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# Global Existence and Uniqueness of the Weak Solution in Keller Segel Model

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*Abstract-* This paper deals with the global existence, uniqueness and boundedness of the weak solution for the chemotaxis system **(P)** defined as

$$\begin{cases} u_t - \Delta u + \operatorname{div}(u\nabla c) = 0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ -\Delta c + \tau c = 0 & x \in \Omega \\ u = 0,\ c = g & \Gamma \\ u(0,x) = u_0 & x \in \Omega \end{cases} \quad \text{(P)}$$

The system **(P)** is under homogeneous Dirichlet boundary conditions in a convex smooth bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\Gamma$ ( $\in H^{\frac{3}{2}}(\Gamma)$and $u_0 \in H^{\frac{1}{2}}(\Omega)$ Based on Galerkin's method, Lax -Milgran's and maximum principle, a prove of the existence and uniqueness of a global solution for the system **(P)** is determined. Moreover we show that the unique solution is positive.

*Keywords and Phrases: chemotaxis, global existence, boundedness, positive solution.*

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*Strictly as per the compliance and regulations of*

Ref
[9] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. J. Theor. Biology. 26 (1970) 399-415.

# Global Existence and Uniqueness of the Weak Solution in Keller Segel Model

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## I. Introduction

Chemotaxis is an important means for cellular communication. It is the influence of chemical substances in the environment on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. The movement towards a higher concentration of the chemical substance is called positive chemotaxis whereas the movement towards regions of lower chemical concentration is called negative chemotactical movement.

The classical chemotaxis model — the so-called Keller–Segel model — system defined in (0.1) was first introduced by Paltak [11] (1953), E. Keller and L. Segel [9] (1970)

$$\begin{aligned} & u_t - \nabla(a \nabla u) + \nabla(\chi u \nabla c) = 0 && (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ & \alpha c_t - \Delta c + \tau c + \beta u = 0 && x \in \mathbb{R}^d \end{aligned} \qquad (0.1)$$

where $u(t,x)$ denotes the density of bacteria in the position $x \in \mathbb{R}^d$ and at time $t$, $c$ the concentration of chemical signal substance, $\alpha \geq 0$ the relaxation time, the parameter $\chi$ the sensitivity of cells to the chemoattractant and $a, \tau, \; \beta$ are given smooth functions. As it can be seen, when $\alpha \neq 0$ the model is

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called Parabolic-Parabolic while it is an Elliptic-Parabolic model when $\alpha = 0$. This modelling is very simple, it exhibits a profound mathematical structure and mostly only dimension 2 is understood, especially chemotactic collapse. The proposed model has been extensively studied in the last few years (see ([7]-[8],[12],[13]) for a recent survey articles).

The Parabolic-Parabolic model has been investigated by many authers (see for examples Refs [8] and [13]), I. Fatkullin [10] had developed numerical method ( a composite particle-grid ) with adaptive time stepping which allows us to resolve and propagates singular solutions when with Neumann boundary condition.

The Elliptic-Parabolic model has been investigated by many authers (see for examples Refs [3] and [4]). This model have been carried out where the main concern is whether the solution of model is bounded or blow-up. It has been proved that the solution strongly depends on the spatial dimension. It does not occur in one-dimensional problems, and it occurs conditionally in higher dimensional situations. More precisley, see [2] in case higher dimensions ($n \geq 3$), if the norm of initial condition $u_0$ is small in space $L^{\frac{n}{2}}(\mathbb{R}^n)$, then there are global weak solutions and if $\left(\int x^2 u_0\right)^{d-2} \leq C \|u_0\|^n_{L^1(\mathbb{R}^n)}$ with $C$ is small ,then there is blow up in a finite time $T^*$. But in two dimension, (see [5]), if $\|u_0\|_{L^1(\mathbb{R}^2)} < \frac{8\pi}{\chi}$ , there are smooth solutions, and if $\|u_0\|_{L^1(\mathbb{R}^2)} > \frac{8\pi}{\chi}$, there is creation of a singular measure (blow-up) in finite time.

In this paper we demonstrate the global existence and uniqueness of weak positive solution for the elliptic-parabolic model's problem defined as

$$(P)\begin{cases} (P_1)\begin{cases} u_t - \Delta u + \operatorname{div}(u\nabla c) = 0 & (t,x) \in \mathbb{R}^+ \times \Omega \\ u = 0 & \Gamma \\ u(0,x) = u_0 & x \in \Omega \end{cases} \\ (P_2)\begin{cases} -\Delta c + \tau c = 0 & x \in \Omega \\ c = g & \Gamma \end{cases} \end{cases}$$

Where $u(t,x)$ is a function denotes the density of bacteria in the position $x \in \Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, $\Omega$ is a bounded convex domain with smooth boundary $\Gamma$, $c$ denotes the concentration of chemical signal that stimulates the bacteria. The parameter $\tau$ is a time constant and it is expressed on the one hand the movement of bacteria (representing a random distribution side and a deterministic drift in the direction of high concentrations) and secondly the diffusion degradation of $c$.

To simplify the solution of the system (P), a decomposition of (P) into two subsystems ($P_1$) and ($P_2$) are adopted. Lax-Milgram's Theorem is very important theorem which we help us to demonstrate the existence and uniqueness of a weak solution for the system ($P_2$). However this theorem can not be applied directly because it is a nonhomogenous system. For this raison an adoptation of Trace Theorem is used to simplify the system ($P_2$), and together with Galerking method we can demonstrate the existence and uniquness of a weak solution for the system ($P_1$). Therefore we have the existence and uniquness for the problem (P). Moreover we show that the solution is positive. The following initial-boundary conditions on $u_0$ and $g$ assumptions are used to prove the proposed solution of (P)

$H_1$ $\quad g \in L^{\frac{1}{2}}(\Gamma)$
$H_2$ $\quad g \in L^{\frac{3}{2}}(\Gamma)$
$H_3$ $\quad u_0 \in L^2(\Omega)$
$H_4$ $\quad u_0 \geq 0$ and $g \geq 0$.

Ref

[8] T. Hillen, K. Painter, Global existence for a parabolic chemotaxis model with prevention of overcrowding. Adv. Appl. Math. 26 (2001) 280-301.

Notes

If the hypothesis $H_1$ is satisfies and using the theorem of trace, one can find a lifting of this trace which we denote $R(g) \in H_0^1(\Omega)$. Thus by definition it verifies $\gamma_0(R(g)) = g$. Now looking for $c$ having the form $c = \widetilde{c} + R(g)$ reduces the problem $(P_2)$ to $\widetilde{c}$.

$$\left(\widetilde{P}_2\right)\begin{cases} -\Delta\widetilde{c} + \tau\widetilde{c} - \Delta R(g) + \tau R(g) = 0 & \text{in } x \in \Omega \\ \widetilde{c} = 0 & \text{on } \Gamma \end{cases}$$

**Definition 1** *We say $(u, \widetilde{c}) \in L^2\left(0, T; H_0^1(\Omega)\right) \times H_0^1(\Omega)$ with $u_t \in L^2\left(0, T; H^{-1}(\Omega)\right)$ is a weak solution of the problem (P) if and only if*

$$\langle u_t, v\rangle + B(u, v, t) = 0 \qquad (0.2)$$

$$a(\widetilde{c}, q, t) = l(q) \qquad (0.3)$$

*where*

$$\begin{cases} B(u, v, t) = \int_\Omega (\nabla u \nabla v + \nabla c \nabla uv + \tau cuv)\, dx \\ a(\widetilde{c}, q) = \int_\Omega (\nabla\widetilde{c}\nabla q + \tau\widetilde{c}q)\, dx \\ l(q) = -\int_\Omega (\nabla R(g)\nabla q + \tau R(g)q)\, dx \end{cases}$$

*for all* $(v, q) \in \left(H_0^1(\Omega)\right)^2$, $0 \le t \le T$, *and*

$$u(0, x) = u_0 \in L^2(\Omega) \qquad (0.4)$$

**Remark 2** *Note that $u \in C\left([0, T]; L^2(\Omega)\right)$ as $u \in L^2\left(0, T; H_0^1(\Omega)\right)$ and $u_t \in L^2\left(0, T; H^{-1}(\Omega)\right)$. Then equality (0.4) makes sense.*

## II. Existence of Weak Solution of the Problem (P)

In this section, use the Theorem of Lax- Milgran to study the existence and uniqueness of weak solution of problem $(P_2)$, which its variational formulat is given by equation (0.3) and use the method of Galerking to study the existence and uniqueness of weak solution of problem $(P_1)$, which its variational formulat is given by equation (0.2). So we have the existence and uniqueness of weak solution of problem (P).

### a) *Existence of weak solution of the problem (P2)*

**Theorem 3** *If the hypothesis $H_1$ holds. Then the problem $(P_2)$ has only one solution $c \in H^1(\Omega)$ for any $q \in H^1(\Omega)$.*

By applying the Theorem of Lax-Milgran, the solution $\widetilde{c}$ of the problem (0.3) exists and it is unique. So $(P_2)$ has unique solution.

**Remark 4** *Elliptic regularity Theorem remains valid provided that the boundary condition $g$ is in the space $L^{\frac{3}{2}}(\Gamma)$ which is the image by the operator trace space $H^2(\Omega)$.*

**Remark 5** [6] *If $c \in H^2(\Omega)$ and ($c$ is a solution of problem $(P_2)$ ) this implies that $c \in W^{1,q}(\Omega)$ $(H^2(\Omega) \hookrightarrow W^{1,q}(\Omega)$ for $1 \le q \le 2^*)$.*

Using the Maximum Principle one can show that the solution of the problem $(P_2)$ is positive as follows. Multiplying the first equation of $(P_2)$ by $q \in H_0^1(\Omega)$, we obtain other variational formulat for problem $(P_2)$

$$\left(\widetilde{P}_3\right) \int_\Omega (\nabla c \nabla q + \tau cq)\, dx = 0.$$

**Proposition 6** [1] *If* $g \in L^{\frac{3}{2}}(\Gamma)$ *and* $c \in H^1(\Omega) \cap C(\overline{\Omega})$ *then the problem* $\left(\widetilde{P}_3\right)$ *have a positive solution c.*

**Proof.** As $\Gamma$ is smooth enough and $g \in L^{\frac{3}{2}}(\Gamma)$ then $c \in H^2(\Omega)$. And as $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, by embedding of Sobolev spaces ( $H^2(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$) this implies that $c \in C(\overline{\Omega})$. If $c = g \geq 0$ on $\Gamma$, then $c^- = \min(c, 0) \in H_0^1(\Omega)$. So, we have

$$\begin{aligned} \int_\Omega cc^- dx &= \int_\Omega (c^-)^2 dx \\ \int_\Omega \nabla c \nabla c^- dx &= \int_\Omega (\nabla c^-)^2 dx, \end{aligned}$$

Since the support of functions $c^-$and $c^+ = \max(c, 0)$ is set $A(x) = \{x/u(x) = 0\}$. This implies that $\nabla u = 0$ on $A(x)$. As $c = c^+ + c^-$, thus we have

$$0 = \int (\nabla c^-)^2 + \tau (c^-)^2 dx \geq \min(1, \tau) \|c^-\|^2_{H_0^1(\Omega)}$$

Finally, we find $c^- = 0$.

*b) Existence of a weak solution of the problem* ***(P1)***

Before proving the existence and uniqueness of weak solution of problem ($P_1$), we need the following lemma

**Lemma 7** *i) For all* $v \in H_0^1(\Omega)$ *then* $B(.,.,t)$ *is continuous in* $H_0^1(\Omega) \times H_0^1(\Omega)$, *there exists a constant positive C such that*

$$|B(u, v, t)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \tag{1.1}$$

*ii) For any* $u \in H_0^1(\Omega)$ *and* $H_2$ *is hold. Then there exists a constant positive* $\beta$ *such that*

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B(u, u, t)$$

**Proof.** i) We use the Cauchy-Shwarz inequality and $c \in H^2(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in \left[1, \frac{2n}{n-2}\right[$ with $n = 2$ or $n = 3$, we obtain i) as follows

$$\begin{aligned} B(u, v, t) &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\nabla c\|_{L^4(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^4(\Omega)} \\ &\quad + \tau \|c\|_{L^4(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^4(\Omega)} \\ &\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned}$$

ii) Making use of $-\Delta c + \tau c = 0$ the expression of $B(u, u, t)$ becomes

$$\begin{aligned} B(u, u, t) &= \int (\nabla u)^2 + \frac{\nabla c}{2} \nabla u^2 + \tau c u^2 dx \\ &= \int (\nabla u)^2 + \left(\tau c - \frac{\Delta c}{2}\right) u^2 dx \\ &= \int (\nabla u)^2 + \frac{1}{2} \tau c u^2 dx \geq \|\nabla u\|^2_{L^2(\Omega)}. \end{aligned}$$

Notes

Finally, by Poincarre inequality yields

$$B(u,u,t) \geq \beta \|u\|^2_{H_0^1(\Omega)}.$$

To demonstrate the existence of weak solution of (P$_1$) via the method of Galerking, we assume $w_k = w_k(x)$ are smooth functions verifying

$$\{w_k\}_{k=1}^{\infty} \quad \text{is an orthogonal basis of } H_0^1(\Omega) \tag{1.2}$$

and

$$\{w_k\}_{k=1}^{\infty} \quad \text{is an orthonormal basis of } L^2(\Omega). \tag{1.3}$$

Consider a positive integer $m$. We will look for a function $u_m : [0\ T] \to H_0^1(\Omega)$ of the form

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t)\, w_k \tag{1.4}$$

which satisfies

$$d_m^k(0) = (u_0, w_k) \tag{1.5}$$

and

$$\langle u_m', w_k\rangle + B(u_m, w_k, t) = 0, \quad 0 \leq t \leq T \text{ and } \quad k = 1, ..., m \tag{1.6}$$

where $u' = u_t$ and here $(.,.)$ denotes the scalar product in $L^2(\Omega)$.

**Theorem 8** *(construction of the approximate solution) For each integer $m$, there exists a unique function $u_m$ of the form* (1.4) *satisfying* (1.5) *and* (1.6).

**Proof.** Assuming $u_m$ has the structure (1.4). Substituting (1.4) into (1.5) and using (1.3) we obtained

$$d_m'^k(t) + \sum_{l=1}^{m} d_m^l B(w_l, w_k, t) = 0 \qquad 0 \leq t \leq T \text{ and } k = 1, ..., m \tag{1.7}$$

According to standard existence theory for ordinary differential equations, there exists a unique absolutely continuous functions $d_m(t) = \left(d_m^1, d_m^2, ..., d_m^m\right)$ satisfying (1.5) and (1.7). So $u_m$ of the form (1.4) satisfies (1.5) and (1.6) for all $t \in [0\ T]$.

### c) Energy estimates

We propose now to send $m$ to infinity and show a subsequence of our solutions $u_m$ of the approximation problems (1.5) and (1.6) converges to a weak solution of (P$_1$). For this we will need some uniform estimates.

**Theorem 9** (*Energy estimates*) [6]. *There exists a constant $C$, depending only on $\Omega$, $T$ and $c$, such that*

$$\max_{0\leq t\leq T} \|u_m\|_{L^2(\Omega)} + \|u_m\|_{L^2\left(0,T;\ H_0^1(\Omega)\right)} + \|u_m'\|_{L^2(0,T;\ H^{-1}(\Omega))} \leq C\,\|u_0\|_{L^2(\Omega)} \tag{1.8}$$

*for* $m = 1, 2, ...$

**Proof.** 1. Multiplying equation (1.6) by $d_m^k(t)$, summing for $k = 1, ..., m$, and then recalling (1.4) we find

$$(\acute{u}_m, u_m) + B(u_m, u_m, t) = 0 \tag{1.9}$$

Notes

for all $0 \leq t \leq T$. From Lemma 7, there exists constant $\beta > 0$ such that

$$\beta \left\| u_m \right\|^2_{H^1_0(\Omega)} \leq B\left(u_m, u_m, t\right) \tag{1.10}$$

for all $0 \leq t \leq T$. Consequently (1.10) yields the inequality

$$\frac{d}{dt}\left(\left\| u_m \right\|^2_{L^2(\Omega)}\right) + \beta \left\| u_m \right\|^2_{H^1_0(\Omega)} \leq 0 \quad \text{for all } 0 \leq t \leq T. \tag{1.11}$$

This implies that

$$\left\| u_m \right\|^2_{L^2(\Omega)} \leq \left\| u_m\left(0\right) \right\|^2_{L^2(\Omega)} \leq \left\| u_0 \right\|^2_{L^2(\Omega)} \quad \text{for all } 0 \leq t \leq T. \tag{1.12}$$

So we have

$$\max_{0 \leq t \leq T} \left\| u_m \right\|_{L^2(\Omega)} \leq \left\| u_0 \right\|_{L^2(\Omega)}. \tag{1.13}$$

2. Integrate inequality (1.11) from 0 to $T$ and we employ the inequality (1.13) to find

$$\left\| u_m \right\|^2_{L^2\left(0,T;\ H^1_0(\Omega)\right)} = \int_0^T \left\| u_m \right\|^2_{H^1_0(\Omega)} dt \leq C \left\| u_0 \right\|^2_{L^2(\Omega)}.$$

3. Fix any $v \in H^1_0(\Omega)$, with $\left\| v \right\|^2_{H^1_0(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in span\left(w_k\right)^{k=m}_{k=1}$, and $\left(v^2, w_k\right) = 0 \ (k = 1, ..., m)$. We use (1.6), we deduce for all $0 \leq t \leq T$ that

$$\left(u'_m, v^1\right) + B\left(u_m, v^1, t\right) = 0.$$

Then (1.4) implies

$$\left\langle u'_m, v \right\rangle = \left(u'_m, v\right) = \left(u'_m, v^1\right) = -B\left(u_m, v^1, t\right),$$

consequently

$$\left| \left\langle u'_m, v \right\rangle \right| \leq C \left\| u_m \right\|_{H^1_0(\Omega)}.$$

Since $\left\| v^1 \right\|^2_{H^1_0(\Omega)} \leq \left\| v \right\|^2_{H^1_0(\Omega)} \leq 1$. Thus

$$\left\| u'_m \right\|_{H^{-1}(\Omega)} \leq C \left\| u_m \right\|_{H^1_0(\Omega)},$$

and therefore

$$\left\| u'_m \right\|^2_{L^2(0,T;\ H^{-1}(\Omega))} = \int_0^T \left\| u'_m \right\|^2_{H^{-1}(\Omega)} dt \leq C \int_0^T \left\| u_m \right\|^2_{H^1_0(\Omega)} dt \leq C \left\| u_0 \right\|^2_{L^2(\Omega)}.$$

### d) *Existence and uniqueness*

Next we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem ($P_1$).

**Theorem 10** *(Existence of weak solution). Under hypothesis $H_2$ and $H_3$, there exists a weak solution of* ($P_1$).

***Proof.*** *1. According to the energy estimates (1.8), we see that the sequence $\{u_m\}^\infty_{m=1}$ is bounded in $L^2\left(0,T;H^1_0\left(\Omega\right)\right)$ and $\{u'_m\}^\infty_{m=1}$ is bounded in $L^2\left(0,T;H^{-1}\left(\Omega\right)\right)$. Consequently there exists a subsequence which is also noted by $\{u_m\}^\infty_{m=1}$ and a function $u \in L^2\left(0,T;H^1_0\left(\Omega\right)\right)$, with $u' \in L^2\left(0,T;H^{-1}\left(\Omega\right)\right)$, such that*

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{weakly in } L^2\left(0,T;H^1_0\left(\Omega\right)\right) \\ u'_m &\rightharpoonup u' \quad \text{weakly in } L^2\left(0,T;H^{-1}\left(\Omega\right)\right). \end{aligned} \tag{1.14}$$

*2. Next fix an integer $N$ and choose a function $v \in C^1\left(0,T;H^1_0\left(\Omega\right)\right)$ having the form*

$$v\left(t\right) = \sum_{k=1}^{N} d^k\left(t\right) w_k \tag{1.15}$$

Notes

Notes

*where* $\{d^k\}_{k=1}^N$ *are given smooth functions. We choose* $m \geq N$, *multiply equation* (1.6) *by* $d^k(t)$, *sum for* $k = 1, ..., N$, *and then integrate with respect to* $t$ *to find*

$$\int_0^T \langle u'_m, v \rangle + B(u_m, v, t)\, dt = 0. \tag{1.16}$$

*We recall* (1.14) *to find upon passing to weak limits that*

$$\int_0^T \langle u', v \rangle + B(u, v, t)\, dt = 0 \qquad \forall v \in L^2\left(0, T; H_0^1(\Omega)\right). \tag{1.17}$$

*As functions of the form* (1.15) *are dense in* $L^2\left(0, T; H_0^1(\Omega)\right)$. *Hence in particular*

$$\langle u', v \rangle + B(u, v, t)\, dt = 0 \qquad \forall v \in H_0^1(\Omega) \text{ and } \forall t \in [0\ T], \tag{1.18}$$

*and from Remark 2 we have* $u \in C\left(0, T; L^2(\Omega)\right)$.
*3. In order to prove* $u(0) = u_0$, *we first note from* (1.17) *that*

$$\int_0^T -\langle u, v' \rangle + B(u, v, t)\, dt = (u(0), v(0)) \tag{1.19}$$

*for each* $v \in C^1\left(0, T; H_0^1(\Omega)\right)$ *with* $v(T) = 0$. *Similary, from* (1.16) *we deduce*

$$\int_0^T -\langle u_m, v' \rangle + B(u_m, v, t)\, dt = (u_m(0), v(0)). \tag{1.20}$$

*We use again* (1.14), *we obtain*

$$\int_0^T -\langle u, v' \rangle + B(u, v, t)\, dt = (u_0, v(0)), \tag{1.21}$$

*since* $u_m(0) \to u_0$ *in* $L^2(\Omega)$. *Comparing* (1.19) *and* (1.21), *we conclude* $u(0) = u_0$.

**Theorem 11** *(Uniqueness of weak solutions ) A weak solution of* $(P_1)$ *is unique.*
***Proof.*** *We suppose there exists two weak solution* $u_1$ *and* $u_2$. *We put*

$$U = u_2 - u_1$$

*then* $U$ *is also a solution of* $(P_1)$*with* $U_0 = (u_2 - u_1)(0) \equiv 0$. *Setting* $v = U$ *in identity* (1.18) *we have*

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2_{L^2(U)}\right) + B(U, U, t) = 0.$$

*From Lemma 7 we have* $B(U, U, t) \geq \beta \|U\|^2_{H_0^1(U)} \geq 0$, *so* $\frac{d}{dt}\left(\frac{1}{2}\|U\|^2_{L^2(U)}\right) \leq 0$, *then integrate with respect to* $t$ *to find*

$$\|U\|^2_{L^2(\Omega)} \leq \|U_0\|^2_{L^2(\Omega)} = 0,$$

*thus* $U \equiv 0$.

### *e) Global solution of problem (P)*

Our main results in this paper are stated as follows.

**Theorem 12** *i) if $c \geq c_0 > 0$. Then la solution $(u, c)$ of problem (P) is global*
*ii) if $c \geq c_0 > 0$. Then la solution $(u, c)$ of problem (P) is global. Furthermore there exists $\tau_0 > 0$ such that $\|u\|_{L^2} \leq e^{-\tau_0 t} \|u_0\|_{L^2}$.*

**Proof.** We put

$$E(t) = \frac{1}{2} \int_{\Omega} u^2 dx \tag{1.22}$$

We derivate the equation (1.22) and we use firsts equations of ($P_1$) and ($P_2$) to find
i) We have

$$\frac{dE}{dt} = -B(u, u, t) \leq 0,$$

therefore

$$E(t) \leq E(0).$$

ii) We have

$$\frac{dE}{dt} = -B(u, u, b, t) = -\int \left( (\nabla u)^2 + \frac{1}{2} \tau c u^2 \right) dx \leq \frac{-1}{2} \tau c_0 \|u\|^2_{L^2(\Omega)} = -\tau_0 E(t).$$

This implies that

$$E(t) \leq E(0) e^{-\tau_0 t}.$$

**Proposition 13** [1] *Let $u_0 \in L^2(\Omega)$ and $u \in C\left([0, T];\ L^2(\Omega)\right) \cap L^2\left(0, T;\ H_0^1(\Omega)\right)$ is the unique weak solution of $(P_1)$. If $u_0 \geq 0$ in $\Omega$, then $u \geq 0$ in $]0, T[ \times \Omega$.*
***Proof.*** *If $u_0 \geq 0$ on $\Gamma$. Therefore $u^- = \min(u, 0) \in L^2\left(]0, T[;\ H_0^1(\Omega)\right)$. A reasoning similar to the Proposition 6, we obtain for all $0 \leq t \leq T$*

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^-)^2 dx + \int_{\Omega} B(u^-, u^-, t)\, dx = 0.$$

*Using the Lemma 7 and integrating with respect to $\tau$ from 0 to $t$, we get*

$$\frac{1}{2} \int_{\Omega} (u^-)^2 dx + \beta \int_0^t \|u(s)\|^2_{H_0^1(U)} ds \leq \frac{1}{2} \int_{\Omega} (u^-(0))^2 dx = 0.$$

*Since $u^-(0) = (u_0)^- = 0$. So $u^- = 0$.*

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