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On the Valuation Credit Risk Via Reduced- Form Approach

By Fadugba S. E. & Edogbanya O. H.
Ekiti State University, Nigeria

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[3] F. Black and M. Scholes, The Pricing of Options and Coporate Liabilities *Journal Of Political Economy*, **Vol. 81**, (1973), 637-654.

On the Valuation Credit Risk Via Reduced-Form Approach

Fadugba S. E. [α] & Edogbanya O. H. [σ]

Abstract- This paper presents the valuation of credit risk via reduced-form approach. Credit risk arises whenever a borrower is expecting to use future cash flows to pay a current debt. It is closely tied to the potential return of investment, the most notable being that the yields on bonds correlate strongly to their perceived credit risk. Credit risk embedded in a financial transaction, is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to decline in creditworthiness of the counter-party to the transaction or perhaps of some third party. Reduced-form approach is known as intensity-based approach. This is purely probabilistic in nature and technically speaking it has a lot in common with the reliability theory. Here the value of firm is not modeled but specifically the default risk is related either by a deterministic default intensity function or more general by stochastic intensity.

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I. Introduction

The main emphasis in the intensity-based approach is put on the modelling of the random time of default, as well as evaluating condition expectations under a risk-neutral probability of functionals of the default time and corresponding cash follows. Typically, the random default time is defined as the jump time of some one-jump process.

In recent years, we see a spectacular growth in trading, especially in derivative instruments. There is also an increase complexity of products in the financial markets with the growing complexity and trading size of financial markets, mathematical models have come to play an increasingly important role in financial decision making, especially in the context of pricing and hedging of derivative instruments. Models have become indispensable tools in the development of new financial products and the management of their risks.

The importance of valuation and hedging models in derivatives markets cannot be over-emphasized. The financial risk can therefore be categorized into four (4) types namely: Market risk, Liquidity risk, Operational risk and Credit risk.

The first category of credit risk models are the ones based on the original framework developed by [14]. Using the principles of option pricing [3]. In such a framework, the default process of a company is driven by the value of the company's assets and the risk of a firm's default is

Author α: Department of Mathematical Sciences Ekiti State University Ado Ekiti, Nigeria. e-mail: emmasfad2006@yahoo.com
Author σ: Department of Mathematics Federal University, Lokoja Kogi State, Nigeria. e-mail: helyna4christ@yahoo.com

therefore explicitly linked to the variability of the firm's asset value. The basic intuition behind the Merton model is that; default occurs when the value of a firm's assets (the market value of the firm) is lower than that of its liabilities. [14] derived an explicit formula for risky bonds which can be used both to estimate the probability of default of a firm and to estimate the yield differential between a risk bond and a default-free bonds.

In addition to [14], first generation structure-firm models include [2], they try to refine the original Merton framework by removing one or more of the unrealistic assumptions. [2] introduce the possibility of more complex capital structure with subordinated debts.

Reduced-form models somewhat differ from each other by the manner in which the recovery rate is parameterized. For example, [12] assumed that, at default, a bond would have a market value equals to an exogenous specified fraction of an otherwise equivalent default-free bond. [7] would have a market value equals to an exogenously specified fraction of an otherwise equivalent default-free bond. [8] followed with a model that when market value at default (recovery rate) is exogenously specified, allows for closed-form solutions for term-structure of credit spreads.

For mathematical background, valuation of credit risk, some numerical method for options valuation and stochastic analysis based on the Ito integral, see ([1], [4], [5], [6], [9], [10], [11], [13], [15], [16], [17] and [18]), just to mention few. In this paper we shall consider reduced-form approach for the valuation of credit risk.

II. Reduced-form Model

In this approach, the value of the firm's assets and its capital structure are not model at all, and the credit events are specified in terms of some exogenously specified jump process (as a rule, the recovery rates at default are also given exogenously). We can distinguish between the reduced-form models that are only concerned with the modelling of default time, and that are henceforth referred to as the intensity-based models, and the reduced form models with migrations between credit rating classes called the credit migration models.

The main emphasis in the intensity-based approach is put on the modelling of the random time of default, as well as evaluating condition expectations under a risk-neutral probability of functionals of the default time and corresponding cash follows. Typically, the random default time is defined as the jump time of some one-jump process. As well shall see, a pivotal role in evaluating respective conditional expectations is played by the default intensity process.

Modelling of the intensity process which is also known as the hazard rate process, is the starting point in the intensity approach.

a) Hazard Function

Before going deeper in the analysis of the reduced-form approach, we shall first examine a related technical question. Suppose we want to evaluate a conditional expectation $E_p(1_{\{\tau>s\}}Y|\mathcal{G}_t)$, where τ is a stopping time on a probability space (Ω, G, p), with respect to some filtration $G = (\mathcal{G}_t)_{t\geq 0}$ and Y is an integrable, $\mathcal{G}_s$-measurable random variable for some $s > t$.

In financial applications, it is quite natural and convenient to model the filtration G as $G = FVH$, where h is the filtration that carries full information about default events (that is, events such as

Ref

[14] R.C. Merton , *On The Pricing of Coporate Debt: The Risk structure of Interest Rates*, Journal of Finance **Vol. 29**, (1974), 449-470.

$\{\tau \le t\}$), whereas the **reference** filtration F carries information about other relevant financial and economic processes, but, typically, it does not carry full information about default event. The first question we address is how to compute the expectation

$$E_p(1_{\{\tau > s\}} Y | \mathcal{G}_t) \tag{2.1}$$

Notes

Using the intensity of τ with respect to F.

b) Hazard Function of a Random Time

We study the case where the reference filtration F is trivial, so that it does not carry any information whatsoever. Consequently, we have that $G = h$. Arguably, this is the simplest possible used in practical financial applications, as it leads to relatively easy calibration of the model.

We start by recalling the notion of a hazard function of a random time. Let τ be a finite, non-negative random time.

Let τ be a finite, non-negative, variable on a probability space $(\Omega, \mathcal{G}, p)$, referred to as the **random time**. We assume that $p\{T = 0\} = 0$ and τ is unbounded;

$$p\{\tau > t\} > 0 \ \text{ for every } \ t \in R_+ \tag{2.2}$$

The right continuous cumulative distribution function F of τ satisfies

$$F(t) = p\{\tau \le t\} < 1 \ \text{ for every } \ t \in R_+ \tag{2.3}$$

We also assume that $p\{\tau < \infty\} = 1$ so that τ is a Markov time.

We introduce the right-continuous jump process $H_t = 1_{\{\tau \le t\}}$ and we write

Lemma 1

For any $\mathcal{G}$-measurable (integrable) random variable Y we have

$$E_p(Y|\mathcal{H}_t) = 1_{\{\tau \le t\}} E_p(Y|\tau) + 1_{\{\tau > t\}} \frac{E_p(1_{\{\tau > t\}} Y)}{p\{\tau > t\}} \tag{2.4}$$

For any $\mathcal{H}_t$-measurable random variable Y we have

$$Y = 1_{\{\tau \le t\}} E_p(Y|\tau) + 1_{\{\tau > t\}} \frac{E_p(1_{\{\tau > t\}} Y)}{p\{\tau > t\}} \tag{2.5}$$

that is, $Y = h(\tau)$ for a Borel measurable $h : R \to R$ which is constant on the interval (t, ∞).

The hazard function is introduced through the following definition.

$h = (\mathcal{H}_t)_{t \ge 0}$ to denote the (right continuous and p-completed) filtration generated by the process H. Of course, τ is an h-stopping time.

We shall assume throughout that all random variables and processes that are used in what follows satisfy suitable integrability conditions. We begin with the following simple and important result.

Definition 1: The increasing right-continuous function $\Gamma : R_+ \to R_+$ given by the formula

$$\Gamma(t) = -\ln(1 - F(t), \ \ \forall\, t \in R_+ \tag{2.6}$$

is called the **hazard function** of a random time τ.
If the distribution function F is an absolutely continuous function, i.e., if we have

$$F(t)=\int_0^t f(u)du$$

for some function $f: R_+ \to R_+$, then we have

$$\begin{aligned} F(t) &= 1-e^{-\Gamma(t)} \\ &= 1-e^{-\int_0^t \gamma(u)du} \end{aligned}$$

where we set

$$\gamma(t)=\frac{f(t)}{1-F(t)}$$

$\gamma : R_+ \to R$ is a non-negative function and it satisfies $\int_0^\infty \gamma(u)du=\infty$.
The function γ is called the **hazard rate or intensity of** τ sometimes, in order to emphasize relevance of the measure p the terminology p**-hazard** rate and p**-intensity** is used. The next result follows from definition 2

Definition 2: The dividend process D of a defaultable contingent claim $(X,C,\tilde{X},Z,\tau)$, which settles at time T, equals

$$D_t = X^d(T)1_{\{t\geq T\}^{-1}} + \int_{(0,t]}(1-Hu)diCv + \int_{(0,t]} Z_u dHu$$

D is a process of finite variation and

$$\begin{aligned} \int_{(0,t]}(1-Hu)dcu &= \int_{(0,t]} 1_{\{\tau>u\}}dcu \\ &= C_\tau - 1_{\{\tau\leq t\}} + C_t 1_{\{\tau>t\}}. \end{aligned}$$

Note that if default occurs at some date t, the promised dividend $C_t - C_t-$, which is due to be paid at this date, is not received by the holder of a defaultable claim. Furthermore, if we set $\tau \wedge t = \min\{\tau,t\}$ then

$$\int_{(0,t]} Z_u dHu = Z_{\tau\wedge t}1_{\{\tau\leq t\}} = Z_{\{\tau=t\}} \tag{2.7}$$

Remark: In principle, the promised payoff X could be incorporated into the promised dividends process C. However, this would inconvenient, since in practice the recovery rules concerning the promised dividend C as the promised claim X are different, in general. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost, but a strictly positive fraction of the face value is usually received by the bondholder.

Corollary 2: For any $\mathcal{G}$-measurable random variable Y we have

$$E_p(1_{\{\tau>t\}}Y|\mathcal{H}_t) = 1_{\{\tau>t\}}e^{\Gamma(t)}E_p(1_{\{\tau>t\}}Y) \tag{2.8}$$

Corollary 3: Let Y be $\mathcal{H}_\infty$-measurable, so that $Y=h(\tau)$ for some function $h: R_+ \to R$. If the hazard function Γ is continuous then

Notes

$$E_p(Y|\mathcal{H}_t) = 1_{\{\tau \le t\}} h(\tau) + 1_{\{\tau > t\}} \int_t^{\infty} h(u) e^{\Gamma(t)-\Gamma(u)} d\Gamma(u) \tag{2.9}$$

If, in addition, the random time τ admits the hazard rate function γ then we have

$$E_p(Y|\mathcal{H}_t) = 1_{\{\tau \le t\}} h(\tau) + 1_{\{\tau > t\}} \int_t^{\infty} h(u)\gamma(u) e^{-\int_t^u \gamma(v)dv} du \tag{2.10}$$

Notes

In particular, for any $t \le s$ we have:

$$p\{\tau > s|\mathcal{H}_t\} = 1_{\{\tau > t\}} e^{-\int_t^s \gamma(v)dv} \tag{2.11}$$

and

$$p\{t < \tau < s|\mathcal{H}_t\} = 1_{\{\tau > t\}} \left(1 - e^{-\int_t^s \gamma(v)dv}\right) \tag{2.12}$$

Lemma 4: The process L, given by the formula

$$\begin{aligned} Lt &:= 1_{\{\tau > t\}} e^{\Gamma(t)} \\ &= \frac{1 - H_t}{1 - F(t)} \\ &= (1 - H_t) e^{\Gamma(t)} \quad \forall \ t \in R_+ \end{aligned}$$

is an h-Martingale.

c) Martingales Associated with Continuous Hazard Function

The h-adapted process of finite variation L given by last formula is an h-martingale (for Γ continuous or a discontinuous function).

We examine further important examples of martingales associated with the hazard function, with the assumption that the hazard function Γ of a random time τ is continuous. Also we assume that the cumulative distribution function F is absolutely continuous function, so that the random time τ admits the intensity function γ, our goal is to establish a martingale characterization of γ.

More specifically, we shall check directly that the process $\hat{M}$, defined as:

$$\begin{aligned} \hat{M}_t &= H_t - \int_0^t Y(u) 1_{\{\tau \le t\}} du \\ &= H_t - \int_0^{t\wedge\tau} \gamma(u) du \\ &= H_t - \Gamma(t \wedge \tau), \end{aligned}$$

follows and h-martingale. To this end,

$$E_p(H_s - H_t|\mathcal{H}_t) = 1_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)}$$

On the other hand, if we denote

$$\begin{aligned} Y &= \int_t^s \gamma(u) 1_{\{\tau \le t\}} du \\ &= \int_{t\wedge\tau}^{s\wedge\tau} \frac{f(u)}{1 - F(u)} du \\ &= \ln \frac{1 - F(t \wedge \tau)}{1 - F(\tau \wedge \tau)} \\ Y &= 1_{\{\tau > t\}} Y. \end{aligned}$$

Let us set $A = \{\tau > t\}$. Using the Fubini's theorem, we obtain

$$E_p(Y|\mathcal{H}_t) = E_p(1_A Y|\mathcal{H}_t) = 1_A \frac{E_p(Y)}{pA} \tag{2.13}$$

This shows that the process $\hat{M}$ follows an h-Martingale.

d) Martingale Hazard Function

Lemma 5: Assume that F (and this also the Hazard function Γ) is continuous function. Then the process

$$M_t = H_t - \Gamma(t \wedge \tau) \tag{2.14}$$

is h-Martingale.

In view of the Martingale in Lemma 5, the following definition is natural.

Definition 3: A function $\bigwedge : R_+ \to R$ is called a martingale hazard function of a random time τ with respect to the filtration if and only if the process

$$H_t - \wedge(t \wedge \tau) \ \text{ is an } h\text{-martingale.}$$

Remarks: Since the bounded, increasing process H is constant after time τ its compensation is constant after τ as well. This explains why the function $\wedge$ has to be evaluated at time $t \wedge \tau$, rather than at time t. H is thus a bounded h-submartingale.

It happen that the martingale hazard function can be found explicitly. In fact, we have the following.

Proposition 6: The unique Martingale hazard function of τ with respect to the filtration h is the right-continuous increasing function $\wedge$ given by the formula

$$\bigwedge(t) = \int_{[0,t]} \frac{dF(u)}{1 - F(u-)} \tag{2.15}$$

$$= \int_{(0,t]} \frac{dp\{\tau \le u\}}{1 - p\{\tau > u\}}. \tag{2.16}$$

Observe that the martingale hazard function $\wedge$ is continuous if and only if F is continuous. In this case, we have

$$\wedge(t) = -ln(1 - F(t)) \tag{2.17}$$

We conclude that the Martingale hazard function $\wedge$ coincides with the hazard function Γ if and only if F is a continuous function.

In general, we have

$$e^{-\Gamma(t)} = e^{-\wedge^c(t)} \prod_{0 \le u \le t} (1 - \triangle \wedge (u)) \tag{2.18}$$

where

$$\wedge^c(t) = \wedge(t) - \sum_{0 \le u \le t} \triangle \wedge (u) \ \text{ and } \ \triangle \wedge (u) = \wedge(u) - \wedge(u-) \tag{2.19}$$

Notes

Notes

e) Default Table Bonds : Deterministic Intensity

In order to value a defaultable claim, we need, of course, to specify the unit in which we would like to express all prices. Formally, this is done through a choice of discount factor (a numeraire). For the sake of simplicity, we shall take the savings account

$$B_t = e^{\int_0^t \gamma_r dv} \quad \forall\, t \in [0, T^*] \tag{2.20}$$

as the numraire, where r is the short term interest rate process.

We also postulate that some probability measure Q^* is a martingale measure relative to this nomeraire. This assumption means, in particular, that the price of any contingent claim Y which settles at time T is given as the conditional expectation.

In accordance with our assumption that the reference filtration is trivial, we also assume that:

- the default time τ admits the Q^*-intensity function
- the short-term interest rate $r(t)$ is a deterministic function of time.

In view of the latter assumption, the price at time t of a unit default-free zero-coupon bond of maturity T equals

$$B(t,T) = e^{-\int_t^T r(v)dv} \tag{2.21}$$

In the market practice, the interest rate (more precisely, the yield curve) can be derived from the market price of the zero-coupon bond. In a similar way the hazard rate can be deduced from the prices of the corporate zero-coupon bonds, or from the market values of other actively traded credit derivatives.

In view of our earlier notation for defaultable claims adopted, for the corporate unit discount bond we have $C \equiv 0$ and $X = L = 1$. And since the reference filtration is assumed trivial, we have that $G = h$.

f) Zero Recovery

Consider first a corporate zero-coupon bond with unit face value, the maturity date T, and zero recovery at default (that is, $\tilde{X} = 0$ and $Z \equiv 0$). Finally, the bond can thus be identified with a claim of the form $1_{\{\tau > T\}}$ which settle at T. It is clear that a corporate bond with zero recovery becomes worthless as soon as default occurs. Its time t price is defined as

$$D^0(t,T) = B_t E_{Q^*}(B_T^{-1} 1_{\{\tau > T\}} | \mathcal{H}_t)$$

The price $D^0(t,T)$ can be represented as follows:

$$D^0(t,T) = 1_{\{\tau > t\}} \tilde{D}^0(t,T) \tag{2.22}$$

where $\tilde{D}^0(t,T)$ is the bond's pre-default value, and is given by the formula

$$\tilde{D}^0(t,T) = e^{-\int_t^T (r(v)+\gamma(v))} dv = B(t,T) e^{-\int_t^T \gamma(v)dv} \tag{2.23}$$

f) Hazard Function

According to this convention, we have $\tilde{X} = 0$ and the recovery process Z satisfy $Z_t = \delta$ for some constant recovery rate $\delta \in [0,1]$. This means that under FRPV the bondholder receives at time of default a fixed fraction of bond's par value.

Using Corollary 3, we check that the pre-default value $\tilde{D}^{\delta}(t,T)$ of a unit corporate zero-coupon bond with FRPV equals

$$\tilde{D}^{\delta}(t,T) = \delta \int_t^T e^{-\int_t^u \tilde{r}(v)dv}\gamma(u)du + e^{-\int_t^\tau \tilde{\sigma}(v)dv} \tag{2.24}$$

where $\tilde{r} = r + \gamma$ is the default risk-adjusted interest rate. Since the fraction of the par value is received at the time of default, in the case of full recovery, that is, for $\delta =$, we do not obtain the equality $\tilde{D}^{\delta}(t,T) = B(t,T)$ but rather the inequality $\tilde{D}^{\delta}(t,T) > B(t,T)$ (at least when the interest rate is strictly positive, so that $B(t,T) < 1$ for $t < T$.

g) Fractional Recovery of Treasury Value (FRTV)

Assume now that $\tilde{X} = 0$ and that the recovery process equal $Z = \delta B(t,T)$. This means that the recovery payoff at the time of default τ represent a fraction of the price of the (equivalent) Treasury bond. The price of a corporate bond which is subject to this recovery scheme equals

$$S_t = B(t,T)(\delta Q^*\{t < \tau \leq T|\mathcal{H}_t\} + Q^*\{\tau > T|\mathcal{H}_t\}).$$

Let us denote by $\hat{D}^{\delta}(t,T)$ the pre-default value of a unit corporate bond subject to the FRTV scheme. Then

$$\hat{D}^{\delta}(t,T) = \int_t^T \delta B(t,T)e^{-\int_t^v \gamma(v)dv}\gamma(u)du + e^{-\int_t^T \tilde{r}(v)dv}$$

or equivalently,

$$\hat{D}^{\delta}(t,T) = B(t,T)\left(\delta\left(1 - e^{-\int_t^T \gamma(v)dv}\right) + e^{-\int_t^T \gamma(v)dv}\right) \tag{2.25}$$

In the case of full recovery, that is, for $\delta = 1$, we obtain $\hat{D}^{\delta}(t,T) = B(t,T)$ as expected.

Remarks. Similar representations can be derived also in the case when the reference filtration F is not trivial, and under the assumption that market risk and credit risk are independent that is:

- the default time admits the F-intensity process γ,
- the interest rate process r is independent of the filtration F.

III. Hazard Processes

In the previous section, it was assumed that the reference filtration F carries no information. However, for practical purposes it is important to study the situation where the reference filtration is not trivial. This section presents some results to this effect.

We assume that a martingale measure Q is given, and examine the valuation of defaultable contingent claims under this probability measure. Note that the defaultable market is incomplete if there are no defaultable assets traded on the market that are sensitive to the same default risk as the defaultable contingent claim we wish to price. Thus, the martingale measure may not be unique.

a) Hazard Process of a Random Time

Let $\tau : \Omega \to R_+$ be a finite, non-negative random variable on a probability space $(\Omega, \mathcal{G}, p)$. Assume $\mathcal{G} = \mathcal{F}_t V \mathcal{H}_t$ for some reference filtration F, so that $G = FVh$.

Notes

We start by extending some definitions and results to the present framework. We denote $F_t = p\{\tau \leq t|\mathcal{F}_t\}$, so that $G_t = 1 - F_t = p\{\tau > t|\mathcal{F}_t\}$ is the survival process with respect to F. F is a bonded non-negative, F-submartingale. As a submartingale, this process admits a Doob-Meter decomposition as $F_t = Z_t + A_t$ where A is an F-predictable increasing process. Assume, in addition, that $F_t < 1$ for every $t \in R_+$.

Notes

Definition 4: The F-hazard process Γ of a random time τ is defined through the equality $1 - F_t = e^{-\Gamma_t}$, that is, $\Gamma_t = \ln G_t$.

Notice that the existence of Γ implies that τ is not an F-stopping time. If the event $\{\tau > t\}$ belongs to the σ-field $\mathcal{F}_t$ for some $t > 0$ then $p\{\tau > t|\mathcal{F}_t\} = 1_{\{\tau>t\}} > 0$ (p-almost surely) and this $\tau = \infty$.

If the hazard process is absolutely continuous, so that $\Gamma_t = \int_0^t \gamma_u du$, for some process γ, then γ is called the F-intensity of τ. Thus the case only if the process Γ is increasing and thus γ is always non-negative. Note that if the reference filtration F is trivial, then the hazard process Γ is the same as the hazard function $\Gamma(\cdot)$. In this case, if T is absolutely continuous, then we have $\gamma_t = \gamma(t)$.

b) Terminal Payoff

The valuation of the terminal payoff $X^d(T)$ is based on the following generalization of Lemma 1.

The question is how to compute $F_p(1_{\{\tau>s\}}Y|\mathcal{G}_t)$ for and $\mathcal{F}_s$-measurable random variable Y?

Lemma 7: For any $\mathcal{G}$-measurable (integrable) random variable Y an arbitrary $s \geq t$ we have

$$E_p(1_{\{\tau>s\}}Y|\mathcal{G}_t) = 1_{\{\tau>t\}}\frac{E_p(1_{\{\tau>s\}}Y|\mathcal{F}_t)}{p\{\tau > t|\mathcal{F}_t\}} \tag{3.1}$$

If, in addition, Y is $\mathcal{F}_s$-measurable then

$$E_p(1_{\{\tau>s\}}Y|\mathcal{G}_t) = 1_{\{\tau>t\}}E_p(e^{\Gamma_t-\Gamma_s}Y|\mathcal{F}_t) \tag{3.2}$$

Assume that Y is $\mathcal{G}_t$-measurable. Then there exists on $\mathcal{F}_t$-measurable random variable $\tilde{Y}$ such that $1_{\{\tau>t\}}Y = 1_{\{\tau>t\}}\tilde{Y}$.

The latter property can be extended to stochastic process: for any G-predictable process X there exists an F-predictable process $\tilde{X}$ such that the equality

$$1_{\{\tau>t\}}X_t = 1_{\{\tau>t\}}\tilde{X}_t \tag{3.3}$$

is valid for every $t \in R_+$, that both processes coincides on the random interval $[0, t)$.

c) Recovery Process

The following extension of Corollary 3 appears to be useful in the valuation of the recovery payoff Z_τ (Note that the payoff occurs at time τ).

Lemma 8: Assume that the hazard process Γ is a continuous, increasing process, and let Z be a bonded, F-predictable process. Then for any $t \leq s$ we have:

$$E_p(Z_\tau 1_{\{t<\tau>s\}}|\mathcal{G}_t) = 1_{\{\tau>t\}}E_p\left(\int_t^s Z_u e^{\Gamma_t-\Gamma_u}d\Gamma u|\mathcal{F}_t\right) \tag{3.4}$$

d) Promised Dividends

To value the promised dividends (that are paid prior to τ, it is convenient to make use of the following result.

Lemma 9: Assume that the hazard process Γ is continuous. Let C be a bounded, F-predictable process of finite variation. Then for event $t \leq s$

$$E_p\left(\int_{(t,s)}(1-H_u)dC_u|\mathcal{G}_t\right) = 1_{\{\tau>t\}}E_p\left(\int_{(t,s]}e^{\Gamma_t-\Gamma_u}dC_u|\mathcal{F}_t\right) \tag{3.5}$$

e) Valuation of Defaultable Claims

We assume that τ is given on a filtered probability spaces (Ω, G, Q^*), where $G = FVh$ and $Q^*\tau > t|\mathcal{F}_t\} > 0$ for every $t \in R_+$ so that the F-hazard process Γ of τ under Q^* is well define. A default time τ is thus a G-stopping time, but it is an$F$-stopping time.

The probability Q^* is assumed to be a martingale measure relative to saving account process B, which is given by (3) for some F-progressively measurable process r. In some sense, this probability, and thus also the F-hazard process Γ of τ under Q^*, are given by the market via calibration.

The ex-dividend price S_t of a defaultable claim $(X, C, \tilde{X}, Z, \tau)$ is given by definition 5 below,

Definition 5: For any date $t \in (0,T)$, the ex-dividend price of the defaultable claim $(X, C, \tilde{X}, Z, \tau)$ is given as

$$S_t = B_t E_{p^*}\left(\int_{(t,T]}B_u^{-1}dDu|\mathcal{F}_t\right) \tag{3.6}$$

we always set $S_T = X^d(T)$. With p^* substituted with Q^* and F replaced by G. We postulate in particular, that the processes Z and C are F-predictable, and the random variable X and$\tilde{X}$ are $\mathcal{F}_T$-measurable and $\mathcal{G}_T$-measurable, respectively. Using Lemmas 7, 8, 9 and the fact that the savings account process B is F-adapted, a convenient representation for the arbitrage price of a defaultable claim in terms of the F-hazard process Γ is derived.

Proposition 10: The value process of a defaultable claim $(X, C, \tilde{X}, Z, T)$ admits the following representation for $t < T$

$$S_t = 1_{\{\tau>t\}}\mathcal{G}_t^{-}B_tE_{Q^*}\left(\int_{(t,T]}B_u^{-1}(\mathcal{G}_u dC_u - Z_u d\mathcal{G}_u)|\mathcal{F}_t\right)$$
$$+1_{\{\tau>t\}}\mathcal{G}_t^{-1}B_tE_{Q^*}(\mathcal{G}_T B_T^{-1}X|\mathcal{F}_t) + B_tE_{Q^*}(B_T^{-1}1_{\{\tau>T\}}\tilde{X}|\mathcal{G}_t)$$

If the hazard process Γ is an increasing, continuous process, then

$$\begin{aligned} S_t &= 1_{\{\tau>t\}}B_tE_{Q^*}\left(\int_{(t,T]}B_u^{-1}e^{\Gamma_t-\Gamma_u}(dC_u + Z_u d\Gamma_u)|\mathcal{F}_t\right) \\ &\quad +1_{\{\tau>t\}}B_tE_{Q^*}(B_T^{-1} - e^{\Gamma_t-\Gamma_T}X|\mathcal{F}_t) + B_tE_{Q^*}(B_T^{-1}1_{\{\tau\leq T\}}\tilde{X}|\mathcal{G}_t) \end{aligned}$$

Notes

Corollary 11: Assume that the F-hazard process Γ is a continuous, increasing process. Then the value process of a defaultable contingent claim $(X, C, \tilde{X}, Z, \tau)$ coincides with the value process of a claim $(X, \hat{C}, \tilde{X}, 0, \tau)$, where we set $\hat{C}_t = C_t + \int_0^t Z_u d\Gamma_u$.

f) Defaultable Bonds : Stochastic Intensity

Consider a defaultable zero-coupon bond with the par (face) value L and maturity date T. First, we re-examine the following recovery schemes: the fractional recovery of par value and the fractional recovery of Treasury value. Subsequently, we shall deal with the fractional recovery of pre-default value, but in this section using the stochastic intensity instead of the deterministic intensity used earlier. We assume that τ has the E-intensity γ.

Notes

g) Functional Recovery of Par Value

Under this scheme, a fixed fraction of the face value of the bond is paid to the bondholders at the time of default. Formally, we deal here with a defaultable claim $(X, 0, 0, Z, \tau)$, which settle at time T. With the promised payoff $X = L$, where L stands for the bond's face value, and with the recovery process $Z = \delta L$, where $\delta \in [0, 1]$ is a constant. The value at time $t < T$ of the bond is given by the expression

$$S_t = LB_t E_{Q^*}(\delta B_\tau^{-1} 1_{\{t<\tau>T\}} + B^{-1}_{T\{\tau>T\}} | \mathcal{G}_t) \tag{3.7}$$

If τ admits the F-intensity γ, the pre-default value of the bond equals

$$\tilde{D}^\delta(t, T) = L\tilde{B}_t E_{Q^*}\left(\delta \int_t^T \tilde{B}_u^{-1} \gamma_u du + B_T^{-1} | \mathcal{F}_t\right) \tag{3.8}$$

Remarks. The above setup is a special case of the fractional recovery of par value scheme with a general F-predictable recovery process $Z_t = \delta_t$, where the process δ_t satisfies $\delta_t \in [0, 1]$, for every $t \in [0, T]$. A general version of formula (3.8) is given by

$$\tilde{D}^\delta(t, T) = L\tilde{B}_t E_{Q^*}\left(\int_t^T \tilde{B}_u^{-1} \delta_u \gamma_u du + \tilde{B}_T^{-1} | \mathcal{F}_t\right) \tag{3.9}$$

h) Fractional Recovery of Treasury Value

Here, in the case of default, the fixed fraction of the face value is paid to bondholders at maturity date T. A corporate zero-coupon bond is now represented by a defaultable claim $(X, 0, 0, Z, \tau)$ with the promised payoff $X = L$ and the recovery process $(Z_t = \delta LBU, T)$. $B(t, T)$ stands for the price at time t of unit zero-coupon Treasury bond with Maturity T. The corporate bond is now equivalent to a single contingent claim Y, which settle at time T and equals

$$Y = L(1_{\{\tau>T\}} + \delta 1_{\{\tau\le T\}}) \tag{3.10}$$

The price of this claim oat time $t < T$ equals

$$S_t = LB_t E_{Q^*}(B_T^{-1}(\delta 1_{\{\tau\le T\}} + 1_{\{\tau>T\}}) | \mathcal{G}_t) \tag{3.11}$$

or equivalently,

$$S_t = LB_t E_{Q^*}(\delta B_T^{-1} B(t, T) 1_{\{t<\tau\le T\}} + (B_T^{-1} 1_{\{\tau>T\}} | \mathcal{G}_t) \tag{3.12}$$

The pre-default value $\hat{D}^\delta(t, T)$ of defaultable bond with the fractional recovery of Treasury value equals

$$\hat{D}^\delta(t, T) = L\tilde{B}_t E_{Q^*}\left(\delta \int_t^T \tilde{B}_u^{-1} B(u, T) \gamma_u du + \tilde{B}_T^{-1} | \mathcal{F}_t\right) \tag{3.13}$$

Again, the last formula is special case of the general situation where $Z_t = \delta_t$ with some predictable recovery ratio process $\delta_t \in [0, 1)$.

i) Fractional Recovery of Pre-default value

Assume that δ_t is some predictable recovery ratio process $\delta_t \in [0, 1)$ and let us set $X = L$. The pre-default value of the bond equals

$$D_M^{\delta}(t,T) = LE_{Q^*}\left(e^{-\int_t^T (r_u+(1-\delta_t)\gamma_u)du}|\mathcal{F}_t\right) \quad (3.14)$$

where

$$\hat{B}_t = \exp\left(\int_0^t (r_u + (1-\delta_u)\gamma_u)du\right) \quad (3.15)$$

j) Choice of a Recovery Scheme

A challenging practical problem is the calibration of statistical properties of both the recovery process δ and the intensity process γ. The empirical evidence strongly suggests that the amount recovered at default is best modelled by the recovery of par value scheme. However, we conclude that recovery concept that specifies the amount recovered as fraction of appropriately discounted par value, that is, the fractional recovery of treasury value, has broader empirical support.

IV. Conclusion

We conclude this section by giving few comments on the reduced-form approach to the modeling of credit risk. The advantages and disadvantages listed below are mainly relative to the alternative structural approach. It also worth noting that some of the disadvantages listed below disappear in the hybrid approach to credit risk modeling.

Advantages

- The specifications of the value-of-the firm process and the default-triggering barrier are not needed.
- The level of the credit risk is reflected in a single quantity: the risk-neutral default intensity.
- The random time of default is an unpredictable stopping time, and thus the default event comes as an almost total surprise.
- The valuation of defaultable claims is rather straightforward. It resembles the valuation of default-free contingent claims in term structure models, through well understood techniques.
- Credit spreads are much easier to quantify and manipulate than in structural models of credit risk. Consequently, the credit spreads are more realistic and risk premia are easier to handle.

Disadvantages

- Typically, current data regarding the level of the firm's assets and the firm's leverage are not taken into account.
- Specific features related to safety covenants and debt's seniority are not easy to handle.

Notes

- All (important) issues related to the capital structure of a firm are beyond the scope of this approach.
- Most practical approaches to Portfolio's credit risk are linked to the value-of-the-firm approach.

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Notes

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