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By Bright O. Osu., Jonathan O. Egemba & Philip U. Uzoma

Abia State University, Nigeria

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I. INTRODUCTION

Asset-liability management is a means of managing the risk that can arise from the changes in the relationship between assets and liabilities. In cases such as in portfolio containing option as well as credit portfolio (i.e. wealth distributions that are highly skewed), it is reasonable to consider asymmetric risk measures since individuals are typically loss averse. Value-at-risk (VaR) and tail conditional expectation (TCE) have also emerged in recent years as standard tools for measuring and controlling the risk of trading portfolios. In some dynamical settings however, the limits of TCE can be transformed into the limits of VaR and conversely even though TCE is more preferable to VaR since it is coherent and VaR is not. We recently discovered in literature that the law of the iterated logarithm (LIL) obeys these coherencies.

The law of the iterated logarithm (LIL) is one of the most important results on the asymptotic behaviour of finite-dimensional standard Brownian motion (Dvoretzky and Erdos, 1951). Its classical laws as fundamental limit theorems in probability theory plays an important role in the development of probability theory and its application. The original statement of LIL obtained by (Khinchine 1924) is for a class of Bernoulli random variables. Kolmogorov (1929) and Hartman-Winter (1941) extended Khinchine's result to large class of independent random variables. Levy (1937) extended Khinchine's

Authors α σ: Department of Mathematics, Abia State University P M B 2000, Uturu, Nigeria. e-mails: megaobrait@yahoo.com, megaobrait@gmail.com, Jonathan.egemba@gmail.com

Author ρ: Department of Mathematics and Statistics, Federal Polytechnic Nekede, P. M. B. 1036, Owerri, Nigeria.
e-mail: Uzomaphilip@gmail.com

result to martingales, an important class of dependent random variables. Strassen(1964) extended Hartman-wintner's result to large classes of functional random variable. After that the research activity of LIL has enjoyed a rich classical period and a modern resurgence (Stout, 1974). To extend the LIL, a lot of fairly neat methods have been found (De Acosta, 1983). However, the key in the proofs of LIL is the additivity of the probabilities and expectations. In practice, such additivity assumption is not feasible in many areas of application because the uncertainty phenomenon cannot be modeled using additive probabilities or probability expectations. As an alternative to the traditional expectations or probability, capacities or non-linear probabilities /expectations have been studied in many fields such as statistics, finance and economics. In statistics, capacities have been applied in robust statistics (Huber 1981), under the assumption of two alternating capacity (Huber and strassen, 1973). Financial risk management is vital to the survival of financial institutions and the stability of the financial system. A fundamental task in risk management is to measure the risk-tailed by a decision, such as the choice of a portfolio (Osu et al., 2013). Recently, the substitution of variance as a risk measure in the standard Markowitz (1952) mean-variance problem has been emphasized, because it makes no distinction between positive and negative deviations from the mean. Variance is a good measure of risk only for distributions that are (approximately) symmetric around the mean such as the normal distribution or more generally, elliptical distributions (Frey and Embrechts, 2006). In capital requirement Logistics is the single most powerful force on risk management in finance. Because it is the intersection of the virtual and physical world of finance that allows one to keep up to date information of where everything and anything are or is going at a particular moment (Achi et al, 2013). Money can be invested and produce more money. However, investing money involves different level of risks depending on the choice of the investment and a high rate or value of money at risk bring about high target of positive expected returns. (Gerber 1979). The LIL assumption can be represented as the assumption of an expected rate of returns on high target, that is the best guess estimate of tomorrow's return level. Since there is no relevant information available at time t that could help forecast returns at time $t + 1$. It is well known in finance that an important framework is calculating the price of uncertainty option claim.

The objective of this work is to investigate if the classical law of iterated logarithm can be centralized for the contingent hedging capacities which depends on its completeness and uniqueness and to show how one should calculate returns of high diversified portfolio to maximize the capital growth in returns by measuring the risk involved to know the future returns on target.

II. FRAME WORK OF LIL HEDGING PRICING CAPACITY (RESULT)

Consider a sequence of independent and identically distributed (iid) random variable $X_1, X_2, \dots, X_n$ with $E(X_n) = 0, Var(X_n) = \sigma^2, \sigma > 0$. Then

$$P \left\{ \log_{n \rightarrow \infty} \sup \frac{S_n}{(2\sigma^2 \log \log n)^{\frac{1}{2}}} = 1 \right\} = 1 \quad (2.1)$$

This implies that with probability one and for

$$\lim_{n \rightarrow \infty} \sup \frac{S_n}{(2\sigma^2 \log \log n)^{\frac{1}{2}}} = \lim_{n \rightarrow \infty} \sup \frac{Z_n}{(2 \log \log n)^{\frac{1}{2}}}, \quad (2.2)$$

then $Z_n > (c \log \log n)^{\frac{1}{2}}$ for infinitely many n if $c < 2$, but for only finitely many n if $c > 2$.

Put in another way, let $\{X_n, n \in W\}$ be a sequence of iid random variable on a probability space $\{\Omega, \mathcal{F}, P\}$, let $S_n = X_1 + X_2 + \dots$ and set $Z_n = \frac{S_n - \mu n}{\sigma \sqrt{n}}$ (where μ is the expectation and σ is the standard deviation). Then we define the law of iterated logarithm for a stationary independent process thus:

$$\lim_{n \rightarrow \infty} \sup \frac{Z_n}{(2 \log \log n)^{\frac{1}{2}}} = 1. \quad (2.3)$$

Similarly with probability one,

$$\lim_{n \rightarrow \infty} \sup \frac{Z_n}{(2 \log \log n)^{\frac{1}{2}}} = -1. \quad (2.4)$$

Since a supremum expectation (super hedge) of LIL is sublinear it is continuous, hence complete (unique) which makes it have a hedging pricing capacity. A good hedging pricing capacity model according to Artzner et al, (1997) must be complete, if it is sublinear and continuous. Completeness implies Uniqueness and continuous implies completeness.

Given a set P of multiple prior probability measure on $(\Omega, \mathcal{F})$, let X be the set of random variable on $(\Omega, \mathcal{F})$, where Ω = sample space and $\mathcal{F}$ is the increasing sequence of Ω . For any $\xi \in X$, we define a pair of maximum (super hedge) and minimum (subhedge) expectation as $(E, \underline{E})$ by (Peng 2006-2009):

$$E(\xi) = \sup_{Q \in P} E_Q(\xi) \Rightarrow \text{Super hedge pricing} \quad (2.5)$$

$$\underline{E}(\xi) = \inf_{Q \in P} E_Q(\xi) \Rightarrow \text{Super hedge pricing}, \quad (2.6)$$

where $E_P(\cdot)$ denotes the classical expectation under probability measure P . Let $\xi = I_A$ for $A \in \mathcal{F}$, immediately, a pair of (V, v) of capacities is given by $V(A) := \sup_{P \in P} P(A)$, $v(A) := \inf_{P \in P} P(A)$, $\forall A \in \mathcal{F}$.

According to Peng(2007) E is called sub-linear expectation in the sense that A functional E on $X \rightarrow (-\infty, +\infty)$ is called a sub-linear expectation, if it satisfies the following properties for all $x, y \in X$. (coherent properties)

1. Monotonicity: $x > y$ implies $E(x) \geq E(y)$.
2. Constant preserving: $E(c) = c \forall c \in R$.
3. Sub-additivity: $E[x + y] \leq E(x) + E(y)$ hedging property.
4. Positive homogeneity: $E(\lambda x) = \lambda E(x), \forall \lambda \geq 0$.

Note: A sublinear expectation is a supremum expectation (Cheng, 2009).

Remark

If a market is complete and self financing, then there exist a neutral probability measure P such that the pricing of any discounted contingent claim ξ in this market is given by $E(\xi)$ then by LIL $\mu = E_P(\xi)$ and variance $\sigma^2 = E_P[(\xi - \mu)^2]$ with probability one.

$\mu = \lim_{n \rightarrow \infty} \frac{1}{n} S_n$, $\sigma = \lim_{n \rightarrow \infty} \sup (2_n \log \log n)^{-\frac{1}{2}} |S_n - n\mu|$ where S_n is the sum of the first n of a sample (X_i) with mean μ and variance σ^2 .

Note; Every sublinear expectation is a supremum expectation /continuous expectation.

The question is can the classical supremum/ superhedge expectation of LIL be centralized for contingent hedging pricing capacity? By the definition, a pricing hedging capacity is called continuous capacity if it satisfies the following desirable axioms or properties. (Wasserman and Kadane 1990).

Given a set function $P: f \rightarrow [0,1]$ then it is a continuous capacity if it satisfies the following:

1. Stability property: the system or function is stable if
2. $P(\phi) = 0, P(\Omega) = 1$;
3. If there exist any positive even bounded continuous function $P(x)$ where $x \in \mathbb{R}$, then for every $a, b \in \mathbb{R}$, $P(A) \leq P(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$ the function $P(x)$ is a self-financing value which completely determines the distribution x and also has a mathematical properties of the (2,1,3 and 4) (very important property of completeness).
4. $P(A_n) \uparrow P(A)$, if $A_n \uparrow A$, $\uparrow \Rightarrow$ Superhedging (Supremum expectation)
5. $P(A_n) \downarrow P(A)$, if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$ $\downarrow \Rightarrow$ Subhedging.

III. THE MODEL

Assuming $P(\xi)$ to be the risk neutral asset, that is the self financing value. If the simple European security V_b is hedgeable then for any positive bounded continuous function, there assume a portfolio process whose self-financing value process $P(\xi)$ of LIL supremum expectation that satisfies the continuous capacity property $P(\xi) \leq V_b(x)$ where X_i is adapted at time t for $V_b(x)$. if the result is satisfied, then it is complete and also a martingale.

a) *Lemma*

Suppose ξ is distributed to G normal $N(0; [\underline{\sigma}^2, \bar{\sigma}^2])$, where $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$. Let ϕ be a bounded continuous function. Furthermore, if ϕ is a positively even function, then for any $b \in \mathbb{R}$

$$e^{-b^2/2\sigma^2} \mathbb{E}[\phi(\xi)] \leq \mathbb{E}[\phi(\xi - b)] \quad (3.1)$$

(see Chen and Hu, 2013 for prove).

It has been shown by Mao(1997) that if X is the solution of the d - dimensional equation

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), t \geq 0, \quad (3.2)$$

and if there exist positive real numbers ρ, k such that for all $x \in \mathbb{R}^d$ and $t \geq 0$, $x^T f(x, t) \leq \rho$, and $\|g(x, t)\|$, then;

$$\lim_{t \rightarrow \infty} \sup \frac{|X(t)|}{\sqrt{2t \log \log t}} \leq k\sqrt{e}, \quad a.s. \quad (3.3)$$

Appleby and Wu (2008) had shown also that for X a unique continuous adapted process which obeys (3.2). Let $A := \{\omega: \lim_{t \rightarrow \infty} X(t, \omega) = \infty\}$. If

$$\lim_{x \rightarrow \infty} xf(x) = L_\infty; g(x) = \sigma, x \in \mathbb{R}, \quad (3.4)$$

where $\sigma \neq 0$ and $L_\infty > \frac{\sigma^2}{2}$, then $P[A] > 0$ and X satisfies for super hedge;

$$\lim_{t \rightarrow \infty} \sup \frac{|X(t)|}{\sqrt{2t \log \log t}} = |\sigma|, a.s. on A, \quad (3.5) \text{ and for sub hedge;}$$

$$\lim_{t \rightarrow \infty} \sup \frac{\log \frac{X(t)}{\sqrt{t}}}{\log \log t} = -\frac{1}{\frac{2L_\infty}{\sigma^2} - 1}, \text{ a.s. on } A \quad (3.6)$$

Theorem 1

If X (the capital allocation to the individual risk with $X = X_1 + X_2 + \dots + X_n$, where $X_1 + X_2, \dots, X_n$ are copies of X) obeys (3.2) and if there exist positive real numbers ρ and C_2 such that for $k \in K^d$ and $t \geq 0$, $xf(x, t) = P$ and $\|g(x, t)\| \leq C_2 K$ (where $\|\cdot\|_{op}$ denotes the operator norm), then and if in addition $\psi(t) = (2 \log t + C_k \log_2 t)^{1/2}$, $-C_2 = \frac{1}{1-c}$ and $n = \log \log t_2$. Then

$$E(\xi) = P(F_n) = \frac{1}{1-c} (\log_2)^{\frac{1-c}{4}} \begin{cases} c > 1 := \text{super hedge} \\ c < 1 := \text{sub hedge} \end{cases} \quad (3.7)$$

Proof;

Following the method variation of Brownian motion result (Osu, 2003) define the event

$$F_n = \{w: S(h_n) < h_{n-1}^{1/2} \phi(1/h_{n-1})\}, \quad (3.8)$$

where $h_n = e^{-n\rho}$, and $0 < \rho < \frac{1}{2}$.

Then the event

$$\{S_i(h_n) < h_{n-1}^{1/2} \psi(1/h_{n-1})\}, \quad (3.9)$$

for the independent and identically distributed (iid) random variables

$$S_i(h_n) = \sup R \left\{ X; t, t+h \right\}, i = 0, 1, \dots, \left\lfloor \frac{1}{2h_n} \right\rfloor;$$

are independent and have equal probabilities.

Moreover since

$$F_n \subseteq \bigcap_{i=0}^{\lfloor 1/2h_n \rfloor} \left\{ S_i(h_n) < h_{n-1}^{1/2} \psi \left(\frac{1}{h_{n-1}} \right) \right\}$$

then

$$P(F_n) \leq \left(P \left\{ S_0(h_n) < h_{n-1}^{1/2} \psi \left(\frac{1}{h_{n-1}} \right) \right\} \right)^{\lfloor 1/2h_n \rfloor} \quad (3.10)$$

By Kochen and Stone (1964), and the scaling property, we have

$$\left\{ S_0(h_n) < h_{n-1}^{1/2} \psi \left(\frac{1}{h_{n-1}} \right) \right\} = P \left\{ S(1) < h_n^{-1/2} h_{n-1}^{1/2} \psi \left(\frac{1}{h_{n-1}} \right) \right\} \leq 1 - C_0 \lambda_n e^{-\lambda_n^{2/2}}, \quad (3.11)$$

$$\text{where } \lambda_n = \left(\frac{h_{n-1}}{h_n} \right)^{\frac{1}{2}} \psi \left(\frac{1}{h_{n-1}} \right).$$

Hence

$$P(F_n) \leq \left(1 - C_0 \lambda_n e^{-\lambda_n^{2/2}} \right)^{\lfloor 1/2h_n \rfloor} = (1-u)^N, \quad (3.12)$$

say, where $u = C_0 \lambda_n e^{-\lambda_n^{2/2}}$ and $N = \lfloor 1/2h_n \rfloor$. And because $\log(1-u) < -u$, then

$$(1-u)^N = e^{\log(1-u)N} = e^{N \log(1-u)} < e^{-Nu}. \quad (3.13)$$

But

$$\begin{aligned} \lambda_n^2 &= \left(\frac{h_{n-1}}{h_n} \right) \left(2 \log \frac{1}{h_{n-1}} + C \log_2 \frac{1}{h_{n-1}} \right) \\ &= \left(\frac{e^{(n-1)\rho}}{e^{-n\rho}} \right) \left(2 \log e^{(n-1)\rho} + C \log_2 e^{(n-1)\rho} \right) \end{aligned}$$

$$\begin{aligned}
 &= \{1 + 0(n^{\rho-1})\} \{2n^{\rho} (1 + 0(n^{-1}) + C_{\rho}(\log_n + 0(n^{-1}))\} \\
 &= 2n^{\rho} + C_{\rho} \log_n + 0(n^{2\rho-1}) \\
 &= 2n^{\rho} + C_{\rho} \log_n + 0(1) \text{ since } \rho < \frac{1}{2}.
 \end{aligned} \tag{3.14}$$

Therefore $\lambda_n \sim C_{1n^{1/2\rho}}$ and

$$NU \sim \frac{1}{e} e^{n\rho} \cdot C_0 C_1 n^{\frac{1}{2\rho}} \exp \left\{ -n^{\rho} - \frac{1}{2} C_{\rho} \log n + 0(1) \right\} = C_2 n^{\delta},$$

where

$$\delta = \frac{1}{2} (1 - C) \rho > 0 \text{ if } C < 1 \tag{3.15}$$

Therefore

$$E(\xi) = P(F_n) < e^{-C_2 n^{\delta}}. \tag{3.16}$$

for large n , so that $\sum_{n=1}^{\infty} P(F_n) < \infty$, and, by Ugbebor (1980), F_n happens only finitely often.

Using equation (3.15), we have (for $\rho = \frac{1}{2}$) equation (3.7) as required.

IV. APPLICATION

We refer to equation (3.7) as the LIL measure. All investment involve some element of risk, but we are predicting a measure that will help attain a high target on expected returns on a risky portfolio by raising performances and cost. Banks meet their target by focusing on the high rate of returns. Hence LIL measure focus on the high rate of returns, because higher target implies higher investment which also implies high expected returns on target. Hence raises challenges of performance which value at risk and conditional value at risk ignore.

Banks expected returns are the risk free rate of capital plus a market premium. That is risk free rate + a market premium;

$$E(B) = R(x) + M(x) \tag{4.1}$$

and risk free rate equals solvency capital minus capital requirement for the risk. Capital requirement is the capital required in respect of a random variable (risk) with the view to avoiding insolvency or shortfall. The reason of solvency is to make sure that the bank have the financial means to meet its future obligation, to pay the present and future claims related to the policy holders and regulators. In order to avoid insolvency over the specific horizon at some given level of risk tolerance they should hold asset of value that is enough or small enough. Solvency capital requirement for the risk = Assets – Liabilities

$$A(x) - L(x) = S(x) \tag{4.2}$$

At least for values greater than the relevant VaR with probability function $f(y)$ then $CVaR$ for the normal distribution is shown to be;

$$CVAR_{\alpha} = \frac{\sigma}{1-\alpha} + \mu \Phi \left(\frac{Q_{\alpha} - \mu}{\sigma} \right). \tag{4.3}$$

For two and three parameter Weibull, Osu and Ogwo (2012) had shown that

$$CVaR_{\alpha} = \frac{x}{1-\alpha} e^{-Q_{\alpha}}, \tag{4.4}$$

$$CVaR_{\alpha} = \frac{e^{-Q_{\alpha}}}{(1-\alpha)}, \tag{4.5}$$

respectively. Equation (4.3) implies that CVAR is a little bigger than VAR and it can be adjusted for by adding an inverse of a decay constant (Klygman, 2004).

V. EMPIRICAL EXAMPLE

Calculate the CVAR of 1 million portfolio on a 100 basis point per day standard deviation, suppose that the daily returns are normally distributed with $\mu = 0$ on a 100 basis point per day.

Solution

Using (4.3), we have $\text{CVAR}_{5\%} = 20 + 16450 = 16470$.

Expected returns = $1\text{m} - 16470 = 983530$ meaning that there is 5% chance that the daily loss on 1m portfolio is equal or exceed only 16470 and a 95% chance that it will worth 983530 or more tomorrow. $\text{CV}_a R_{1\%} = \Phi_{0.99} = 2.326 \therefore \frac{2.326}{100} = 0.02326 \times 1\text{m} = 23260$ which is $\text{VAR}_{-(1\%)}$ on 1m portfolio. Hence, $1\text{m} - 23260 = 976740$ is the expected returns. Which means that there is 1% chance that the daily loss on 1m portfolio is equal or exceed only 23260 and 99% chance of being worth 976740 or more tomorrow.

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Using (3.7), we have the expected returns $20\text{m} - (-5128205.12) = 25128205.12$

VI. CONCLUSION

It shows that the classical LIL can be centralized for hedging pricing capacity as the supremum/sublinear expectation is continuous in the interval $[0,1]$. Hence investigating LIL for capacities shows that the supremum limit points of it lie with probability capacity one between the lower and upper standard bound and also satisfies the desirable axioms under the hedging pricing continuous capacity. Here laws of iterated logarithm (LIL) has been represented as the assumption of an ERR on a target of high diversified portfolio in bank's capital requirements as it utilizes information on the whole distribution, have a continuous hedging capacity, hence complete and unique. Which CVAR ignores useful information on. The measure on ERR curbs rates of returns on target, provides incentives for risk managers by raising challenges of performances and cost. Making it an optimal computational method to increase performances in hedging and banks attaining their targets on focus as it's application is a measure of a multifractal returns on banks portfolios.

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