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Combined Richtmyer-Meshkov and Kelvin-Helmholtz Instability with Surface Tension

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Abstract- A nonlinear theoretical model of the combined Richtmyer-Meshkov and Kelvin-Helmholtz instability between two different density fluids with surface tension is proposed. The model is based on the extended Layzer's potential flow model. It is observed that, the surface tension decreases the velocity but does not affect the curvature of the bubble tip, provided surface tension is greater than a critical value. Under a certain condition it is also observed that surface tension stabilized the motion with nonlinear oscillations. The nonlinear oscillations depend on surface tension and relative velocity shear of two fluids. All results are obtained theoretically and supported by numerical technique.

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Abstract- A nonlinear theoretical model of the combined Richtmyer-Meshkov and Kelvin-Helmholtz instability between two different density fluids with surface tension is proposed. The model is based on the extended Layzer's potential flow model. It is observed that, the surface tension decreases the velocity but does not affect the curvature of the bubble tip, provided surface tension is greater than a critical value. Under a certain condition it is also observed that surface tension stabilized the motion with nonlinear oscillations. The nonlinear oscillations depend on surface tension and relative velocity shear of two fluids. All results are obtained theoretically and supported by numerical technique.

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I. Introduction

The instability of two fluid interface under an acceleration by an incident shock was predicted theoretically by Richtmyer[1] and then Meshkov[2] confirmed experimentally Richtmyer's prediction. Since then, this interfacial instability has been referred to as the Richtmyer-Meshkov instability. Such instability is observed in supernova explosions, inertial confinement fusion, shock tube experiment etc.[3, 4]. On the other hand, the Kelvin-Helmholtz instability[5] arises when two fluids are separated by an interface across which the tangential velocity is discontinuous. Such a flow is unstable under a sinusoidal perturbation of the interface. The Kelvin- Helmholtz instability plays an important role in many astrophysical and experimental situations[6,7,8]. The Kelvin- Helmholtz instability and shear flow effects in general are also of practical importance in a number of high energy density systems. They should be considered in multi shock experiments for direct drive capsule for inertial confinement fusion, since Kelvin- Helmholtz instability may accelerate the growth of turbulent mixing layer at the interface between the ablator and solid deuterium-tritium nuclear fuel. In high energy density and astrophysical system, it has been seen that structures driven by shear flow appear on the high density spikes produced by Richtmyer-Meshkov instability.

In the linear theory, Richtmyer-Meshkov and kelvin-Helmholtz instabilities are well understood. In the nonlinear stage, the interface becomes finger like structure. The structure is called a bubble if the lighter fluid pushes across the unperturbed surface into the heavier fluid and a spike if the opposite takes place. The dynamics of such Richtmyer-Meshkov and kelvin-Helmholtz instabilities generated nonlinear structures have been studied under different physical situation using an expression near the tip of the bubble second order in the transverse coordinate to unperturbed surface following Layzers[9, 11] approach. In the domain of linear theory, Chandrasekhar[12] has investigated the problem of Kelvin-Helmholtz instability taking the effect of surface tension and Mikaelian[13] has studied the same effect on Richtmyer-Meshkov instability.

The present article deals with the problem of the time development of the nonlinear interfacial structure caused by combined Richtmyer-Meshkov and Kelvin-Helmholtz instability in presence of surface tension. The dynamics of the bubble tip is investigated under the nonlinear potential flow

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model and obtained a condition for oscillatory stabilization of the interface between two fluids. A analytic expression of the velocity of the bubble tip is also obtained for asymptotic stage.

II. Basic Hydrodynamic Model

To describe the nature of the bubble tip in presences of velocity shear and surface tension, we consider two incompressible, inviscid fluids separated by an initial horizontal interface situated at $y = 0$ in a two dimensional $x - y$ plane. The classical combined RM and KH instability refers to the following density and velocity profile:

$$\rho = \begin{cases} \rho_h & : y > 0 \\ \rho_l & : y < 0 \end{cases} \tag{1}$$

$$U = \begin{cases} U_h & : y > 0 \\ U_l & : y < 0 \end{cases} \tag{2}$$

Initially, the system is subjected to a sudden and very brief acceleration $g(t)$, and the evolution of the fluid flow and of the interface deformation is studied. As is customary in impulsive models, the time-dependent acceleration is represented by a Dirac function, under the form:

$$g(t) = \Delta U \ \delta(t) \tag{3}$$

whereΔU is the speed change imparted to the fluids by the shock. The equation of the interface is defined by function η, under the parabolic form:

$$y = \eta(x,t) = \eta_0(t) + \eta_2(t)(x - \eta_1(t))^2 \tag{4}$$

The nonlinear perturbed interface forms a bubble or spike according to $\eta_0(t) > 0$, $\eta_2(t) < 0$ or $\eta_0(t) < 0$, $\eta_2(t) > 0$. Here, at time t, the position of the bubble tip is given by $(\eta_1(t), \eta_0(t))$ and $\eta_2(t)$ gives the curvature of the tip of the bubble. In presence of streaming motion of the fluids, the tip of the bubble moves parallel to unperturbed interface with velocity $\dot{\eta_1}(t)$.

Each fluid is inviscid and incompressible and so flows irrotationally. As a result, velocity potentials ϕ_h and ϕ_l may be constructed in upper and lower fluids, respectively Each velocity potential must satisfy Laplace's equation, so that

$$\nabla^2 \phi_h = 0 \text{ in } y > \eta(x,t) \tag{5}$$

$$\nabla^2 \phi_l = 0 \text{ in } y < \eta(x,t) \tag{6}$$

According to the extended Layzer model [9, 11, 14, 17, 16], the velocity potentials describing the motion for the upper and lower fluids are given by,

$$\phi_h(x,y,t) = a_1(t)\cos\left(k(x - \eta_1(t)\right)e^{-k(y-\eta_0(t))} + a_2(t)\sin\left(k(x - \eta_1(t)\right)e^{-k(y-\eta_0(t))} - xU_h \tag{7}$$

$$\phi_l(x,y,t) = b_0(t)y + b_1(t)\cos\left(k(x-\eta_1(t)\right)e^{k(y-\eta_0(t))} + b_2(t)\sin\left(k(x-\eta_1(t)\right)e^{k(y-\eta_0(t))} - xU_l \quad (8)$$

where k is the perturbed wave number.

The boundary condition that the normal velocity is continuous at the interface $y = \eta(x,t)$ can be written as

$$\frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}\frac{\partial \phi_h}{\partial x} = -\frac{\partial \phi_h}{\partial y} \quad (9)$$

$$\frac{\partial \eta}{\partial x}\left(\frac{\partial \phi_h}{\partial x} - \frac{\partial \phi_l}{\partial x}\right) = \frac{\partial \phi_h}{\partial y} - \frac{\partial \phi_l}{\partial y} \quad (10)$$

The dynamical boundary condition (Bernoulli's equation) at the interface $y = \eta(x,t)$ is of the form,

$$-\rho_{h(l)}\frac{\partial \phi_{h(l)}}{\partial t} + \frac{1}{2}\rho_{h(l)}(\vec{\nabla}\phi_{h(l)})^2 + \rho_{h(l)}gy = -p_{h(l)} + f_{h(l)}(t) \quad (11)$$

The pressure boundary condition at two fluid interface including surface tension[15] is

$$p_h - p_l = \frac{T}{R} \quad (12)$$

where T is surface tension and R is the radius of curvature.

Plugging the boundary condition (12) at the interface $y = \eta(x,t)$ in Eq.(11), we obtain the following equation,

$$\rho_h\left[-\frac{\partial \phi_h}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi_h)^2\right] - \rho_l\left[-\frac{\partial \phi_l}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi_l)^2\right] + \Delta U\ \delta(t)(\rho_h - \rho_l)y = -\frac{T}{R} + f_h - f_l \quad (13)$$

Here we have studied the dynamics of the peak of the perturbed structure where $|k(x-\eta_1(t))| \ll 1$. Thus we can neglect the terms of $O(|x-\eta_1|^i)$ $(i \geq 3)$ [14] – [17]. With this point of view,

$$\frac{1}{R} = 2\eta_2\left(1 + 4\eta_2^2(x-\eta_1)^2\right)^{-\frac{3}{2}} \approx 2\eta_2\left(1 - 6\eta_2^2(x-\eta_1)^2\right) \quad (14)$$

Substituting all the fluid parameters η, ϕ_h and ϕ_l in the boundary conditions (9),(10) and (13), and equating the coefficients of $(x-\eta_1)^i$,$(i = 0,1,2)$, we obtain the flowing nonlinear equations

$$\frac{d\xi_1}{d\tau} = \xi_4 \quad (15)$$

$$\frac{d\xi_2}{d\tau} = V_h - \frac{\xi_5(2\xi_3 + 1)}{2\xi_3} \quad (16)$$

$$\frac{d\xi_3}{d\tau} = -\frac{1}{2}(6\xi_3 + 1)\xi_4 \quad (17)$$

$$\frac{kb_0}{\sqrt{k\Delta U}} = -\frac{12\xi_3\xi_4}{6\xi_3 - 1} \tag{18}$$

$$\frac{k^2 b_1}{\sqrt{k\Delta U}} = \frac{6\xi_3 + 1}{6\xi_3 - 1}\xi_4 \tag{19}$$

$$\frac{k^2 b_2}{\sqrt{k\Delta U}} = \frac{(2\xi_3 + 1)\xi_5 - 2\xi_3(V_h - V_l)}{2\xi_3 - 1} \tag{20}$$

$$\frac{d\xi_4}{d\tau} = \frac{N_1(\xi_3, r)}{D_1(\xi_3, r)}\frac{\xi_4^2}{(6\xi_3 - 1)} + \frac{24(1+r)\xi_3^3(6\xi_3 - 1)\sigma}{D_1(\xi_3, r)} + \frac{N_2(\xi_3, r)}{D_1(\xi_3, r)}\frac{(6\xi_2 - 1)\xi_5^2}{2\xi_3(2\xi_3 - 1)^2}$$
$$+\frac{2(4\xi_3 - 1)(6\xi_3 - 1)}{D_1(\xi_3, r)(2\xi_3 - 1)^2}[(V_h - V_l)^2\xi_3 - (V_h - V_l)(2\xi_3 + 1)\xi_5 \tag{21}$$

and

$$\frac{d\xi_5}{d\tau} = -\frac{(2\xi_3 - 1)r\xi_4\xi_5}{2\xi_3 D_2(\xi_3, r)} + \frac{\xi_4(6\xi_3 + 1)}{2D_2(\xi_3, r)(6\xi_3 - 1)(2\xi_3 - 1)}[4(V_h - V_l)(4\xi_3 - 1) - \frac{\xi_5}{\xi_3}(28\xi_3^2 - 4\xi_3 - 1)] \tag{22}$$

where $r = \frac{\rho_h}{\rho_l}$; $\xi_1 = k\eta_0$; $\xi_2 = k\eta_1$; $\xi_3 = \frac{\eta_2}{k}$; $\xi_4 = \frac{ka_1}{\Delta U}$; $\xi_5 = \frac{ka_2}{\Delta U}$; $\tau = t(k\Delta U)$; $\sigma = \frac{Tk}{(\rho_h+\rho_l)(\Delta U)^2}$ and $V_{h(l)} = \frac{U_{h(l)}}{\Delta U}$ are corresponding dimensionless quantities. The functions $N_1(\xi_3, r)$, $N_2(\xi_3, r)$, $D_1(\xi_3, r)$ and $D_2(\xi_3, r)$ are given by

$$N_1(\xi_3, r) = 36(1 - r)\xi_3^2 + 12(4 + r)\xi_3 + (7 - r);$$
$$D_1(\xi_3, r) = 12(r - 1)\xi_3^2 + 4(r - 1)\xi_3 - (r + 1) \tag{23}$$

and

$$N_2(\xi_3, r) = 16(1 - r)\xi_3^3 + 12(1 + r)\xi_3^2 - (1 + r);$$
$$D_2(\xi_3, r) = 2(1 - r)\xi_3 + (r + 1) \tag{24}$$

The above set of five equations (15)(17), (21) and (22) together with Eqs. (23) and(24) which define the different functions describe the combined effect of RM and KH instability.

III. Linear Approximation

In this section, we establish that the usual KH instability growth rate (without surface tension and shock) is recovered on linearization of Eqs. (15)-(17),(20) and (22). Let us consider

$$\frac{d\eta_1}{dt} = \alpha_h U_h + \alpha_l U_l \tag{25}$$

where

$$\alpha_{h(l)} = \frac{\rho_{h(l)}}{\rho_h+\rho_l} \tag{26}$$

Then Eq. (16) gives, after linearization,

$$k^2 a_2 = 2\alpha_l (U_h - U_l)\eta_2 \tag{27}$$

Linearizing Eqs. (17), (21) and (22) we get

$$\frac{d\eta_2}{dt} = -\frac{1}{2}k^3 a_1 \tag{28}$$

$$k\frac{da_1}{dt} = -2\alpha_h \alpha_l (U_h - U_l)^2 \eta_2 \tag{29}$$

$$\frac{da_2}{dt} = -\rho_h (U_h - U_l) k a_1 \tag{30}$$

Eliminating η_2 from Eqs. (28) and (29)

$$\frac{d^2 a_1}{dt^2} = k^2 \alpha_h \alpha_l (U_h - U_l)^2 a_1 \tag{31}$$

Thus the growth rate is given by

$$\gamma(k) = k\sqrt{\alpha_h \alpha_l (U_h - U_l)^2} \tag{32}$$

This result agrees with the result obtained by Chandrasekhar [12] and Mikaelian [18]. Note that Eq.(27) connecting η_2 and a_2 provides the consistency condition.

IV. Result and Discussion

The growth, curvature and growth rate of the peak height of the bubble is obtained by numerical integration of Eqs. (15), (16), (17), (21) and (22). The initial perturbed interface is assumed to be $y = \eta_0(t = 0)cos(kx)$. The expansion of the cosine function gives $(\xi_2)_{initial} = 0$, $(\xi_3)_{initial} = -\frac{1}{2}(\xi_1)_{initial}$ and $(\xi_1)_{initial}$ is the arbitrary initial amplitude. If the shock incidence is oblique then the normal component generates velocity shear and causes KH instability [18]. The shock generated initial values of ξ_4 and ξ_5 are obtained from the impulsive accelerations. From the linear formula $\xi_1(\tau) = \xi_1(0)A\tau$, we set $(\xi_4)_{initial} = (\xi_1)_{initial}A$ and $(\xi_5)_{initial} = 0$, where $A = \frac{\rho_h - \rho_l}{\rho_h + \rho_l}$ is the Atwood number. The obtained numerical results are shown in figures.

The growth rate contributed in absence of velocity shear and surface tension,i.e; by normally incident shock induced Richtmyer-Meshkov instability varies as $\frac{1}{t}$[15]. However in presence of velocity shear the growth rate due to combined influence of Richtmyer-Meshkov and Kelvin-Helmholtz instability approaches finite saturation value asymptotically.

$$[(\xi_3)_{asymp}]_{bubble} = -\frac{1}{6} \tag{33}$$

$$[(\xi_4)_{asymp}]_{bubble} = \sqrt{\frac{5}{16}\left(\frac{1-A}{1+A}\right)(V_h - V_l)^2 - \frac{2}{9}\left(\frac{\sigma}{1+A}\right)} \tag{34}$$

and

$$[(\xi_5)_{asymp}]_{bubble} = 0 \tag{35}$$

These asymptotic values are obtained by setting $\frac{d\xi_3}{d\tau} = 0$, $\frac{d\xi_4}{d\tau} = 0$ and $\frac{d\xi_5}{d\tau} = 0$. Note that the above asymptotic values exist if the surface tension is less than a critical value T_c, given by

$$T_c = \frac{45}{16}\frac{\rho_l}{k}(V_h - V_l)^2(\Delta U)^2 \tag{36}$$

Here the critical value is depended on the magnitude of relative velocity shear of two fluids, and the density of the lower fluid only. The growth (ξ_1) and velocity (ξ_4) of the tip is reduced if $T \to T_c - 0$. This feature exhibit in figure 1. Moreover, the asymptotic velocity of the bubble tip becomes large if there is a large velocity shear or large shock strength, which produce a large velocity jump after the shock impedance.

Figure (2) and (3) describe the nonlinear oscillation behavior of the perturbed interface. The nonlinear oscillation occurs if $T > T_c$. It is clear from the figure (2) that,the amplitude and period of oscillation of the interface decreases for large surface tension. Under this condition the self generated transverse velocity (ξ_5) also becomes oscillatory. On the other hand, $\xi_5 \to 0$ asymptotically when $T < T_c$. The oscillatory behavior also depends upon the magnitude of the relative velocity shear. Figure (3) shows that the amplitude and period of oscillation increases with magnitude of the relative velocity.

Further, the equilibrium is attained when $T = T_c$, i.e,

$$\dot{\xi}_3 = \dot{\xi}_4 = \dot{\xi}_5 = 0 \text{ when } \xi_3 = -\frac{1}{6} \text{ and } \xi_4 = \xi_5 = 0 \tag{37}$$

This feature shows by solid line in figure (2). Thus the combined Richtmyer-Meshkov and Kelvin-Helmholtz instability is stabilized when $T \geq T_c$.

V. Conclusion

In this article, we have studied the effect of surface tension on the interfacial structure of two fluids interface induced by combined action of Richtmyer-Meshkov and Kelvin-Helmholtz instability under nonlinear potential flow model. The analytic expressions for bubble tip growth rate at asymptotic stage are obtained for arbitrary Atwood number and velocity shear. In absence of surface tension the growth rate is reduced for small velocity shear and initial velocity induced by shock. Surface tension becomes a stabilizing factor of the instability, provided it is larger than a critical value. In this case, oscillatory behavior of motion is described by numerical integration of governing equations. The nature of oscillations depends on both surface tension and relative velocity shear of two fluids. On the other hand, below the critical value, surface tension dominates the growth and growth rate of the instability. This is a theoretical work and may be helpful for experiential study of the two fluid instability in future.

VI. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

VII. Acknowledgement

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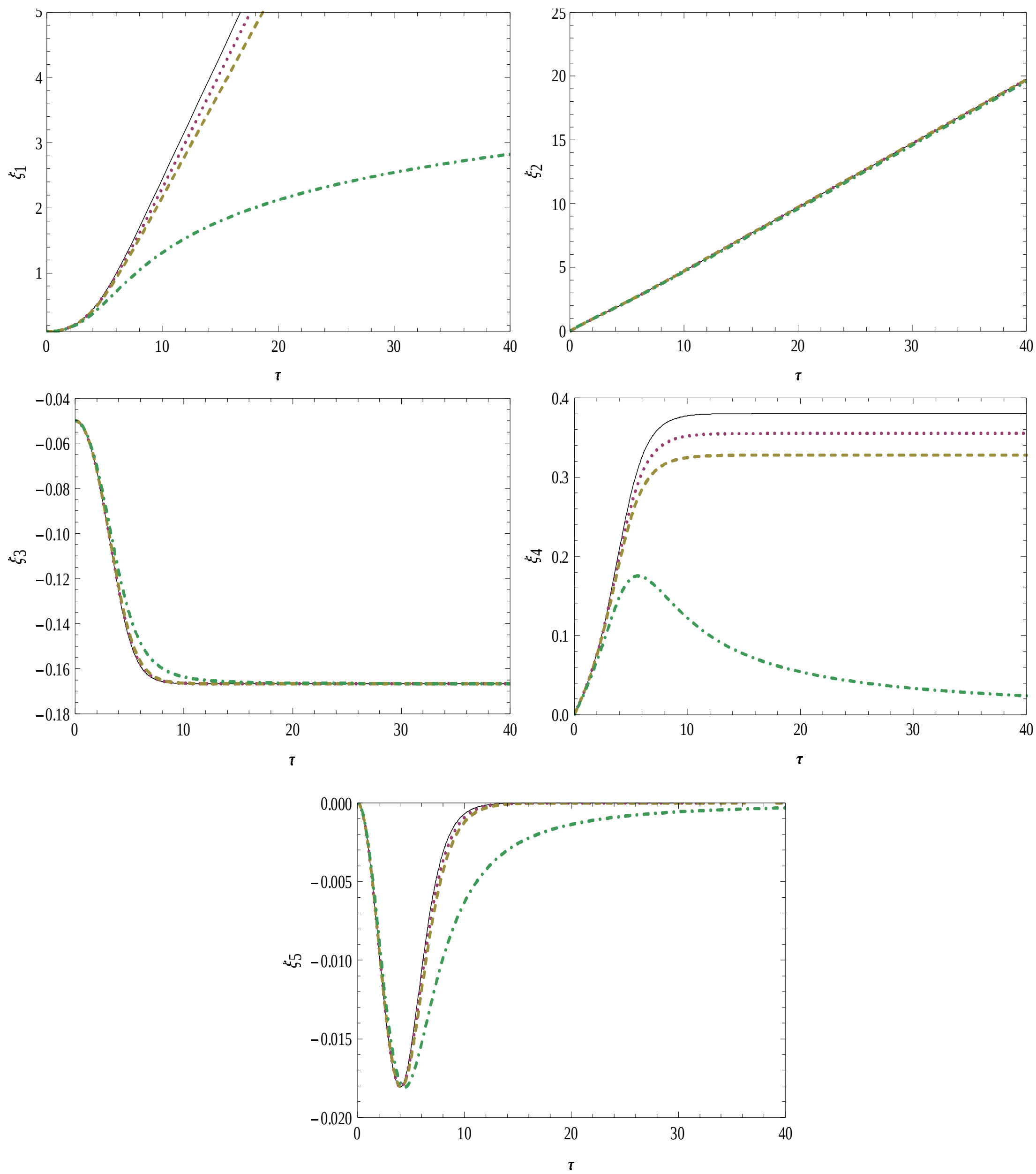

Figure 1 : Variation of ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 with τ. Initial value ξ_1 =0.1, ξ_2 =0, ξ_3 =-0.05, ξ_4 = 0.02, and ξ_5 = 0 with r = 1.5, V_h=0.6,V_l = 0.1, σ=0 (line),0.1 (Dot), 0.2 (Dash), 0.25 (Dash-Dot).

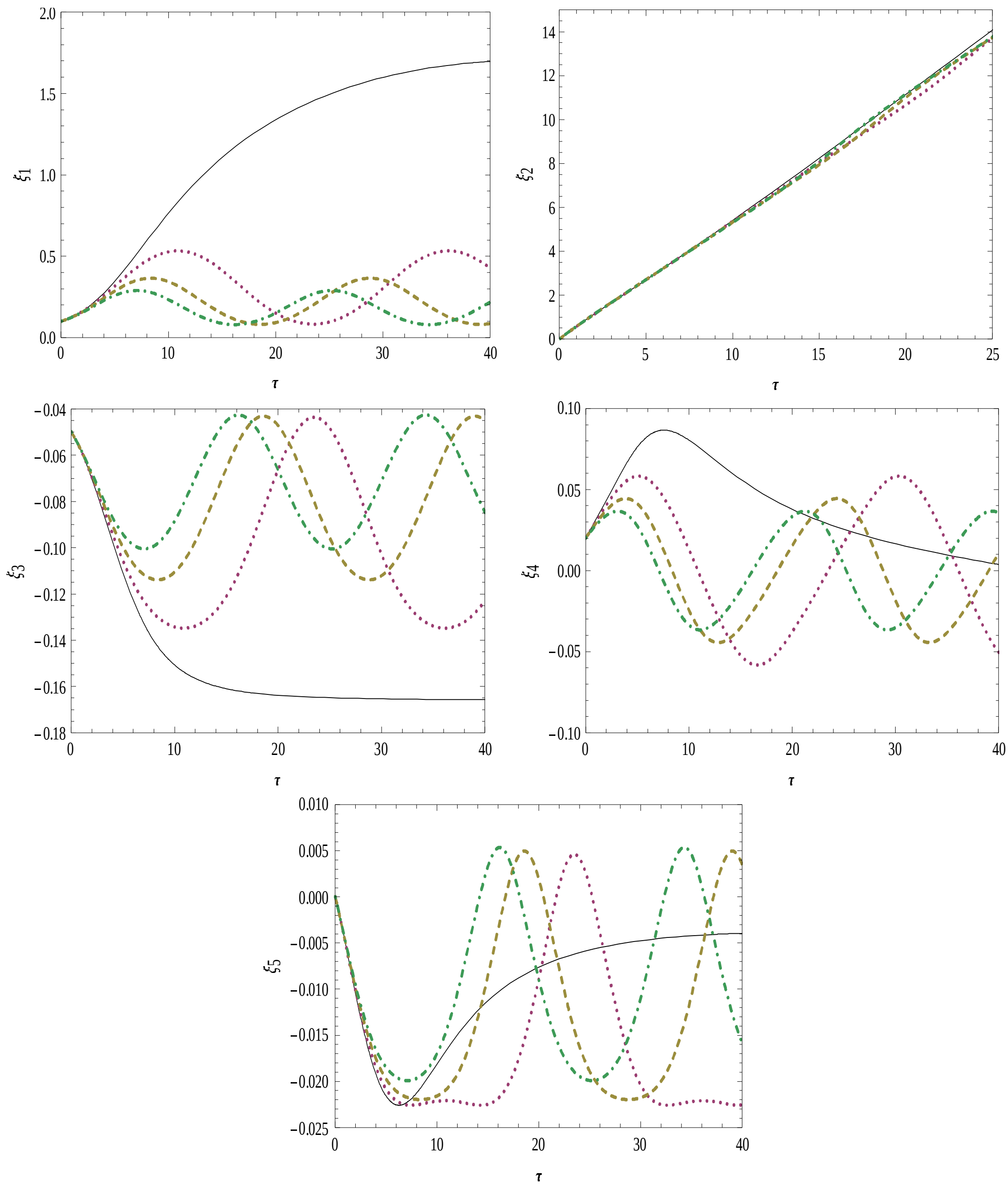

Figure 2 : Variation of ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 with τ. Initial value ξ_1 =0.1, ξ_2 =0, ξ_3 =-0.05, $\xi_4 = 0.02$, and $\xi_5 = 0$ with $r = 1.5$, V_h=0.6,$V_l = 0.1$, σ=9/32 (line),0.5 (Dot), 0.75 (Dash), 1 (Dash-Dot).

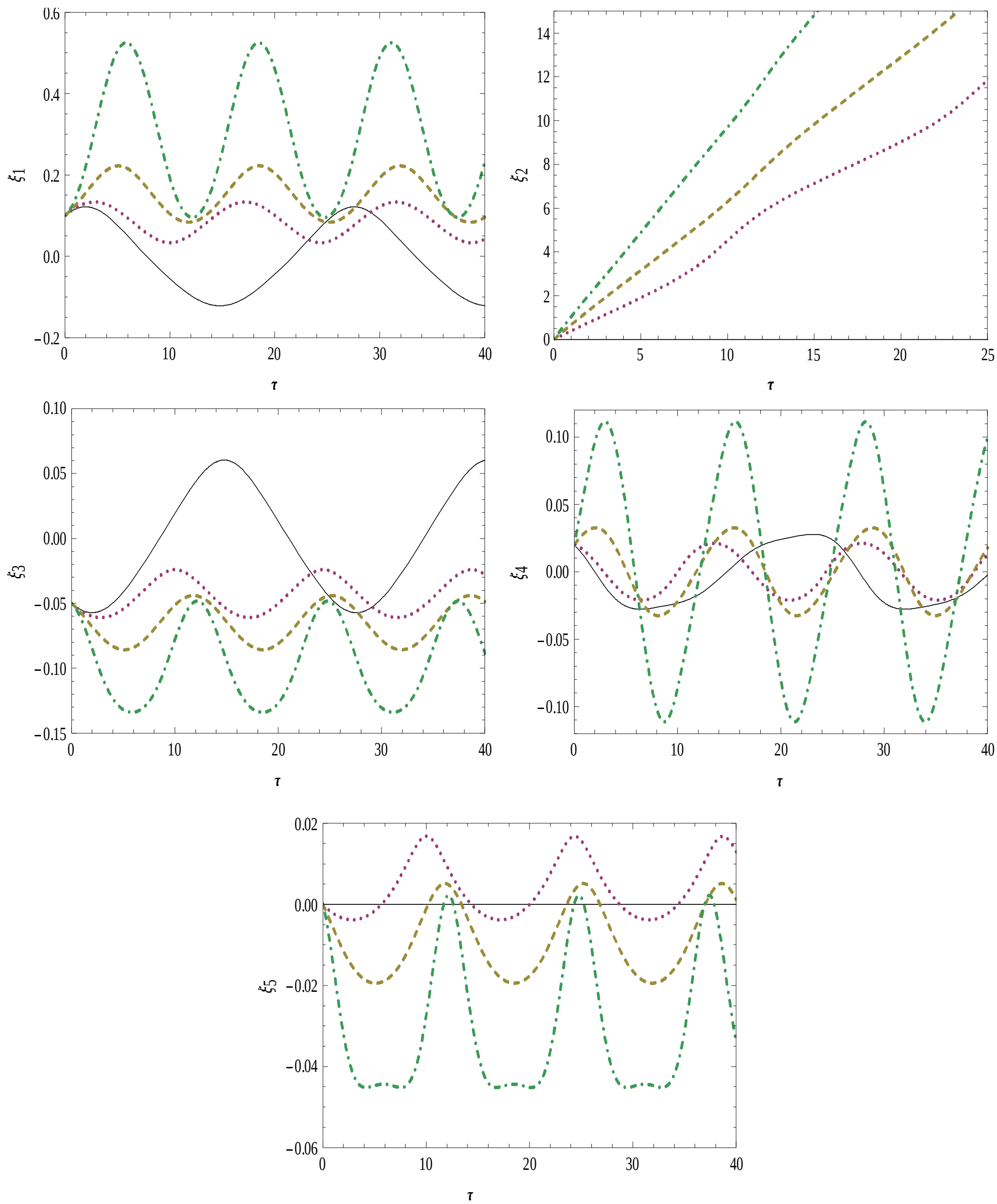

Figure 3 : Variation of ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 with τ. Initial value ξ_1 =0.1, ξ_2 =0, ξ_3 =-0.05, ξ_4 = 0.02, and ξ_5 = 0 with r = 1.5, σ=2, V_h=0,V_l = 0 (line),V_h=0.4,V_l = 0.1(Dot), V_h=0.7,V_l = 0.1 (Dash),V_h=1.1,V_l = 0.1 (Dash-Dot).