Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 14 Issue 1 Version 1.0 Year 2014
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Construction of a Mixed Quadrature Rule using Three Different Well-Known Quadrature Rules

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GJSFR-F Classification : MSC 2010: 65D30, 65D32

CONSTRUCTIONOFAMIXEDQUADRATURERULEUSINGTHREEDIFFERENTWELL-KNOWNQUADRATURERULES

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Construction of a Mixed Quadrature Rule using Three Different Well-Known Quadrature Rules

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Abstract- This paper deals with construction of a mixed quadrature rule of precision nine by using Gauss-Legendre 3-point rule, Lobatto 4-point rule and Clenshaw-Curtis 5-point rule, each having precision five. This mixed rule is successfully tested on different real definite integrals.

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I. Introduction

Real definite integral of the form

$$I(f)=\int_a^b f(x)dx \tag{1.1}$$

can be approximated by using (i) Gauss-Legendre quadrature rule,(ii) Lobatto quadrature rule,(iii)Clenshaw-Curtis quadrature rule. Among these three quadrature rules, Lobatto and Clenshaw-Curtis quadrature rules are of closed type where as Gauss-Legendre quadrature rule[2] is of open type. An n-point Clenshaw-Curtis rule[4] is of degree of precision n, while an n-point Gauss-Legendre rule and an n-point Lobatto rule are of degree of precision $2n$-1 and $2n$-3 respectively. That means Gauss-Legendre rule needs less number of nodes to maintain a particular precision than Lobatto and Clenshaw-Curtis rule. Usually by increasing the value of 'n' in these above three rules, we can optimize the accuracy of approximation of the real definite integral (1.1).The nodes of n-point Gauss-Legendre rule and n-point Lobatto rule are zeros of $P_n(x),\ n\geq 2$ and $P'_{n-1}(x),\ n\geq 3$ respectively. The nodes of n-point Clenshaw-Curtis rule are obtained from the following equation.

$$x_i=\cos\frac{i\pi}{n} \qquad i=0,1,2,\ldots n \tag{1.2}$$

In the Gauss-Legendre and Lobatto rules, the computational complexity for the evaluation of the zeros of $P_n(x)$ and $P'_{n-1}(x)$ increases for large n. Also the computational complexity may arise to find the nodes of Clenshaw-Curtis rule for large n. In these three rules, as we move from lower order rule to higher order rule, almost all the information obtained in computing the former gets discarded because the nodes and weights are different for different values of n.

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Keeping these facts in view, we desire to construct a mixed quadrature rule[3] of precision nine which is a linear combination of Gauss-Legendre 3-point rule, Lobatto 4-point rule and Clenshaw-Curtis 5-point rule each having precision five. The construction of the mixed quadrature rule is out-lined in the following section.

II. Construction of the Mixed Quadrature Rule of Precision Nine

Notes

We choose the Gauss-Legendre 3-point rule $\left(R_{GL_3}(f)\right)$:

$$I(f)=\int_a^b f(t)dt=\int_{-1}^{1} f(x)dt \approx R_{Gl_3}(f)$$

$$=\frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{2.1}$$

the Clenshaw-Curtis 5-point rule $\left(R_{CC_5}(f)\right)$:

$$I(f)=\int_a^b f(t)dt=\int_{-1}^{1} f(x)dt \approx R_{cc_5}(f)$$

$$=\frac{1}{15}\left[f(-1)+8f\left(\frac{-1}{\sqrt{2}}\right)+12f(0)+8f\left(\frac{1}{\sqrt{2}}\right)+f(1)\right] \tag{2.2}$$

and the Lobatto 4-point rule $\left(R_{L_4}(f)\right)$:

$$I(f)=\int_a^b f(t)dt=\int_{-1}^{1} f(x)dt \approx R_{L_4}(f)$$

$$=\frac{1}{6}\left[f(-1)+5f\left(\frac{-1}{\sqrt{5}}\right)+5f\left(\frac{1}{\sqrt{5}}\right)+f(1)\right] \tag{2.3}$$

Each of these rules (2.1), (2.2) and (2.3) is of precision 5. Let $E_{GL_3}(f)$, $E_{CC_5}(f)$ and $E_{L_4}(f)$ denote the errors in approximating the integral $I(f)$ by the rules (2.1), (2.2) and (2.3) respectively .Then

$$I(f)=R_{GL_3}(f)+E_{GL_3}(f) \tag{2.4}$$

$$I(f)=R_{CC_5}(f)+E_{CC_5}(f) \tag{2.5}$$

and $$I(f)=R_{L_4}(f)+E_{L_4}(f) \tag{2.6}$$

Assuming $f(x)$ to be sufficiently differentiable in $-1\le x\le 1$, using Maclaurin's expansion of the function $f(x)$ we can express the errors associated with the quadrature rules under reference as

$$E_{GL_3}(f)=\frac{8}{7!\times 25}f^{(vi)}(0)+\frac{88}{9!\times 125}f^{(viii)}(0)+\frac{656}{11!\times 625}f^{(x)}(0)+\dots$$

$$E_{CC_5}(f)=\frac{2}{7!\times 15}f^{(vi)}(0)+\frac{1}{9!\times 5}f^{(viii)}(0)+\frac{1}{11!\times 6}f^{(x)}(0)+\dots$$

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and $E_{L_4}(f)=\frac{-32}{7!\times 75}f^{(vi)}(0)-\frac{128}{9!\times 125}f^{(viii)}(0)-\frac{3136}{11!\times 1875}f^{(x)}(0)-...$

Now multiplying the Eqs(2.4),(2.5) and (2.6)by60,-32 and 35respectively, then adding the results we obtain,

$$I(f)=\frac{1}{63}\left[60R_{GL_3}(f)+35R_{L_4}(f)-32R_{CC_5}(f)\right]+$$

$$\frac{1}{63}\left[60E_{GL_3}(f)+35E_{L_4}(f)-32E_{CC_5}(f)\right]$$

or $\quad I(f)=R_{GL_3L_4CC_5}(f)+E_{GL_3L_4CC_5}(f)$ (2.7)

where $\quad R_{GL_3L_4CC_5}(f)=\frac{1}{63}\left[60R_{GL_3}(f)+35R_{L_4}(f)-32R_{CC_5}(f)\right]$ (2.8)

and $\quad E_{GL_3L_4CC_5}(f)=\frac{1}{63}\left[60E_{GL_3}(f)+35E_{L_4}(f)-32E_{CC_5}(f)\right]$ (2.9)

Eq (2.8), expresses the desired mixed quadrature rule for the approximate evaluation of $I(f)$ and Eq (2.9) expresses the error generated in this approximation.

so, $E_{GL_3L_4CC_5}(f)=\frac{-16}{11!\times 1125}f^{(x)}(0)-...$ (2.10)

As the first term of $E_{GL_3L_4CC_5}(f)$ contains 10th order derivative of the integrand, so the degree of precision of the mixed quadrature rule is 9. It is called a mixed type rule as it is constructed from three different types of rules of equal precision.

III. Error Analysis of the Mixed Quadrature Rule

An asymptotic error estimate and an error bound of the mixed quadrature rule (2.8) is given in theorems 3.1 and 3.2 respectively.

Theorem-3.1

Let *f(x)* be sufficiently differentiable function in the closed interval [-1,1].Then the error $E_{GL_3L_4CC_5}(f)$ associated with the mixed quadrature rule $R_{GL_3L_4CC_5}(f)$ is given by

$$\left|E_{GL_3L_4CC_5}(f)\right|\cong\frac{16}{11!\times 1125}\left|f^{(x)}(0)\right|$$

Proof:

The proof Follows from Eq(2.10).

Theorem-3.2

The bound for the truncation error

$$E_{GL_3L_4CC_5}(f)=I(f)-R_{GL_3L_4CC_5}(f)$$

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is given by

$$E_{GL_3L_4CC_5}(f) \le \frac{4M}{33075}$$

where, $M = \max_{-1\le x\le 1}\left|f^{(vii)}(x)\right|$

Proof:

We have

$$E_{GL_3}(f) = \frac{8}{7!\times 25} f^{(vi)}(\eta_1) \qquad \eta_1 \in [-1,1]$$

$$E_{L_4}(f) = \frac{-32}{7!\times 75} f^{(vi)}(\eta_2) \qquad \eta_2 \in [-1,1]$$

$$E_{CC_5}(f) = \frac{2}{7!\times 15} f^{(vi)}(\eta_3) \qquad \eta_3 \in [-1,1]$$

(Refer to Conte and Boor [1])

$$\text{so, } E_{GL_3L_4CC_5}(f) = \frac{1}{63}\left[60E_{GL_3}(f) + 35E_{L_4}(f) - 32E_{CC_5}(f)\right]$$

$$= \frac{96}{7!\times 315} f^{(vi)}(\eta_1) - \frac{224}{7!\times 945} f^{(vi)}(\eta_2) - \frac{64}{7!\times 945} f^{(vi)}(\eta_3)$$

Let $K = \max_{x\in[-1,1]}\left|f^{(vi)}(x)\right|$ and $k = \min_{x\in[-1,1]}\left|f^{(vi)}(x)\right|$. As $f^{(vi)}(x)$ is continuous and [-1,1] is compact, hence there exist points b and a in the interval [-1,1] such that $K = f^{(vi)}(b)$ and $k = f^{(vi)}(a)$. Thus

$$E_{GL_3L_4CC_5}(f) \le \frac{96}{7!\times 315} f^{(vi)}(b) - \frac{224}{7!\times 945} f^{(vi)}(a) - \frac{64}{7!\times 945} f^{(vi)}(a)$$

$$= \frac{2}{33075}\left[f^{(vi)}(b) - f^{(vi)}(a)\right]$$

$$= \frac{2}{33075}\int_a^b f^{(vii)}(x)\,dx$$

$$= \frac{2}{33075}(b-a) f^{(vii)}(\xi) \qquad \text{for some } \xi \in [-1,1]$$

by Mean value theorem[1]

Hence by choosing $\left|(b-a)\right| \le 2$

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we have, $E_{GL_3L_4CC_5}(f) \le \frac{2}{33075}\left|(b-a)\right|\left|f^{(vii)}(\xi)\right| \le \frac{4M}{33075}$

where, $M = \max_{x\in[-1,1]}\left|f^{(vii)}(x)\right|$

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IV. Numerical Verification

For the Numerical verification of the mixed quadrature rule $\left(R_{GL_3L_4CC_5}(f)\right)$, the following integrals are considered.

Table :1

Exact value of $I_1(f)=\int_0^{\frac{\pi}{4}}\cos^2(x)dx=0.6426990818$

Quadrature/Mixed quadrature rule	Aproximate value of $I_1(f)$
$R_{GL_3}(f)$	**0.6427**011120
$R_{L_4}(f)$	**0.6426**963791
$R_{CC_5}(f)$	**0.642699**93278
$R_{GL_3L_4CC_5}(f)$	**0.642699081**69

Table:2

Exact value of $I_2(f)=\int_0^{\frac{\pi}{4}}e^{\cos x}dx=1.9397348506$

Quadrature/Mixed quadrature rule	Aproximate value of $I_2(f)$
$R_{GL_3}(f)$	**1.9397**36725
$R_{L_4}(f)$	**1.9397**323524
$R_{CC_5}(f)$	**1.9397**356328
$R_{GL_3L_4CC_5}(f)$	**1.93973485066**7

Table : 3

Exact value of $I_3(f)=\int_0^1 e^{-x^2}dx=0.7468241328$

Quadrature/Mixed quadrature rules	Aproximate value of $I_3(f)$
$R_{GL_3}(f)$	**0.7468**145841
$R_{L_4}(f)$	**0.7468**365980
$R_{CC_5}(f)$	**0.7468**198579
$R_{GL_3L_4CC_5}(f)$	**0.74682413**53

Table: 4

Exact value of $I_4(f)=\int_0^1\frac{1}{1+e^x}dx=0.3798854936$

Quadrature/Mixed quadrature rules	Aproximate value of $I_4(f)$
$R_{GL_3}(f)$	**0.3798853**08
$R_{L_4}(f)$	**0.37988**57384
$R_{CC_5}(f)$	**0.3798854**149
$R_{GL_3L_4CC_5}(f)$	**0.37988549**304

Table : 5

Exact value of $I_5(f)=\int_0^1(1+x^2)^{\frac{3}{2}}dx=1.5679519622$

Quadrature/Mixed quadrature rules	Aproximate value of $I_5(f)$
$R_{GL_3}(f)$	**1.5679**493894
$R_{L_4}(f)$	**1.5679**552310
$R_{CC_5}(f)$	**1.5679**507093
$R_{GL_3L_4CC_5}(f)$	**1.56795196**43

Table: 6

Exact value of $I_6(f)=\int_0^1 x^2e^{-x}dx=0.1606027942$

Quadrature/Mixed quadrature rules	Aproximate value of $I_6(f)$
$R_{GL_3}(f)$	**0.1605**953868
$R_{L_4}(f)$	**0.1606**126841
$R_{CC_5}(f)$	**0.1605**997226
$R_{GL_3L_4CC_5}(f)$	**0.16060279**415

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Table: 7

Exact value of $I_7(f)=\int_0^1 \frac{1}{1+x^4}dx=0.8669729870$

Quadrature/Mixed quadrature rules	Aproximate value of $I_7(f)$
$R_{GL_3}(f)$	**0.867**51846
$R_{L_4}(f)$	**0.866**2609
$R_{CC_5}(f)$	**0.867**21664
$R_{GL_3L_4CC_5}(f)$	**0.86697**313

Table: 8

Exact value of $I_8(f)=\int_0^{\frac{\pi}{4}} e^{3x}\sin(2x)dx=2.5886286324$

Quadrature/Mixed quadrature rules	Aproximate value of $I_8(f)$
$R_{GL_3}(f)$	**2.589**258
$R_{L_4}(f)$	**2.58**778613
$R_{CC_5}(f)$	**2.588**8872
$R_{GL_3L_4CC_5}(f)$	**2.58862863**8

Table: 9

Exact value of $I_9(f)=\int_1^{1.5} x^2\ln(x)dx=0.1922593577$

Quadrature/Mixed quadrature rules	Aproximate value of $I_9(f)$
$R_{GL_3}(f)$	**0.19225937**
$R_{L_4}(f)$	**0.1922593**316
$R_{CC_5}(f)$	**0.1922593**658
$R_{GL_3L_4CC_5}(f)$	**0.1922593577**326

Table: 10

Exact value of $I_{10}(f)=\int_0^1 \frac{4}{1+x^2}dx=\pi\cong 3.141592654$

Quadrature/Mixed quadrature rules	Aproximate value of $I_{10}(f)$
$R_{GL_3}(f)$	**3.141**068
$R_{L_4}(f)$	**3.14**2276
$R_{CC_5}(f)$	**3.14**135
$R_{GL_3L_4CC_5}(f)$	**3.14159**272

Table: 11

Exact value of $I_{11}(f)=\int_3^{3.5} \frac{x}{\sqrt{x^2-4}}dx=0.6362133458$

Quadrature/Mixed quadrature rules	Aproximate value of $I_{11}(f)$
$R_{GL_3}(f)$	**0.63621**31959
$R_{L_4}(f)$	**0.63621**35467
$R_{CC_5}(f)$	**0.636213**2845
$R_{GL_3L_4CC_5}(f)$	**0.63621334**577

V. Conclusion

In this article, we construct a new mixed quadrature rule of precision nine for evaluating the real definite integrals. An asymptotic error estimate and an error bound of the mixed quadrature rule are given. Finally we give some numerical examples to show the superiority of the mixed quadrature rule with respect to the usual Gauss-Legendre 3-point rule, Clenshaw-Curtis 5-point rule and Lobatto 4-point rule.

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VI. Acknowledgement

The research is supported by the ministry of Social Justice and Empowerment, Govt. of India, under a central sector scheme of R.G.N.F .

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