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Generalized Methods for Generating Moments of Continuous Distribution

By Oyeka ICA & Okeh UM
Nnamdi Azikiwe University Awka, Nigeria

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Generalized Methods for Generating Moments of Continuous Distribution

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I. Introduction

The purpose of this paper is to enable the reader and researcher learn some novel topics that may enable further studies in the areas of probability distributions and applied statistics. The paper presents more generalized methods of estimating moments of continuous random variables and of some continuous probability distributions including the beta and gamma families of distributions and their properties using more generalized methods. These moments are often tedious, cumbersome or impossible to obtain using some of the existing traditional methods.

Now some of the traditional methods often used to estimate moments of random variables and probability distributions are either based on the first principle definition of expected values of random variables, the concept of factorial moments and more generally on the theory of moment generating function. However moment generating functions of probability distributions do not always exist, and even if they exist they are sometimes difficult and tedious to evaluate in practical applications, especially if the estimation of higher moments is of interest. Furthermore if the powers or order of the random variables whose moments are to be determined are negative or fractional instead of merely whole numbers, then the method of moment generating functions may in general not be available.

The approach in estimation of moments of random variables and probability distributions has often been divided into methods of moments about the 'origin' or zero and moments about the mean of the distribution, an approach that is rather unnecessary and time consuming.

In this chapter we develop and present generalized methods for generating all conceivable moments of random variables and probability distributions about an

Author α : Department of Statistics, Nnamdi Azikiwe University Awka, Nigeria. Abakaliki, Nigeria. e-mail: uzomaokey@ymail.com
Author σ : Department of Industrial Mathematics and Applied Statistics, Ebonyi State University Abakaliki Nigeria.

arbitrarily chosen real valued constant. This approach homogenizes methods for moments about the 'origin' and moments about the mean treating them in essentially the same way. Moments about the 'origin' and moments about the mean are presented as only special cases of the more generalized methods. Methods are presented for both uni-variate and bi-variate random variables and probability distributions. Under specified conditions the methods may be used to determine moments of reciprocals or inverses and ratios of random variables. In the case of bi-variate distributions the method would enable the researcher estimate all conceivable moments of marginal probability distributions. The methods presented in this paper are available for use with only continuous random variables and probability distributions. Similar methods may also be developed as appropriate for discrete random variables and probability distributions.

Notes

The generating nth moment of X^c *about* λ

Generally $E\left(X^c+\lambda\right)^n$ is interpreted as the nth moment or expected value of the distribution of X^c about some real number λ where n and c are usually non-negative integers and λ is either 0 or $-\mu$ where μ is the mean of the random variable or probability distribution.

However for the present methods while 'n' may still be any non-negative integer, 'c' and λ may be any real numbers that are not necessarily integers or whole numbers. We will here still base the generalized method on the definition of expected values of random variables. To differentiate this method from the conventional moment generating function(mgf) of the random variable X usually designated by $M_X(t)=E\left(e^tX\right)$, we here refer to the present method as the generalized moment generating function (gmgf) designated by $g_n(c,\lambda)$ (which is read as g_n of c about λ) termed the nth moment of X^c *about* λ, for n=0,1,2,…; $-\infty\le c\le\infty$ *and* $-\infty\le\lambda\le\infty$. To develop the method, suppose X is a continuous random variable with probability density function (pdf), $f(x)$ *for* $-\infty\le x\le\infty$. Now

$$g_n(c;\lambda)=E\left(X^c+\lambda\right)^n \tag{1}$$

Expanding Equation 1 binomially and integrating we have

$$g_n(c;\lambda)=E\left(X^c+\lambda\right)^n=\int_{-\infty}^{\infty}\left(X^c+\lambda\right)^n f(x)\,dx$$

$$=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}(x^c)^r f(x)dx$$

$$g_n(c;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}x^{cr}f(x)dx=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr}f(x)dx$$

$$=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr}f(x)dx=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\mu_r(c)' \tag{2}$$

$$\mu_r(c)'=\int_{-\infty}^{\infty}x^{cr}f(x)dx \tag{3}$$

is the rth moment of X^c about zero, or the (c,r)th moment of X about zero(Oyeka et al, 2013).

Note that $g_n(c;\lambda)$ as defined in Equations 2 and 3 generates all conceivable moments of the distribution of X^c for all real values of 'c'.

Properties of $g_n(c;\lambda)$

Notes

Some of the properties of generalized moment generating function; $g_n(c;\lambda)$ from its definition include

$$g_0(c;\lambda) = 1 \tag{4}$$

$$g_1(c;\lambda) = E\left(X^c + \lambda\right) = \lambda + \mu_r(c)' \tag{5}$$

The first moment of the distribution of X^c *about* λ,

If $\lambda = 0$,that is if the nth moment of X^c is taken about the origin,zero,then

$$g_n(c;0) = E\left(X^c - 0\right)^n = \mu_r(c)' \tag{6}$$

If $\lambda = -\mu_1(c)' = -\mu$ the nth moment or mean value of X^c about the origin zero

$$g_n(c;-\mu) = E\left(X^c - \mu\right)^n = \mu_n(c), \tag{7}$$

the nth moment of X^c about its mean $\mu, \mu = \mu_1(c)'$.

Generalized moment generating functions, like the traditional moment generating functions when they exist, uniquely determine and define probability distributions. In particular, the moment generating function, $\mu_r(c)'$ of Equation 3 generates the same rth moment of the distribution of X^c about zero as would the corresponding traditional or regular moment generating function $M_X(t)$ when it exists. The later however is often more difficult to obtain and evaluate in practical applications.

Under specified conditions as illustrated later $g_n(c;\lambda)$ may be used to obtain all possible moments of the distribution of X^c ,where 'c' is some non-positive real number thereby enabling one obtain moments of random variables with negative and fractional values.

Note that the above properties of $g_n(c;\lambda)$ are quite consistent with existing theories of probability distributions. For example as can be seen from 4 the sum of all probability values over its range of definition is always 1.0.Equation 5 in particular also conforms with the known fact that first moments of distributions about then mean, $\lambda = -\mu_1(c)'$ is always zero. If in Equation 7, we let n=2,that is if the second moment of a distribution is taken about its mean, the resulting value is the variance of that distribution.

As noted above, generalized moment generating function $g_n(c;\lambda)$ may be used to obtain all conceivable moments of a continuous distribution. For example the variance, third and fourth moments of the distribution of $Y = X^c$ are obtained from Equation 2 by setting $\lambda = -\mu_1(c)' = -\mu$ where μ is the mean of X^c .Thus

$$\mu_2(c) = g_2(c;-\mu), \tag{8}$$

$$\text{the variance of } X^c$$

$$\mu_3(c) = g_3(c;-\mu), \tag{9}$$

The third moment of X^c about its mean; and

$$\mu_4(c) = g_4(c;-\mu), \tag{10}$$

The fourth moment of the distribution of X^c about its mean. Hence the Skewness Sk(c) and Kurtosis, Ku(c) of the distribution of X^c are obtained as respectively

$$Sk(c) = \frac{\mu_3(c)}{(\mu_2(c))^{3/2}} = \frac{g_3(c;-\mu)}{(g_2(c;-\mu))^{3/2}} \tag{11}$$

And

$$Ku(c) = \frac{\mu_4(c)}{(\mu_2(c))^2} = \frac{g_3(c;-\mu)}{(g_2(c;-\mu))^2} \tag{12}$$

We now illustrate the use of $g_n(c;\lambda)$ with some examples.

Example 1

Suppose the random variable X has the probability density function (pdf),f(x)=2x,0¡x¡1.Interest is to find an expression for the estimation of all conceivable moments of the random variable Y=X. Note that conventionally the mean and variance of Y=X is by definition. $\mu = \mu_1' = \frac{2}{3}$ *and* $\sigma^2 = \frac{1}{18}$

Now to develop a more generalized expression for obtaining these moments and more, we have from Equation 2 that

$$g_n(c;-\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{2}{r+2}, where\ \mu_r(c)' = \frac{2}{cr+2} = \frac{2}{r+2},$$

Since in the present example c=1.

The first moment (n=1) of Y=X about λ is from the above expression

$$g_1(1;\lambda) = \frac{\lambda+2}{1+2} = \lambda + \frac{2}{3}$$

If now $\lambda = 0, then\ g_1(1;0) = 0 + \frac{2}{3} = \frac{2}{3} = \mu_1' = \mu$ the mean of Y=X as earlier obtained.

Hence if $\lambda = -\frac{2}{3}$ then as expected $g_1(1;-\frac{2}{3}) = 0.$ If now we set n=2,that is if interest is in determining the second moment of Y=X about λ we have

$$g_2(1;-\lambda) = \lambda^2 + 2\lambda.\frac{2}{3} + \frac{2}{4} = \lambda^2 + \frac{4\lambda.}{3} + \frac{1}{2}.$$

If we now let $\lambda = -\mu_1(1)' = -\mu = -\frac{2}{3}$, then we would have that

$\left(-\frac{2}{3}\right)^2 + \frac{4}{3}\left(-\frac{2}{3}\right) + \frac{1}{2} = \frac{4}{9} - \frac{8}{9} + \frac{1}{2} = \frac{1}{1}\frac{}{8} = \sigma^2$, the variance of Y=X as obtained earlier. If we

Notes

had chosen $c=\frac{1}{2}$, that is if interest is in determining the moments of $Y=X^{\frac{1}{2}}$ *about* λ is $g_1(\frac{1}{2};\lambda)=\lambda+\frac{2}{\frac{1}{2}+2}=\lambda+\frac{4}{5}$ so that if $\lambda=0$, then, $g_1(\frac{1}{2};0)=\frac{4}{5}$.

Notes

If $\lambda=-\mu_1(\frac{1}{2})'=-\mu=-\frac{4}{5}$ *then* $g_1(\frac{1}{2};-\frac{4}{5})=0$.

The second moment of $Y=X^{\frac{1}{2}}$ *about* λ is $g_2(\frac{1}{2};\lambda)=\lambda^2+2\lambda(\frac{4}{5})+\frac{2}{3}$. Hence, if

$$\lambda=-\mu_1(\frac{1}{2})'=-\mu=-\frac{4}{5},$$

$g_2(\frac{1}{2};-\mu)=g_2(\frac{1}{2};-\frac{4}{5})=\left(-\frac{4}{5}\right)^2+2\left(-\frac{4}{5}\right)\left(\frac{4}{5}\right)+\frac{2}{3}=\frac{2}{3}-\frac{16}{25}=\frac{50-48}{75}=\frac{2}{75}=\sigma^2$, the variance of the distribution of $Y=X^{\frac{1}{2}}$ as obtained using the traditional method. If the researcher had wanted to estimate these moments using moment generating function we would have that the corresponding moment generating function for Y=X is $M_Y(t)=M_X(t)=\frac{te^t-(e^t-1)}{t^2}=\frac{1+te^t-e^t}{t^2}$, which is fairly cumbersome even if differential with respect to t, the resulting derivative do not exist at t=0.Hence the method of moment generating function cannot possible be used to obtain the moments of the distribution of the random variable Y=X and other similarly specified distributions.

Example 2.Beta family of Distributions

Suppose interest is in finding the gmgf of the distribution of the random variable $Y=X^c$,where X has the beta distribution with parameters α *and* β and pdf ,f(x) given as

$$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).\Gamma(\beta)}.x^{\alpha-1}.(1-x)^{\beta-1},0<x<1;\alpha>0,\beta>0.$$

To obtain the required gmgf we have from Equation 3 that

$$\mu_r(c)'=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).\Gamma(\beta)}\int_0^1 x^{cr}x^{\alpha-1}.(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).\Gamma(\beta)}\int_0^1 x^{cr+\alpha-1}.(1-x)^{\beta-1}dx$$

Or

$$\mu_r(c)'=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).}.\frac{\Gamma(cr+\alpha)}{\Gamma(cr+\alpha+\beta)} \quad (13)$$

Hence from Equation 2,we have that the gmgf of the beta family of distributions represented by the random variable $Y=X^c$ is

$$g_n(c;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).}.\frac{\Gamma(cr+\alpha)}{\Gamma(cr+\alpha+\beta)} \quad (14)$$

All desired moments of the beta family of distributions may be obtained using Equation 14.For instance, the first moment of X^c *about* λ is

$$g_1(c;\lambda) = \lambda + \mu_r(c)' = \lambda + \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}.\frac{\Gamma(cr+\alpha)}{\Gamma(cr+\alpha+\beta)}$$

If c=1 that is if interest is in the first moment or mean of Y=X then we have $g_1(1;\lambda) = \lambda + \frac{\alpha}{\alpha+\beta}$ *so that if* $\lambda = 0$ *that is if the moment is taken about zero,*

then $g_1(1;0) = \mu_r(1)' = \frac{\alpha}{\alpha+\beta} = \mu,$

The mean of the beta distribution:If n=2 and c=1 then

$$g_2(1;\lambda) = \lambda^2 + 2\lambda\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Hence if $\lambda = -\mu_1(c)' = -\mu = \frac{-\alpha}{\alpha+\beta}$, *then we have*

$$g_2\left(1;\frac{\alpha}{\alpha+\beta}\right) = \left(-\frac{\alpha}{\alpha+\beta}\right)^2 + 2\left(-\frac{\alpha}{\alpha+\beta}\right)\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha.\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \sigma^2,$$

Which is the variance of the beta distribution. Other moments such as the skewness and kurtosis (Equation 11 and 12) of the various forms of the beta family of distributions can be similarly obtained. This is left as an exercise for the reader. If in Equation 14 we set $\alpha = \beta = 1$then we obtain the gmgf of the uniform distribution in a generalized form as

$$g_n(c;\lambda) = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}.\frac{\Gamma(cr+1)}{\Gamma(cr+2)} = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{1}{(cr+1)}. \quad (15)$$

Note that the moments of the beta family of distributions which are easily obtained using the present method namely $g_n(c;\lambda)$, cannot possibly be obtained using the method of moment generating function because the usual or traditional moment generating function do not exist for the better family of distributions.

Example 3.The Gamma family of distributions

Suppose the random variable X has the gamma distribution with parameters α *and* β and pdf,f(x) given as

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1}.e^{-\frac{x}{\beta}}; x \geq 0; \alpha > 0; \beta > 0.$$

Interest is in determining the gmgf of the random variable $Y = X^c$ where X has the gamma distribution.To do this we have from Equation 3 that

Notes

$$\mu_r(c)' = \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_0^{\infty} x^{cr} x^{\alpha-1} e^{-\frac{x}{\beta}}.dx = \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_0^{\infty} x^{cr+\alpha-1} e^{-\frac{x}{\beta}}.dx$$

Letting $V = \frac{x}{\beta}$, integrating and simplifying we have that

Notes

$$\mu_r(c)' = \beta^{cr}.\frac{\Gamma cr+\alpha}{\Gamma\alpha.} \quad (16)$$

Hence using Equation 16 in Equation 2 yields the gmgf of the gamma family of distributions represented by the random variable $Y = X^c$,as

$$g_n(c;\lambda), = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}.\frac{\Gamma cr+\alpha}{\Gamma cr}. \quad (17)$$

As usual all conceivable moments of the gamma family of distributions are obtained using Equation 17.For example the variance of $Y = X^c$ is

$$g_2(c;\lambda), = \lambda^2 + 2\lambda.\frac{\Gamma c+\alpha.\beta^c}{\Gamma\alpha} + \frac{\Gamma 2c+\alpha.\beta^{2c}}{\Gamma\alpha}$$

If c=1 then

$$g_2(1;\lambda), = \lambda^2 + 2\lambda.\alpha\beta + \alpha(\alpha+1).\beta^2$$

Hence if $\lambda = -\mu = -\alpha\beta$ *where* $\mu = \alpha\beta$ is the mean of the usual gamma distribution, then $g_2(1;\alpha\beta) = (-\alpha\beta)^2 + 2(-\alpha\beta)(\alpha\beta) + \alpha(\alpha+1).\beta^2 = \alpha\beta^2 = \sigma^2$, the variance of the usual gamma distribution. Note that the third moment of the gamma family of distributions about λ is obtained from Equation 9 as

$$g_n(n;\lambda) = \lambda^3 + 3\lambda^2.\beta^c.\frac{\Gamma c+\alpha}{\Gamma\alpha} + 3\lambda.\beta^{2c}\frac{\Gamma 2c+\alpha}{\Gamma\alpha} + \beta^{3c}\frac{\Gamma 3c+\alpha}{\Gamma\alpha}$$

If in particular $c = 1$ *and* $\lambda = -\alpha.\beta$ *where* $\alpha.\beta$ is the mean of the gamma distribution, then we have that

$$g_3(1;\alpha\beta) = (-\alpha\beta)^3 + 3(-\alpha\beta)^2.(\alpha\beta) + 3(-\alpha\beta)\alpha(\alpha+1).\beta^2 + \alpha(\alpha+1)(\alpha+2).\beta^3 = 2\alpha$$

Hence the Skewness of the beta distribution is easily obtained using Equation 11 as

$$Sk(1) = \frac{g_3(1;-\alpha\beta)}{g_2(1;-\alpha\beta)^{\frac{3}{2}}} = \frac{2\alpha\beta^3}{(\alpha\beta^2)^{\frac{3}{2}}} = \frac{2}{\alpha^{\frac{1}{2}}}$$

Similarly the fourth moment of the beta distribution about its mean is

$$g_4(1;\alpha\beta) = (-\alpha\beta)^4 + 4(-\alpha\beta)^3(\alpha\beta) + 6(-\alpha\beta)^2\alpha(\alpha+1).\beta^2 + $$
$$+4(-\alpha\beta)\alpha(\alpha+1)(\alpha+2).\beta^3 + \alpha(\alpha+1)(\alpha+2)(\alpha\beta)\beta^4 = 6\alpha\beta^4$$

Hence the corresponding Kurtosis is

$$Ku(1;-\alpha\beta) = \frac{g_4(1;\alpha\beta)}{(g_4(1;\alpha\beta))^2} = \frac{6\alpha\beta^4}{(\alpha\beta^2)^2} = \frac{6}{\alpha}$$

Notes

Setting $\alpha = 1$ in Equation 17 gives the gmgf of all forms of the exponential distribution as

$$g_n(n;\lambda) = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}.\beta^{cr}.\overline{)cr} + 1 = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}.\beta^{cr}(cr).\overline{)cr} \qquad (18)$$

Similarly setting $\beta = 2 \text{ and } \alpha = \frac{k}{2}$ where k=1,2,...gives the gmgf of the chi-square distribution with k degrees of freedom as

$$g_n(c;\lambda) = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}.2^{cr}.\frac{\overline{)cr + \frac{k}{2}}}{\overline{)\frac{k}{2}}} \qquad (19)$$

Now as noted earlier gmgf s' can be used to obtain moments of powers of random variables with negative indeces.This is possible for example with the gamma distribution if in Equation 17, $cr + \alpha > 0$; that is if the real number c is such that $'c' \geq \frac{-\alpha}{r}$, for r=1,2,.. and some specified value of $\alpha > 0$. For example if in Equation 19,we choose $'c' = -\frac{3}{2} \text{ and } k = 10$, and interest is in determining all possible moments of the random variable $Y = X^{-\frac{3}{2}}$, where X has the chi-square distribution with 10 degree of freedom. Then it is possible to generate moments up to the third moment of this random variable. Specifically the possible moments of $Y = X^{-\frac{3}{2}}$, are obtained from Equation 19 as

$$g_1(-\frac{3}{2};\lambda) = \lambda + 2^{-\frac{3}{2}}\frac{\overline{)-\frac{3}{2}+5}}{\overline{)5}} = \lambda + 2^{-\frac{3}{2}}\frac{\overline{)\frac{7}{2}}}{24} = \lambda + \frac{5\overline{)\pi}\,2^{-\frac{3}{2}}}{64}$$

Hence setting $\lambda = 0$ we have that the mean of the random variable $Y = X^{-\frac{3}{2}}$, where X has the chi-square distribution with degrees of freedom is $\mu = \frac{5\overline{)\pi}\,2^{-\frac{3}{2}}}{64} = \frac{5\overline{)2\pi}}{256}$

If n=2 the the variance of $Y = X^{-\frac{3}{2}}$, is obtained from the expression

$$g_2(-\frac{3}{2};\lambda)=\lambda^2+2\lambda\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)+2^{-3}\frac{\sqrt{-3}+8}{\sqrt{5}}=\lambda^2+2\lambda\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)+\frac{2^{-3}}{24},\ \textit{so that setting}\ \lambda=-\mu=\frac{-5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}=\frac{5\sqrt{2\pi}}{256},\ \textit{gives}$$

$$g_2(-\frac{3}{2};-\mu)=\left(\frac{-5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)^2+2\left(\frac{-5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)+\frac{2^{-3}}{24}=\frac{1}{192}-\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)^2=\frac{1}{192}-\frac{78.575}{32768}=0.005-0.002=0.003$$

Notes

Although the usual or traditional moment generating function about the origin or zero and about the mean $\alpha\beta$ of the gamma family of distributions namely

$$M(t)=(1-\beta t)^{-\alpha}\ \textit{and}\ M(t;\alpha\beta)=e^{-\alpha\beta t}(1-\beta t)^{-\alpha}$$

Respectively exist they are often relatively difficult to differentiate in practical applications and evaluate the required moments. Furthermore these moments cannot possibly be used if 'c' is not a whole number.

Example 4: The Normal Distribution

To obtain the gmgf of the random variable $Y=X^c$ where X has the normal distribution with parameters μ *and* σ^2 with pdf,f(x) given by

$$f(x)=\frac{1}{\sigma^2\sqrt{2\pi}}e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2},-\infty<x<\infty;-\infty<\mu<\infty;\sigma^2>0.$$

We have from Equation 3 that the rth moment of X^c about the origin or zero is

$$\mu_r(c)'=\frac{1}{\sigma^2\sqrt{2\pi}}\int_{-\infty}^{\infty}x^{cr}.e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2}dx$$

Now letting $'V'=\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2$, solving for x, expanding binomially and simplifying gives

$$\mu_r(c)'=\sum_{t=0}^{cr}\binom{cr}{t}\mu^{cr-t}\left(2\sigma^2\right)^{\frac{t}{2}}\frac{1}{\sqrt{\pi}}\int_0^{\infty}V^{\frac{t}{2}+\frac{1}{2}-1}e^{-V}dV$$

Or

$$\mu_r(c)'=\sum_{t=0}^{cr}\binom{cr}{t}\mu^{cr-t}\left(2\sigma^2\right)^{\frac{t}{2}}\frac{\sqrt{\frac{t}{2}+\frac{1}{2}}}{\sqrt{\pi}}\qquad(20)$$

For t=0,2,4;etc,that is for all even numbers. Hence the gmgf of the normal distribution represented by the random variable $Y=X^c$, where X is normally distributed with mean μ and variance σ^2 is from Equation 2

$$g_n(c;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\sum_{t=0}^{cr}\binom{cr}{t}\mu^{cr-t}\left(2\sigma^2\right)^{\frac{t}{2}}\frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\Gamma\pi} \qquad (21)$$

For all even numbers t,that is for t=0,2,3.That is provided we set $\left(2\sigma^2\right)^{\frac{t}{2}}\frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\Gamma\pi}=0$ for all odd values of t. Since with $V=\frac{x-\mu}{\sigma}$ we have that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}V^t e^{-\frac{V^2}{2}}dV=0$ for all odd values of t, that is for t=1,3,5…as may be easily verified. As with other gmgfs Equation21 may be used to generate all conceivable moments of all forms of the normal distribution represented by the random variable $Y=X^c$ for all real valued numbers 'c' provided $\mu_r(c)'$ is evaluated for all even values of 's'that is for all s=0,2,4,…etc.

For example the fourth moment of the random variable $Y=X^c$ about λ where X has the normal distribution with parameters μ *and* σ^2 is from Equation 21 with 'c'=1

$$g_4(1;\lambda)=\lambda^4+4\lambda^3\mu+6\lambda^2\left(\mu^2+\sigma^2\right)+4\lambda\left(\mu^3+3\mu\sigma^2\right)+\left(\mu^4+6\mu^2\sigma^2+3\sigma^4\right)$$

If the moment is now taken about the mean, that is if we set $\lambda=-\mu$ we obtain the fourth moment of the normal distribution about its mean μ as

$$g_4(1;-\mu)=\mu^4-4\mu^4+6\mu^4+6\mu^2\sigma^2-4\mu^4-12\mu^2\sigma^2+\mu^4+6\mu^2\sigma^2+3\sigma^4=3\sigma^4$$

Hence from Equation 12 we obtain the Kurtosis of the normal distributions

$$Ku(1)=\frac{g_4(1;-\mu)}{\left(g_4(1;-\mu)\right)^2}=\frac{3\sigma^4}{\left(\sigma^2\right)^2}=3, \exp ected.$$

These results are relatively much more difficult to obtain using the usual conventional moment generating function of the normal distribution. Thus even if one uses the moment generating function of the normal distribution about its mean μ namely $e^{\frac{1}{2}\sigma^2 t^2}$, differentiation and evaluation of this function at t=0 up to four times is clearly tedious and cumbersome. The generalized moment generating functions (gmgf) of some other continuous probability distributions are similarly obtained using Equation 2.The results are summarized in Table 1 below.

Table 1 : Generalized Moment Generating Functions (gmgf) of Some Continuous probability Distributions

S/No	Distribution	Gmgf $g_n(c;\lambda)$	Mgf about mean $(\mu)\left(M^t_{X-\mu g}\right)$
1	Beta $\left(B(\alpha;\beta)\right)$	$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\cdot\frac{\Gamma(\alpha+\beta)\Gamma(cr++\alpha)}{\Gamma(\alpha)\Gamma(cr++\alpha+\beta)}$	-
2	Uniform $\left(\beta(1;1)\right)$	$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{\Gamma(cr+1)}{\Gamma(cr+2)}$	-

Notes

Notes

3	Gamma $(G(\alpha;\beta))$	$\sum_{r=1}^{n}\binom{n}{r}\lambda^{n-r}\beta^{cr}\frac{\Gamma cr.\alpha}{\Gamma\alpha}$	$e^{-\alpha\beta t}.(1-\beta t)^{-\alpha}$
4	Exponential $(G(\alpha\beta))$	$\sum_{r=1}^{n}\binom{n}{r}\lambda^{n-r}\beta^{cr}\Gamma cr.+1$	$e^{-\beta t}.(1-\beta t)^{-1}$
5	Square (k) $\left((\frac{k}{2};2)\right)$	$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}2^{cr}\frac{\Gamma cr.+\frac{k}{2}}{\Gamma\frac{k}{2}}$	$e^{-kt}.(1-2t)t\frac{k}{2}$

Relationship between Generalized moment Generating Function and Factorial Moment generating Function about λ

The factorial moment generating function of the random variable X is usually defined as $F_X(t)=Et^X$ for some positive real value 't'. This expression is however only able to provide estimates of the moments of the distribution of the random variable X about zero or the origin. But just as is the case with generalized moment generating functions, a generalized approach to factorial moment generating functions would be more instructive and helpful in obtaining these moments about any desired real number, such as λ as used above. This would enable easier and quicker calculation of higher and more complicated moments of the distribution of the random variable X.

Thus, specifically suppose interest is in finding a generalized factorial moment generating function of the random variable $Y=X^C$ about an arbitrary chosen real number λ which is not necessarily zero or the mean of X^c,where 'c' is any real number. Furthermore suppose that X is for the present purposes a continuous random variable with probability density function, f(x),for $-\infty<x<\infty$. Then the generalized factorial moment generating function for X^C *about* λ is given by

$$F_X(c,t;\lambda)=Et^{(X^C+\lambda)}=\int_{-\infty}^{\infty}t^{(X^C+\lambda)}.f(x)dx \qquad (22)$$

Now Equation 22 can be alternatively expressed as

$$F_X(c,t;\lambda)=\int_{-\infty}^{\infty}t^{l\,\mathrm{int}^p}.f(x)dx=\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(l\,\mathrm{int})^n}{n!\,p^n}f(x)dx \qquad (23)$$

where $p=x^c+\lambda.$

$$F_X(c,t;\lambda)=\sum_{n=0}^{\infty}\frac{(l\,\mathrm{int})^n}{n!}\int_{-\infty}^{\infty}(x^c+\lambda)^n=\sum_{n=0}^{\infty}\frac{(l\,\mathrm{int})^n}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr}.f(x)$$

That is

$$F_X(c,t;\lambda)=\sum_{n=0}^{\infty}\frac{(l\,\mathrm{int})^n}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\mu_r(c)' \qquad (24)$$

Hence the nth factorial moment of the distribution of X^c about λ is the coefficient of $\frac{(l\,\text{int})^n}{n!}$ in the series expansion of Equation 23 or the nth derivative of this Equation, that is of $F_X(c,t;\lambda)$ with respect to t, evaluated at t=1.The result is seen to actually be the same as the generalized moment generating function, $g_n(c;\lambda)$ of the distribution of the random variable X^C about λ as already obtained above (Equation 2).Thus with the present approach, generalized moment generating functions are essentially the same yielding the same results, and need not be treated as different concepts.

II. Summary and Conclusion

We have presented in this paper method of obtaining the moment of some continuous bi-variate distributions with parameters α_1, β_1 *and* α_2, β_2 in finding the nth moment of the variable $x^c y^d\left(c \geq 0, d \geq 0\right)$ where X and Y are continuous random variables having the joint pdf, f(x,y).The proposed methods were the so called $g_n(c,d)$ defined $g_n(c,d) = E\left(X^c Y^d + \lambda\right)^n$, the nth moment of expected value of the t distribution of the cth power of X and dth power of Y about the constant λ.These moments are obtained by the use of bi-variate moment generating functions, when they exist. The proposed $g_n(c,d)$ exists for all continuous probability distributions unlike some of its competitors such as factorial moments of moment generating function which do not always exist. The results obtained using $g_n(c,d)$ are the same as results obtained using such other methods as moment generating functions of available. The proposed method is available and easy to use without the need for any modifications even when the powers of the random variable being considered are non-negative real numbers that do not need to be integers. The results obtained using $g_n(c,d)$ are the same as results obtained using other methods such as moment generating functions when they exist.

Notes

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