then we obtain the following relation:

$$\int_0^\infty v(x)\, D_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] dx = \int_0^\infty f(x)\, R_{\gamma_n}^{\alpha,\beta}\left[v(x)\right] dx. \tag{3.5}$$

Proof. The result (3.5) can easily be obtained in view of equations (2.1) and (2.2).

IV. INVERSION FORMULAE

Theorem 4.1 . If $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$; or $f(x) \in M_p(0,\infty)$, $p > 2$, and satisfies the following conditions:

$$p^{-1}+q^{-1}=1,\ \Re\left(\alpha + t\, a_j \frac{f_{j'i'j}}{F_{j'i'j}} + t\sum_{i=1}^{n} u^{(i)} \frac{c_j^{(i)}}{\psi_j^{(i)}}\right) > -q^{-1},\ \Re\left(\beta + t\, b_j \frac{f_{j'i'j}}{F_{j'i'j}} + t\sum_{i=1}^{n} v^{(i)} \frac{c_j^{(i)}}{\psi_j^{(i)}}\right) > -q^{-1},$$

for $j = \left(\overline{1,u_r}\right)$ and the integrals are absolutely convergent, then we obtain the following relation:

$$D_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] = v_1(x), \tag{4.1}$$

Then it gives the following result:

$$f(x) = \int_0^\infty y^{-1}\left[v_1(y)\right]\left[h_1\left(x\,y^{-1}\right)\right] dy, \tag{4.2}$$

where

$$h_1(x) = \frac{1}{2\pi i}\int_L \frac{x^{-s}}{R(s)}\, ds, \tag{4.3}$$

$$R(s) = R_{\gamma_n}^{\alpha-s+1,\beta}[1]. \tag{4.4}$$

Proof. On taking Mellin transform of (4.1) and using the relation (3.1), then we arrive at

$$M\{f(x)\} = \frac{M\{v_1(x)\}}{R(s)},$$

Notes

© 2014 Global Journals Inc. (US)

which on inverting leads to

$$f(x)=\frac{1}{2\pi i}\int_L x^{-s}\frac{M\{v_1(x)\}}{R(s)}ds$$

$$=\frac{1}{2\pi i}\int_L \frac{x^{-s}}{R(s)}\left\{\int_0^\infty y^{s-1}\left[v_1(y)\right]dy\right\}ds .$$

Notes

On interchanging the order of integration and using the (4.3), we get

$$f(x)=\int_0^\infty \frac{v_1(y)}{y}\left\{\frac{1}{2\pi i}\int_L\left(\frac{x}{y}\right)^{-s}\frac{1}{R(s)}\,ds\right\}dy .$$

This completes the proof of (4.2).

Theorem 4.2. If $f(x)\in L_p(0,\infty)$, $1\le p\le 2$; or $f(x)\in M_p(0,\infty)$, $p>2$, and satisfies the following conditions:

$$p^{-1}+q^{-1}=1,\ \Re\left(\delta+t\,a_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^n u^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-p^{-1},\ \Re\left(\beta+t\,b_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^n v^{(i)}\frac{c_j^{(i)}}{\psi_j^{(i)}}\right)>-q^{-1},$$

for $j=\left(\overline{1,u_r}\right)$ and the integrals are absolutely convergent, then we obtain the following relation:

$$R_{\gamma_n}^{\delta,\beta}\left[f(x)\right]=v_2(x), \tag{4.5}$$

Then it gives the following result:

$$f(x)=\int_0^\infty y^{-1}\left[v_2(y)\right]\left[h_2\left(x\,y^{-1}\right)\right]dy, \tag{4.6}$$

where

$$h_2(x)=\frac{1}{2\pi i}\int_L\frac{x^{-s}}{D(s)}ds, \tag{4.7}$$

$$D(s)=D_{\gamma_n}^{\delta+s-1,\beta}\left[1\right]. \tag{4.8}$$

Proof. By taking Mellin transform of (4.5) and using the relation (3.3), then we get

$$M\{f(x)\}=\frac{M\{v_2(x)\}}{D(s)},$$

which on inverting leads to

$$f(x)=\frac{1}{2\pi i}\int_L x^{-s}\frac{M\{v_2(x)\}}{D(s)}ds$$

$$=\frac{1}{2\pi i}\int_L\frac{x^{-s}}{D(s)}\left\{\int_0^\infty y^{s-1}\left[v_2(y)\right]dy\right\}ds .$$

© 2014 Global Journals Inc. (US)

Further, on interchanging the order of integration and using the (4.7), we get

$$f(x)=\int_0^\infty \frac{v_2(y)}{y}\left\{\frac{1}{2\pi i}\int_L\left(\frac{x}{y}\right)^{-s}\frac{1}{D(s)}\,ds\right\}dy.$$

This completes the proof of (4.6).

V. General Properties

The relations given in this section can be established with the help of the definitions (2.1) and (2.2).

$$x^{-1}\,D_{\gamma_n}^{\alpha,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=R_{\gamma_n}^{\alpha,\beta}\left[f(x)\right], \qquad (5.1)$$

$$x^{-1}\,R_{\gamma_n}^{\delta,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=D_{\gamma_n}^{\delta,\beta}\left[f(x)\right], \qquad (5.2)$$

$$x^{u}\,D_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=D_{\gamma_n}^{\alpha-u,\beta}\left[x^{u}f(x)\right], \qquad (5.3)$$

$$x^{u}\,R_{\gamma_n}^{\delta,\beta}\left[f(x)\right]=R_{\gamma_n}^{\delta+u,\beta}\left[x^{u}f(x)\right], \qquad (5.4)$$

The properties given below express the homogeneity of the operators D and R respectively.

If $D_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=v_1(x)$, then $D_{\gamma_n}^{\alpha,\beta}\left[f(cx)\right]=v_1(cx)$.

If $R_{\gamma_n}^{\delta,\beta}\left[f(x)\right]=v_2(x)$, then $R_{\gamma_n}^{\delta,\beta}\left[f(cx)\right]=v_2(cx)$.

VI. Special Cases

(i) If we reduce the general class of polynomials to unity and also taking $k=1=r$, the I-function reduces to Fox's H-function [8] and further specifying the parameters appropriately in view of the relationship[3].

$$\frac{t^{rq-\upsilon-1}}{\Gamma(r)}H_{1,2}^{1,1}\left[-at^{q}\left|\begin{matrix}(1-r,1)\\(0,1),(\upsilon-rq+1,q)\end{matrix}\right.\right]=G_{q,\upsilon,r}\left[a,t\right].$$

Then, we obtain the following pair of fractional integral operators in terms of Lorenzo-Hartely G-function [6]:

$$D_{\gamma_n,\varepsilon}^{\alpha,\beta}\left[f(x)\right]=t\,\Gamma(\varepsilon)\,x^{-\alpha-t\left\{\beta-\left(\frac{a}{b}+1\right)(\varepsilon b-\upsilon-1)\right\}-1}\int_0^x y^{-\alpha-\frac{ta}{b}(\varepsilon b-\upsilon-1)}\left(x^t-y^t\right)^{\beta-\varepsilon b+\upsilon+1}$$

$$\times H\begin{bmatrix}\gamma_1\upsilon\\ \vdots\\ \gamma_n\upsilon\end{bmatrix}G_{b,\upsilon,\varepsilon}\left[z,\left(\frac{y^t}{x^t}\right)^{\frac{a}{b}}\left(1-\frac{y^t}{x^t}\right)\right]\psi\left(\frac{y^t}{x^t}\right)f(y)\,dy, \qquad (6.1)$$

and

© 2014 Global Journals Inc. (US)

$$R_{\gamma_n,\varepsilon}^{\delta,\beta}\left[f(x)\right]=t\,\Gamma(\varepsilon)\,x^{\delta-\frac{ta}{b}(\varepsilon b-\upsilon-1)}\int_0^x y^{\delta-t\left\{\beta-\left(\frac{a}{b}+1\right)(\varepsilon b-\upsilon-1)\right\}-1}\left(y^t-x^t\right)^{\beta-\varepsilon b+\upsilon+1}$$

$$\times H\begin{bmatrix}\gamma_1\upsilon\\ \vdots\\ \gamma_n\upsilon\end{bmatrix} G_{b,\upsilon,\varepsilon}\left[z,\left(\frac{x^t}{y^t}\right)^{\frac{a}{b}}\left(1-\frac{x^t}{y^t}\right)\right]\psi\left(\frac{x^t}{y^t}\right)f(y)\,dy. \qquad (6.2)$$

Notes

(ii) Further, taking $\varepsilon=1$ in (6.1) and (6.2), in view of the relationship [3]

$$t^{q-\upsilon-1}H_{1,2}^{1,1}\left[-at^{\upsilon}\,\middle|\,\begin{matrix}(0,1)\\ (0,1),(\upsilon-q+1,q)\end{matrix}\right]=R_{q,\upsilon}[a,t],$$

then, we get the following pair of fractional integral operators containing Lorentzo-Hartley R-function [6]:

$$D_{\gamma_n,1}^{\alpha,\beta}\left[f(x)\right]=t\;x^{-\alpha-t\left\{\beta-\left(\frac{a}{b}+1\right)(\varepsilon b-\upsilon-1)\right\}-1}\int_0^x y^{-\alpha-\frac{ta}{b}(\varepsilon b-\upsilon-1)}\left(x^t-y^t\right)^{\beta-b+\upsilon+1}$$

$$\times H\begin{bmatrix}\gamma_1\upsilon\\ \vdots\\ \gamma_n\upsilon\end{bmatrix} R_{b,\upsilon}\left[z,\left(\frac{y^t}{x^t}\right)^{\frac{a}{b}}\left(1-\frac{y^t}{x^t}\right)\right]\psi\left(\frac{y^t}{x^t}\right)f(y)\,dy, \qquad (6.3)$$

and

$$R_{\gamma_n,1}^{\delta,\beta}\left[f(x)\right]=t\;x^{\delta-\frac{ta}{b}(\varepsilon b-\upsilon-1)}\int_0^x y^{\delta-t\left\{\beta-\left(\frac{a}{b}+1\right)(b-\upsilon-1)\right\}-1}\left(y^t-x^t\right)^{\beta-b+\upsilon+1}$$

$$\times H\begin{bmatrix}\gamma_1\upsilon\\ \vdots\\ \gamma_n\upsilon\end{bmatrix} R_{b,\upsilon}\left[z,\left(\frac{x^t}{y^t}\right)^{\frac{a}{b}}\left(1-\frac{x^t}{y^t}\right)\right]\psi\left(\frac{x^t}{y^t}\right)f(y)\,dy. \qquad (6.4)$$

(iii) If we reduce H-function of several complex variables transforms to the generalized Lauricella function of several complex variables [17] and, we get the following fractional integral operators:

$$X_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=B\,t\,x^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta}$$

$$\times F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left[\begin{matrix}\left[1-(a):\theta',\ldots,\theta^{(n)}\right],\left[1-(b'):\phi'\right];\ldots;\left[1-(b)^{(n)}:\phi^{(n)}\right];\\ \left[1-(c):\psi',\ldots,\psi^{(n)}\right],\left[1-(d'):\delta'\right];\ldots;\left[1-(d)^{(n)}:\delta^{(n)}\right];\end{matrix}\;-\gamma_1\upsilon,\ldots,-\gamma_n\upsilon\right]$$

$$\times\prod_{j=1}^{k} I_{p'_{ij},q'_{ij};r}^{m'_j,n'_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\,\middle|\,\begin{matrix}\left(e_{p'_{ij}},E_{p'_{ij}}\right)\\ \left(f_{q'_{ij}},F_{q'_{ij}}\right)\end{matrix}\right]$$

© 2014 Global Journals Inc. (US)

$$\times \prod_{i=1}^{r} S_{N_i}^{M_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right) f(y)\, dy\ , \tag{6.5}$$

and

$$Y_{\gamma_n}^{\delta,\beta}\left[f(x)\right]=B\,t\,x^{\delta}\int_x^{\infty} y^{-\delta-t\beta-1}\left(y^t-x^t\right)^{\beta}$$

$$\times F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left[\begin{matrix}\left[1-(a):\theta',\ldots,\theta^{(n)}\right],\left[1-(b'):\phi'\right];\ldots;\left[1-(b)^{(n)}:\phi^{(n)}\right];\\ \left[1-(c):\psi',\ldots,\psi^{(n)}\right],\left[1-(d'):\delta'\right];\ldots;\left[1-(d)^{(n)}:\delta^{(n)}\right];\end{matrix}\ -\gamma_1\mu,\ldots,-\gamma_n\mu\right]$$

$$\times \prod_{j=1}^{k} I_{p'_{ij},q'_{ij};r}^{m'_j,n'_j}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\left|\begin{matrix}\left(e_{p'_{ij}},E_{p'_{ij}}\right)\\ \left(f_{q'_{ij}},F_{q'_{ij}}\right)\end{matrix}\right.\right]$$

$$\times \prod_{i=1}^{r} S_{N_i}^{M_i}\left[z_i\left(\frac{x^t}{y^t}\right)^{g_i}\left(1-\frac{x^t}{y^t}\right)^{h_i}\right]\psi\left(\frac{x^t}{y^t}\right) f(y)\, dy\ , \tag{6.6}$$

where

$$B=\frac{\prod_{j=1}^{A}\Gamma\left(1-a_j\right)\prod_{j=1}^{B'}\Gamma\left(1-b'_j\right)\ldots\prod_{j=1}^{B^{(n)}}\Gamma\left(1-b_j^{(n)}\right)}{\prod_{j=1}^{C}\Gamma\left(1-c_j\right)\prod_{j=1}^{D'}\Gamma\left(1-d'_j\right)\ldots\prod_{j=1}^{D^{(n)}}\Gamma\left(1-d_j^{(n)}\right)}\,. \tag{6.7}$$

The operators earlier defined by Saxena et al. [13], Chaurasia and Srivastava [1], can also be easily derived by assigning suitable values to the parameters occurring in (2.1) and (2.2).

VII. ACKNOWLEDGEMENTS

One of the author (Jitendra Daiya) is grateful to NBHM (National Board of Higher Mathematics) for granting a Post-Doctoral Fellowship.

REFERENCES RÉFÉRENCES REFERENCIAS

[1]. V.B.L. Chaurasia and A. Srivastava , A unified approach to fractional calculus pertaining to H-function, Soochow Journal of Mathematics, 33(2) (2007), 211-221.

[2]. A. Erdélyi , On some functional transformations, Univ. Politec. Torino, Rend. Sem. Mat., 10 (1950), 217-234.

[3]. K.C. Gupta, New relationships of the H-function with functions of practical utility in Fractional Calculus, Ganita Sandesh, 15(2001), 63-66.

[4]. H. Kober ,On fractional integrals and derivatives, Quart. J. Math., 1 (1940), 193-211.

[5]. D. Kumar and J. Daiya, Generalized fractional differentiation of the $\overline{H}$-function involving general class of polynomials, Int. J. Pure Appl. Sci. Technol., 16(1) (2013), 42-53.

Notes

© 2014 Global Journals Inc. (US)

Notes

[6]. C.F. Lorenzo, and T.T. Hartley, Generalized functions for the fractional calculus, NASA/TP 209424, (1999), 1-17.

[7]. E. R. Love, Some integral equations involving hypergeometric functions, J. Edinburgh Math. Soc., 3(15) (1967), 169-198.

[8]. A.M. Mathai, R.K. Saxena and H.J. Haubold, The H-function: Theory and Applications, Springer, New York (2010).

[9]. J. Ram and D. Kumar, Generalized fractional integration involving Appell hypergeometric of the product of two H-functions, Vijanana Prishad Anusandhan Patrika, 54(3) (2011), 33-43.

[10]. M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep., College General Ed. Kyushu Univ., 11 (1978), 135-143.

[11]. M. Saigo, R. K. Saxena and J. Ram , On the fractional calculus operator associated with the H-function, Ganita Sandesh, 6(1) (1992), 36-47.

[12]. R.K. Saxena, V. S. Kiryakova and O. P. Dave, A unified approach to certain fractional integration operators, Math. Balkanica, 8 (1994), 211-219.

[13]. R. K. Saxena and V. S. Kiryakova, On relation between the two-dimensional H-transforms in terms of Erdélyi-Kober operators, Math. Balkanica, 6 (1992), 133-140.

[14]. R.K. Saxena, J. Ram and D. Kumar, Generalized fractional integration of the product of Bessel functions of the first kind, Proceeding of the 9th Annual Conference, SSFA, 9 (2010), 15-27.

[15]. V.P. Saxena, The I-Function, Anamaya Publishers, New Delhi, (2008).

[16]. H.M. Srivastava, A contour integral involving Fox's H-function, Indian J. Math., (1972), 14, 1-6.

[17]. H.M.Srivastava and M.C.Daust, Certain generalized Neumann expansions associated with the Kampe de Feriet function, Nederl. Akad. Wetensch. Proc. Ser. A-72, Indag Math., 31 (1969), 449-457.

[18]. H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. I, Comment. Math. Univ. St. Paul., 24 (1975), 119-137.

[19]. H.M. Srivastava, and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine, Angew, Math. 283/284 (1976), 265-274.

[20]. H.M. Srivastava and N.P. Singh, The integration of certain product of the multivariable H-function with a general class of Polynomials, Rend. Circ. Mat. Palermo, 32 (1983), 157-187.

© 2014 Global Journals Inc. (US)

Notes

This page is intentionally left blank

© 2014 Global Journals Inc. (US)