Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 14 Issue 7 Version 1.0 Year 2014
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Numerical Simulation of Wave Equation

By Md. Saiduzzaman, Md. Yeadul Islam Shaikh & SM Sala Uddin Samrat
International University of Business Agriculture and Technology, Bangladesh

Abstract- Wave equation is a very important equation in applied mathematics. This equation is used to simulate large destructive waves in fjord, lake, or the ocean generated by slides, earthquakes, subsea volcanoes, meteorites. It has analytical solution but it is time consuming. Therefore one needs to use numerical methods for solving this equation. For this we investigate finite difference method and present explicit upwind difference scheme for one dimensional wave equation, central difference scheme for second order wave equation. We implement the numerical scheme by computer programming for initial boundary value problem and verify the qualitative behavior of the numerical solution of the wave equation.

Keywords: *linear advection equation, equation of continuity, wave equation, central difference scheme, explicit upwind difference scheme.*

GJSFR-A Classification : *FOR Code: 029999*

NUMERICALSIMULATIONOFWAVEEQUATION

Strictly as per the compliance and regulations of :

Numerical Simulation of Wave Equation

Md. Saiduzzaman α, Md. Yeadul Islam Shaikh σ & SM Sala Uddin Samrat ρ

Abstract- Wave equation is a very important equation in applied mathematics. This equation is used to simulate large destructive waves in fjord, lake, or the ocean generated by slides, earthquakes, subsea volcanoes, meteorites. It has analytical solution but it is time consuming. Therefore one needs to use numerical methods for solving this equation. For this we investigate finite difference method and present explicit upwind difference scheme for one dimensional wave equation, central difference scheme for second order wave equation. We implement the numerical scheme by computer programming for initial boundary value problem and verify the qualitative behavior of the numerical solution of the wave equation.

Keywords: linear advection equation, equation of continuity, wave equation, central difference scheme, explicit upwind difference scheme.

I. Introduction

Wave equation is used to understanding of tsunamis, assists warning systems, assists building of harbor protection (break maters), recognize critical coastal areas, hind cost historical tsunami (assist geologists). It has analytical solution but it is time consuming. How over with rapid developments of numerical methods and computer technology the system can be solved numerically. Numerical simulation[1] is very much challenging. Many scientific groups are involved in dealing with this problem. The aim of this thesis will be provide an easy way to solving wave equation. In this thesis we use finite difference scheme known as central difference scheme [3], [4], explicit upwind difference scheme [8],[6].

In the first section we introduced symbols and notations. In the second section we introduced the first and second order wave equation, method of characteristics, D'Alembert solution and analytical solution of the wave equation. In the third section we discussed numerical methods. In the last we discussed about our numerical experiments and results.

II. Mathematical Models and Methods

a) Symbols and Notation

Let $\Omega \subset R^d, d \in \{1,2\}$ be a region occupied by a fluid flow, and let $[t_0, T]$ be a time interval with $0 \le t_0 \le T$. An arbitrary point in Ω is denoted by $X = (x_1, x_2, \ldots\ldots\ldots, x_d)^T$. For the description of a general unsteady compressible fluid flow, we introduce the quantities:

The density $\rho = \rho(x,t)$, the velocity vector $V = V(x,t) = (v_1(x,t), \ldots\ldots v_d(x,t))^T$, the pressure $P = P(x,t)$ the energy density $E = E(x,t)$. We denote the density of the external force by $f = f(x,t) = (f_1(x,t), f_2(x,t), \ldots. f_d(x,t))^T$, the mass flux by $q = q(x,t)$ for the description of the viscous flow, let λ and μ denote the coefficient of kinetic viscosity respectively.

b) The equation of continuity [5] is given by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

c) First order wave equation

The first -order wave (advection) equation [6] is $(c > 0)$

$$\frac{\partial u}{\partial t} + c\frac{\partial y}{\partial x} = 0, u(x,0) = u_0(x)$$

d) Wave equation in 1d

One dimensional wave equation [2] is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Initial condition:

$$u(0,x) = u_0(x) \quad \text{and} \quad u_t(0,x) = v_0(x)$$

Formally, we can write Laplace equation as:

$$u_{tt}(t,x) - c^2 u_{xx}(t,x) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(t,x) \qquad (2)$$

e) D'Alembert's solution

We get the characteristics system from (2) as

$$\frac{d}{dx}(x) = \pm c \qquad (3)$$

Integrating both side of (3) then we get

Author α: Lecturer, Department of Mathematics, International University of Business Agriculture and Technology, Dhaka, Bangladesh. e-mail: saiduzzaman.ju@gmail.com

Author σ: Lecturer, Department of Mathematics, Sheikh Lutfor Rahman Adarsha Govt. college, Kotalipara, Gopalgonj, Bangladesh. e-mail: yeadul_ju@yahoo.com

Author ρ: Student, Jahangir nagar University, Dhaka, Bangladesh. e-mail: samrat1273@gmail.com

$$x = ct + \xi \text{ (when c is positive)} \tag{4}$$

Solving (2.24) and (2.25) then we get

$$x = \frac{1}{2}(\eta + \xi) \quad \text{and} \quad t = \frac{1}{2c}(\eta - \xi)$$

$$x = -ct + \eta \text{ (when c is negative)} \tag{5}$$

Introducing new variable $\eta = -x + ct$ and $\xi = x - ct$

We consider,

$$w(\eta,\xi) = u(t,x) = u[\frac{1}{2c}(\eta - \xi), \frac{1}{2}(\eta + \xi)] \tag{6}$$

$$\frac{\partial w}{\partial \eta} = \frac{\partial u}{\partial t}.\frac{\partial t}{\partial \eta} + \frac{\partial u}{\partial x}.\frac{\partial x}{\partial \eta} = \frac{1}{2c}\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = -\frac{1}{4c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{4c}\frac{\partial^2 u}{\partial x \partial t} - \frac{1}{4c}\frac{\partial^2 u}{\partial x \partial t} + \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$$

$$= -\frac{1}{4c^2}[\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}] = 0 \text{ (using 1)} \tag{7}$$

Integrating (6) with respect to ξ we get,

$$w_\xi = f(\eta)$$

Again Integrating (6) with respect to η we get,

$$w = f(\eta) + g(\xi) \tag{8}$$

Now we get from (6) and (8)

$$w = f(\eta) + g(\xi)$$
$$= f(x+ct) + g(x-ct) \tag{9}$$

This is the general form of solution of (1).

We use initial condition to determine f and g. For t=0 we get,

$$u(0,x) = u_0(x) = f(x) + g(x) \tag{10}$$

$$u_t(0,x) = v_0(x) = [\frac{\partial f}{\partial \eta}.\frac{\partial \eta}{\partial t} + \frac{\partial g}{\partial \xi}.\frac{\partial \xi}{\partial t}]_{t=0}$$

$$cf'(x) - cg'(x) \tag{11}$$

Integrating (11) we get,

$$f(x) - g(x) = \frac{1}{c}\int_0^x v_0(s)ds + k \tag{12}$$

Solving (10)and(12)we get,

$$f(x) = \frac{1}{2}u_0(x) + \frac{1}{2c}\int_0^x v_0(s)ds + k \tag{13}$$

$$f(x) = \frac{1}{2}u_0(x) + \frac{1}{2c}\int_0^x v_0(s)ds + k \tag{14}$$

From (9) we get,

$$u(t,x) = \frac{1}{2}\{u_0(x+ct) + u_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)ds$$

This is called D'Alemberts solution for 1-D wave equation [7].

f) Method of characteristic

Consider the IVP

$$u_x(t,x) + cu_x(t,x) = 0, -\infty < x < \infty, t > 0$$
$$u(0,x) = f(x), -\infty < x < \infty,$$

Where c=constant. If we measure the rate of change of u from moving position given by the chain rule,

$$\frac{d}{dt}u(t,x(t)) = u_t(x(t),t) + u_x(t,x(t))x'(t)$$

The first term on the right-hand side above is the change in u at a fixed point x while the second one is the change in u resulting from the movement of the observation position.

Assuming that $x'(t) = c, c$ from the PDE we see that

$$\frac{d}{dt}u(t,x(t)) = u_t(t,x(t)) + cu_x(t,x(t)) = 0$$

That is $u =$ constant as perceived from the moving observation point. The position of this point is obtained by integrating its velocity $x'(t) = c \;\; c$

$$x = x_0 + ct, x_0 = x(0)$$

This formula defines a family of lines in the (x,t)-plane, which are called characteristics (Fig.1.1). As mentioned above, the characteristics have the property that $u(t,x)$ takes a constant value along each one of them (but in the integral, different constant values on different characteristics)

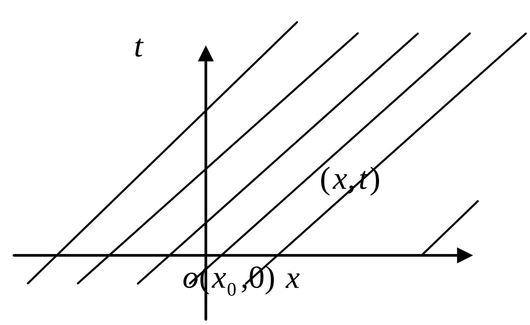

Figure 1.1 : Characteristic plane

Hence, to find the value of the solution u at (x,t) we consider the characteristics through (x,t) of equation $x = ct + x_0$ which intersects the x axis at $(x_0,0)$ .Since u is a constant on this line, its value at (x,t) is the same as at $(x_0,0)$ .But the latter is known from the IC, so

$$u(t,x_0) = f(x_0)$$

The parameter is now replaced from the equation of the characteristics line: $x_0 = x - ct$ .Thus the solution of the given IVP is

$$u(t,x) = f(x - ct)$$

This formula shows that at a fixed time t the shape of the solution is the same as at $t = 0$,0, but is shifted by ct along the x axis. In other words, the shape of the initial data function travels in the positive (negative) x direction with velocity c if $c > 0 (c < 0)$ which means that the solution is a wave.

III. Numerical Methods

a) Explicit Upwind Difference Scheme

We now describe the explicit upwind difference scheme for example we take linear advection equation.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad (15)$$

With initial condition $u(x,0) = u_0(x)$

and left-hand boundary condition $u(a,t) = u_a(t)$

b) Central difference scheme for Wave equation

In this chapter we investigate the finite difference scheme for the Wave equation [7].For a given velocity u the general form of wave equation is

$$u_{tt}(t,x) = c^2 u_{xx}(t,x), 0 \le x \le a, 0 \le t \le b \qquad (16)$$

With boundary condition $u(0,t) = 0$ and $u(a,t) = 0$ for $0 \le t \le b$ and initial condition is $u(x,0) = f(x)$ and $u_t(0,x) = g(x)$ for $0 < x < a$.

For equidistance grid, with temporal step size Δt and spatial step size Δx the discretization of $u_{tt}(t,x)$ and $u_{xx}(t,x)$ we use the central difference formula then we get

$$u_{tt}(t,x) = \frac{u(t+\Delta t,x) - 2u(t,x) + u(t-\Delta t,x)}{\Delta t^2} + o(\Delta t^2) \qquad (17)$$

And

$$u_{xx}(t,x) = \frac{u(t,x+\Delta x) - 2u(t,x) + u(t,x-\Delta x)}{\Delta x^2} + o(\Delta x^2) ....(18)$$

We consider uniform grid that is $x_{i+1} = x_i + \Delta x$ and $t^{n+1} = \Delta t$ .Drop the terms $o(\Delta t^2)$ and $o(\Delta x^2)$ and use the approximation u_i^n for $u(x_i, t^n)$ in (17) and (18). Then for equation (15) we get

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$u_i^{n+1} = (2 - 2\gamma^2)u_i^n + \gamma^2 (u_{i+1}^n + u_{i-1}^n) - u_i^{n-1}$$

Where $\gamma = c\dfrac{\Delta t}{\Delta x}$

This is the central difference scheme for wave equation.

Stencil:

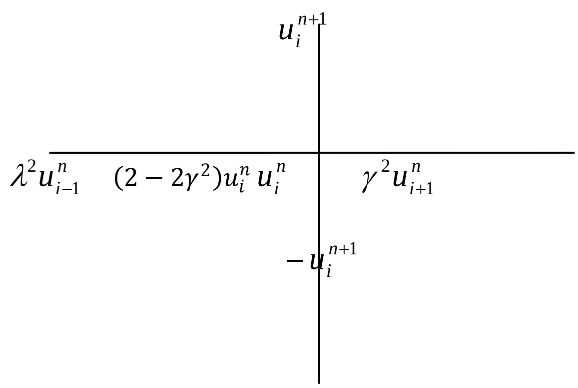

Figure 1.2 : Stencil of wave equation

IV. Numerical Experiments and Results

We develop a computer program (code) in Matlab scientific programming language and the implement the central difference scheme for wave equation.

a) Inserting data

We implemented the central difference scheme for the numerical simulation for the wave equation. We implemented the scheme for artificial initial and boundary data and verify the qualitative behavior of the numerical solution of wave equation. The main parts of the implementation of our numerical schemes are given in the following algorithm.

Input: nt and nx the number of grid points of time and space respectively.

If T the right end point of [0,T]

x_b the right end point of [0,b]

u_0 the initial velocity apply as initial condition.

u_b the velocity at the boundary point, apply as a boundary value.

Output: $u(t,x)$ the solution matrix.

Initialization: $dt = \dfrac{t-0}{nt}$ the temporal grid size.

$dx = \dfrac{b-0}{nx}$ the spatial grid size.

$c = 0.5$ a constant

$gm = c * \dfrac{dt}{dx}.$

$gma = 2 - 2 * gm * gm.$

Step 1: calculation for the scheme

for $j = 3 : nt\ nt$

for $i = 2 : nx - 1$

$$u(j,i) = gma * u(j-1.i) + gm * gm * (u(j-1,i+1)$$
$$u(j-1,i-1)) - u(j-2,i)$$

Step 2: Output $u(t,x).$

Step 3: Graph presentation.

Step 3: Stop.

b) Results

To test the accuracy of the implementation of the numerical scheme for wave equation, we discuss our experiment and results under generating the cases.

Case 1: In this case we considered the first order wave equation (linear advection) $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$.

We considered the analytical solution of one dimensional first order wave equation and performed the experiment in matlab. We choose the parameters as $\Delta t = 0.8, \Delta x = 6.28, c = 0.5, x = (0,2\pi), t = (0,101)$ with initial condition $u(x,0) = \sin x$ and boundary condition $u(x,0) = t^2$ and $u(x,t) = 0$ and run the propagation for 101 ts. In this case we see the figure in plot form and mesh form as follows.

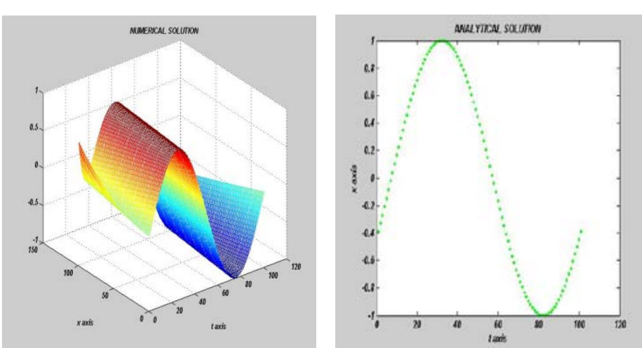

Figure 1 : Analytical solution of first order wave equation in plot form and mesh form

Case 2: In this case we consider the first order wave equation(linear advection) $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$ and perform the numerical experiment for three time step. We choose $\Delta t = 0.8, \Delta x = 6.28, c = 0.5, x = (0,2\pi), t = (0,101)$ with initial condition $u(x,0) = \sin x$ and boundary condition $u(x,0) = t^2$ and $u(x,t) = 0$ and run the propagation for 101 ts. We present the solution for the three different time step as shown in the figure. As expected we observer that the initial value in moving forward with respect to time.

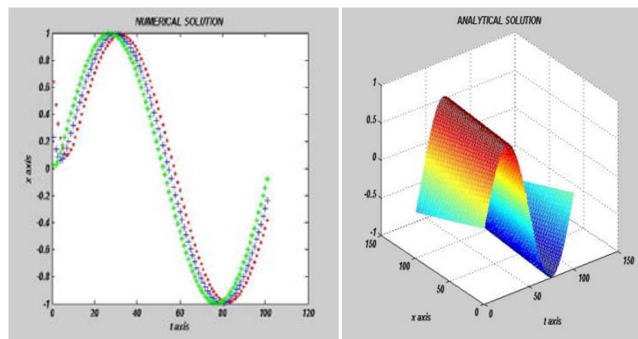

Figure 2 : Numerical solution of first order wave equation in plot form and mesh form using explicit upwind difference scheme

Case 3: In this case we consider the analytical solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c\dfrac{\partial^2 u}{\partial x^2}$ and perform the experiment in matlab. We choose the parameters as $\Delta t = 0.1, \Delta x = 6.28, c = 0.5, x = (0,2\pi), t = (0,101)$ with initial condition $u(x,0) = \sin x$ and boundary condition $u(x,0) = t^2$ and $u(x,t) = 0$ and run the propagation for 101 ts. In this case we see the figure in plot form and mesh form as follows.

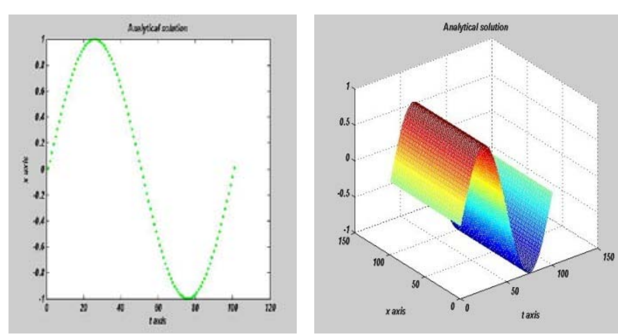

Figure 3 : Analytical solution of 2D wave equation in plot form and mesh form

Case 4: In this case we perform the numerical experiment for the equation $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$ and perform the experiment in matlab. We choose the parameters as $\Delta t = 0.1, \Delta x = 6.28, c = 0.5, x = (0,2\pi), t = (0,101)$ with initial condition $u(x,0) = \sin x$ and boundary condition $u(x,0) = t^2$ and $u(x,t) = 0$. In this case the initial configuration is not moving forward but only smears out which is typical behavior of the wave equation.

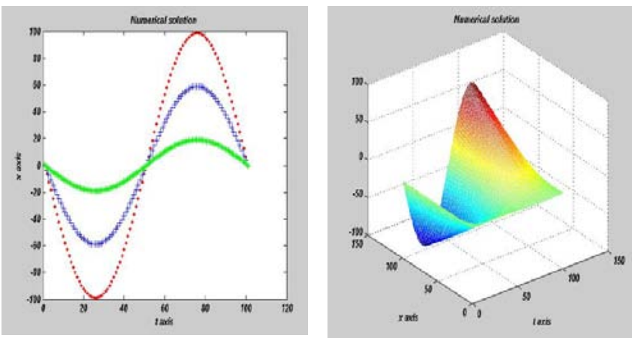

Figure 4 : Numerical solution of 2D wave equation in plot form and mesh form using central difference scheme

V. Conclusion

In this paper we considered wave equation which is a fundamental partial differential equation in fluid mechanics. First we showed equation of continuity, first order wave equation, second order wave equation, method of characteristics, D'Alembert solution. Finally we showed the numerical result of first order wave equation based on explicit upwind difference scheme and second order wave equation based on central difference scheme agrees with basic qualitative behavior of this equation.

References Références Referencias

1. Mathematical modeling and numerical simulation of the cardiovascular system-Alfio quarteroni and Luca Formaggia.
2. Knut Andress Lie, "The Wave Equation in 1D and 2D" Dept. of Informatics, University of Oslo.
3. John C. Strikwedra, "Finite Difference Schemes and Partial Differential Equation" Second Edition.
4. Peter J. Olver, Numerical Analysis Lecture Note.
5. http://community.dur.ac.uk/suzanne.fielding/teaching /BLT/sec1.pdf
6. Numerical Methods for Hyperbolic Systems-Eric Sonnendrucker-2013
7. Partial Differential Equations-Stepan Tersian, April-2005.
8. L. S. Andellah, "Finite Difference Method-Explicit Upwind Difference Scheme" Lecture note 2007.

This page is intentionally left blank