Parallel Surfaces Satisfying the Properties of Ruled Surfaces in Minkowski 3-Space

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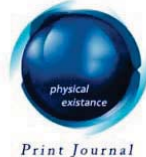
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Parallel Surfaces Satisfying the Properties of Ruled Surfaces in Minkowski 3-Space

Yasin Ünlütürk [α] & Cumali Ekici [σ]

Abstract- In this study, some properties of timelike parallel surfaces have been investigated in Minkowski 3-space. The main motivation of this work is to study the conditions under which parallel surfaces of timelike ruled surfaces with timelike ruling become timelike ruled surfaces. Also some characterizations of ruled surfaces such as distribution parameter, striction curve and orthogonal trajectory have been given for timelike parallel ruled surfaces.

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I. Introduction

Parallel surfaces and ruled surfaces are some of the main topics in both the classical and the modern differential geometry. These surfaces have many applications especially in physics and engineering [2,8,17,18,19]. It is possible to see many interesting papers which study on these two fields in terms of differential geometry such as [3,4,6,7,9,10,11,12,15,17,21,23,24].

If we mention briefly these studies: Craig studied to find the parallel of ellipsoid [4]. Çöken et al. studied parallel timelike ruled surfaces with timelike rulings in Dual space D_1^3 [3]. Görgülü and Çöken gave the dupin indicatrices for parallel pseudo Euclidean hypersurfaces in semi Euclidean space R_1^n [9]. Eisenhart studied parallel surfaces within a chapter of his book [6]. Nizamoğlu investigated a parallel ruled surface as a one-parameter curve using E. Study theorem and obtained some geometric characterizations of such a surface [15]. Güneş studied the relations among curves under parallel map preserving the connection [11]. Park examined offsets of ruled surfaces in Euclidean space [17]. Küçük and Gürsoy researched Bertrand offsets of trajectory ruled surfaces in view of their integral invariants [12]. Tarakçı and Hacısalihoğlu dealt with parallel surfaces as surfaces at a constant distance from the edge of regression on a surface in the general sense, [21]. Ekici and Çöken gave the parallel timelike ruled surface with a timelike ruling and its geometric invariants in terms of the main surface in Dual space D_1^3 [7]. Ünlütürk studied parallel ruled surfaces in Minkowski 3-space in detail [23, 24].

In this study, we have given some properties of timelike parallel surfaces in Minkowski 3-space. We have also studied the conditions under which parallel

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surfaces of timelike ruled surfaces with timelike ruling become timelike ruled surfaces. Furthermore we obtained some characterizations of ruled surfaces such as distribution parameter, striction curve and orthogonal trajectory have been given for timelike parallel ruled surfaces.

II.   PRELIMINARIES

Let E_1^3 be the three-dimensional Minkowski space, that is, the three-dimensional real vector space E^3 with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2$$

where $\mathbf{x} = (x_1, x_2, x_3)$ denotes the canonical coordinates in E^3. A vector $\mathbf{x}$ of E_1^3 is called to be spacelike, timelike and lightlike, respectively, if it satisfies $< \mathbf{x}, \mathbf{x} >> 0$ or $\mathbf{x} = \mathbf{0}$, $< \mathbf{x}, \mathbf{x} >< 0$, $< \mathbf{x}, \mathbf{x} >= 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or null vector in E_1^3 is said to be causal. The norm of $\mathbf{x} \in E_1^3$ is defined by $\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, then the vector $\mathbf{x}$ is called a spacelike or timelike unit vector if it satisfies $< \mathbf{x}, \mathbf{x} >= 1$ or $< \mathbf{x}, \mathbf{x} >= -1$, respectively. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [16]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of E_1^3, the inner product is the real number $< \mathbf{x}, \mathbf{y} >= x_1 y_1 + x_2 y_2 - x_3 y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2 y_3 - x_3 y_2), (x_3 y_1 - x_1 y_3), -(x_1 y_2 - x_2 y_1))$ [14].

A one-parameter family of lines $\{\alpha(u), X(u)\}$, the parameterized surface

$$\varphi(u, v) = \alpha(u) + vX(u), \qquad u \in I, \qquad v \in R \qquad (1)$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$ where $\alpha(u)$ is a point in E_1^3 and a vector $X(u) \in E_1^3$. The normal vector of the surface is denoted by N. Let us take timelike ruled surface φ with a spacelike directrix and timelike ruling. So the system $\{T, X, N\}$ establishes an orthonormal frame such that $T = \alpha'(u)$. Therefore

$$< T, T >=< X, X >= -1, \ < N, N >= 1 \text{ and } < D_T N, N >= 0. \qquad (2)$$

Derivative equations of the frame $\{T, X, N\}$ are

$$D_T T = aX + bN, \ \ D_T X = aT + cN, \ \ D_T N = -bT + cX. \qquad (3)$$

Also the cross products of these vectors are as follows:

$$T \wedge X = N, \ T \wedge N = X, \ X \wedge N = T. \qquad (4)$$

The distribution parameter is expressed as

$$\lambda = \frac{\det(\alpha', X, X')}{|X'|^2} \qquad (5)$$

where, as usual, (α', X, X') is a short for $\langle \alpha' \wedge X, X' \rangle$ [22].

Notes

Theorem 2.1. A surface in Minkowski 3-space is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal vector on the surface is a spacelike vector [1].

The coefficients which belong to the parametric equation of the surface given in (1) are as follows:

$$E = \langle X_u, X_u \rangle, \qquad F = \langle X_u, X_v \rangle, \qquad G = \langle X_v, X_v \rangle,$$

where the differentiable functions $E, F, G : U \to R$ are called the coefficients of the first fundamental form I. So the first fundamental form is

$$I = E du^2 + 2F du dv + G dv^2.$$

The following differentiable functions

$$
\begin{aligned}
l &= -\langle X_u, N_u \rangle = \langle N, X_{uu} \rangle \\
m &= -\langle X_u, N_v \rangle = -\langle X_v, N_u \rangle = \langle N, X_{uv} \rangle \\
n &= -\langle X_v, N_v \rangle = \langle N, X_{vv} \rangle
\end{aligned}
$$

are called the coefficients of the second fundamental form II. So the second fundamental form is

$$II = l du^2 + 2m du dv + n dv^2,$$

[14].

Theorem 2.2. Up to Lorentzian motions, a ruled surface is uniquely determined by the quantities

$$Q = \langle \alpha', X \wedge X' \rangle, \quad J = \langle X, X'' \wedge X' \rangle, \quad F = \langle \alpha', X \rangle \qquad (6)$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [13].

Theorem 2.3. The Gauss K and mean H curvatures of a timelike ruled surface φ with timelike ruling in terms of the parameters Q, J, F, D in E_1^3 are obtained as

$$K = -\frac{Q^2}{D^4} \quad \text{and} \quad H = \frac{1}{2D^3}(-QF - Q^2 J - Jv^2 + vQ'), \qquad (7)$$

where $D = \sqrt{-\varepsilon Q^2 + \varepsilon v^2}$, respectively [5].

Theorem 2.4. The parameter curves are lines of curvature if and only if $F = m = 0$, where the coefficients F and m belong, respectively, to the first and second fundamental forms in E_1^3 [14].

Definition 2.5. Let M and M^r be two surfaces in E_1^3. The function $f : M \to M^r$, $f(p) = p + r\mathbf{N}_p$ is said to be the parallelization function between M and M^r and furthermore M^r is said to be a parallel surface to M in E_1^3 where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [9].

Notes

Theorem 2.6. Let M be a surface and M^r be a parallel surface of M in E_1^3. Let $f : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$, we have the following relations:

1. $f_*(X) = X + rS(X)$
2. $S^r(f_*(X)) = S(X)$
3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1 + rk} f_*(X) \tag{8}$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction of X [9].

Definition 2.7. Let M be a hypersurface of $\overline{M}$- manifold and M^r be a parallel surface of M in E_1^3. If σ is a curve passing through p on M and T is the tangent vector field of σ on M, then $\sigma^r = f \circ \sigma$ is a curve passing through a point $f(p)$ on M^r and $f_*(T) \in T_{f(p)}M^r$ is a tangent of σ^r at $f(p)$. The connection D^r belongs to the parallel surface M^r of M and the vector $\mathbf{N}^r$ is the unit normal vector of M^r, where $\langle \mathbf{N}^r, \mathbf{N}^r \rangle = \varepsilon = \pm 1$. Therefore the Gauss equation is as follows:

$$\overline{D}_{f_*(T)} f_*(T) = D^r_{f_*(T)} f_*(T) - \varepsilon \left\langle S^r(f_*(T)), f_*(T) \right\rangle \mathbf{N}^r \tag{9}$$

[11, 16].

Theorem 2.8. Let $\varphi(u, v)$ be a surface in E_1^3 with the normal vector N. Then the shape operator S of φ is given in terms of the basis $\{\varphi_u, \varphi_v\}$ by

$$\begin{aligned}
-S(\varphi_u) = N_u &= \frac{mF - lG}{EG - F^2}\varphi_u + \frac{lF - mE}{EG - F^2}\varphi_v \\
-S(\varphi_v) = N_v &= \frac{nF - mG}{EG - F^2}\varphi_u + \frac{mF - nE}{EG - F^2}\varphi_v
\end{aligned} \tag{10}$$

[20].

III. Timelike Parallel Surfaces

The representation of points are obtained on M^r by using the representations of points on M. Let φ be the position vector of a point P on M and φ^r be the position vector of a point $f(P)$ on the parallel surface M^r. Then $f(P)$ is at a constant distance r from P along the normal to the surface M. Therefore the parameterization for M^r is given by

$$\varphi^r(u, v) = \varphi(u, v) + r\mathbf{N}(u, v) \tag{11}$$

where r is a constant scalar and $\mathbf{N}$ is the unit normal vector field on M. In E_1^3, let the parallel surface of a timelike surface $\varphi(u, v)$ be as given in (11), is defined in E_1^3 as where $\mathbf{N}$ is the unit normal vector on the surface $\varphi(u, v)$ such that $\langle \mathbf{N}, \mathbf{N} \rangle = 1$ and $r \in R$. Fundamental forms' coefficients of timelike parallel surfaces can be given relative to ones of timelike surface as follows:

Notes

$$E^r = E - 2rl + r^2 \langle \mathbf{N}_u, \mathbf{N}_u \rangle, \qquad l^r = l - r \langle \mathbf{N}_u, \mathbf{N}_u \rangle$$
$$F^r = F - 2rm + r^2 \langle \mathbf{N}_u, \mathbf{N}_v \rangle, \quad m^r = m - r \langle \mathbf{N}_u, \mathbf{N}_v \rangle \qquad (12)$$
$$G^r = G - 2rn + r^2 \langle \mathbf{N}_v, \mathbf{N}_v \rangle, \quad n^r = n - r \langle \mathbf{N}_v, \mathbf{N}_v \rangle,$$

where E, F, G, l, m, n, are fundamental forms' coefficients of the surface φ, and $E^r, F^r, G^r, l^r, m^r, n^r$, are fundamental forms' coefficients of the parallel surface φ^r [23].

To get explicit formulas for H^r and K^r, we work in a parallel surface M^r. Let $\varphi^r(u, v)$ be a timelike parallel surface with first and second fundamental forms

$$E^r du^2 + 2F^r du dv + G^r dv^2 \quad \text{and} \quad l^r du^2 + 2m^r du dv + n^r dv^2,$$

respectively. Define 2×2 matrices $\mathcal{F}_{I^r}$ and $\mathcal{F}_{II^r}$ by

$$\mathcal{F}_{I^r} = \begin{bmatrix} E^r & F^r \\ F^r & G^r \end{bmatrix}, \quad \mathcal{F}_{II^r} = \begin{bmatrix} e^r & f^r \\ f^r & g^r \end{bmatrix},$$

also the matrix of S_p^r, with respect to the basis $\{\varphi_u^r, \varphi_v^r\}$ of $T_p M^r$, is

$$\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}. \qquad (13)$$

Definition 3.1. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Let $\mathbf{N}^r$ and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian and mean curvature functions are defined as $K^r : M^r \to R, K^r(f(P)) = \det S_{f(P)}^r$ and $H^r : M^r \to R, H^r(f(P)) = \dfrac{1}{2} \operatorname{tr} S_{f(P)}^r$ where $P \in M, f(P) \in M^r$ and $\langle \mathbf{N}, \mathbf{N} \rangle = 1$, respectively.

Theorem 3.2. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Let $\mathbf{N}^r$ and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian K^r and mean H^r curvatures are given in terms of the coefficients of the fundamental forms I^r and II^r as follows:

$$K^r = \frac{l^r n^r - m^{r2}}{E^r G^r - F^{r2}} \quad \text{and} \quad H^r = \frac{l^r G^r - 2m^r F^r + n^r E^r}{2(E^r G^r - F^{r2})}, \qquad (14)$$

respectively.

Proof. Since $S_p^r = \mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}$ for the matrices $\mathcal{F}_{I^r}$ and $\mathcal{F}_{II^r}$, using the Definition 3.1 and the equation (13) we get

$$K^r = \det(\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}) = \frac{\det \mathcal{F}_{II^r}}{\det \mathcal{F}_{I^r}} = \frac{l^r n^r - m^{r2}}{E^r G^r - F^{r2}}.$$

For the mean curvature H^r, first the matrix $\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}$ is obtained as

$$
\begin{aligned}
\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r} &= \frac{1}{E^r G^r - F^{r2}} \begin{bmatrix} G^r & -F^r \\ -F^r & E^r \end{bmatrix} \begin{bmatrix} l^r & m^r \\ m^r & n^r \end{bmatrix} \\
&= \frac{1}{E^r G^r - F^{r2}} \begin{bmatrix} l^r G^r - m^r F^r & m^r G^r - n^r F^r \\ m^r E^r - l^r F^r & n^r E^r - m^r F^r \end{bmatrix}.
\end{aligned}
$$

From Definition 3.1, the mean curvature is found as

$$H^r = \tfrac{1}{2}\mathrm{tr}(\mathcal{F}_{I^r}^{-1}\mathcal{F}_{II^r}) = \frac{l^r G^r - 2m^r F^r + n^r E^r}{2(E^r G^r - F^{r2})}.$$

Lemma 3.3. Let M be a timelike surface and M^r be its parallel surface in E_1^3. The surface M is a timelike one if and only if the surface M^r is a timelike parallel surface.

Proof. $(\Rightarrow)$: If M is a timelike surface, then by Theorem 2.1, the unit normal vector $\mathbf{N}$ of M has to be as follows

$$\langle \mathbf{N}_P, \mathbf{N}_P \rangle > 0. \tag{15}$$

Between the unit normal vectors of the surfaces M and M^r, there is the following relation:

$$\mathbf{N}_P = \mathbf{N}_{f(P)}^r. \tag{16}$$

By substituting (16) into (15), we get

$$\langle \mathbf{N}_{f(P)}^r, \mathbf{N}_{f(P)}^r \rangle > 0. \tag{17}$$

The inequality (17) means that M^r is a timelike surface in accordance with Theorem 2.1.

$(\Leftarrow)$: If M^r is a timelike parallel surface, in accordance with Theorem 2.1, we get

$$\langle \mathbf{N}_{f(P)}^r, \mathbf{N}_{f(P)}^r \rangle > 0. \tag{18}$$

By substituting (16) into (18), we have

$$\langle \mathbf{N}_P, \mathbf{N}_P \rangle > 0. \tag{19}$$

The inequality (19) means that M^r is a timelike surface in accordance with Theorem 2.1.

Theorem 3.4. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Then we have

$$K^r = \frac{K}{1 + 2rH + r^2 K} \quad \text{and} \quad H^r = \frac{H + rK}{1 + 2rH + r^2 K} \tag{20}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [23].

Corollary 3.5. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Then we have

$$K = \frac{K^r}{1 - 2rH^r + r^2 K^r} \quad \text{and} \quad H = \frac{H^r - rK^r}{1 - 2rH^r + r^2 K^r} \tag{21}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [23].

Theorem 3.6. Let M be a timelike surface and M^r be a its parallel surface in E_1^3. The curves on the timelike parallel surface M^r which correspond to the lines of curvature on the timelike surface M are also the lines of curvature.

Notes

Proof. If the lines of curvature on M are chosen as parameter curves, then in accordance with Theorem 2.4, we have

$$F = m = 0. \tag{22}$$

It suffices to see $F^r = m^r = 0$ such that the curves on M^r which correspond to the lines of curvature on M are the lines of curvature. The parametric representation of M^r is as in (11). From Weingarten equations given in (10), by using the equation (22), we get

$$N_u = -\frac{l}{E}\varphi_u \quad \text{and} \quad N_v = -\frac{n}{G}\varphi_v. \tag{23}$$

If the values of F^r and m^r are used in the equations (22) and (23), then we have

$$F^r = F - 2rm + r^2 \langle N_u, N_v \rangle = r^2 \langle N_u, N_v \rangle = r^2 \frac{ln}{EG} \langle \varphi_u, \varphi_v \rangle = 0 \tag{24}$$

and

$$m^r = m - r \langle N_u, N_v \rangle = -r \langle N_u, N_v \rangle = -r\frac{ln}{EG}F = 0. \tag{25}$$

In (24) and (25), it is seen that the coefficients F^r and m^r vanish. So the curves on M^r are the lines of curvature.

IV. Timelike Parallel Ruled Surfaces

Theorem 4.1. Let M be a timelike ruled surface with a timelike ruling and M^r be a parallel surface of M in E_1^3. A parallel surface of a timelike developable ruled surface is again a timelike ruled surface.

Proof. Let the timelike ruled surface M with a timelike ruling be given as

$$\varphi(u,v) = \alpha(u) + vX(u), \ \langle \alpha', \alpha' \rangle = 1, \ \langle X, X \rangle = -1, \ \langle X', X' \rangle = 1. \tag{26}$$

We get its normal vector as follows:

$$N = \alpha' \wedge X + vX' \wedge X. \tag{27}$$

For the normal vector of a timelike developable ruled surface which is constant along its ruling and is independent from the parameter v, we infer that the expressions $\alpha' \wedge X$ and $X' \wedge X$ in (27) are linearly dependent, that is

$$\alpha' \wedge X = \lambda X' \wedge X,$$

for $\lambda \in \mathbb{R}$. Also, from (26), we obtain the normal vector of the surface M as

$$N = (\lambda + v)X' \wedge X. \tag{28}$$

In the end, the unit normal vector of the surface M is as follows

$$\mathbf{N} = X' \wedge X. \tag{29}$$

We get the parallel surface of the ruled surface $\varphi(u,v) = \alpha(u) + vX(u)$ as

$$\varphi^r(u,v) = \alpha(u) + rX'(u) \wedge X(u) + vX(u). \tag{30}$$

We call this surface obtained in (30) as timelike parallel ruled surface. The ruling of timelike parallel ruled surface is

$$f_*(X) = f_*(T) \wedge N^r = (T - rbT) \wedge N = (1 - rb)X. \tag{31}$$

And also we find

$$f \circ \alpha(u) = \alpha(u) + rN(u) = \alpha(u) + rX'(u) \wedge X(u). \tag{32}$$

The coefficient n^r of the second fundamental form of the surface M^r is calculated as

$$n^r = -\langle \varphi_v^r, \mathbf{N}_v \rangle = -\langle X, 0 \rangle = 0.$$

The drall of timelike parallel ruled surface is obtained as

$$P^r = < \frac{df \circ \alpha}{du}, f'_*(X) \wedge f_*(X) > . \tag{33}$$

From (33), we find

$$P^r = \langle \alpha' + rX'' \wedge X, (1 - rb)^2 X' \wedge X \rangle = 0. \tag{34}$$

Finally, timelike parallel ruled surface given in (30) is a developable ruled surface.

The coefficients of the first I^r and second II^r fundamental forms for timelike parallel ruled surface M^r parameterized in (30) are given by

$$
\begin{aligned}
E^r = {}& \langle \alpha', \alpha' \rangle + 2r \langle \alpha', X'' \wedge X \rangle + 2v \langle \alpha', X' \rangle + r^2 \langle X'' \wedge X, X'' \wedge X \rangle \\
& + 2rv \langle X', X'' \wedge X \rangle + v^2 \langle X', X' \rangle .
\end{aligned}
\tag{35}
$$

Since $\langle X', X' \rangle = 1$ and $\langle X'', X' \rangle = 0$, X'' lies in the plane spanned by the vectors X and $X' \wedge X$. Therefore

$$X'' = wX + yX' \wedge X, \tag{36}$$

where $w, y \in \mathbb{R}$. By using (36), we have

$$X'' \wedge X = (wX + yX' \wedge X) \wedge X = (yX' \wedge X) \wedge X = -yX'. \tag{37}$$

Substituting (37) into (35), the coefficients E^r, F^r and G^r are found as

$$
\begin{aligned}
E^r &= 1 + (ry - v)^2 \\
F^r &= \langle \varphi_u^r, \varphi_v^r \rangle = \langle \alpha' + rX'' \wedge X + vX', X \rangle = \langle \alpha', X \rangle \\
G^r &= \langle \varphi_v^r, \varphi_v^r \rangle = \langle X, X \rangle = -1.
\end{aligned}
$$

Also, the normal vector of the surface is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

Additionally, the coefficients l^r, m^r and n^r of the second fundamental form are computed as follows:

$$
\begin{aligned}
l^r &= -\langle \varphi_u^r, N_u^r \rangle = -\langle \alpha', \alpha'' \wedge X \rangle + \langle X', \alpha'' \wedge X \rangle (ry - v) + v^2 y - rvy^2 \\
m^r &= \langle -\varphi_u^r, N_v^r \rangle = -\langle \alpha', X' \wedge X \rangle \\
n^r &= -\langle \varphi_v^r, N_v^r \rangle = -\langle X, X' \wedge X \rangle = 0.
\end{aligned}
$$

Notes

Corollary 4.2. Let M^r be a timelike parallel ruled surface. Then the causal characters of the directrix and the ruling of M^r are a spacelike curve and a timelike vector, respectively.

Proof. The ruling of timelike parallel ruled surface M^r given in (30) is timelike since $\langle X, X \rangle = -1$. The causal character of the directrix is seen by the following computation:

$$\left\langle \frac{df \circ \alpha(u)}{du}, \frac{df \circ \alpha(u)}{du} \right\rangle = \langle \alpha' + rX'' \wedge X, \alpha' + rX'' \wedge X \rangle. \qquad (38)$$

By using $\langle X', X' \rangle = 1$ and $\langle X', X'' \rangle = 0$, from (38), the following result is obtained that

$$\left\langle \frac{df \circ \alpha(u)}{du}, \frac{df \circ \alpha(u)}{du} \right\rangle = 1 + r^2 y^2 > 0 \qquad (39)$$

which means that the causal character of the directrix is spacelike.

Theorem 4.3. Let M^r be a timelike parallel ruled surface with timelike ruling and $f_*(T)$, $f_*(X)$ and N^r be the tangent vector field of the directrix, tangent vector field of the ruling and the normal vector field of the surface M^r, respectively. Hence we have

$$f_*(T) \wedge \mathbf{N}^r = f_*(X), \quad f_*(T) \wedge f_*(X) = (1\text{-}rb)\mathbf{N}^r, \quad f_*(X) \wedge \mathbf{N}^r = f_*(T). \qquad (40)$$

Proof. Frenet equations for timelike developable ruled surface M are obtained in (3) by taking $c = 0$. And also the cross products of the unit vectors T, X, $\mathbf{N}$ for timelike developable ruled surface M are as in (4). By means of these information, we have the following results:

$$f_*(T) \wedge \mathbf{N}^r = (T - rbT) \wedge \mathbf{N} = (1 - rb)X = f_*(X)$$

$$f_*(T) \wedge f_*(X) = (T\text{-}rbT) \wedge (X\text{-}rbX) = (1\text{-}rb)^2 \mathbf{N} = (1\text{-}rb)^2 \mathbf{N}^r \qquad (41)$$

$$f_*(X) \wedge \mathbf{N}^r = (1 - rb)X \wedge \mathbf{N} = (1 - rb)T = f_*(T).$$

Theorem 4.4. The vectors $f_*(T)$, $f_*(X)$ and $\mathbf{N}^r$ for timelike parallel ruled surface M^r are spacelike, timelike and spacelike vectors, respectively, while the unit vectors T, X, $\mathbf{N}$ for timelike developable ruled surface M with timelike ruling are spacelike, timelike and spacelike vectors, respectively.

Proof. The unit normal vector $\mathbf{N}^r$ of the timelike parallel ruled surface M^r is a spacelike vector because

$$\langle \mathbf{N}^r, \mathbf{N}^r \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = 1.$$

The tangent vector field of the directrix is a spacelike vector since

$$\langle f_*(T), f_*(T) \rangle = (1 - rb)^2 > 0.$$

From (31), the vector $f_*(X)$ is a timelike vector because of

$$\langle f_*(X), f_*(X) \rangle = \langle (1 - rb)X, (1 - rb)X \rangle = -(1 - rb)^2 = -1 < 0.$$

Notes

We get the position vector of the striction curve on the timelike parallel ruled surface M^r as

$$\overrightarrow{O\gamma} = \overrightarrow{Of \circ \alpha} + \overrightarrow{\theta f_*(X)}. \tag{42}$$

Using $f_*(X) = X^r$ in (42), we have the striction curve as

$$(u) = f \circ \alpha(u) + \theta X^r(u) \text{ and } \theta = \theta(u). \tag{43}$$

By (43), we obtain the value θ as follows:

$$\theta = -\frac{\left\langle \dfrac{df \circ \alpha}{du}, \dfrac{dX^r}{du} \right\rangle}{\left\langle \dfrac{dX^r}{du}, \dfrac{dX^r}{du} \right\rangle} = -\frac{\langle \alpha', X' \rangle + r \langle X'' \wedge X, X' \rangle}{(1 - rb)\langle X', X' \rangle}. \tag{44}$$

Hence using (43) and (44), we find the striction curve as

$$(u) = \alpha(u) + rX'(u) \wedge X(u) - \frac{\langle \alpha', X' \rangle + r \langle X'' \wedge X, X' \rangle}{\langle X', X' \rangle} X. \tag{45}$$

After some arrangements in (45), it becomes

$$(u) = \alpha(u) + rX'(u) \wedge X(u) - \frac{(1 - rya)}{a} X. \tag{46}$$

Corollary 4.5. The striction curve of timelike parallel ruled surface with a timelike ruling is also the directrix provided that $\langle \alpha', X' \rangle = 0$ and $\langle X'' \wedge X, X' \rangle = 0$.

Proof. Straightforward calculation by using (45).

Corollary 4.6. The striction curve of timelike parallel ruled surface with a timelike ruling is also the directrix provided that $1 - rya = 0$.

Proof. Straightforward calculation by using (46).

Theorem 4.7. The striction curve of timelike parallel ruled surface M^r with a timelike ruling is a timelike curve.

Proof. The normal vector N^r of timelike parallel ruled surface M^r with a timelike ruling is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

For $v = 0$, we have

$$N^r(u, 0) = \alpha'(u) \wedge X(u). \tag{47}$$

From (47), we get

$$\begin{aligned} \langle N^r(u,0), N^r(u,0) \rangle &= \langle \alpha'(u) \wedge X(u), \alpha'(u) \wedge X(u) \rangle \\ &= F^2 + 1 > 0. \end{aligned} \tag{48}$$

The result (48) means that the striction curve is a timelike one because the vector, which is normal to it, is a spacelike vector.

Theorem 4.8. The striction curve of timelike parallel ruled surface M^r with a timelike ruling does not depend on the choice of the base curve $f \circ \alpha$.

Proof. Let $f \circ \alpha$ and ρ be two different directrices of timelike parallel ruled surface with a timelike ruling. Then timelike parallel ruled surface is given as

Notes

$$\varphi^r(u,v) = f \circ \alpha(u) + vX^r(u) = \rho(u) + sX^r(u) \qquad (49)$$

for some function $s = s(v)$. Assume that the curves $\gamma(u)$ and $\overline{\gamma}(u)$ are the striction curves of the surfaces in (49). Then as analogous to (45) from (49) we obtain

$$\gamma(u) - \overline{\gamma}(u) \;\; = (v-s)X^r - \frac{\langle (v-s)X^{r\prime}, X' \rangle}{\langle X', X' \rangle} X(u) = 0. \qquad (50)$$

The proof is completed by the result obtained in (50).

Theorem 4.9. Given timelike parallel ruled surface M^r which is parallel to timelike developable ruled surface M with a timelike ruling. There exists a unique orthogonal trajectory of M^r through each point of M. This orthogonal trajectory in terms of the magnitudes of timelike ruled surface M with timelike ruling is as follows:

$$\beta(s) = \alpha(s) + rX'(s) \wedge X(s) + g(s)X(s).$$

Here, the function $g(s)$ has been taken instead of $v(1-rb)$.

Proof. Let

$$\begin{aligned} \varphi^r: \;\; & I \times J \longrightarrow E_1^3 \\ & (u,v) \longrightarrow \varphi^r(u,v) \;\; = f \circ \alpha(u) + vX^r(u) \\ & \qquad\qquad\qquad\qquad = \alpha(u) + rX'(u) \wedge X(u) + v(1-rb)X. \end{aligned}$$

An orthogonal trajectory of M^r is given by

$$\begin{aligned} \beta: \;\; & \widetilde{I} \longrightarrow M^r \\ & s \longrightarrow \beta(s) = f \circ \alpha(s) + g(s)X^r(s). \end{aligned} \qquad (51)$$

We may assume $\widetilde{I} \subset I$. Since

$$\langle \beta'(s), X^r(s) \rangle \;\; = \;\; \langle \alpha'(s), X(s) \rangle - g'(s) = 0, \qquad (52)$$

we have

$$g(s) = \int \langle \alpha'(s), X(s) \rangle \, ds + h,$$

where h is a real constant. So $h = g(s_0) - F(s_0)$, where

$$-\int \langle \alpha'(s), X(s) \rangle \, ds = F(s).$$

Therefore we find that the orthogonal trajectory of the surface M^r through the point P_0 is unique. Thus, we have $\widetilde{I} = I$ since the orthogonal trajectory of M^r meets each one of the rulings of the surface M^r.

Corollary 4.10. Let M^r be a timelike parallel ruled surface with timelike ruling. The Gaussian and mean curvatures K^r and H^r of the surface M^r are as follows:

$$K^r = \frac{-Q^2}{D^4 - rQFD - rQ^2JD + rvQ'D - rv^2JD - r^2Q^2} \qquad (53)$$

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Notes

$$H^r = \frac{-QFD - Q^2 JD - v^2 JD + vQ'D - 2rQ^2}{2D^4 - 2rQFD - 2rQ^2 JD + 2rvQ'D - 2rv^2 JD - 2r^2 Q^2},$$

respectively, in terms of the parameters Q, J, F, D.

Proof. Using (7) in (20), the values of Gauss curvature K^r and mean curvature H^r are obtained as in (53).

Theorem 4.11. Let $\varphi(u,v)$ be a timelike ruled surface in E_1^3 with $F = m = 0$. Then the parallel surface

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v)$$

is a timelike developable ruled surface while one of the parameters of parallel surface is constant and the other is variable.

Proof. Every surface $u = u_0$ (a constant) is a ruled one as it is the union of the straight lines given by $v = $constant. This surface is developable provided that the curve $(u) = \varphi(u, v_0)$ is a line of curvature of M, i.e., if φ_v is a principal vector. This is true since the matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal. Similarly for the surfaces $v = $constant. If $F = m = 0$, then $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal. So $\mathcal{F}_I^{-1} \mathcal{F}_{II}$, the matrix of Weingarten map, depends on the basis $\{\varphi_u, \varphi_v\}$. It means that the principal vectors φ_u and φ_v are lines of curvature. Hence, the ruled surface M is a developable ruled surface. From Theorem 4.1., $\varphi^r(u,v)$ is a developable timelike ruled surface.

Theorem 4.12. Let M^r be a timelike parallel ruled surface with timelike ruling. The rulings of M^r are both an asymptotic and a geodesic line in M^r.

Proof. Let $f_*(X) \in \chi(M^r)$ be a tangent vector field for a ruling of M^r while $\overline{D} \in \chi(E_1^3)$, $D \in \chi(M)$ and $D^r \in \chi(M^r)$. Each one of the rulings is geodesic since each one of the rulings is a straight line in E_1^3. Thus we have

$$\overline{D}_{f_*(X)} f_*(X) = 0. \tag{54}$$

The Gauss equation for M^r is

$$\overline{D}_{f_*(X)} f_*(X) = D^r_{f_*(X)} f_*(X) - S^r(f_*(X)), f_*(X)\rangle N^r, \tag{55}$$

where $\overline{D}$ is Levi-Civita connection on M^r. By using (54), the equation (55) becomes

$$D^r_{f_*(X)} f_*(X) = \langle S^r(f_*(X)), f_*(X) \rangle N^r.$$

Furthermore, since

$$D^r_{f_*(X)} f_*(X) \in \chi(M^r) \text{ and } \langle S^r(f_*(X)), f_*(X) \rangle N^r \in \chi^\perp(M^r),$$

we get the following results:

$$D^r_{f_*(X)} f_*(X) = 0 \text{ or } \langle S^r(f_*(X)), f_*(X) \rangle N^r = 0.$$

Also, for the normal vector N^r of the surface M^r, we write

$$\chi(E_1^3) = \chi(M^r) \oplus \chi^\perp(M^r) \text{ and } \chi(M^r) \cap \chi^\perp(M^r) = \{0\}.$$

Then, we obtain

Notes

$$D^r{}_{f_*(X)}f_*(X) = 0 \text{ and } \langle S^r(f_*(X)), f_*(X) \rangle = 0.$$

The last equations completes the proof.

Theorem 4.13. Let M^r be a timelike parallel ruled surface. Then the Gaussian curvature $K^r(f(P))$ of M^r satisfies

$$K^r \geq 0$$

at each point $f(P) \in M^r$.

Proof. Let $f_*(X)$ be the timelike vector field of the rulings through the point $f(P) \in M^r$. We get an orthogonal base $\{f_*(X), f_*(Y)\}$ of $\chi(M^r)$ in which $f_*(Y)$ is a spacelike vector field. We obtain the matrix corresponding to the shape operator of M^r derived from N^r as follows:

$$S^r = \begin{bmatrix} -\langle S(X), X \rangle & \langle S^r(f_*(X)), f_*(Y) \rangle \\ -\langle S^r(f_*(Y)), f_*(X) \rangle & \langle S(Y), f_*(Y) \rangle \end{bmatrix}.$$

Using $\langle S^r(f_*(X)), f_*(X) \rangle = \langle S(X), X \rangle = 0$ by means of Theorem 4.12 and Definition 3.1, we have the Gaussian curvature K^r as follows:

$$\begin{aligned} K^r &= -\det S^r \\ &= \langle S^r(f_*(Y)), f_*(X) \rangle^2 = \langle S(Y), X \rangle^2 \geq 0. \end{aligned}$$

Example 4.14. A hyperbolic cylinder has the parameterization

$$\varphi(u, v) = (\cosh u, \sinh u, v). \tag{56}$$

It is easily seen that its base curve is a spacelike curve and its ruling is a timelike vector. A hyperbolic cylinder is a developable ruled surface since its drall λ vanishes. The unit normal vector of hyperbolic cylinder is found as

$$\mathbf{N} = \left(-\frac{5 \cosh u}{\sqrt{2 \cosh^2 u - 1}}, \frac{5 \sinh u}{\sqrt{2 \cosh^2 u - 1}}, 0\right).$$

By using the expression $\varphi^r = \varphi + r\mathbf{N}$ for $r = 5$, parallel surface of hyperbolic cylinder can be parameterized as

$$\varphi^r(u, v) = \left(-\frac{5 \cosh u}{\sqrt{2 \cosh^2 u - 1}} + \cosh u, \frac{5 \sinh u}{\sqrt{2 \cosh^2 u - 1}} + \sinh u, v\right), \tag{57}$$

where the base curve C^r is

$$C^r = \left(-\frac{5 \cosh u}{\sqrt{2 \cosh^2 u - 1}} + \cosh u, \frac{5 \sinh u}{\sqrt{2 \cosh^2 u - 1}} + \sinh u, 0\right)$$

and the ruling X^r is $(0, 0, 1)$. The surface in (57) is a ruled surface because it can be written in the form $\varphi^r = C^r + vX^r$. Also, parallel surface of a hyperbolic cylinder is a developable ruled surface since its drall λ^r vanishes. It means that the surface given with the parametrization in (57) is a timelike parallel ruled surface with timelike ruling, so the red and blue surfaces in (Fig. 1) show timelike hyperbolic cylinder and its timelike parallel ruled surface, respectively.
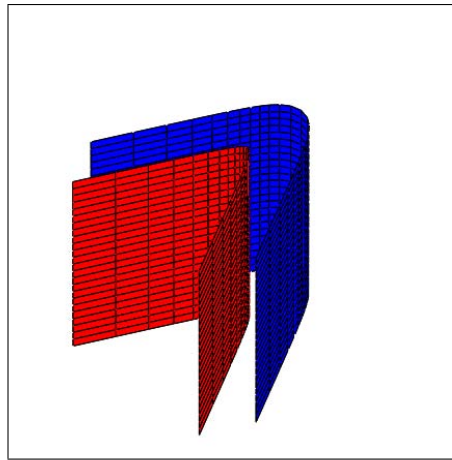
Figure 1 : Hyperbolic cylinder and its parallel surface

Also by using the equation (46), the striction curve of timelike parallel ruled surface with timelike ruling is found as $(u) = (\cosh u, \sinh u, -1)$ by taking $r = -5$, $y = 1$ and $a = 0$. Its orthogonal trajectory is calculated as

$$\beta(u) = (\cosh u, \sinh u, v(1 + 5b))$$

by means of Theorem 4.9. The Gaussian curvature K^r of timelike parallel ruled surface vanishes because the main surface is developable, therefore timelike parallel ruled surface is also developable from Theorem 4.1. Nevertheless, the vanishing of the Gaussian curvature can be seen by computing the coefficients of the first and second fundamental forms of the surface given in in (56) or by calculating the values of Q, J, F, D in (6) and then putting them into (53) in Corollary 4.10. For instance, the values of Q, J, F, D are as follows:

$$Q = 0, \ J = 0, \ F = 0, \ D = \sqrt{\varepsilon}\,|v|$$

for the surface given in (56). As a result, the accuracy of Theorem 4.13 is seen.

Example 4.15. The helicoid of the 3 rd kind has the parametrization

$$\varphi(u, v) = (v \cosh u, v \sinh u, u). \tag{58}$$

It is easily seen that its base curve is a spacelike curve and its ruling is a timelike vector. The helicoid of the 3 rd kind is not a developable ruled surface since its drall λ doesn't vanish. Hence, the parallel surface of the helicoid of the 3 rd kind can not become a ruled surface because of Theorem 4.1. The unit normal vector for the helicoid of the 3 rd kind is found as

$$\overrightarrow{\mathbf{N}} = \left(\frac{5\sinh u}{\sqrt{2\cosh^2 u - 1 - v^2}}, -\frac{5\cosh u}{\sqrt{2\cosh^2 u - 1 - v^2}}, -\frac{v}{\sqrt{2\cosh^2 u - 1 - v^2}}\right).$$

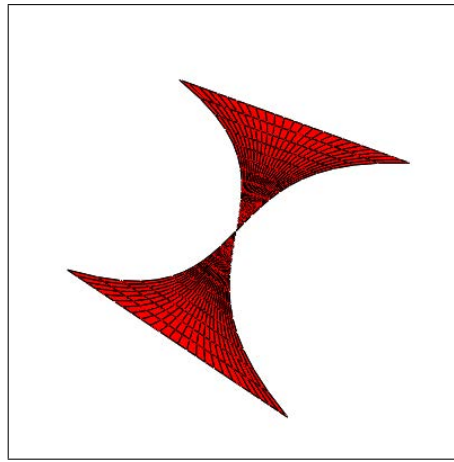The helicoid of the 3 rd kind in Minkowski 3-space is seen in (Fig. 2):

Notes

Figure 2 : The Helicoid of the 3rd kind

By using the expression $\varphi^r = \varphi + r\,\mathbf{N}$ for $r = -1$, parallel surface of the helicoid of the 3 rd kind can be parameterized as

$$\varphi^r(u,v) \;=\; \left(\frac{5\sinh u}{\sqrt{2\cosh^2 u - 1 - v^2}} + v\cosh u, \; -\frac{5\cosh u}{\sqrt{2\cosh^2 u - 1 - v^2}} + v\sinh u, \right.$$
$$\left. -\frac{v}{\sqrt{2\cosh^2 u - 1 - v^2}} + u\right). \tag{59}$$

Again we can state that the surface in (58) is not a ruled surface because both it can not be written in the form $\varphi^r = C^r + vX^r$ and the main surface is, as stated previously, not developable one. The parallel surface of the helicoid of the 3 rd kind and the two surfaces together are seen in (Fig. 3) and (Fig. 4), respectively.
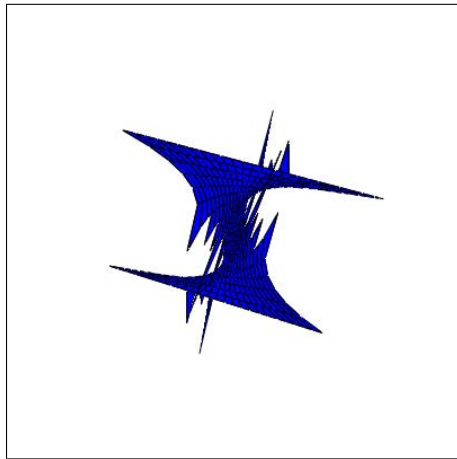

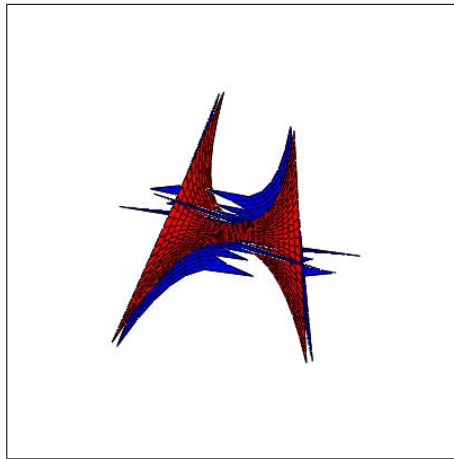*Figure 3 :* The parallel surface of the helicoid of the 3rd kind

Figure 4 : The two surfaces together

V. CONCLUSION

In this paper, we have constructed timelike parallel ruled surfaces by using the elements of differential geometry in Minkowski 3-space. Furthermore, we have presented some characterizations of timelike parallel ruled surfaces whose original surfaces are timelike ruled surfaces with timelike ruling. Researchers can try to see the results we obtained in this work, in Euclidean and

Lorentzian n-spaces. The results have a number of applications in computer-aided design and manufacturing of sculptured surfaces.

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