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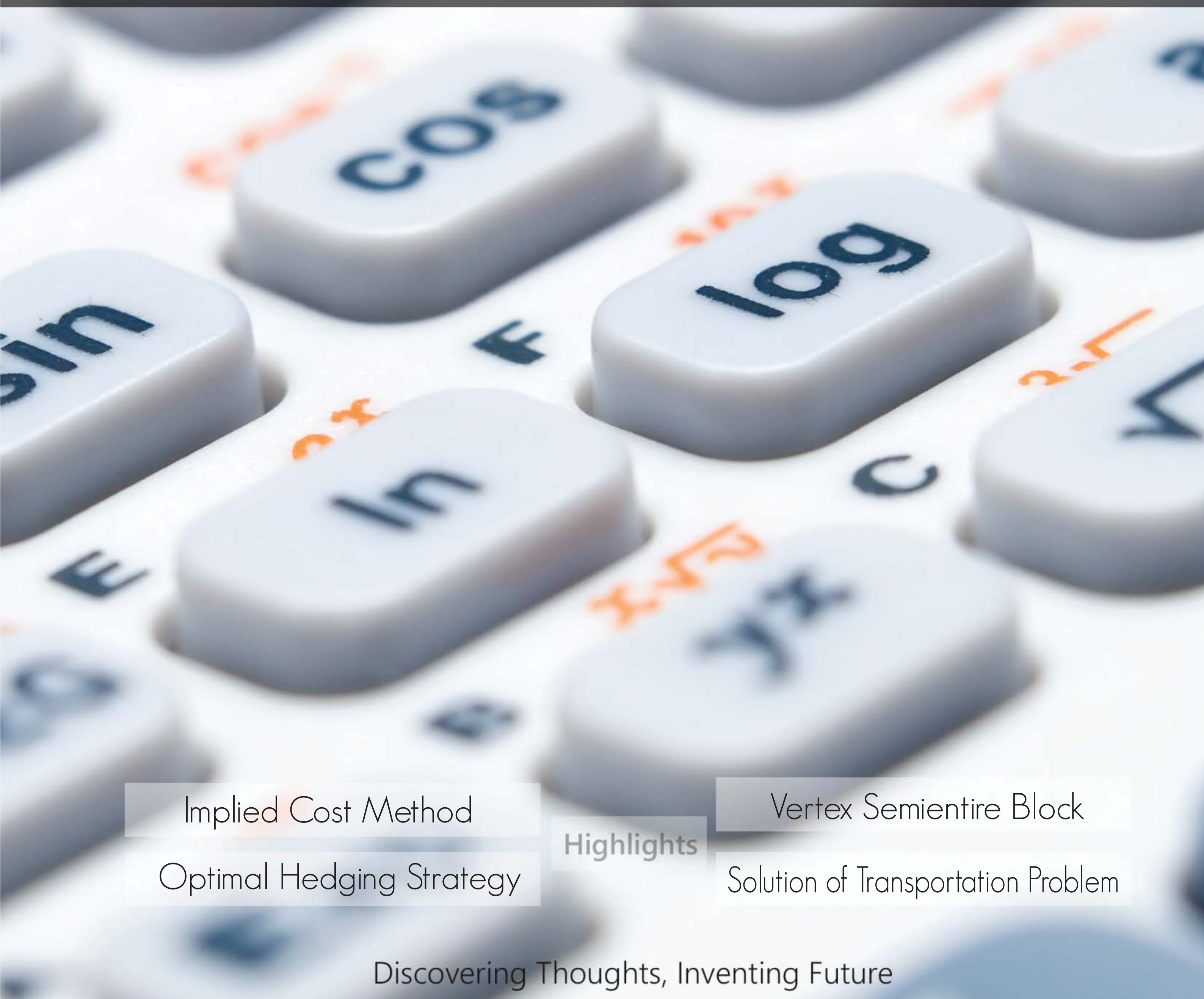

Highlights

Implied Cost Method

Optimal Hedging Strategy

Vertex Semientire Block

Solution of Transportation Problem

Discovering Thoughts, Inventing Future

VOLUME 14 ISSUE 1 VERSION 1.0

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Vertex Semientire Block Graph

By Venkanagouda M Goudar & Rajanna N E
Sri Siddhartha Institute of Technology, India

Abstract- In this communications, the concept of the vertex semientire block graph is introduced. We present characterization of graphs whose vertex semientire block graph is planar, outerplanar and minimally non-outerplanar. Also we establish a characterization of graphs whose vertex semientire block graph is Eulerian.

Keywords: inner vertex number, line graph, outerplanar, vertex semientire graph.

GJSFR-F Classification : MSC 2010: 05C76

VERTEXSEMIENTIREOCKGRAPH

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Vertex Semientire Block Graph

Venkanagouda M Goudar [α] & Rajanna N E [σ]

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Keywords: inner vertex number, line graph, outerplanar, vertex semientire graph.

I. Introduction

By graph, we mean a finite, undirected graph without loops or multiple edges. We refer the terminology of [5].

The inner vertex number i(G) of a planar graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally non-outerplanar if i(G) = 1.

A new concept of a graph valued functions called the pathos vertex semientire graph $Pe_v(G)$ of a plane graph G was introduced [6] and is defined as the graph whose vertex set is V(T) b_i r and the two vertices are adjacent if and only if they are adjacent vertices, vertices lie on the path of pathos and vertices lie on the regions. Since the system of pathos for a tree is not unique, the corresponding vertex semientire block graph is also not unique.

Blockdegree is the number of vertices lies on a block. Blockpath is a path in which each edge in a path becomes a block. Degree of a region is the number of vertices lies on a region.

Now we define the vertex semientire block graph. The vertex semientire block graph denoted by $\mathbf{e}_{vb}(G)$ is the graph whose vertex set is V(T) b_i r and the two vertices are adjacent if and only if they are adjacent vertices, vertices lie on the blocks and vertices lie on the regions.

II. Preliminaries

We need the following results to prove further results.

Theorem 1 [Ref 4]. If G be a connected plane graph then $\mathbf{e}_v(G)$ is planar if and only if G is a tree.

Theorem 2 [Ref 3]. Every maximal outerplanar graph G with p vertices has 2p – 3 edges.

Author α: Department of Mathematics, Sri Siddhartha Institute of Technology, Tumkur, Karnataka.
e-mail: vmgouda@gmail.com
Author σ: Department of Mathematics Adichunchanagiri Institute of Technology, Chikmagalur, Karnataka.
e-mail: ethinamane@gmail.com

Ref

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III. Vertex Semientire Block Graph

We start with a preliminary result.

Remark 1. For any graph G, $G \subseteq e_v(G) \subseteq e_{vb}(G)$

Remark 2. For any (p, q) graph G the degree of a vertex in vertex semientire block graph $e_{vb}(T)$ is 2p+1.

In the following theorem we obtain the number of vertices and edges in a vertex semientire block graph.

Theorem 3. For any (p, q) graph G with b blocks and r regions vertex semientire block graph $\mathbf{e}_{vb}(G)$ has (p + b +r) vertices and $q+\sum_{i=1}^{k} d(b_i)+\sum_{j}^{l} d(r_j)$ edges, where $d(b_i)$ is the block degree of a block bi and $d(r_j)$ is the degree of a region r_j.

Proof. By the definition of vertex semientire block graph $\mathbf{e}_{ev}(G)$, the number of vertices is the union of the vertices, blocks and the regions of G. Hence the number of vertices of vertex semientire block graph $\mathbf{e}_{vb}(G)$is (p + k + 1).

Further, by remark 1, the graph G is a sub graph of $\mathbf{e}_{vb}(G)$, hence all the edges of G are present $\mathbf{e}_{vb}(G)$. Also by the Remark 2, the number of vertices is the degree of a block. Lastly, the degree of regionvertex is the number of vertices lies on the region and is the number of edges in $\mathbf{e}_{vb}(G)$. Hence the number of edges vertex semientire block graph $\mathbf{e}_{vb}(G)$ is $q+\sum_{i=1}^{k} d(b_i)+\sum_{j}^{l} d(r_j)$

Theorem 4. For any tree T, vertex semientire block graph $\mathbf{e}_{vb}(G)$ is always nonseparable.

Proof. Consider a graph G. We have the following cases

Case 1. Suppose G be a tree. All internal vertices of G are the cut vertices C_i. These cut vertices lies on the region as well as on two blocks. Clearly C_i is not a cut vertex in $\mathbf{e}_{vb}(G)$. Hence $\mathbf{e}_{vb}(G)$is nonseparable.

Case 2. Suppose G be any graph with at least one cut vertex. Since cut vertex C_i lies on at least two blocks and one region. Hence in $\mathbf{e}_{vb}(G)$, C_i becomes non-cut vertex. Hence $\mathbf{e}_{vb}(G)$ is nonseparable.

Theorem 5. For any graph G, vertex semientire block graph $e_{vb}(G)$ is planar if and only if G is a tree.

Proof. Suppose $\mathbf{e}_{vb}(G)$ is planar. Assume that G is a graph other than a tree, say a cycle C_n, for Without loss of generality we take n=3. Clearly all vertices of C3 lies on a block b_i. By the definition of vertex semientire block graph $\mathbf{e}_{vb}(G)$, C_3 along with bi form a graph K_4. Further all vertices of C3lies on both regions vertices r_1 and r_2. Clearly r_1 and r_2 are adjacent to all vertices of C_3 to form a graph which is homeomorphic to K_5 and is non planar, a contradiction. Hence G must be a tree.

Conversely suppose a graph G is a tree. By definition, for each edge of a tree G, there is a $K_4 - e$ in $\mathbf{e}_{vb}(G)$. Clearly $\mathbf{e}_{vb}(G)$ is planar.

Theorem 6. For any tree T the vertex semientire block graph $\mathbf{e}_{vb}(G)$outerplanar if and only if T is a path P_n.

Proof. Suppose vertex semientire block graph $\mathbf{e}_{vb}(G)$ is outerplanar. Assume that G is a tree which is not a path P_n. Since each edge is a block and both end vertices lies on a block. These end vertices and a blockvertex form a graph K3 in $\mathbf{e}_{vb}(G)$. Further the regionvertex is adjacent to all vertices of G to form a graph such that it has at least two inner verteices, which is non outerplanar, a contradiction.

Notes

Conversely, suppose a graph G is a path P_n. By definition of $\mathbf{e}_{vb}(G)$, the regionvertex r is adjacent to two vertices v_1 ,v_2 to form K_3 and a pathosvertex Pi is adjacent to two vertices v1,v_2 fo K_3 to form $K_4 - x$, which is outerplanar.

Theorem 7. For any graph G, vertex semientire block graph $\mathbf{e}_{vb}(G)$ is not minimally non outerplanar .

Proof. Proof follows from the Theorem 6.

Notes

Theorem 8. For any graph G, vertex semientire block graph $\mathbf{e}_{vb}(G)$ is maximal outerplanar if G is a path P_n.

Proof. Suppose vertex semientire block graph $\mathbf{e}_{vb}(G)$ is maximal outerplanar, then $\mathbf{e}_{vb}(G)$ is connected. Let G be a path P_n, it contains p vertices and p-1 edges. Given that $\mathbf{e}_{vb}(G)$ is maximal outerplanar, by Theorem 2, it has 2p – 3 edges. We know that $V[\mathbf{e}_{eb}(G)] = 2p$ and $E[e_{vb}(G)] = 4p - 3$.

$$\Rightarrow 2(2p) - 3 = q = 4p - 3$$

$$\Rightarrow 2p - 3 = 4p - 3 \quad \textit{is satisfied.}$$

Clearly, G=P_n is a nonempty path. Hence necessity is proved.

Theorem 9. For any graph G vertex semientire block graph $\mathbf{e}_{vb}(G)$ is Eulerian if and only if following conditions hold:

i. G is a graph without tree
ii. Each region contains even number of vertices.
iii. number of vertices in each block is even and
iv. the number of vertices in a graph G is even.

Proof. Suppose $\mathbf{e}_{vb}(G)$ is Eulerian. We have the following cases.

Case 1. Assume that G is a tree. Clearly a tree contains at least two vertices vi and vj of odd degree and at least one vertex v_k of even degree. By the Remark 2, in evb(G), deg(v_i) and deg(v_j) becomes even and deg(v_k) becomes odd. Clearly $\mathbf{e}_{vb}(G)$ is non Eulerian, a contradiction.

Case 2. Assume that degree each region contains odd number of vertices. By definition of $\mathbf{e}_{vb}(G)$, the degree of regionvertex in $\mathbf{e}_{vb}(G)$ is becomes odd. Clearly $\mathbf{e}_{vb}(G)$ is non Eulerian, a contradiction.

Case 3. Assume that the number of vertices in each block is odd. By definition of $\mathbf{e}_{vb}(G)$, the degree of blockvertex in $\mathbf{e}_{vb}(G)$ is becomes odd. Clearly $\mathbf{e}_{vb}(G)$ is non Eulerian, a contradiction.

Case 4. Assume that the number of vertices in a graph G is odd. Clearly it follows that G has either at least two vertices of even degree and at least two vertices of odd degree or all vertices of even degree. If all vertices of odd degree then G must be a complete graph. Without loss of generality G is either K_3 or K_4. If G is K_3, then inner regionvertex is of odd degree. By the case 2, $\mathbf{e}_{vb}(G)$ is non Eulerian. Also if G is K4, then each region have odd degree. By the case 2, $\mathbf{e}_{vb}(G)$ is non Eulerian, which is a contradiction.

Conversely suppose G satisfies all the conditions of the Theorem. For a graph G with degree of regionvertex, degree of blockvertex and all vertices of G is even, then the corresponding vertices in $\mathbf{e}_{vb}(G)$ is even. Hence $\mathbf{e}_{vb}(G)$ is Eulerian.

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Implied Cost Method (ICM): An Alternative Approach to Find the Feasible Solution of Transportation Problem

By Md. Ashraful Babu, Md. Abu Helal, Mohammad Sazzad Hasan & Utpal Kanti Das

University of Business Agriculture and Technology, Bangladesh

Abstract- Transportation Algorithm uses for solving Transportation Problem to find feasible solution which may optimal or not. This is now a challenge that developed such an algorithm which gives optimal solution without using Optimality Methods like as MODI, Stepping-Stone. For this reason there are several transportation algorithm exists for solving TP like as North West Corner Rule (NWC), Least Cost Method (LCM), Vogel's Approximation Method (VAM) etc. where VAM provides the feasible solution which is lower than NWC and LCM and for some case of TP it coincides with optimal solution. In this paper we proposed a new approach named "Implied Cost Method (ICM)" where feasible solutions are lower than VAM and very close to optimal solution or sometimes coincides with optimal solution.

Keywords: transportation problem (TP), ICM, LCM, VAM, feasible solution, optimal solution.

GJSFR-F Classification : *MSC 2010: 03D32*

IMPLIEDCOSTMETHODICMANALTERNATIVEAPPROACHTOFINDTHEFEASIBLESOLUTIONOFTRANSPORTATIONPROBLEM

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Implied Cost Method (ICM): An Alternative Approach to Find the Feasible Solution of Transportation Problem

Md. Ashraful Babu [α], Md. Abu Helal [σ], Mohammad Sazzad Hasan [ρ] & Utpal Kanti Das [ω]

Abstract- Transportation Algorithm uses for solving Transportation Problem to find feasible solution which may optimal or not. This is now a challenge that developed such an algorithm which gives optimal solution without using Optimality Methods like as MODI, Stepping-Stone. For this reason there are several transportation algorithm exists for solving TP like as North West Corner Rule (NWC), Least Cost Method (LCM), Vogel's Approximation Method (VAM) etc. where VAM provides the feasible solution which is lower than NWC and LCM and for some case of TP it coincides with optimal solution. In this paper we proposed a new approach named "*Implied Cost Method (ICM)*" where feasible solutions are lower than VAM and very close to optimal solution or sometimes coincides with optimal solution.

Keywords: transportation problem (TP), ICM, LCM, VAM, feasible solution, optimal solution.

I. Introduction

Transportation Model is one of the most important branches in Production Management. To obtain best possible profit in a business it's not only a matter of profit maximization but also cost should be minimized in optimistic way.

Transportation cost is the most important part of the total expenditure of a company along with the production cost. In generally a business company has some factories for manufacturing products and some retail centers for distributing products which are known as sources and destinations respectively in transportation model. For a transportation problem we consider m sources and n destinations where c_{ij} is the unit cost of i^{th} source to j^{th} destinations. The main output of the transportation problem is x_{ij} the amount of products transferred from i^{th} source to j^{th} destination so that total transportation cost will be minimized. In this paper we proposed a new algorithm to modify Least Cost Method (LCM) to obtain the feasible solution of Transportation problem which is more efficient than other existing algorithms. Also transportation problem is known as special types of linear programming problem, the general linear programming formulation of transportation is given below:

Author α: Faculty, Department of Mathematics, IUBAT-International University of Business Agriculture and Technology Dhaka-1230, Bangladesh. e-mail: ashraful388@gmail.com

Author σ: Department of Mathematics, IUBAT-International University of Business Agriculture and Technology, Dhaka-1230, Bangladesh. e-mail: abu.helal@iubat.edu

Author ρ: Lecturer, Department of Population Sciences, University of Dhaka, Dhaka-1000, Bangladesh. e-mail: shfuad2011@gmail.com

Author ω: Department of Computer Science & Engineering, IUBAT-International University of Business Agriculture and Technology, Dhaka-1230, Bangladesh, e-mail: ukd@iubat.edu

$$\text{Minimize:} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} \le s_i \ , \qquad for\ i = 1,2,3 \ldots\ldots\ldots\ldots\ldots\ldots m$$

$$\sum_{i=1}^{m} x_{ij} \ge d_j \ , \qquad for\ j = 1,2,3 \ldots\ldots\ldots\ldots\ldots\ldots n$$

$$x_{ij} \ge 0, \qquad for\ all\ i, j$$

where s_i be the amount of supply capacity of i^{th} source and d_j be the amount of demand of the j^{th} destination.

II. Methodology

In the beginning we noticed that there are some existing transportation algorithms such as North West Corner Rule (NWC), Least Cost Method (LCM), Vogel's Approximation Method (VAM) etc. where VAM is the better than others. In this paper we developed a new algorithm "ICM" which provides feasible solution lower than VAM.

a) Existing Algorithm of Vogel's Approximation Method (VAM)

Vogel's Approximation Method (VAM) is known as the most efficient algorithm of Transportation Problem. The existing algorithm of VAM is given below:

Step-1: Indentify the boxes having minimum and next to minimum transportation cost in each row and write the difference (Penalty) along the side of the table against the corresponding row.

Step-2: Indentify the boxes having minimum and next to minimum transportation cost in each column and write the difference (Penalty) along the side of the table against the corresponding column.

If minimum cost appear in two or more times in a row or column then select these same cost as a minimum and next to minimum cost and penalty will be zero.

Step-3: i) Indentify the row and column with the largest penalty, breaking ties arbitrarily. Allocate as much as possible to the variable with the least cost in the selected row or column. Adjust the supply and demand and cross out the satisfied row or column. If a row and column are satisfies simultaneously, only one of them is crossed out and remaining row or column is assigned a zero supply or demand.

ii) If two or more penalty costs have same largest magnitude, then select any one of them (or select most top row or extreme left column).

Step-4:

i. If exactly one row or one column with zero supply or demand remains uncrossed out, stop.
ii. If only one row or column with positive supply or demand remains uncrossed out, determine the basic variables in the row or column by the Least-Cost Method.
iii. If all uncrossed out rows or column have (remaining) zero supply or demand, determined the zero basic variables by the Least-Cost Method. Stop.
iv. Otherwise, go to Step-1.

Notes

Notes

b) Proposed Algorithm for Implied Cost Method (ICM)

In this proposed algorithm, determining implied cost by multiplying unit transport cost and maximum possible amount of commodity according to the availability of supply and demand. We allocate the maximum possible amount of commodity to the lowest implied cost. The goal of the transportation problem is transfer maximum possible amount into the lowest cost roots so that total cost will be minimized. For this reason we choose the concept of implied cost which ensures that cost of each and every allocation is minimum that's why total transportation cost will be minimized.

The algorithm of Implied Cost Method (ICM) is given below:

Step-1: Balanced the supply and demand by adding dummy supply or demand

$$if \sum_i s_i > \sum_j d_j \ \ or \ \ if \sum_i s_i < \sum_j d_j$$

Step-2: Determine implied cost for each cell by the product of unit transportation cost and maximum possible amount of commodity according to the availability of supply and demand.

Step-3: Identify the lowest implied cost and allocate maximum possible amount of x_{ij}.

Step-4: If two or more implied costs are equal then select that cell where the allocation is maximum.

Step-5: Adjust the supply and demand and cross out the satisfied row or column. If a row and column are satisfies simultaneously, only one of them is crossed out and remaining row or column is assigned a zero supply or demand.

Step-6: If exactly one row or one column with zero supply or demand remains uncrossed out then Stop. Otherwise, go to Step-2.

III. Numerical Illustration

Consider some transportation problem and solve these by Implied Cost Method (ICM) and compare results with the other existing methods.

Example-1:

Table-1.1 Consider a Mathematical Model of a Transportation Problem in Table-1.1

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5	9	11	**30**
S2	4	3	8	6	**25**
S3	3	8	10	5	**20**
S4	2	6	7	3	**15**
Demad	**30**	**30**	**20**	**10**	

a) Solv ing Example -1 by Implied Cost Method (ICM)

Costs are indicating in the top-right corner, implied costs are indicating in bottom-left corner and allocations are indicating in bottom-right corner.

Iteration-1: Determine implied cost by multiplying unit transport cost and maximum possible amount of supply and demand for each cell and placed it in left-bottom corner of each cell. Here cell (S4, D1) and (S4, D4) has the lowest implied cost and (S4,D1) has maximum allocation that is 15. So allocate 15 in (S4,D1) cell which is placed in right-bottom corner.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7 *210*	5 *150*	9 *180*	11 *110*	30
S2	4 *100*	3 *75*	8 *160*	6 *60*	25
S3	3 *60*	8 *160*	10 *200*	5 *50*	20
S4	2 *30* 15	6 *90*	7 *105*	3 *30*	
Demand	15	30	20	10	

Adjust supply and demand and crossed out S4 row.

Iteration 2 : Compute implied cost for all cells in uncrossed out rows and columns.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7 *105*	5 *150*	9 *180*	11 *110*	30
S2	4 *60*	3 *75*	8 *160*	6 *60*	25
S3	3 *45* 15	8 *160*	10 *200*	5 *50*	5
S4	2 *30* 15	6	7	3	
Demand		30	20	10	

Here cell (S3,D1) has the lowest implied cost, allocate 15 in (S3,D1) cell and adjust supply and demand and crossed out D1 column.

Iteration 3 : Compute implied cost for all cells in uncrossed out rows and columns.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 *150*	9 *180*	11 *110*	30
S2	4	3 *75*	8 *160*	6 *60*	25
S3	3 *45* 15	8 *40*	10 *50*	5 *25* 5	
S4	2 *30* 15	6	7	3	
Demand		30	20	5	

Here cell (S3,D4) has the lowest implied cost, allocate 5 in (S3,D4) cell and adjust supply and demand and crossed out S3 row.

Notes

Iteration 4 : Compute implied cost for all cells in uncrossed out rows and columns.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 *150*	9 *180*	11 *55*	30
S2	4	3 *75*	8 *160*	6 *30* **5**	20
S3	3 *45* **15**	8	10	5 *25* **5**	
S4	2 *30* **15**	6	7	3	
Demand		30	20		

Here cell (S2,D4) has the lowest implied cost, allocate 5 in (S2,D4) cell and adjust supply and demand and crossed out D4 column.

Iteration 5 : Compute implied cost for all cells in uncrossed out rows and columns.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 *150*	9 *180*	11	30
S2	4	3 *60* **20**	8 *160*	6 *30* **5**	
S3	3 *45* **15**	8	10	5 *25* **5**	
S4	2 *30* **15**	6	7	3	
Demand		10	20		

Here cell (S2,D2) has the lowest implied cost, allocate 20 in (S2,D2) cell and adjust supply and demand and crossed out S2 row.

Iteration 6 : Compute implied cost for all cells in uncrossed out rows and columns.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 *50* **10**	9 *180*	11 *55*	20
S2	4	3 *60* **20**	8	6 *30* **5**	
S3	3 *45* **15**	8	10	5 *25* **5**	
S4	2 *30*	6	7	3	

Notes

	15				
Demand			20		

Here cell (S1,D2) has the lowest implied cost, allocate 10 in (S1,D2) cell and adjust supply and demand and crossed out D2 column.

Iteration-7: Finally one allocation has been left. Allocate 20 in (S1,D3) cell adjust supply and demand and crossed out S1 row and set zero demand inD3 column.

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 *50* **10**	9 *180* **20**	11	
S2	4	3 *60* **20**	8	6 *30* **5**	
S3	3 *45* **15**	8	10	5 *25* **5**	
S4	2 *30* **15**	6	7	3	
Demand			0		

Final table of feasible solution:

Table :1.2

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 *50* **10**	9 *180* **20**	11	**30**
S2	4	3 *60* **20**	8	6 *30* **5**	**25**
S3	3 *45* **15**	8	10	5 *25* **5**	**20**
S4	2 *30* **15**	6	7	3	**15**
Demand	**30**	**30**	**20**	**10**	

Total Transportation Cost:

Sum of all implied costs for allocated cells: **50 + 180 + 60 + 30 + 45 + 25 + 30 = 420**

A.2. Solution of Example-1 using Vogel's Approximation Method (VAM):

Source	Destinations				Supply
	D1	D2	D3	D4	
S1	7	5 **30**	9	11	**30**
S2	4	3	8	6	**25**

Notes

Table: 1.3

	25	0			
S3	3 5	8	10 5	5 10	20
S4	2	6	7 15	3	15
Demand	30	30	20	10	

Notes

Total Transportation Cost:

$$(5 \times 30) + (4 \times 25) + (3 \times 0) + (3 \times 5) + (10 \times 5) + (5 \times 10) + (7 \times 15) = 470$$

A. Example-2:

Consider a Mathematical Model of a Transportation Problem in Table-2.1:

Source	Destinations					Supply
	D1	D2	D3	D4	D5	
S1	7	6	4	5	9	40
S2	8	5	6	7	8	30
S3	6	8	9	6	5	20
S4	5	7	7	8	6	10
Demand	30	30	15	20	5	

Table-2.1

B.1. Solution of Example-2 using Implied Cost Method (ICM):

Costs are indicating in the top-right corner, implied costs are indicating in bottom-left corner and allocations are indicating in bottom-right corner.

Source	Destinations					Supply
	D1	D2	D3	D4	D5	
S1	7 *35* 5	6	4 *60* 15	5 *100* 20	9	40
S2	8	5 *150* 30	6	7 *0* 0	8	30
S3	6 *90* 15	8	9	6	5 *25* 5	20
S4	5 *50* 10	7	7	8	6	10
Demand	30	30	15	20	5	

Table -2.2

Total Transportation cost: $35 + 60 + 100 + 150 + 90 + 25 + 50 = 510$

B.2. Solution of Example-2 using Vogel's Approx imation Method (VAM):

Source	Destinations					Supply
	D1	D2	D3	D4	D5	
S1	7 5	6 0	4 15	5 20	9	40
S2	8	5 30	6	7	8	30

S3	6 **15**	8	9	6	5 **5**	**20**
S4	5 **10**	7	7	8	6	**10**
Demand	**30**	**30**	**15**	**20**	**5**	

Tabl e-2.3

Total Transportation cost: $\mathbf{35 + 0 + 60 + 100 + 150 + 90 + 25 + 50 = \ 510}$

IV. Result Analysis

In above two examples we observed that Implied Cost Method (ICM) provides lowest feasible solutions which are lower than VAM or equal to VAM and very close to optimal solution. The comparison table is given below:

Transportation Problems	**Optimal Solution**	**Methods**			
		ICM	**VAM**	**LCM**	**NWC**
Example-1	**410**	**420**	470	435	540
Example-2	**510**	**510**	**510**	**510**	635

V. Conclusion

In this paper we proposed a new algorithm named "Implied Cost Method (ICM)" which is easier than other algorithms. ICM provides better feasible solution than others which are very close to optimal solution and sometimes it is equal to optimal solution. But it is not grantee that all time ICM provides least feasible solution but most of the times it gives better approach.

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The Nature of Points in Countable Boolean Lattice Measures

By D. V. S. R. Anil Kumar, Y. V. Seshagiri Rao & Y. Narasimhulu

T. K. R. College of Engineering and Technology, India

Abstract- This paper describes a class of null sets; point, lattice measure of a point and lattice semi-finite measure were introduced. Here it has been derived a result that in a countable Boolean lattice the lattice measure of any two points are either disjoint or identical also the class of all points in countable Boolean lattice is countable and proved that Any union countable of null partial lattices is null partial lattice, also established that the class of points in countable Boolean lattice is countable. It has been obtained a theorem that if a countable Boolean lattice is pointless if and only if every non empty set in countable Boolean lattice contains countable number of disjoint non-empty sets. Finally it has been observed that some elementary nature of points in a countable Boolean lattice.

Keywords: *Lattice, measure of a point, partial lattice measure, semi-finite measure.*

GJSFR-F Classification : *ASM:03G10, 28A05, 28A12.*

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The Nature of Points in Countable Boolean Lattice Measures

D. V. S. R. Anil Kumar [α], Y. V. Seshagiri Rao [σ] & Y. Narasimhulu [ρ]

Abstract- This paper describes a class of null sets; point, lattice measure of a point and lattice semi-finite measure were introduced. Here it has been derived a result that in a countable Boolean lattice the lattice measure of any two points are either disjoint or identical also the class of all points in countable Boolean lattice is countable and proved that Any countable union of null partial lattices is null partial lattice, also established that the class of points in countable Boolean lattice is countable. It has been obtained a theorem that if a countable Boolean lattice is pointless if and only if every non empty set in countable Boolean lattice contains countable number of disjoint non-empty sets. Finally it has been observed that some elementary nature of points in a countable Boolean lattice.

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I. Introduction

The origin of a lattice concept can be traced back to Boole's analysis of thought and Dedekind's study of divisibility, Schroder and Pierce contributed substantially to this area. Though some of the work in this direction was done around 1930, much momentum was gained in 1967 with the contributions of Birkhoff's [3]. In 1963, Gabor szasz [9] introduced the generalization of the lattice measure concepts. To study σ- additive set functions on a lattice of sets, Gena A. DE Both [4] introduced σ - lattice in 1973. The concept of partial lattices was introduced by George Gratzer [6] in 1978. In 2000, Pao - Sheng Hus [8] characterized outer measures associated with lattice measure. The Hann decomposition theorem of a signed lattice measure by Jun Tanaka [10] defined a signed lattice measure on a lattice σ - algebras and the concept of sigma algebras are extensively studied by [5]. D.V.S.R. Anil Kumar etal [1] introduce the concept of measurable Borel lattices, σ - lattice and δ –lattice to characterize a class of Measurable Borel Lattices. This paper is organized as follows. Section 2 presents the preliminaries definitions and results.

In Section 3, a class of null sets, lattice measure of a point and lattice semi-finite measure. It has been established a result that in a countable Boolean lattice the lattice measure of any two points are either disjoint or identical. In fact it has been proved that the class of all points in countable Boolean lattice is countable. Finally it has been observed various elementary natures of points in countable Boolean lattice.

Author α: Teegala Krishna Reddy College of Engineering and Technology, Meerpet, Hyderabad, A.P. India.
e-mail: anilkumardaita@yahoo.in
Author σ: Teegala Krishna Reddy Engineering College, Meerpet, Hyderabad, A.P. India.
e-mail: yangalav@gmail.com
Author ρ: Professor of Mathematics, Director, Academic Staff College, University of Hyderabad, A.P., India.
e-mail: yakkalan@uohyd.ernet.in

Ref

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II. Preliminaries

Consider a lattice (L,$\wedge$,$\vee$) with the operations meet $\wedge$ and join $\vee$ and usual ordering $\leq$, where L is a collection of subset of a non empty set X. Now this lattice (L,$\wedge$,$\vee$) is denoted by L and satisfy the commutative law, the associative law and the absorption law. A lattice L is called distributive if the distributive law is satisfied. The zero and one elements of the lattice L are denoted by 0 and 1 respectively. A distributive lattice L is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that $x \vee x^c = 1$ and $x \wedge x^c = 0$. An operator C: L $\to$ L, where L is a lattice is called a lattice complement in L if the law of complementation, the law of contra positive and the law of double negation are satisfied. The following are very important examples of Boolean lattice. Let ({0,1},$\leq$) be the set consisting of the two elements 0,1 equipped with the usual order relation $0 \leq 1$.This poset is a Boolean lattice with respect to the operations presented in the tables below (at the left the lattice operations and at the right the complementation)

a	b	$a \wedge b$	$a \vee b$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

X	X^c
0	1
1	0

B= ({0, 1},$\vee$, $\wedge$, ^{c}, 0, 1). The power set P(X) of a universe X a Boolean lattice if we choose the set theoretic complement A^c = X\A:={$x\in$X: $x\in$X and $x\notin$A} as the complement of a given set A in the universe X. Such a Boolean lattice is P = (P(X),$\vee$, $\wedge$, ^{c} , ϕ , X). E = (2^X ,$\vee$, $\wedge$,^{c} , 0 , 1) is the collection 2^X of all two valued functional on the universe X is a Boolean lattice if we choose the functional $\chi^c = 1-\chi$ as the complement of a given functional χ. Let (D, $\vee$, $\wedge$, ^{c} ,1,70) is a Boolean lattice where D={1,2,5,7,10,14,35,70} is the set of all divisors of 70, $x \wedge y$ = Greatest Common Devisor of x and y, $x \vee y$ = Least Common Multiple of x and y and $x^c = \frac{70}{x}$.

A Boolean lattice L is called a countable Boolean lattice if L is closed under countable joins and is denoted by σ (L). {empty setϕ, X}, Power set of X, and Let $X = \Re$, L = {measurable subsets of $\Re$} with usual ordering ($\leq$) are all countable Boolean lattices. The entire set X together with countable Boolean lattice is called lattice measurable space and is denoted by the ordered pair (X, σ (L)). $X = \Re$, where $\Re$ is extended real number system and σ (L) = {All Lebesgue measurable sub sets of $\Re$}, ($\Re$,σ (L)) is a lattice measurable space. If μ: σ (L) $\to R \cup \{\infty\}$ satisfies the following properties (i) $\mu(\phi) = \mu(0) = 0$ (ii) for all h, g $\in \sigma$ (L), such that μ (h), μ (g) ≥ 0; $h \leq g \Rightarrow \mu(h) \leq \mu(g)$ (iii) for all h, g $\in \sigma$ (L): $\mu(h \vee g) + \mu(h \wedge g) = \mu(h) + \mu(g)$ (iv) If $h_n \in \sigma$ (L), n $\in$ N such that $h_1 \leq h_2 \leq ... \leq h_n \leq ....$, then $\mu(\bigvee_{n=1}^{\infty} h_n) = \lim \mu(h_n)$ then μ is called a lattice measure on the countable Boolean lattice σ (L)[2].

Definition2.1. Let σ (L) be a countable Boolean lattice, H $\subseteq \sigma$ (L), and restrict $\wedge$ and $\vee$ to H as follows. For a, b, c $\in$ H, if $a \wedge b = c$ (dually, $a \vee b = c$), then we say that in H, $a \wedge b$ (dually $a \vee b$) is defined and it equals c, if, for a, b $\in$ H, $a \wedge b \notin$ H(dually $a \vee b \notin$ H), then we say that $a \wedge b$(dually $a \vee b$) is not defined in H. Thus (H, $\wedge$, $\vee$) is a set with two binary partial operations. (H, $\wedge$, $\vee$) is called a partial Boolean lattice of σ (L).

Notes

Note2.1. Here onwards we call partial Boolean lattice by simply partial lattice.

Observation2.1. Every subset of a countable Boolean lattice determines a partial lattice. Every Boolean sub lattice of σ (L) is a partial lattice and the converse need not be true.

A set A is said to be measurable partial lattice, if A is in σ (L). ($\Re$, σ (L)) be lattice measurable space. Then the interval (a, ∞) is a measurable partial lattice under usual ordering.

Notes

III. Nature of Points in Countable Boolean Lattice

Definition3.1. Let X be a lattice measurable space, let μ be a lattice measure on X, and let N be a measurable partial lattice. If μ is a positive lattice measure, then N is null partial lattice if its lattice measure μ (N) = 0.

Example3.1. The empty set is always a null partial lattice.

Observation3.1. Any countable union of null partial lattices is null partial lattice.

Observation3.2. Any partial lattice of a null partial lattice is itself a null partial lattice.

Definition3.2. Let (X, σ (L)) be a lattice measurable space. A nonempty class N of sets, where N is contained in σ (L) is called a class of null partial lattice of σ (L).If

For $E \in N$, $F \in \sigma(L)$, then $E \wedge F \in N$, and also for any $E_n \in N$, n=1, 2, 3...., $\bigvee_{n=1}^{\infty} E_n \in N$.

Definition3.3. Let (X, σ (L), μ) be a lattice measure space. A set E in σ (L) is called a μ-point if $\mu(E) > 0$ and $F \in \sigma(L)$ such that F is contained in E, then either μ (E-F) = 0 or μ (F) = 0.

Example3.2. Consider the set X={1, 2, ..., 9, 10} and let σ (L) be the power set of X. Define the lattice measure μ of a set to be its cardinality, that is, the number of elements in the set. Then, each of the singletons {i}, for i=1, 2, ... 9, 10 is a μ- point.

Definition3.4. Let σ (L) be the countable Boolean lattice .A partial lattice E is said to be a point of σ (L) if $E \neq \phi$ and F in σ (L), F is contained in E implies $F = \phi$ or F = E.

Example3.3. Let σ (L) = {All Lebesgue measurable subsets of a real line $\Re$}. Here σ (L) is a countable Boolean lattice. Clearly {1} is a member of σ (L). Put {1} = E. It can be easily verified that E is a point of σ (L).

({1} $\neq \phi$ that is $E \neq \phi$ also $F \in \sigma(L)$, F < E, then $F = \phi$ or F = E).

Example3.4. Suppose X consists of five points a, b, c, d, and e. Suppose E consists of two sets, E_1 = {a, b, c} and E_2 = {c, d, e}. We can find the countable Boolean lattice by E.

Let F_1 = {a, b} = $E_1 \cap E_2^c$ and F_2 = {c} = $E_1 \cap E_2$,

F_3 = {d, e} = $E_1^c \cap E_2$ and F_4 = {a, b, d, e} = $F_1 \cup F_3$.

Clearly σ (E) consists of the sets $\emptyset$, F_1, F_2, F_3, $F_1 \cup F_3$, $F_1 \cup F_2 = E_1$, $F_2 \cup F_3 = E_2$, X. The partial lattices F_1, F_2, F_3 are the points of the countable Boolean lattice. Every member of σ (E) is a union of some collection (possibly empty) of F_i. The only partial lattices of F_i are the empty set and F_i itself.

Note3.1. A countable Boolean lattice σ (L) of X is said to be pointless if there are no points of σ (L).

Example3.5. Consider the Lebesgue lattice measure on the real line. This lattice measure has no points.

Definition3.5. Lattice semi-finite measure: A lattice measure μ on a countable Boolean lattice σ (L) of X is said to be semi-finite if $F \in \sigma(L)$, $\mu(F) = \infty$ implies there exists $E \in \sigma(L)$ such that E is contained in F and $0 < \mu(E) < \infty$.

Note3.2. A lattice σ –finite measure is a lattice semi-finite but converse is not true.

Example3.6. An example of lattice measure which is lattice semi finite but not lattice σ –finite.

Consider an infinite set X.

Put $\mu(A) = |A|$ (the number of elements) if A is finite and

$\mu(A) = 0$ if A is infinite.

Definition3.6. A partially ordered set X is said to satisfy the countable chain condition (CCC), if every strong antichain in X is countable. In other words no two elements have a common lower bound.

Example3.7. The partially ordered set of non-empty open partial lattices of X satisfies the countable chain condition. That is every pair wise disjoint collection of non-empty open partial lattices of X is countable.

Result3.1. Let $(X, \sigma(L), \mu)$ be a lattice measure space. If E_1 and E_2 are points, then either $\mu(E_1 \Delta E_2) = 0$ or $\mu(E_1 \wedge E_2) = 0$ or (the lattice measure of any two points are either disjoint or identical).

Proof. Let E_1 and E_2 are points.

Since E_1 is a point by definition3.3, $E_2 \in \sigma(L)$ such that E_2 is contained in E_1.

This implies $\mu(E_1-E_2) = 0$ or $\mu(E_2) = 0$.

Since E_2 is a point $\mu(E_2) \neq 0$, $\mu(E_1-E_2) = 0$.

By similar argument we have that $\mu(E_2-E_1) = 0$.

Now consider $E_1 \Delta E_2 = (E_1-E_2) \vee (E_2-E_1)$

This implies $\mu(E_1 \Delta E_2) = \mu(E_1-E_2) + \mu(E_2-E_1)$.

Which leads to $\mu(E_1 \Delta E_2) = 0$.

Also evidently $(E_1 \vee E_2) = (E_1 \wedge E_2) \vee (E_1 \Delta E_2)$.

This implies $\mu(E_1 \vee E_2) = \mu(E_1 \wedge E_2) + \mu(E_1 \Delta E_2)$.

Which leads to $\mu(E_1 \vee E_2) = \mu(E_1 \wedge E_2)$ (since $\mu(E_1 \Delta E_2) = 0$).

Again if $\mu(E_1-E_2) \neq 0$, then $\mu(E_2) = 0$.

Now $E_1 \wedge E_2 \leq E_2$.

Hence $\mu(E_1 \wedge E_2) \leq \mu(E_2)$.

Which implies $\mu(E_1 \wedge E_2) \leq 0$.

But $\mu(E_1 \wedge E_2) \geq 0$ (by definition3.3).

Therefore $\mu(E_1 \wedge E_2) = 0$.

If $E_2 - E_1 \neq 0$ similarly we get $\mu(E_1 \wedge E_2) = 0$.

Result3.2. Let $(X, \sigma(L), \mu)$ be a lattice measure space and μ is lattice σ – finite measure. Then the class A of all points in $\sigma(L)$ is countable.

Proof. Let $E_1, E_2 \in A$ be any two points of $\sigma(L)$ by result 3.1.

We have either $\mu(E_1 \Delta E_2) = 0$ or $\mu(E_1 \wedge E_2) = 0$.

If $\mu(E_1 \Delta E_2) = 0$, then the set $(E_1 \wedge E_2)$ represents a point or if $\mu(E_1 \wedge E_2) = 0$ then (E_1-E_2) and (E_2-E_1) represents two disjoint points.

This implies two disjoint sets in $\sigma(L) - N$.

Continuing this process for E_1, E_2 ……, we get a countable collection of disjoint sets in $\sigma(L) - N$. Which leads to $\sigma(L) - N$ is countable.

Theorem3.1. Any countable union of null partial lattices is null partial lattice.

Proof. Let $A_1, A_2, \ldots\ldots\ldots$ be a sequence of null partial lattices.

That is for each positive integer n, we have $B \in \sigma(L)$ such that $A_n < B_n$ and $\mu(B_n) = 0$.

Now clearly $\bigvee_{n=1}^{\infty} A_n < \bigvee_{n=1}^{\infty} B_n$. Since each $B_n \in \sigma(L)$ by the definition of $\sigma(L)$,

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$\overset{\infty}{\underset{n=1}{\vee}} B_n \in \sigma$ (L).Now $\mu\left(\overset{\infty}{\underset{n=1}{\vee}} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} 0 = 0$

Therefore $\overset{\infty}{\underset{n=1}{\vee}} A_n$ is the largest partial lattice such that whose lattice measure is zero.

Hence $\overset{\infty}{\underset{n=1}{\vee}} A_n$ is also a null partial lattice.

Notes

Theorem3.2. Let μ be a lattice semi-finite measure on countable Boolean lattice σ (L) of X. Let N denotes the collection of partial lattice of μ - measure zero. Then σ (L) – N satisfies countable chain condition (CCC) if and only if μ is lattice σ – finite measure.

Proof. If μ is lattice σ – finite measure, it is obvious that σ (L) – N satisfies countable chain condition (CCC) (by result 3.2).

Conversely, if $\mu(X) < \infty$, then there is nothing to prove.

If $\mu(X) = \infty$, choose E_1 in σ (L) such that $0 < \mu(E_1) < \infty$.

Choose E_2 in σ (L) such that E_2 is contained in $X - E_1$ and $0 < \mu(E_2) < \infty$.

Continuing this process we get a sequence of disjoint partial lattices $E_1, E_2, \ldots,$ in σ (L) such that E_i in σ (L) – N and $\mu(E_i) < \infty$. If $\mu(X - \overset{\infty}{\underset{i=1}{\vee}} E_i) < \infty$, then we have a decomposition of X.

Which implies that μ is σ – finite. Hence $\mu(X - \overset{\infty}{\underset{i=1}{\vee}} E_i) = \infty$. Choose E_α in σ (L) such that E_α is contained in $X - \overset{\infty}{\underset{i=1}{\vee}} E_i$ and $0 < \mu(E_\alpha) < \infty$, where α is the first countable ordinal.

Proceeding inductively, since σ (L) – N satisfies countable chain condition (CCC), there exists a countable ordinal β such that $\mu(X - \underset{\alpha<\beta}{\vee} A_\alpha) < \infty$.

This implies that μ is lattice σ – finite measure.

Theorem3.3. Let σ (L) is a countable Boolean lattice of a set X. Then σ (L) is pointless if and only if every non empty set in σ (L) contains countable number of disjoint non empty sets in σ (L).

Proof. Let E in σ (L) is non empty set.

Fix $x \in E$.

We can choose E_1 in E such that $x \notin E_1$.

Now E_1 is non empty and E_1 is contained in E.

Choose E_2 in E such that $x \notin E_2$.

Now E_2 is non empty and E_2 is contained in E - E_1.

Choose E_3 in E such that $x \notin E_3$.

Continuing this process we get a family $\{E_\alpha / \alpha < \beta\}$ of non empty disjoint sets contained in ß where β is the first uncountable ordinal.

The converse part is trivial.

Theorem3.4. Let σ (L) is a countable Boolean lattice of a set X. Then it satisfies countable chain condition (CCC) if and only if σ (L) is isomorphic to the power set.

Proof. We can prove this theorem by using theorem 3.1 and theorem 3.2.

If σ (L) satisfies countable chain condition (CCC), then the number of points of σ (L) is countable.

From X remove all points of σ (L).

In the view of above theorem 3.2, the remaining part is empty.

Hence it is isomorphic.

Remark 3.1 Take the numbers 0, 1 and the fractions $\frac{m}{n}$, $0 < \frac{m}{n} < 1$

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That is

$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\ldots\ldots\ldots$ order as follows

$0 < \frac{m}{n} < 1$ for all $\frac{m}{n}$; $\frac{m}{n} \leq \frac{r}{s}$ only if $\max(m, r) = r$; $\frac{m}{n}, \frac{r}{s}$ in comparable if $n \neq s$.

Clearly the fractions from 0 to 1 have a countable number of points and of counter points.

IV. Conclusion

A crucial result is obtained that the lattice measures of any two points are either disjoint or identical. By defining a class of null sets, lattice measure of a point and lattice semi- finite measure, it was observed scrupulously that the class of all points in a countable Boolean lattice is countable and various elementary nature of points in a countable Boolean lattice have been identified.

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Optimal Hedging Strategy of Asset Returns on Target in Finance Logistics using the Law of Iterated Logarithm (Lil) Measure

By Bright O. Osu., Jonathan O. Egemba & Philip U. Uzoma
Abia State University, Nigeria

Abstract- The world of finance works better through logistics and there are more to a risk measurement and hedging than being coherent. Thus, several predictable assumptions hast been made in other to make risk calculation and hedging tractable which both Value-at-risk (VaR) and Conditional tail expectation (CTE or CVAR) ignore useful information on target . The question is can the classical law of iterated logarithm(LIL)be centralized for risky and contingent hedging capacities? Here we find the imposition of the law of iterated logarithm (LIL) constraint unique and complete, hence continuous for the QUEST as it utilizes information in the whole distribution, curbs rate of returns on target, provides incentives for risk management and raises challenges of performances and cost.

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GJSFR-F Classification : MSC 2010: 33B30, 90B06

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Abstract- The world of finance works better through logistics and there are more to a risk measurement and hedging than being coherent. Thus, several predictable assumptions hast been made in other to make risk calculation and hedging tractable which both Value-at-risk (VaR) and Conditional tail expectation (CTE or CVAR) ignore useful information on target . The question is can the classical law of iterated logarithm(LIL)be centralized for risky and contingent hedging capacities? Here we find the imposition of the law of iterated logarithm (LIL) constraint unique and complete, hence continuous for the QUEST as it utilizes information in the whole distribution, curbs rate of returns on target, provides incentives for risk management and raises challenges of performances and cost.

Keywords: law of iterated logarithm (lil), hedging, sublinear expectation, cvar, capital requirement, expected returns.

I. Introduction

Asset-liability management is a means of managing the risk that can arise from the changes inthe relationship between assets and liabilities. In cases such as in portfolio containing option as well as credit portfolio (i.e wealth distributions that are highly skewed), it is reasonable to consider asymmetric risk measures since individualsare typically loss averse.Value-at-risk (VaR) and tail conditionalexpectation (TCE) have also emerged in recent years as standard tools for measuring andcontrolling the risk of trading portfolios. In some dynamical settings however, the limits of TCEcan be transformed into the limits ofVaR and conversely even though TCE is more preferableto VaRsince it is coherent and VaR is not.Werecently discovered in literature that the law of the iterated logarithm (LIL) obeys these coherencies.

The law of the iterated logarithm(LIL) is one of the most important results on the asymptoticbehaviour of finite-dimensional standard Brownian motion (Dvoretzky and Erdos, 1951). Its classical laws as fundamental limit theorems in probability theory plays an important role in the development of probability theory and its application. The original statement of LIL obtained by (Khinchine 1924) is for a class of Bernoulli random variables. Kolmogorov(1929) and Hartman-Winter (1941) extended Khinchine's result to large class of independent random variables. Levy (1937) extendedKhinchine's

Authors α σ: Department of Mathematics, Abia State University P M B 2000, Uturu, Nigeria. e-mails: megaobrait@yahoo.com, megaobrait@gmail.com, Jonathan.egemba@gmail.com

Author ρ: Department of Mathematics and Statistics,Federal Polytechnic Nekede, P. M. B. 1036, Owerri, Nigeria. e-mail: Uzomaphilip@gmail.com

result to martingales, an important class of dependent random variables. Strassen(1964) extended Hartman-wintner's result to large classes of functional random variable. After that the research activity of LIL has enjoyed a rich classical period and a modern resurgence (Stout, 1974).To extend the LIL, a lot of fairly neat methods have been found (De Acosta, 1983). However, the key in the proofs of LIL is the additivity of the probabilities and expectations. In practice, such additivity assumption is not feasible in many areas of application because the uncertainty phenomenon cannot be modeled using additive probabilities or probability expectations. As an alternative to the traditional expectations or probability, capacities or non-linear probabilities /expectations have been studies in many fields such as statistics, finance and economics. In statistics, capacities have been applied in robust statistics (Huber 1981), under the assumption of two alternating capacity (Huber and strassen, 1973). Financial risk management is vital to the survival of financial institutions and the stability of the financial system. A fundamental task in risk management is to measure the riskentailed by a decision, such as the choice of a portfolio (Osu et al., 2013). Recently, thesubstitution of variance as a risk measure in the standard Markowitz (1952) mean-varianceproblem has been emphasized, because it makes no distinction between positive and negativedeviations from the mean. Variance is a good measure of risk only for distributions thatare (approximately) symmetric around the mean such as the normal distribution or moregenerally, elliptical distributions (Frey and Embrechts, 2006).In capital requirement Logistics is the single most powerful force on risk management in finance. Because it is the intersection of the virtual and physical world of finance that allows one to keep up to date information of where everything and anything are or is going at a particular moment (Achi etal, 2013). Money can be invested and produce more money. However, investing money involves different level of risks depending on the choice of the investment and a high rate or value of money at risk bring about high target of positive expected returns. (Gerber 1979).The LIL assumption can be represented as the assumption of an expected rate of returns on high target, that is the best guess estimate of tomorrow's return level. Since there is no relevant information available at time t that could help forecast returns at time t + 1. It is well known in finance that an important framework is calculating the price of uncertainty option claim.

The objective of this work is to investigate if the classical law of iterated logarithm can be centralized for the contingent hedging capacities which depends on its completeness and uniqueness and to show how one should calculate returns of high diversified portfolio to maximize the capital growth in returns by measuring the risk involved to know the future returns on target.

II. Frame work of Lil Hedging Pricing Capacity (Result)

Consider a sequence of independent and identically distributed (iid) random variable $X_1, X_2, \ldots, X_n$ with $E(X_n) = 0, Var(X_n) = \sigma^2, \sigma > 0$. Then

$$P\left\{\log_{n\to\infty} sup\frac{S_n}{(2\sigma^2 \log logn\,)^{\frac{1}{2}}} = 1\right\} = 1 \quad . \qquad (2.1)$$

This implies that with probability one and for

$$\lim_{n\to\infty} sup\frac{S_n}{(2\sigma^2 \log logn\,)^{\frac{1}{2}}} = lim_{n\to\infty} sup\frac{Z_n}{(2\, log\, logn\,)^{\frac{1}{2}}}, \qquad (2.2)$$

then$Z_n > (c\, log\, logn)^{\frac{1}{2}}$ for infinitely many n if $c < 2$, but for only finitely many n if $c > 2$.

Put in another way, let $\{X_n, n \in W\}$ be a sequence of iid random variable on a probability space $\{\Omega, \mathfrak{F}, P\}$, let $S_n = X_1 + X_2 + \cdots$ and set $Z_n = \frac{S_n - \mu n}{\sigma n^{\frac{1}{2}}}$ (where μ is the expectation and σ is the standard deviation). Then we define the law of iterated logarithm for a stationary independent process thus:

Notes

$$lim_{n\to\infty}\, sup \frac{Z_n}{(2\, log\, logn\,)^{\frac{1}{2}}} = 1. \qquad (2.3)$$

Similarly with probability one,

$$lim_{n\to\infty}\, sup \frac{Z_n}{(2\, log\, logn\,)^{\frac{1}{2}}} = -1. \qquad (2.4)$$

Since a supremum expectation (super hedge) of LIL is sublinear it is continuous, hence complete (unique) which makes it have a hedging pricing capacity.
A good hedging pricing capacity model according to Artzner etal, (1997) must be complete, if it is sublinear and continuous. Completeness implies Uniqueness and continuous implies completeness.

Given a set P of multiple prior probability measure on $(\Omega\,, f)$, let X be the set of random variable on $(\Omega\,, f)$, where Ω = sample space and f is the increasing sequence of Ω. For any ξ Є X, we define a pair of maximum (super hedge) and minimum (subhedge) expectation as $(\mathbb{E}, \epsilon)$by (Peng 2006-2009):

$$E\,(\xi) = \underset{Q\in P}{Sup} E_Q(\xi)\ \Rightarrow\ Super\ hedge\ pricing \qquad (2,5)$$

$$E\,(\xi) = \underset{Q\in P}{Inf} E_Q(\xi)\ \Rightarrow\ Super\ hedge\ pricing, \qquad (2.6)$$

where $E_P(.)$ denotes the classical expectation under probability measure P. Let $\xi = I_A$for $A \in f$, immediately, a pair of (V, v) of capacities is given by $V(A) := \sup_{p\in P} P(A)$, $v(A) := sup_{p\in P} P(A)$, $\forall A \in f$.

According to Peng(2007)E is called sub-linear expectation in the sense that A functional$E$ on $X \longrightarrow (-\infty, +\infty)$ is called a sub-linear expectation, if it satisfies the following properties for all x, y Є X. (coherent properties)

1. Monotonicity: $x > y$implies $E(x) \geq E(y)$.
2. Constant preserving: $E(c) = C\ \forall\ C \in R$.
3. Sub-addivity: $E[x + y] \leq E(x) + E(y)$ hedging property.
4. Positive homogeneity: $E(\lambda x) = \lambda\, E(x), \forall \lambda \geq 0$.

Note: A sublinear expectation is a supremum expectation (Cheng, 2009).

Remark

If a market is complete and self financing, then there exist a neutral probability measure P such that the pricing of any discounted contingent claim ξ in this market is given by $\mathbb{E}$ (ξ) then by LIL $\mu = E_p(\xi)$ *and variance* $\sigma^2 = E_p[(\xi - \mu)^2]$ with probability one.

$\mu = \lim_{n\to\infty} \frac{1}{n} S_n$, $\sigma = \lim_{n\to\infty} Sup(2_n \log\log n)^{-\frac{1}{2}} |S_n - n\mu|$where$S_n$ is the sum of the first n of a sample (X_i) with mean μ and variance σ^2.

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Note; Everysublinear expectation is a supremum expectation /continuous expectation.

The question is can the classical supremum/ superhedge expectation of LIL be centralized for contingent hedging pricing capacity? By the definition, a pricing hedging capacity is c alled continuous capacity if it satisfies the following desirable axioms or properties.(Wasserman and Kadane 1990).

Given a set function $P: f \longrightarrow [0,1]$ then it is a continuous capacity if it satisfies the following:

1. Stability property: the system or function is stable if
2. $P(\phi) = 0, P(\Omega) = 1;$
3. If there exist any positive even bounded continuous function P(x) where x ЄR, thenfor every $a, b \in R. P(A) \leq P(B)$ whenever $A \subset B$ and $A, B \in f$the function P(x) is a self-financing value which completely determines the distribution x and also has a mathematical properties of the (2,1,3 and 4)(very important property of completeness).
4. $P(A_n) \uparrow P(A), if\ A_n \uparrow A,$ $\uparrow \Rightarrow$Superhedging (Supremum expectation)
5. $P(A_n) \downarrow P(A), if\ A_n \downarrow A, where\ A_n, A$Єf $\downarrow \Rightarrow$ Subhedging .

III. The Model

Assuming $P(\xi)$ to be the risk neutral asset, that is the self financing value.If the simple European security V_b is hedgeble then for any positive bounded continuous function, there assume a portfolio process whose self-financing value process $P(\xi)$ of LIL supremum expectation that satisfies the continuous capacity property $P(\xi) \leq$ $V_b(x)$ *where* $X_i$is adapted at time $t$for $V_b(x)$. if the result is satisfied, then it is complete and also a martingale.

a) Lemma

Suppose ξ is distributed to G normal $N(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, where $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$. Let ϕ be a bounded continuous function. Furthermore, if ϕ is a positively even function, then for any $b \epsilon R$

$$e^{-b^2/2\sigma^2} \epsilon[\phi(\xi)] \leq \in [\phi(\xi - b)] \quad . \tag{3.1}$$

(see Chen and Hu, 2013 for prove).

It has been shown by Mao(1997) that if X is the solution of the d- dimensional equation

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), t \geq 0, \tag{3.2}$$

and if there exist positive real numbers ρ, k such that for all $x \in R^d$ and $t \geq 0$, $x^T f(x,t) \leq \rho$, and $\|g(x,t)\|$, then;

$$\lim_{t\to\infty} \sup \frac{|X(t)|}{\sqrt{2tloglogt}} \leq k\sqrt{e}, \quad a.s. \tag{3.3}$$

Appleby and Wu (2008) had shown also that for X a unique continuous adapted process which obeys (3.2). Let $A := \{\omega: \lim_{t\to\infty} X(t,\omega) = \infty\}$. If

$$\lim_{x\to\infty} xf(x) = L_\infty ; g(x) = \sigma, x \in R, \tag{3.4}$$

where $\sigma \neq 0$and $L_\infty > \frac{\sigma^2}{2}$, then $P[A] > 0$ and X satisfies for super hedge;

$\lim_{t\to\infty} \sup \frac{|X(t)|}{\sqrt{2tloglogt}} = |\sigma|, a.s. on\ A,$ (3.5) and for sub hedge;

Notes

$$\lim_{t\to\infty} \sup \frac{log\frac{X(t)}{\sqrt{t}}}{loglogt} = -\frac{1}{\frac{2L_\infty}{\sigma^2}-1}, a.s. on\ A \quad (3.6)$$

Theorem 1

Notes

If X (the capital allocation to the individual risk with $X = X_1 + X_2 + \cdots X_n$, where $X_1 + X_2, \ldots, X_n$ are copies of X) obeys (3.2) and if the exist positive real numbersρ andC_2 such that for $k\epsilon K^d$ and $t \geq 0$,$xf(x,t) = P$ and $\|g(x,t)\|_, \leq C_2 K$ (where $\|\cdot\|_{op}$ denotes the operator norm), then and if in addition $\psi(t) = (2\log t + C_k log\ _2 t)^{1/2}, -C_2 = \frac{1}{1-c}$ and $n = loglogt_2$. Then

$$E(\xi) = P(F_n) = \frac{1}{1-c}(log_2)^{\frac{1-c}{4}} \begin{cases} c > 1 := super\ hegde \\ c < 1 := \ sub\ hegde \end{cases} \quad (3.7)$$

Proof;

Following the method variation of Brownian motion result (Osu, 2003) define the event

$$F_n = \{w: S(h_n) < h_{n-1}^{1/2}\phi(1/h_{n-1})\}, \quad (3.8)$$

where $h_n = e^{-n\rho}$, and $0 < \rho < \frac{1}{2}$.

Then the event

$$\{S_i(h_n) < h_{n-1}^{1/2}\psi(1/h_{n-1})\}, \quad (3.9)$$

for the independent and identically distributed (iid) random variables

$$S_i(h_n) = \sup R\left\{X; t, t+h), i = 0, 1, \ldots, \left[\frac{1}{2h_n}\right];\right.$$

are independent and have equal probabilities.

Moreover since

$$F_n \subseteq \bigcap_{i=0}^{[1/2h_n]} \left\{S_i(h_n) < h_{n-1}^{1/2}\psi\left(\frac{1}{h_{n-1}}\right)\right\}$$

then

$$P(F_n) \leq \left(P\left\{S_0(h_n) < h_{n-1}^{1/2}\psi\left(\frac{1}{h_{n-1}}\right)\right\}\right)^{[1/2h_n]} \quad (3.10)$$

By Kochen and Stone (1964), and the scaling property, we have

$$\left\{S_0(h_n) < h_{n-1}^{1/2}\psi\left(\frac{1}{h_{n-1}}\right)\right\} = P\left\{S(1) < h_n^{-1/2} h_{n-1}^{1/2}\psi\left(\frac{1}{h_{n-1}}\right)\right\} \leq 1 - C_0\lambda_n e^{-\lambda_n^{2/2}}, \quad (3.11)$$

$$\text{where } \lambda_n = \left(\frac{h_{n-1}}{h_n}\right)^{\frac{1}{2}}\psi\left(\frac{1}{h_{n-1}}\right).$$

Hence

$$P(F_n) \leq \left(1 - C_0\lambda_n e^{-\lambda^{2/2}}\right)^{[1/2h_n]} = (1-u)^N, \quad (3.12)$$

say, where $u - C_0\lambda_n e^{-\lambda^{2/2}}$ and $N = [1/2h_n]$. Andbecause $\log(1-u) < -u$, then

$$(1-u)^N = e^{log(1-u)^N} = e^{Nlog\ (1-u)} < e^{-Nu}. \quad (3.13)$$

But

$$\lambda_n^2 = \left(\frac{h_{n-1}}{h_n}\right)\left(2\ log\frac{1}{h_{n-1}} + Clog_2\frac{1}{h_{n-1}}\right)$$

$$= \left(\frac{e^{(n-1)^p}}{e^{-n^p}}\right)\left(2\log e^{(n-1)^p} + Clog\ _2 e^{(n-1)^p}\right)$$

$$= \{1 + 0(n^{\rho-1})\}\left\{2n^{\rho}\left(1 + 0(n^{-1}) + C_{\rho}(log_n + 0(n^{-1})\right)\right\}$$
$$= 2n^{\rho} + C_{\rho} log_n + 0(n^{2\rho-1})$$
$$= 2n^{\rho} + C_{\rho} log_n + 0(1) \text{since } \rho < \tfrac{1}{2}. \qquad (3.14)$$

Therefore $\lambda_n \sim C_{1n^{1/2\rho}}$ and

$$NU \sim \frac{1}{e} e^{n\rho} \cdot C_0 C_1 n^{\frac{1}{2}\rho} \exp\left\{-n^{\rho} - \frac{1}{2} C_{\rho} \log n + 0(1)\right\} = C_2 n^{\delta},$$

where

$$\delta = \frac{1}{2}(1 - C)\rho > 0 \text{if } C < 1 \qquad . \qquad (3.15)$$

Therefore

$$E(\xi) = P(F_n) < e^{-C_2 n^{\delta}}. \qquad (3.16)$$

for large n, so that $\sum_{n=1}^{\infty} P(F_n) < \infty$, and, by Ugbebor (1980),F_n happens only finitely often.

Using equation (3.15), we have (for$\rho = \frac{1}{2}$) equation (3.7) as required.

IV. Application

We refer to equation (3.7) as the LIL measure. All investment involve some element of risk, but we are predicting a measure that will help attain a high target on expected returns on a risky portfolio by raising performances and cost. Banks meet their target by focusing on the high rate of returns. Hence LIL measure focus on the high rate of returns, because higher target implies higher investment which also implies high expected returns on target. Hence raises challenges of performance which value at risk and conditional value at risk ignore.

Banks expected returns are the risk free rate of capital plus a market premium. That is risk free rate + a market premium;

$$E(B) = \mathrm{R(x)} + \mathrm{M(x)} \qquad , \qquad (4.1)$$

and risk free rate equals solvency capital minus capital requirement for the risk. Capital requirement is the capital required in respect of a random variable (risk) with the view to avoiding insolvency or shortfall. The reason of solvency is to make sure that the bank have the financial means to meet its future obligation, to pay the present and future claims related to the policy holders and regulators.In order to avoid insolvency over the specific horizon at some given level of risk tolerance they should hold asset of value that is enough or small enough. Solvency capital requirement for the risk = Assets – Liabilities

$$\mathrm{A(x)} - \mathrm{L(x)} = S(x) \qquad (4.2)$$

At least for values greater than the relevant VaR with probability function $f(y)$ then $CVaR$ for the normal distribution is shown to be;

$$\mathrm{CVAR}_{\alpha} = \frac{\sigma}{1-\alpha} + \mu\phi\left(\frac{\mathrm{Q}_{\alpha} - \mu}{\sigma}\right). \qquad (4.3)$$

For two and three parameter Weibull, Osu and Ogwo (2012) had shown that

$$\mathrm{CVaR}_{\alpha} = \frac{x}{1-\alpha} e^{-\mathrm{Q}_{\alpha}}, \qquad (4.4)$$

$$CVaR_{\alpha} = \frac{e^{-\mathrm{Q}_{\alpha}}}{(1-\alpha)}, \qquad (4.5)$$

respectively. Equation (4.3) implies that CVAR is a little bigger than VAR and it can be adjusted for by adding an inverse of a decay constant (Klygman, 2004).

Notes

V. Emperical Example

Calculate the CVAR of 1 million portfolio on a 100 basis point per day standard deviation, suppose that the daily returns are normally distributed with $\mu = 0$ on a 100 basis point per day.

Solution

Using (4.3), we have CV$AR_{5\%}$ = *20 + 16450 = 16470.*

Expected returns = 1m – 16470 = 983530 meaning that there is 5% chance that the daily loss on 1m portfolio is equal or exceed only 16470 and a 95% chance that it will worth 983530 or more tomorrow. $CV_a R_{1\%} = \phi_{0.99} = 2.326 \therefore \frac{2.326}{100} = 0.02326 \text{ X } 1\text{m} = 23260$ which is$VAR_{-(1\%)}$ on 1m portfolio.Hence, 1m – 23260 = 976740 is the expected returns. Which means that there is 1% chance that the daily loss on 1m portfolio is equal or exceed only 23260 and 99% chance of being worth 976740 or more tomorrow.

Expected returns = 1m – 23360 = 976640. Which means that there is 1% chance that the daily loss on 1m portfolio is equal or exceed only 23360 and 99% chance of being worth 976640 or more tomorrow.

Using (3.7), we have the expected returns 20m—(-5128205.12) =25128205.12

VI. Conclusion

It shows that the classical LIL can be centralized for hedging pricing capacity as the supremum/sublinear expectation is continuous in the interval [0,1]. Hence investigating LIL for capacities shows that the supremum limit points of it lie with probability capacity one between the lower and upper standard bound and also satisfies the desirable axioms under the hedging pricing continuous capacity. Here laws of iterated logarithm (LIL) has been represented as the assumption of an ERR on a target of high diversified portfolio in bank's capital requirements as it utilizes information on the whole distribution, have a continuous hedging capacity, hence complete and unique. Which CVAR ignores useful information on. The measure on ERR curbs rates of returns on target, provides incentives for risk managers by raising challenges of performances and cost. Making it an optimal computational method to increase performances in hedging and banks attaining their targets on focus as it's application is a measure of a multifractal returns on banks portfolios.

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Computation of a Summation Formula

By Salahuddin, M. P. Chaudhary & Vinesh Kumar
Jawaharlal Nehru University. India

Abstract - The main aim of the present paper is to compute a summation formula involving recurrence relation of Gamma function.

Keywords: *contiguous function, recurrence relation, bailey summation theorem and legendre duplication formula.*

GJSFR-F Classification : *MSC 2010: 40A25, 65B10*

COMPUTATIONOFASUMMATIONFORMULA

Strictly as per the compliance and regulations of :

Notes

Computation of a Summation Formula

Salahuddin [α], M. P. Chaudhary [σ] & Vinesh Kumar [ρ]

Abstract- The main aim of the present paper is to compute a summation formula involving recur- rence relation of Gamma function.

Keywords : *contiguous function, recurrence relation, bailey summation theorem and legendre duplication formula.*

I. Introduction

Generalized Gaussian hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ccc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$ (a-b)\ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \\ c & ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ccc} a+1,\ b & ; & \\ c & ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ccc} a,\ b+1 & ; & \\ c & ; & z \end{array}\right] \tag{2} $$

Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\ \Gamma(z) \tag{3} $$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$ \sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{4} $$

$$ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{5} $$

$$ = \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{6} $$

Author α: P.D.M College of Engineering, Bahadurgarh, Haryana, India e-mails: sludn@yahoo.com, vsludn@gmail.com
Author σ: International Scientific Research and Welfare Organization, New Delhi, India. e-mail: mpchaudhary 2000@yahoo.com
Author ρ: Jawaharlal Nehru University, New Delhi, India

Bailey summation theorem [Prud, p.491(7.3.7.8)]is defined as follows

$$ {}_2F_1\left[\begin{matrix} a, 1-a & ; & \frac{1}{2} \\ c & ; & 2 \end{matrix}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} \tag{7} $$

II. Main Result of Summation Formula

Notes

$$ {}_2F_1\left[\begin{matrix} a & , & -a-57 & ; & \frac{1}{2} \\ & c & & ; & 2 \end{matrix}\right] $$

$$ = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+57}}\Bigg[\frac{1}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+57}{2})}\Bigg\{466399323142605149496819323748432740352000000 $$

$$ -907329016457775897789212258687070508154880000a $$

$$ +601460859948470727580060186448457350184960000a^2 $$

$$ -188803621917949331258961363038291864282726400a^3 $$

$$ +30588219059232477286307847889681327097487360a^4 $$

$$ -2350301414679388620818098236412133391230976a^5 $$

$$ +27879111867954013574486523140576965442457a^6 $$

$$ +6922484518387552294683241535450869753574a^7 $$

$$ -253571224837681433534397775156450174579a^8 $$

$$ -10854994499773183450718637148310195424a^9 $$

$$ +455755408380661339944372619176652144a^{10} $$

$$ +145589201260019100940545911010809520a^{11} - 368534064005346223864867177858668a_{12} $$

$$ -148455131024117581464577195085940a^{13} + 703789848420398412314305604430a^{14} $$

$$ +816627382858436132925398053800a^{15} + 737178620075422201572806790a^{16} $$

$$ -154532052964854408687971400a^{17} - 3564866447338277563836900a^{18} $$

$$ -13366558115214311007000a^{19} + 32941890966055701630a^{20} + 445343405624977860a^{21} $$

$$ +1502906526285450a^{22} - 10300901787540a^{23} - 110701049550a^{24} - 342030780a^{25} $$

$$ -52206a^{26} + 1596a^{27} + 2a^{28} + 123497122854418710329010591625119843483648000000c $$

$$ -183991675839092440016296215227039429846630400000ac $$

$$ +98833276767076334863559708934245452810813440000a^2c $$

$$ -25480192157412806624362721615281515889950720000a^3c $$

$$ +3328693717559958060639978952490792365287014400a^4c $$

$$ -185334899163743130094590943302042755666411520a^5c $$

$$ -2199777574208672328220000000994805006483456a^6c $$

$$ +636507086366403919913068309934401111449600a^7c $$

Notes

$$-786946959907484168061439374690130247680a^8c$$
$$-1053333545504392703202030197721492817 92a^9c$$
$$+127210378435098756224417048548724326 4a^{10}c$$
$$+118535841449585106927547371397966080a^{11}c$$
$$-247233008505961828377294660520960a^{12}c-816892409432196242697905908302 72a^{13}c$$
$$-754883858724479220655956538816a^{14}c+2475076534136043515496960960 0a^{15}c$$
$$+531219767676732770129836160a^{16}c+1066729314095318132138 88a^{17}c$$
$$-97230321230337925005376a^{18}c-1019279993673478817280a^{19}c+25910654883521 4080a^{20}c$$
$$+76009425920395968a^{21}c+539477067007424a^{22}c+781695008640a^{23}c-7135623040a^{24}c$$
$$-34856640a^{25}c-47040a^{26}c+13494135444075880964110548638532103003176960000c^2$$
$$-1625757601519294124516001851748297433782681600 0ac^2$$
$$+723399542452159352960297728258041269720186880 0a^2c^2$$
$$-154335584911825080779852794209971445560770560 0a^3c^2$$
$$+161277491380084838730576831057220479059066880a^4c^2$$
$$-59174550117021419737839954938192707856056 32a^5c^2$$
$$-2647238678234832107894003820863271891578 88a^6c^2$$
$$+230966926448064958065569208225129108710 40a^7c^2$$
$$+16658681789095200434138935350012422348 8a^8c^2$$
$$-3613405199728141387355690663303117760 0a^9c^2$$
$$-212949336348315839241353521493266944a^{10}c^2$$
$$+3226476315166131901783344971679792 0a^{11}c^2$$
$$+40433494175922611598927031485785 6a^{12}c^2-142185537878787222987097435019 28a^{13}c^2$$
$$-328225254933201367131430008200a^{14}c^2+1143229647253660219786188240a^{15}c^2$$
$$+9736761500480030377690430 4a^{16}c^2+8201067152309457213799 44a^{17}c^2$$
$$-591192458252736574405 6a^{18}c^2-141960880118220820800a^{19}c^2-72500110202681532 8a^{20}c^2$$
$$+2211323321863656a^{21}c^2+38291954101128a^{22}c^2+138885037200a^{23}c^2+55218800a^{24}c^2$$
$$-622440a^{25}c^2-840a^{26}c^2+850350841888616944888178254733450433003520000c^3$$
$$-8515296832751290559466335704757694266710425600ac^3$$
$$+3176567663094301371561153309912365816768102400a^2c^3$$
$$-561918804935876961195354695777965295750086656a^3c^3$$
$$+46189958201951932823079728295118858792992768a^4c^3$$
$$-869322064993772452952491622404221024337920a^5c^3$$
$$-102991028046491082289579737359791411036160a^6c^3$$

$$+433220703749246454590435183017487774515 2a^7c^3$$

$$+1259820682844207116031850030710508421 12a^8c^3$$

$$-60451637998658036396224830604271616 00a^9c^3$$

$$-14027592915899872770083343725785088 0a^{10}c^3$$

$$+3882856168239761927883033010888704a^{11}c^3+116812278024072710451296861554688a^{12}c^3$$

$$-6824751339549184140725116108 80a^{13}c^3-4901978279536021725113526272 0a^{14}c^3$$

$$-330492273500001618768322560a^{15}c^3+6714367719278375671330816a^{16}c^3$$

$$+116581044196640285982720a^{17}c^3+3028852864684494848 00a^{18}c^3$$

$$-6826246208006184960a^{19}c^3-65078503959332864a^{20}c^3-143929851985920a^{21}c^3$$

$$+745317314560a^{22}c^3+4461649920a^{23}c^3+6522880a^{24}c^3$$

$$+3561261065507197629301363276799707812200448000c^4$$

$$-3010245394488795638590287560253158832985866240ac^4$$

$$+947399864953188793058206727349547698216763392a^2c^4$$

$$-138618021601763633900282415668783181747191808a^3c^4$$

$$+8705958157821068935221924991437973346451456a^4c^4$$

$$-654553238144815784361224603908288659456 0a^5c^4$$

$$-2233490858540569309574759837006431878348 8a^6c^4$$

$$+41385997728409548009117055451510403072 0a^7c^4$$

$$+2925423394280211074290105348782948556 8a^8c^4$$

$$-48501169212439170397162258345559040 0a^9c^4$$

$$-266842921260842831673407899155850 24a^{10}c^4$$

$$+1392461043637448315560927139066 88a^{11}c^4+14601002459767951505475582032768a^{12}c^4$$

$$+859336989584718499311405312 00a^{13}c^4-34076193607103472191304537 60a^{14}c^4$$

$$-52591205727410294231738880a^{15}c^4+5469569146519205674918 4a^{16}c^4$$

$$+6704288546893480857600a^{17}c^4+48990032197516467200a^{18}c^4-427814011162137 60a^{19}c^4$$

$$-2102705477448576a^{20}c^4-9072154291200a^{21}c^4-5764944640a^{22}c^4+39836160a^{23}c^4$$

$$+58240a^{24}c^4+1072222813971144348944661206228083295146475520c^5$$

$$-771955970780867938711218378135730636858589184ac^5$$

$$+205497257561337377900462770972553886277042176a^2c^5$$

$$-24708803793159510844090828036989490393251840a^3c^5$$

$$+11270195676999892253829889121669330711347 20a^4c^5$$

$$+2238769146793340831289778042347001164595 2a^5c^5$$

$$-311210053332190825493414598362766901248 0a^6c^5$$

$$+5124628293853339619904189458261606400a^7c^5$$

Notes

$$+3829773132020314532300280189521756160a^8c^5$$

$$-933508745701830528533431891918848a^9c^5$$

$$-2743566111413385672154074653147136a^{10}c^5-19247072311498272636067394641920a^{11}c^5$$

$$+970964177802757606050320424960a^{12}c^5+15032825475370109826676088832a^{13}c^5$$

$$-85768382005633298871803904a^{14}c^5-3384247255338239724994560a^{15}c^5$$

$$-18375409016677836963840a^{16}c^5+156077581075549765632a^{17}c^5$$

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$$-166865707008a^{21}c^5-266133504a^{22}c^5$$

$$+243890488149341637300824039218934163569639424c^6$$

$$-150321771497018843749746565721892434107957248ac^6$$

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$$-3307414859224703274338630570242969782190080a^3c^6$$

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$$-4025881344973482804545093699908976640a^7c^6$$

$$+316621333331420393869780615398096896a^8c^6$$

$$+3984040841614292103838617498894336a^9c^6-166306661074386710952004844480512a^{10}c^6$$

$$-3038507371758994979751178951680a^{11}c^6+30278304609796953429224494080a^{12}c^6$$

$$+1031766521753165323963149312a^{13}c^6+3470841044718249605731328a^{14}c^6$$

$$-108558108968035386316800a^{15}c^6-1157796012019705051136a^{16}c^6$$

$$-901101796452329472a^{17}c^6+43219410012979200a^{18}c^6+226584758400000a^{19}c^6$$

$$+199364092928a^{20}c^6-993248256a^{21}c^6-1584128a^{22}c^6$$

$$+43355389491055769847463879254244627518062592c^7$$

$$-22932332314025228643707793270129164078284800ac^7$$

$$+4354351621061894603205398821295755488133120a^2c^7$$

$$-340280278205918007749476681007943439613952a^3c^7$$

$$+5268826680927529460010835301966427455488a^4c^7$$

$$+607087424002840331349343636652194529280a^5c^7$$

$$-18437303301372552264659488449434746880a^6c^7$$

$$-603661151100359177840858346740514816a^7c^7$$

$$+16429189593277264282920436663320576a^8c^7+458939103196979469552427872092160a^9c^7$$

$$-5237422369426935938151121059840a^{10}c^7-207974579190169053766694141952a^{11}c^7$$

Notes

$$-296405190586920108836093952a^{12}c^7 + 38723174283616217992396800a^{13}c^7$$

$$+354068653936202771005440a^{14}c^7 - 1229871189063139983360a^{15}c^7$$

$$-32292563516970172416a^{16}c^7 - 130278255330263040a^{17}c^7 + 262376459304960a^{18}c^7$$

$$+2860554977280a^19c^7 + 5018517504a^{20}c^7$$

$$+6171290629603643126170580214228889934757888c^8$$

$$-2802629659556658619881783593724180336476160ac^8$$

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$$-49625299280098543277519119704047616a^7c^8+427094431749423840936631304773632a^8c^8$$

$$+28966115302685253557341551360000a^9c^8 + 17932140456528884843713876992a^{10}c^8$$

$$-8394161385978078190463453184a^{11}c^8 - 69785672710363246659449856a^{12}c^8$$

$$+744918949016407969689600a^{13}c^8 + 12861312108952905861120a^{14}c^8$$

$$+29989104310096588800a^{15}c^8 - 435016831929698304a^{16}c^8 - 2864322226022400a^{17}c^8$$

$$-3278428707840a^{18}c^8 + 12770334720a^{19}c^8 + 22404096a^{20}c^8$$

$$+716136359002421753334636316504433990041600c^9$$

$$-278927665676587737338789279445144305664000ac^9$$

$$+37070372176984097349474493611626385637376a^2c^9$$

$$-1696926592070759457997617833086770216960a^3c^9$$

$$-23453509224301325867233427545925877760a^4c^9$$

$$+3259768163222589195235110000391618560a^5c^9$$

$$+3259768163222589195235110000391618560a^5c^9-406186249061206132900695113203712a^6c^9$$

$$-2729628530242733424233244937420800a^7c^9 - 7089741271975303163177293578240a^8c^9$$

$$+1174373949670553060404796620800a^9c^9 + 10111389834351011106294890496a^{10}c^9$$

$$-199478333790058458479984640a^{11}c^9 - 3134893884117907735183360a^{12}c^9$$

$$+981384285718052536320a^{13}c^9 + 249222060888817336320a^{14}c^9$$

$$+1331407173441945600a^{15}c^9 - 1469053900554240a^{16}c^9 - 27175272284160a^{17}c^9$$

$$-52973240320a^{18}c^9 + 68671834987125330030421072486570267770880c^{10}$$

$$-22882708575763566744543275240786849955840ac^{10}$$

$$+2511663177351930865843795854161162010624a^2c^{10}$$

$$-81036432418679722583828635815766917120a^3c^{10}$$

Notes

$$-2606010011827642918441988956573663232a^{4}c^{10}$$
$$+151675625085089343325830323308265472a^{5}c^{10}$$
$$+169672184521348041468997016033689 6a^{6}c^{10}$$
$$-10466346441969849865667597844480 0a^{7}c^{10}-1226537509620043280063155175424a^{8}c^{10}$$
$$+30480845647972909412775112704a^{9}c^{10}+513919518846713652406136832a^{10}c^{10}$$
$$-19356324718624328783462 40a^{11}c^{10}-768287484993890474229 76a^{12}c^{10}$$
$$-314383187993679028224a^{13}c^{10}+2406525855763341312a^{14}c^{10}+209877215971737 60a^{15}c^{10}$$
$$+30351396421632a^{16}c^{10}-97054543872a^{17}c^{10}-189190144a^{18}c^{10}$$
$$+5497635147596192782359823742199807344640c^{11}$$
$$-1561382656611557490409360890779821670400ac^{11}$$
$$+1397768234886447026048074413956608819 20a^{2}c^{11}$$
$$-281254212125211814985535118316981452 8a^{3}c^{11}$$
$$-17164132780881378263887152020874854 4a^{4}c^{11}$$
$$+51497003723734164506760884256768 00a^{5}c^{11}$$
$$+12233175599739158150716207398912 0a^{6}c^{11}-2713864686690451131253552840704a^{7}c^{11}$$
$$-60745021571699617498179305472a^{8}c^{11}+415445612517627736086282240a^{9}c^{11}$$
$$+14624775143761555693240320a^{10}c^{11}+37100204878937028820992a^{11}c^{11}$$
$$-1096071224208325607424a^{12}c^{11}-7955496728726077440a^{13}c^{11}+25921829103 20640a^{14}c^{11}$$
$$+158110675107840a^{15}c^{11}+346733936640a^{16}c^{11}$$
$$+370280998566765020991674964489126543360c^{12}$$
$$-891843941420315757672732737269451980 80ac^{12}$$
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$$-813573109439563605283287493640192 0a^{4}c^{12}$$
$$+11735857866009354930112399147008 0a^{5}c^{12}+52261287503253017808086878126 08a^{6}c^{12}$$
$$-3935263832102474252394312499 2a^{7}c^{12}-1842369256216054559514673152a^{8}c^{12}$$
$$-19565127288930234925056 00a^{9}c^{12}+261744719713921596325888a^{10}c^{12}$$
$$+16950173458410637885 44a^{11}c^{12}-7558979539619708928a^{12}c^{12}$$
$$-95986355680051200a^{13}c^{12}-173511440793600a^{14}c^{12}+470567485440a^{15}c^{12}$$
$$+1031946240a^{16}c^{12}+21100243617267630080507066093823590400c^{13}$$
$$-4282590123912327053620539705073336320ac^{13}$$
$$+241243015910053286321986048595329024a^{2}c^{13}$$
$$+317392344731618976203762659491840a^{3}c^{13}-2929183908692450114853916796518 40a^{4}c^{13}$$

Notes

$$+114573289216159186949297405952 0a^5c^{13}+15582003342453488885546824499 2a^6c^{13}$$

$$+12606295907735858052071424 0a^7c^{13}-375865918549928141389824 00a^8c^{13}$$

$$-227483303554457870008320a^9c^{13}+2811309253983410847744a^{10}c^{13}$$

$$+29506358779073003520a^{11}c^{13}+126753836000870 40a^{12}c^{13}-595955621560320a^{13}c^{13}$$

$$-1493623111680a^{14}c^{13}+10212118911658137817527282903180902 40c^{14}$$

$$-1732970261008864174227792341237760 00ac^{14}$$

$$+742353483105052463481192745376153 6a^2c^{14}+850288560864889303011510190080 00a^3c^{14}$$

$$-8138561013982787524540382576640a^4c^{14}-33817863753060452180243251200a^5c^{14}$$

$$+3356324308027556616644067328a^6c^{14}+226552020635334286835712 00a^7c^{14}$$

$$-508961813714509096550400a^8c^{14}-530776509681837146112 0a^9c^{14}$$

$$+1250579308532701593 6a^{10}c^{14}+28504770220523520 0a^{11}c^{14}+645716386775040a^{12}c^{14}$$

$$-1520294952960a^{13}c^{14}-3810263040a^{14}c^{14}+420722557176183393337197529871155 20c^{15}$$

$$-59126358308763918853535271223296 00ac^{15}+1840114300012465981259924897792 00a^2c^{15}$$

$$+408920457653206529469243300249 6a^3c^{15}-1736361743587106968593309368 32a^4c^{15}$$

$$-1941669908253312615217889280a^5c^{15}+51429915147772080517283840a^6c^{15}$$

$$+6223106225897005677281 28a^7c^{15}-4000985207569851088896a^8c^{15}$$

$$-697931006907462451 20a^9c^{15}-883049898600038 40a^{10}c^{15}+1498402705637376a^{11}c^{15}$$

$$+4381294460928a^{12}c^{15}+14765542741829842722188381965516 80c^{16}$$

$$-16986892330157810895003309834240 0ac^{16}+3568900932399156837419266867200a^2c^{16}$$

$$+126768410949664960828136226816a^3c^{16}-2765896222956318792434057216a^4c^{16}$$

$$-51423387034574054522880000a^5c^{16}+51712852631141119203737 6a^6c^{16}$$

$$+1001255261276762092339 2a^7c^{16}-5485653566916526080a^8c^{16}-557859563864064000a^9c^{16}$$

$$-1607886168784896a^{10}c^{16}+3344648896512a^{11}c^{16}+9779675136a^{12}c^{16}$$

$$+44110245621889122921426714624000c^{17}-409599864054058667677109452800 0ac^{17}$$

$$+505869793920301334896595435 52a^2c^{17}+289549945897109984511000576 0a^3c^{17}$$

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$$-253406339923968 0a^9c^{17}-8891450523648a^{10}c^{17}+11189941481234911447930739097 60c^{18}$$

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$$-1120277974741919465472 0a^5c^{18}-16774719107706126336a^6c^{18}+713308679464550400a^7c^{18}$$

$$+2698661084528640a^8c^{18}-5027903569920a^9c^{18}-17641766912a^{10}c^{18}$$

$$+2400419584854638398399315968 0c^{19}-137288119877417665025802240 0ac^{19}$$

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$$-1535229549819440912138240a^2c^{19}+675445575549134611415040a^3c^{19}$$
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$$-1753523695458598256640ac^{22}-23198555938561720320a^2c^{22}+261866139942912000a^3c^{22}$$
$$+2140357525504000a^4c^{22}-3299239526400a^5c^{22}-19293798400a^6c^{22}$$
$$+798215672745752002560c^{23}-11337550315860787200ac^{23}-175094783960678400a^2c^{23}$$
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$$+157141042200576c^{26}-206561083392ac^{26}-3623878656a^2c^{26}+420906795008c^{27}+536870912c^{28}\Big\}$$
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$$-7045582209742714179916024848294219848613888 0000ac$$
$$+57986056640905171422632284931380541245685760000a^2c$$
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$$-1553366879870044980110829451196a^{14}c^3 + 6030312781884889180813180440a^{15}c^3$$

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$$+4700423290365271996879552 72a^{16}c^3 + 3899011085768023912944972a^{17}c^3$$

$$-29046409548810679611300a^{18}c^3 - 6857230367941804593 60a^{19}c^3$$

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$$+201405062817115850922745699367543699637534 72000c^4$$

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$$+194552315024994763499 52a^{17}c^5 + 14144410152121190656 0a^{18}c^5 - 127350332457444096a^{19}c^5$$

$$-6100734541533632a^{20}c^5 - 26294845344768a^{21}c^5 - 16695369600a^{22}c^5 + 115524864a^{23}c^5$$

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$$+168896a^{24}c^5 + 19510110923162133147721262805422739809233797712c^6$$
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$$+24324667545290419717315188535957913 6a^2c^{14} + 183796980775398087110335943147520a^3c^{14}$$

$$-29712047142197264815364958584832 0a^4c^{14} + 125600213760664562893581189120 0a^5c^{14}$$

$$+159523441825166765944537612288a^6c^{14} + 108127355219080577010892800a^7c^{14}$$

$$-38783653358950851147202560a^8c^{14} - 23308062237038873935872 0a^9c^{14}$$

$$+29149039836618959093 76a^{10}c^{14} + 305058402088176844 80a^{11}c^{14} + 12969253947310080a^{12}c^{14}$$

$$-617239750901760a^{13}c^{14} - 1546966794240a^{14}c^{14} + 9315990005518305972486606859901337 60c^{15}$$

$$-161064951065689450838436224843120640ac^{15} + 70386628283482312624049043359662 08a^2c^{15}$$

$$+773621732762710430740454005800 96a^3c^{15} - 776056311936078732631747330048 0a^4c^{15}$$

$$-307176633090167385827212001 28a^5c^{15} + 32244282054416704490970808 32a^6c^{15}$$

$$+215138609973362716362670 08a^7c^{15} - 491780583986888169881600a^8c^{15}$$

Notes

$$-5109275964306436521984a^9c^{15}+1215108056164741939 2a^{10}c^{15}+275416785707728896a^{11}c^{15}$$
$$+623814359777280a^{12}c^{15}-1469618454528a^{13}c^{15}-3683254272a^{14}c^{15}$$
$$+36354665714647566301947230662164480c^{16}-51922845991552669877786858 06837760ac^{16}$$
$$+16467971023097603941964010278 0928a^2c^{16}+3587988397955062408574398365696a^3c^{16}$$
$$-156106890071336647983849537536a^4c^{16}-1722678005662129516322488320a^5c^{16}$$
$$+4651038506896440457035776 0a^6c^{16}+5596513193004587962859 52a^7c^{16}$$
$$-363592883351530635264 0a^8c^{16}-631458897296700211 20a^9c^{16}-79661106634948608a^{10}c^{16}$$
$$+135792745198 3872a^{11}c^{16}+3970548105216a^{12}c^{16}+1211564592842492994413829071831040c^{17}$$
$$-14133836411940348155650514 0920320ac^{17}+30258694236024879646721315 96288a^2c^{17}$$

Derivation of the result (8):

Substituting $b=-a-57, z=\frac{1}{2}$ in given result (2),we get

$$(2a+57)\ {}_2F_1\left[\begin{matrix} a\ , & -a-57 & ; \ \frac{1}{2} \\ c & & ; \ \end{matrix}\right]$$
$$=a\ {}_2F_1\left[\begin{matrix} a+1\ , & -a-57 & ; \\ c & & ; \ \frac{1}{2} \end{matrix}\right]+(a+57)\ {}_2F_1\left[\begin{matrix} a\ , & -a-56 & ; \ \frac{1}{2} \\ c & & ; \ \frac{1}{2} \end{matrix}\right]$$

Now involving the derived result of Ref[6], we can prove the main result.

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On the Valuation Credit Risk Via Reduced- Form Approach

By Fadugba S. E. & Edogbanya O. H.
Ekiti State University, Nigeria

Abstract- This paper presents the valuation of credit risk via reduced-form approach. Credit risk arises whenever a borrower is expecting to use future cash flows to pay a current debt. It is closely tied to the potential return of investment, the most notable being that the yields on bonds correlate strongly to their perceived credit risk. Credit risk embedded in a financial transaction, is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to decline in creditworthiness of the counter-party to the transaction or perhaps of some third party. Reduced-form approach is known as intensity-based approach. This is purely probabilistic in nature and technically speaking it has a lot in common with the reliability theory. Here the value of firm is not modeled but specifically the default risk is related either by a deterministic default intensity function or more general by stochastic intensity.

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GJSFR-F Classification : MSC 2010: 62P05, 97M30, 91G40

ONTHEVALUATIONCREDITRISKVIAREDUCED-FORMAPPROACH

On the Valuation Credit Risk Via Reduced-Form Approach

Fadugba S. E. [α] & Edogbanya O. H. [σ]

Abstract- This paper presents the valuation of credit risk via reduced-form approach. Credit risk arises whenever a borrower is expecting to use future cash flows to pay a current debt. It is closely tied to the potential return of investment, the most notable being that the yields on bonds correlate strongly to their perceived credit risk. Credit risk embedded in a financial transaction, is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to decline in creditworthiness of the counter-party to the transaction or perhaps of some third party. Reduced-form approach is known as intensity-based approach. This is purely probabilistic in nature and technically speaking it has a lot in common with the reliability theory. Here the value of firm is not modeled but specifically the default risk is related either by a deterministic default intensity function or more general by stochastic intensity.

Keywords: credit risk, risk-neutral valuation formula, reduced-form approach.

I. Introduction

The main emphasis in the intensity-based approach is put on the modelling of the random time of default, as well as evaluating condition expectations under a risk-neutral probability of functionals of the default time and corresponding cash follows. Typically, the random default time is defined as the jump time of some one-jump process.

In recent years, we see a spectacular growth in trading, especially in derivative instruments. There is also an increase complexity of products in the financial markets with the growing complexity and trading size of financial markets, mathematical models have come to play an increasingly important role in financial decision making, especially in the context of pricing and hedging of derivative instruments. Models have become indispensable tools in the development of new financial products and the management of their risks.

The importance of valuation and hedging models in derivatives markets cannot be over-emphasized. The financial risk can therefore be categorized into four (4) types namely: Market risk, Liquidity risk, Operational risk and Credit risk.

The first category of credit risk models are the ones based on the original framework developed by [14]. Using the principles of option pricing [3]. In such a framework, the default process of a company is driven by the value of the company's assets and the risk of a firm's default is

Ref

[3] F. Black and M. Scholes, The Pricing of Options and Coporate Liabilities *Journal Of Political Economy*, **Vol. 81**, (1973), 637-654.

Author α: Department of Mathematical Sciences Ekiti State University Ado Ekiti, Nigeria. e-mail: emmasfad2006@yahoo.com

Author σ: Department of Mathematics Federal University, Lokoja Kogi State, Nigeria. e-mail: helyna4christ@yahoo.com

therefore explicitly linked to the variability of the firm's asset value. The basic intuition behind the Merton model is that; default occurs when the value of a firm's assets (the market value of the firm) is lower than that of its liabilities. [14] derived an explicit formula for risky bonds which can be used both to estimate the probability of default of a firm and to estimate the yield differential between a risk bond and a default-free bonds.

In addition to [14], first generation structure-firm models include [2], they try to refine the original Merton framework by removing one or more of the unrealistic assumptions. [2] introduce the possibility of more complex capital structure with subordinated debts.

Reduced-form models somewhat differ from each other by the manner in which the recovery rate is parameterized. For example, [12] assumed that, at default, a bond would have a market value equals to an exogenous specified fraction of an otherwise equivalent default-free bond. [7] would have a market value equals to an exogenously specified fraction of an otherwise equivalent default-free bond. [8] followed with a model that when market value at default (recovery rate) is exogenously specified, allows for closed-form solutions for term-structure of credit spreads.

For mathematical background, valuation of credit risk, some numerical method for options valuation and stochastic analysis based on the Ito integral, see ([1], [4], [5], [6], [9], [10], [11], [13], [15], [16], [17] and [18]), just to mention few. In this paper we shall consider reduced-form approach for the valuation of credit risk.

II. Reduced-form Model

In this approach, the value of the firm's assets and its capital structure are not model at all, and the credit events are specified in terms of some exogenously specified jump process (as a rule, the recovery rates at default are also given exogenously). We can distinguish between the reduced-form models that are only concerned with the modelling of default time, and that are henceforth referred to as the intensity-based models, and the reduced form models with migrations between credit rating classes called the credit migration models.

The main emphasis in the intensity-based approach is put on the modelling of the random time of default, as well as evaluating condition expectations under a risk-neutral probability of functionals of the default time and corresponding cash follows. Typically, the random default time is defined as the jump time of some one-jump process. As well shall see, a pivotal role in evaluating respective conditional expectations is played by the default intensity process.

Modelling of the intensity process which is also known as the hazard rate process, is the starting point in the intensity approach.

a) Hazard Function

Before going deeper in the analysis of the reduced-form approach, we shall first examine a related technical question. Suppose we want to evaluate a conditional expectation $E_p(1_{\{\tau>s\}}Y|\mathcal{G}_t)$, where τ is a stopping time on a probability space (Ω, G, p), with respect to some filtration $G = (\mathcal{G}_t)_{t\geq 0}$ and Y is an integrable, $\mathcal{G}_s$-measurable random variable for some $s > t$.

In financial applications, it is quite natural and convenient to model the filtration G as $G = FVH$, where h is the filtration that carries full information about default events (that is, events such as

Ref

[14] R.C. Merton , *On The Pricing of Coporate Debt: The Risk structure of Interest Rates,* Journal of Finance **Vol. 29**, (1974), 449-470.

$\{\tau \leq t\}$), whereas the **reference** filtration F carries information about other relevant financial and economic processes, but, typically, it does not carry full information about default event. The first question we address is how to compute the expectation

$$E_p(1_{\{\tau > s\}} Y | \mathcal{G}_t) \tag{2.1}$$

Using the intensity of τ with respect to F.

Notes

b) Hazard Function of a Random Time

We study the case where the reference filtration F is trivial, so that it does not carry any information whatsoever. Consequently, we have that $G = h$. Arguably, this is the simplest possible used in practical financial applications, as it leads to relatively easy calibration of the model.

We start by recalling the notion of a hazard function of a random time. Let τ be a finite, non-negative random time.

Let τ be a finite, non-negative, variable on a probability space $(\Omega, \mathcal{G}, p)$, referred to as the **random time**. We assume that $p\{T = 0\} = 0$ and τ is unbounded;

$$p\{\tau > t\} > 0 \text{ for every } t \in R_+ \tag{2.2}$$

The right continuous cumulative distribution function F of τ satisfies

$$F(t) = p\{\tau \leq t\} < 1 \text{ for every } t \in R_+ \tag{2.3}$$

We also assume that $p\{\tau < \infty\} = 1$ so that τ is a Markov time.

We introduce the right-continuous jump process $H_t = 1_{\{\tau \leq t\}}$ and we write

Lemma 1

For any $\mathcal{G}$-measurable (integrable) random variable Y we have

$$E_p(Y|\mathcal{H}_t) = 1_{\{\tau \leq t\}} E_p(Y|\tau) + 1_{\{\tau > t\}} \frac{E_p(1_{\{\tau > t\}} Y)}{p\{\tau > t\}} \tag{2.4}$$

For any $\mathcal{H}_t$-measurable random variable Y we have

$$Y = 1_{\{\tau \leq t\}} E_p(Y|\tau) + 1_{\{\tau > t\}} \frac{E_p(1_{\{\tau > t\}} Y)}{p\{\tau > t\}} \tag{2.5}$$

that is, $Y = h(\tau)$ for a Borel measurable $h : R \to R$ which is constant on the interval (t, ∞).

The hazard function is introduced through the following definition.

$h = (\mathcal{H}_t)_{t \geq 0}$ to denote the (right continuous and p-completed) filtration generated by the process H. Of course, τ is an h-stopping time.

We shall assume throughout that all random variables and processes that are used in what follows satisfy suitable integrability conditions. We begin with the following simple and important result.

Definition 1: The increasing right-continuous function $\Gamma : R_+ \to R_+$ given by the formula

$$\Gamma(t) = -\ln(1 - F(t), \ \forall\, t \in R_+ \tag{2.6}$$

is called the **hazard function** of a random time τ.
If the distribution function F is an absolutely continuous function, i.e., if we have

$$F(t) = \int_0^t f(u)du$$

for some function $f : R_+ \to R_+$, then we have

$$\begin{aligned} F(t) &= 1 - e^{-\Gamma(t)} \\ &= 1 - e^{-\int_0^t \gamma(u)du} \end{aligned}$$

where we set

$$\gamma(t) = \frac{f(t)}{1 - F(t)}$$

$\gamma : R_+ \to R$ is a non-negative function and it satisfies $\int_0^\infty \gamma(u)du = \infty$.
The function γ is called the **hazard rate or intensity of** τ sometimes, in order to emphasize relevance of the measure p the terminology p**-hazard** rate and p**-intensity** is used. The next result follows from definition 2

Definition 2: The dividend process D of a defaultable contingent claim $(X, C, \tilde{X}, Z, \tau)$, which settles at time T, equals

$$D_t = X^d(T)1_{\{t \geq T\}^{-1}} + \int_{(0,t]} (1 - Hu)diCv + \int_{(0,t]} Z_u dHu$$

D is a process of finite variation and

$$\begin{aligned} \int_{(0,t]} (1 - Hu)dcu &= \int_{(0,t]} 1_{\{\tau > u\}} dcu \\ &= C_\tau - 1_{\{\tau \leq t\}} + C_t 1_{\{\tau > t\}}. \end{aligned}$$

Note that if default occurs at some date t, the promised dividend $C_t - C_t-$, which is due to be paid at this date, is not received by the holder of a defaultable claim. Furthermore, if we set $\tau \wedge t = \min\{\tau, t\}$ then

$$\int_{(0,t]} Z_u dHu = Z_{\tau \wedge t} 1_{\{\tau \leq t\}} = Z_{\{\tau = t\}} \tag{2.7}$$

Remark: In principle, the promised payoff X could be incorporated into the promised dividends process C. However, this would inconvenient, since in practice the recovery rules concerning the promised dividend C as the promised claim X are different, in general. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost, but a strictly positive fraction of the face value is usually received by the bondholder.

Corollary 2: For any $\mathcal{G}$-measurable random variable Y we have

$$E_p(1_{\{\tau > t\}} Y | \mathcal{H}_t) = 1_{\{\tau > t\}} e^{\Gamma(t)} E_p(1_{\{\tau > t\}} Y) \tag{2.8}$$

Corollary 3: Let Y be $\mathcal{H}_\infty$-measurable, so that $Y = h(\tau)$ for some function $h : R_+ \to R$. If the hazard function Γ is continuous then

Notes

$$E_p(Y|\mathcal{H}_t) = 1_{\{\tau\leq t\}}h(\tau) + 1_{\{\tau>t\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}d\Gamma(u) \tag{2.9}$$

If, in addition, the random time τ admits the hazard rate function γ then we have

$$E_p(Y|\mathcal{H}_t) = 1_{\{\tau\leq t\}}h(\tau) + 1_{\{\tau>t\}}\int_t^\infty h(u)\gamma(u)e^{-\int_t^u \gamma(v)dv}du \tag{2.10}$$

Notes

In particular, for any $t \leq s$ we have:

$$p\{\tau > s|\mathcal{H}_t\} = 1_{\{\tau>t\}}e^{-\int_t^s \gamma(v)dv} \tag{2.11}$$

and

$$p\{t < \tau < s|\mathcal{H}_t\} = 1_{\{\tau>t\}}\left(1 - e^{-\int_t^s \gamma(v)dv}\right) \tag{2.12}$$

Lemma 4: The process L, given by the formula

$$\begin{aligned} Lt &:= 1_{\{\tau>t\}}e^{\Gamma(t)} \\ &= \frac{1-H_t}{1-F(t)} \\ &= (1-H_t)e^{\Gamma(t)} \quad \forall\ t \in R_+ \end{aligned}$$

is an h-Martingale.

c) Martingales Associated with Continuous Hazard Function

The h-adapted process of finite variation L given by last formula is an h-martingale (for Γ continuous or a discontinuous function).

We examine further important examples of martingales associated with the hazard function, with the assumption that the hazard function Γ of a random time τ is continuous. Also we assume that the cumulative distribution function F is absolutely continuous function, so that the random time τ admits the intensity function γ, our goal is to establish a martingale characterization of γ.

More specifically, we shall check directly that the process $\hat{M}$, defined as:

$$\begin{aligned} \hat{M}_t &= H_t - \int_0^t Y(u)1_{\{\tau\leq t\}}du \\ &= H_t - \int_0^{t\wedge\tau}\gamma(u)du \\ &= H_t - \Gamma(t\wedge\tau), \end{aligned}$$

follows and h-martingale. To this end,

$$E_p(H_s - H_t|\mathcal{H}_t) = 1_{\{\tau>t\}}\frac{F(s)-F(t)}{1-F(t)}$$

On the other hand, if we denote

$$\begin{aligned} Y &= \int_t^s \gamma(u)1_{\{\tau\leq t\}}du \\ &= \int_{t\wedge\tau}^{s\wedge\tau}\frac{f(u)}{1-F(u)}du \\ &= \ln\frac{1-F(t\wedge\tau)}{1-F(\tau\wedge\tau)} \\ Y &= 1_{\{\tau>t\}}Y. \end{aligned}$$

Let us set $A = \{\tau > t\}$. Using the Fubini's theorem, we obtain

$$E_p(Y|\mathcal{H}_t) = E_p(1_A Y|\mathcal{H}_t) = 1_A \frac{E_p(Y)}{pA} \tag{2.13}$$

This shows that the process $\hat{M}$ follows an h-Martingale.

d) Martingale Hazard Function

Lemma 5: Assume that F (and this also the Hazard function Γ) is continuous function. Then the process

$$M_t = H_t - \Gamma(t \wedge \tau) \tag{2.14}$$

is h-Martingale.

In view of the Martingale in Lemma 5, the following definition is natural.

Definition 3: A function $\bigwedge : R_+ \to R$ is called a martingale hazard function of a random time τ with respect to the filtration if and only if the process

$$H_t - \wedge(t \wedge \tau) \text{ is an } h\text{-martingale.}$$

Remarks: Since the bounded, increasing process H is constant after time τ its compensation is constant after τ as well. This explains why the function $\wedge$ has to be evaluated at time $t \wedge \tau$, rather than at time t. H is thus a bounded h-submartingale.

It happen that the martingale hazard function can be found explicitly. In fact, we have the following.

Proposition 6: The unique Martingale hazard function of τ with respect to the filtration h is the right-continuous increasing function $\wedge$ given by the formula

$$\bigwedge(t) = \int_{[0,t]} \frac{dF(u)}{1 - F(u-)} \tag{2.15}$$

$$= \int_{(0,t]} \frac{dp\{\tau \leq u\}}{1 - p\{\tau > u\}}. \tag{2.16}$$

Observe that the martingale hazard function $\wedge$ is continuous if and only if F is continuous. In this case, we have

$$\wedge(t) = -ln(1 - F(t)) \tag{2.17}$$

We conclude that the Martingale hazard function $\wedge$ coincides with the hazard function Γ if and only if F is a continuous function.

In general, we have

$$e^{-\Gamma(t)} = e^{-\wedge^c(t)} \prod_{0 \leq u \leq t} (1 - \triangle \wedge (u)) \tag{2.18}$$

where

$$\wedge^c(t) = \wedge(t) - \sum_{0 \leq u \leq t} \triangle \wedge (u) \text{ and } \triangle \wedge (u) = \wedge(u) - \wedge(u-) \tag{2.19}$$

Notes

Notes

e) Default Table Bonds : Deterministic Intensity

In order to value a defaultable claim, we need, of course, to specify the unit in which we would like to express all prices. Formally, this is done through a choice of discount factor (a numeraire). For the sake of simplicity, we shall take the savings account

$$B_t = e^{\int_0^t \gamma_r dv} \quad \forall\, t \in [0, T^*] \tag{2.20}$$

as the numraire, where r is the short term interest rate process.

We also postulate that some probability measure Q^* is a martingale measure relative to this nomeraire. This assumption means, in particular, that the price of any contingent claim Y which settles at time T is given as the conditional expectation.

In accordance with our assumption that the reference filtration is trivial, we also assume that:

- the default time τ admits the Q^*-intensity function
- the short-term interest rate $r(t)$ is a deterministic function of time.

In view of the latter assumption, the price at time t of a unit default-free zero-coupon bond of maturity T equals

$$B(t, T) = e^{-\int_t^T r(v)dv} \tag{2.21}$$

In the market practice, the interest rate (more precisely, the yield curve) can be derived from the market price of the zero-coupon bond. In a similar way the hazard rate can be deduced from the prices of the corporate zero-coupon bonds, or from the market values of other actively traded credit derivatives.

In view of our earlier notation for defaultable claims adopted, for the corporate unit discount bond we have $C \equiv 0$ and $X = L = 1$. And since the reference filtration is assumed trivial, we have that $G = h$.

f) Zero Recovery

Consider first a corporate zero-coupon bond with unit face value, the maturity date T, and zero recovery at default (that is, $\tilde{X} = 0$ and $Z \equiv 0$). Finally, the bond can thus be identified with a claim of the form $1_{\{\tau>T\}}$ which settle at T. It is clear that a corporate bond with zero recovery becomes worthless as soon as default occurs. Its time t price is defined as

$$D^0(t, T) = B_t E_{Q^*}(B_T^{-1} 1_{\{\tau>T\}} | \mathcal{H}_t)$$

The price $D^0(t, T)$ can be represented as follows:

$$D^0(t, T) = 1_{\{\tau>t\}} \tilde{D}^0(t, T) \tag{2.22}$$

where $\tilde{D}^0(t, T)$ is the bond's pre-default value, and is given by the formula

$$\tilde{D}^0(t, T) = e^{-\int_t^T (r(v)+\gamma(v))} dv = B(t, T) e^{-\int_t^T \gamma(v)dv} \tag{2.23}$$

f) Hazard Function

According to this convention, we have $\tilde{X} = 0$ and the recovery process Z satisfy $Z_t = \delta$ for some constant recovery rate $\delta \in [0, 1]$. This means that under FRPV the bondholder receives at time of default a fixed fraction of bond's par value.

Using Corollary 3, we check that the pre-default value $\tilde{D}^{\delta}(t,T)$ of a unit corporate zero-coupon bond with FRPV equals

$$\tilde{D}^{\delta}(t,T) = \delta \int_t^T e^{-\int_t^u \tilde{r}(v)dv}\gamma(u)du + e^{-\int_t^\tau \tilde{\sigma}(v)dv} \tag{2.24}$$

where $\tilde{r} = r + \gamma$ is the default risk-adjusted interest rate. Since the fraction of the par value is received at the time of default, in the case of full recovery, that is, for $\delta =$, we do not obtain the equality $\tilde{D}^{\delta}(t,T) = B(t,T)$ but rather the inequality $\tilde{D}^{\delta}(t,T) > B(t,T)$ (at least when the interest rate is strictly positive, so that $B(t,T) < 1$ for $t < T$.

g) Fractional Recovery of Treasury Value (FRTV)

Assume now that $\tilde{X} = 0$ and that the recovery process equal $Z = \delta B(t,T)$. This means that the recovery payoff at the time of default τ represent a fraction of the price of the (equivalent) Treasury bond. The price of a corporate bond which is subject to this recovery scheme equals

$$S_t = B(t,T)(\delta Q^*\{t < \tau \leq T|\mathcal{H}_t\} + Q^*\{\tau > T|\mathcal{H}_t\}).$$

Let us denote by $\hat{D}^{\delta}(t,T)$ the pre-default value of a unit corporate bond subject to the FRTV scheme. Then

$$\hat{D}^{\delta}(t,T) = \int_t^T \delta B(t,T)e^{-\int_t^v \gamma(v)dv}\gamma(u)du + e^{-\int_t^T \tilde{r}(v)dv}$$

or equivalently,

$$\hat{D}^{\delta}(t,T) = B(t,T)\left(\delta\left(1 - e^{-\int_t^T \gamma(v)dv}\right) + e^{-\int_t^T \gamma(v)dv}\right) \tag{2.25}$$

In the case of full recovery, that is, for $\delta = 1$, we obtain $\hat{D}^{\delta}(t,T) = B(t,T)$ as expected.

Remarks. Similar representations can be derived also in the case when the reference filtration F is not trivial, and under the assumption that market risk and credit risk are independent that is:

- the default time admits the F-intensity process γ,
- the interest rate process r is independent of the filtration F.

III. Hazard Processes

In the previous section, it was assumed that the reference filtration F carries no information. However, for practical purposes it is important to study the situation where the reference filtration is not trivial. This section presents some results to this effect.

We assume that a martingale measure Q is given, and examine the valuation of defaultable contingent claims under this probability measure. Note that the defaultable market is incomplete if there are no defaultable assets traded on the market that are sensitive to the same default risk as the defaultable contingent claim we wish to price. Thus, the martingale measure may not be unique.

a) Hazard Process of a Random Time

Let $\tau : \Omega \to R_+$ be a finite, non-negative random variable on a probability space $(\Omega, \mathcal{G}, p)$. Assume $\mathcal{G} = \mathcal{F}_t V \mathcal{H}_t$ for some reference filtration F, so that $G = FVh$.

Notes

Notes

We start by extending some definitions and results to the present framework. We denote $F_t = p\{\tau \leq t|\mathcal{F}_t\}$, so that $G_t = 1 - F_t = p\{\tau > t|\mathcal{F}_t\}$ is the survival process with respect to F. F is a bonded non-negative, F-submartingale. As a submartingale, this process admits a Doob-Meter decomposition as $F_t = Z_t + A_t$ where A is an F-predictable increasing process. Assume, in addition, that $F_t < 1$ for every $t \in R_+$.

Definition 4: The F-hazard process Γ of a random time τ is defined through the equality $1 - F_t = e^{-\Gamma_t}$, that is, $\Gamma_t = \ln G_t$.

Notice that the existence of Γ implies that τ is not an F-stopping time. If the event $\{\tau > t\}$ belongs to the σ-field $\mathcal{F}_t$ for some $t > 0$ then $p\{\tau > t|\mathcal{F}_t\} = 1_{\{\tau > t\}} > 0$ (p-almost surely) and this $\tau = \infty$.

If the hazard process is absolutely continuous, so that $\Gamma_t = \int_0^t \gamma_u du$, for some process γ, then γ is called the F-intensity of τ. Thus the case only if the process Γ is increasing and thus γ is always non-negative. Note that if the reference filtration F is trivial, then the hazard process Γ is the same as the hazard function $\Gamma(\cdot)$. In this case, if T is absolutely continuous, then we have $\gamma_t = \gamma(t)$.

b) Terminal Payoff

The valuation of the terminal payoff $X^d(T)$ is based on the following generalization of Lemma 1.

The question is how to compute $F_p(1_{\{\tau>s\}}Y|\mathcal{G}_t)$ for and $\mathcal{F}_s$-measurable random variable Y?

Lemma 7: For any $\mathcal{G}$-measurable (integrable) random variable Y an arbitrary $s \geq t$ we have

$$E_p(1_{\{\tau>s\}}Y|\mathcal{G}_t) = 1_{\{\tau>t\}} \frac{E_p(1_{\{\tau>s\}}Y|\mathcal{F}_t)}{p\{\tau > t|\mathcal{F}_t\}} \tag{3.1}$$

If, in addition, Y is $\mathcal{F}_s$-measurable then

$$E_p(1_{\{\tau>s\}}Y|\mathcal{G}_t) = 1_{\{\tau>t\}} E_p(e^{\Gamma_t - \Gamma_s}Y|\mathcal{F}_t) \tag{3.2}$$

Assume that Y is $\mathcal{G}_t$-measurable. Then there exists on $\mathcal{F}_t$-measurable random variable $\tilde{Y}$ such that $1_{\{\tau>t\}}Y = 1_{\{\tau>t\}}\tilde{Y}$.

The latter property can be extended to stochastic process: for any G-predictable process X there exists an F-predictable process $\tilde{X}$ such that the equality

$$1_{\{\tau>t\}}X_t = 1_{\{\tau>t\}}\tilde{X}_t \tag{3.3}$$

is valid for every $t \in R_+$, that both processes coincides on the random interval $[0, t)$.

c) Recovery Process

The following extension of Corollary 3 appears to be useful in the valuation of the recovery payoff Z_τ (Note that the payoff occurs at time τ).

Lemma 8: Assume that the hazard process Γ is a continuous, increasing process, and let Z be a bonded, F-predictable process. Then for any $t \leq s$ we have:

$$E_p(Z_\tau 1_{\{t<\tau>s\}}|\mathcal{G}_t) = 1_{\{\tau>t\}} E_p\left(\int_t^s Z_u e^{\Gamma_t - \Gamma_u} d\Gamma u|\mathcal{F}_t\right) \tag{3.4}$$

d) Promised Dividends

To value the promised dividends (that are paid prior to τ, it is convenient to make use of the following result.

Lemma 9: Assume that the hazard process Γ is continuous. Let C be a bounded, F-predictable process of finite variation. Then for event $t \leq s$

$$E_p\left(\int_{(t,s)}(1-H_u)dC_u|\mathcal{G}_t\right)=1_{\{\tau>t\}}E_p\left(\int_{(t,s]}e^{\Gamma_t-\Gamma_u}dC_u|\mathcal{F}_t\right) \tag{3.5}$$

e) Valuation of Defaultable Claims

We assume that τ is given on a filtered probability spaces (Ω, G, Q^*), where $G = FVh$ and $Q^*\tau > t|\mathcal{F}_t\} > 0$ for every $t \in R_+$ so that the F-hazard process Γ of τ under Q^* is well define. A default time τ is thus a G-stopping time, but it is an$F$-stopping time.

The probability Q^* is assumed to be a martingale measure relative to saving account process B, which is given by (3) for some F-progressively measurable process r. In some sense, this probability, and thus also the F-hazard process Γ of τ under Q^*, are given by the market via calibration.

The ex-dividend price S_t of a defaultable claim $(X, C, \tilde{X}, Z, \tau)$ is given by definition 5 below,

Definition 5: For any date $t \in (0, T)$, the ex-dividend price of the defaultable claim $(X, C, \tilde{X}, Z, \tau)$ is given as

$$S_t = B_t E_{p^*}\left(\int_{(t,T]} B_u^{-1} dDu|\mathcal{F}_t\right) \tag{3.6}$$

we always set $S_T = X^d(T)$. With p^* substituted with Q^* and F replaced by G. We postulate in particular, that the processes Z and C are F-predictable, and the random variable X and$\tilde{X}$ are $\mathcal{F}_T$-measurable and $\mathcal{G}_T$-measurable, respectively. Using Lemmas 7, 8, 9 and the fact that the savings account process B is F-adapted, a convenient representation for the arbitrage price of a defaultable claim in terms of the F-hazard process Γ is derived.

Proposition 10: The value process of a defaultable claim $(X, C, \tilde{X}, Z, T)$ admits the following representation for $t < T$

$$S_t = 1_{\{\tau>t\}}\mathcal{G}_t^{-}B_tE_{Q^*}\left(\int_{(t,T]}B_u^{-1}(\mathcal{G}_u dC_u - Z_u d\mathcal{G}_u)|\mathcal{F}_t\right)$$

$$+1_{\{\tau>t\}}\mathcal{G}_t^{-1}B_tE_{Q^*}(\mathcal{G}_T B_T^{-1}X|\mathcal{F}_t) + B_tE_{Q^*}(B_T^{-1}1_{\{\tau>T\}}\tilde{X}|\mathcal{G}_t)$$

If the hazard process Γ is an increasing, continuous process, then

$$\begin{aligned} S_t &= 1_{\{\tau>t\}}B_tE_{Q^*}\left(\int_{(t,T]}B_u^{-1}e^{\Gamma_t-\Gamma_u}(dC_u + Z_u d\Gamma_u)|\mathcal{F}_t\right) \\ &\quad +1_{\{\tau>t\}}B_tE_{Q^*}(B_T^{-1} - e^{\Gamma_t-\Gamma_T}X|\mathcal{F}_t) + B_tE_{Q^*}(B_T^{-1}1_{\{\tau\leq T\}}\tilde{X}|\mathcal{G}_t) \end{aligned}$$

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Corollary 11: Assume that the F-hazard process Γ is a continuous, increasing process. Then the value process of a defaultable contingent claim $(X, C, \tilde{X}, Z, \tau)$ coincides with the value process of a claim $(X, \hat{C}, \tilde{X}, 0, \tau)$, where we set $\hat{C}_t = C_t + \int_0^t Z_u d\Gamma_u$.

f) Defaultable Bonds : Stochastic Intensity

Consider a defaultable zero-coupon bond with the par (face) value L and maturity date T. First, we re-examine the following recovery schemes: the fractional recovery of par value and the fractional recovery of Treasury value. Subsequently, we shall deal with the fractional recovery of pre-default value, but in this section using the stochastic intensity instead of the deterministic intensity used earlier. We assume that τ has the E-intensity γ.

g) Functional Recovery of Par Value

Under this scheme, a fixed fraction of the face value of the bond is paid to the bondholders at the time of default. Formally, we deal here with a defaultable claim $(X, 0, 0, Z, \tau)$, which settle at time T. With the promised payoff $X = L$, where L stands for the bond's face value, and with the recovery process $Z = \delta L$, where $\delta \in [0, 1]$ is a constant. The value at time $t < T$ of the bond is given by the expression

$$S_t = LB_t E_{Q^*}(\delta B_\tau^{-1} 1_{\{t<\tau>T\}} + B_{T\{\tau>T\}}^{-1} | \mathcal{G}_t) \tag{3.7}$$

If τ admits the F-intensity γ, the pre-default value of the bond equals

$$\tilde{D}^\delta(t, T) = L\tilde{B}_t E_{Q^*}\left(\delta \int_t^T \tilde{B}_u^{-1} \gamma_u du + B_T^{-1} | \mathcal{F}_t\right) \tag{3.8}$$

Remarks. The above setup is a special case of the fractional recovery of par value scheme with a general F-predictable recovery process $Z_t = \delta_t$, where the process δ_t satisfies $\delta_t \in [0, 1]$, for every $t \in [0, T]$. A general version of formula (3.8) is given by

$$\tilde{D}^\delta(t, T) = L\tilde{B}_t E_{Q^*}\left(\int_t^T \tilde{B}_u^{-1} \delta_u \gamma_u du + \tilde{B}_T^{-1} | \mathcal{F}_t\right) \tag{3.9}$$

h) Fractional Recovery of Treasury Value

Here, in the case of default, the fixed fraction of the face value is paid to bondholders at maturity date T. A corporate zero-coupon bond is now represented by a defaultable claim $(X, 0, 0, Z, \tau)$ with the promised payoff $X = L$ and the recovery process $(Z_t = \delta LBU, T)$. $B(t, T)$ stands for the price at time t of unit zero-coupon Treasury bond with Maturity T. The corporate bond is now equivalent to a single contingent claim Y, which settle at time T and equals

$$Y = L(1_{\{\tau>T\}} + \delta 1_{\{\tau \le T\}}) \tag{3.10}$$

The price of this claim oat time $t < T$ equals

$$S_t = LB_t E_{Q^*}(B_T^{-1}(\delta 1_{\{\tau \le T\}} + 1_{\{\tau>T\}}) | \mathcal{G}_t) \tag{3.11}$$

or equivalently,

$$S_t = LB_t E_{Q^*}(\delta B_T^{-1} B(t, T) 1_{\{t<\tau \le T\}} + (B_T^{-1} 1_{\{\tau>T\}} | \mathcal{G}_t) \tag{3.12}$$

The pre-default value $\hat{D}^\delta(t, T)$ of defaultable bond with the fractional recovery of Treasury value equals

$$\hat{D}^\delta(t, T) = L\tilde{B}_t E_{Q^*}\left(\delta \int_t^T \tilde{B}_u^{-1} B(u, T) \gamma_u du + \tilde{B}_T^{-1} | \mathcal{F}_t\right) \tag{3.13}$$

Again, the last formula is special case of the general situation where $Z_t = \delta_t$ with some predictable recovery ratio process $\delta_t \in [0,1)$.

i) Fractional Recovery of Pre-default value

Assume that δ_t is some predictable recovery ratio process $\delta_t \in [0,1)$ and let us set $X = L$. The pre-default value of the bond equals

$$D_M^\delta(t,T) = LE_{Q^*}\left(e^{-\int_t^T (r_u+(1-\delta_t)\gamma_u)du} | \mathcal{F}_t\right) \tag{3.14}$$

where

$$\hat{B}_t = \exp\left(\int_0^t (r_u + (1-\delta_u)\gamma_u)du\right) \tag{3.15}$$

j) Choice of a Recovery Scheme

A challenging practical problem is the calibration of statistical properties of both the recovery process δ and the intensity process γ. The empirical evidence strongly suggests that the amount recovered at default is best modelled by the recovery of par value scheme. However, we conclude that recovery concept that specifies the amount recovered as fraction of appropriately discounted par value, that is, the fractional recovery of treasury value, has broader empirical support.

IV. Conclusion

We conclude this section by giving few comments on the reduced-form approach to the modeling of credit risk. The advantages and disadvantages listed below are mainly relative to the alternative structural approach. It also worth noting that some of the disadvantages listed below disappear in the hybrid approach to credit risk modeling.

Advantages

- The specifications of the value-of-the firm process and the default-triggering barrier are not needed.
- The level of the credit risk is reflected in a single quantity: the risk-neutral default intensity.
- The random time of default is an unpredictable stopping time, and thus the default event comes as an almost total surprise.
- The valuation of defaultable claims is rather straightforward. It resembles the valuation of default-free contingent claims in term structure models, through well understood techniques.
- Credit spreads are much easier to quantify and manipulate than in structural models of credit risk. Consequently, the credit spreads are more realistic and risk premia are easier to handle.

Disadvantages

- Typically, current data regarding the level of the firm's assets and the firm's leverage are not taken into account.
- Specific features related to safety covenants and debt's seniority are not easy to handle.

Notes

- All (important) issues related to the capital structure of a firm are beyond the scope of this approach.
- Most practical approaches to Portfolio's credit risk are linked to the value-of-the-firm approach.

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An Wonderful Summation Formula

By Salahuddin, M. P. Chaudhary & Upendra Kumar Pandit
University of Delhi, India

Abstract- The main aim of the present paper is to develop a summation formula associated to recurrence relation and contiguous relation.

Keywords: *gaussian hypergeometric function, contiguous function, recurrence relation, bailey summation theorem and legendre duplication formula.*

GJSFR-F Classification : *MSC 2010: 40A25*

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An Wonderful Summation Formula

Salahuddin [α], M. P. Chaudhary [σ] & Upendra Kumar Pandit [ρ]

Abstract- The main aim of the present paper is to develop a summation formula associated to recurrence relation and contiguous relation.

Keywords: gaussian hypergeometric function, contiguous function, recurrence relation, bailey summation theorem and legendre duplication formula.

I. Introduction

Generalized Gaussian hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lcc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{lcc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lcc} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$ (a-b)\ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \\ c & ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ccc} a+1,\ b & ; & \\ c & ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ccc} a,\ b+1 & ; & \\ c & ; & z \end{array}\right] \tag{2} $$

Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\ \Gamma(z) \tag{3} $$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$ \sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{4} $$

Author α: P.D.M College of Engineering, Bahadurgarh, Haryana, India. e-mails: sludn@yahoo.com, vsludn@gmail.com
Author σ: International Scientific Research and Welfare Organization, New Delhi, India. e-mail: mpchaudhary 2000@yahoo.com
Author ρ: Former Research Scholar, Department of Mathematics, University of Delhi, New Delhi, India.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{5}$$

$$= \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{6}$$

Bailey summation theorem [Prud, p.491(7.3.7.8)]is defined as follows

$${}_2F_1\left[\begin{array}{ccc} a, 1-a & ; & \frac{1}{2} \\ c & ; & 2 \end{array}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} \tag{7}$$

II. Main Result of Summation Formula

$${}_2F_1\left[\begin{array}{cccc} a & , & -a-56 & ; & \frac{1}{2} \\ & c & & ; & 2 \end{array}\right]$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+56}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+56}{2})}\Big\{-392312867195120535999863402667183046656000000a$$

$$+48766743343525871219470737203011367632896000 0a^2$$

$$-228095684410100725800504919763538436030464000a^3$$

$$+51845398391020438102873122501399352168243200a^4$$

$$-5849348254141838819888217025184181557852160a^5$$

$$+239139167257437895025431683734632206915072a^6$$

$$+10501030990169311014798995758581643165056a^7$$

$$-1069742972529123218285391002112425222976a^8$$

$$-58085318091046803792485552717097883 20a^9$$

$$+19796056983140682830910721585938120 80a^{10}+84224486650476063878278393820518 80a^{11}$$

$$-2166063483233160892282874093541420a^{12} - 256905109007543481086056623140 70a^{13}$$

$$+12263148246423700484660696881 35a^{14} + 285817514973478754983529834 70a^{15}$$

$$-16238629638374524099123588 5a^{16} - 114646946374887129233912 70a^{17}$$

$$-100263562465497518965605a^{18}+10264716050363662387 50a^{19}+2558333903764332193 5a^{20}$$

$$+145491821083438830a^{21} - 641582290366875a^{22} - 12414432376710a^{23} - 56984964855a^{24}$$

$$-28493010a^{25} + 557193a^{26} + 1554a^{27} + a^{28}$$

$$+392312867195120546888732853085535207424000000c$$

$$-125100171165232825846363275153841714102272 0000ac$$

$$+10266100026757779137600800574986139556904960000a^2 c$$

$$-3673776415361892765997954663614150459260928 00a^3 c$$

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$$+6565612312684305661616387720217187926736896 0a^4c$$
$$-557893513703000537793563522006426813492428 8a^5c$$
$$+1059333428047590749465102179076548640071 68a^6c$$
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$$-183670760a^{24}c^2-606060a^{25}c^2-420a^{26}c^2$$
$$+663737526871105715595391119524538938818560000c^3$$

$$-9390202016953520247683081151453071652421632 00ac^3$$

$$+4639531918393230082061132583698356731012710 40a^2c^3$$

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$$-2546078890312792426990361625491865728a^9c^3$$

$$-677164638574265602499977247452608 0a^{10}c^3$$

$$+21604779137479947426182967539524 16a^{11}c^3$$

$$+2131582341357374841313914842236 8a^{12}c^3-9058591559586657646193580934 96a^{13}c^3$$

$$-1777030255698958554931422240 0a^{14}c^3+8151994885281839771292568 0a^{15}c^3$$

$$+4858030925277407826158496a^{16}c^3+35071593918886452352568a^{17}c^3$$

$$-26510345378226103680 0a^{18}c^3-5245877955165115840a^{19}c^3-22662383004669408a^{20}c^3$$

$$+63940348656184a^{21}c^3+858226998720a^{22}c^3+2365279280a^{23}c^3+655200a^{24}c^3$$

$$-3640a^{25}c^3+34779944205072284601212819316481841115955200 0c^4$$

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$$+4979325736800669962400953497024a^{12}c^4-363064422727408328770680345 60a^{13}c^4$$

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$$+2400493464507572569083 52a^{16}c^4+3563235410023309363200a^{17}c^4$$

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Notes

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$$+13899204373084896656724988356932257182 72a^{7}c^{3}$$

$$+8517746141292713559471874097972415027 2a^{8}c^{3}$$

$$-198245563721744397142552096613806067 2a^{9}c^{3}$$

$$-977124518402970410116380747390265 60a^{10}c^{3}$$

$$+876737682093207913608858325481824a^{11}c^{3}+7010514209822603687000693132291 2a^{12}c^{3}$$

$$+3383840829336226828048072582 96a^{13}c^{3}-22362118976493259089885304000a^{14}c^{3}$$

$$-3532478876841715428074792 80a^{15}c^{3}+8830512093863101848789 44a^{16}c^{3}$$

$$+6570843447847981759943 2a^{17}c^{3}+52855409396731427200 0a^{18}c^{3}$$

$$-8987234209769129 60a^{19}c^{3}-36805344722097952a^{20}c^{3}-197508253526584a^{21}c^{3}$$

$$-150896374720a^{22}c^{3}+1939821520a^{23}c^{3}+5751200a^{24}c^{3}+3640a^{25}c^{3}$$

$$+17806305327535988146506816383998539061002240 00c^{4}$$

$$-1475429664357373343398965380464176186733363200ac^{4}$$

$$+44947336305025747701071506869622933133721600 0a^{2}c^{4}$$

$$-6203800509345848891463489395703906708042547 2a^{3}c^{4}$$

$$+3359979718808927117872814805675901145776128a^{4}c^{4}$$

$$+5140356225314135066133380951069451558912 0a^{5}c^{4}$$

$$
\begin{gathered}
-1066843289308164827345327471822232547328a^{6}c^{4}\\
+594463229495211349953754456928812851 20a^{7}c^{4}\\
+158887102682735724069778801226502973 44a^{8}c^{4}\\
-51410200604844707875330671393182720a^{9}c^{4}\\
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+6740345981690786878603026073024a^{12}c^{4}+1049894515337405221810537702 40a^{13}c^{4}\\
-904614468961873878081793920a^{14}c^{4}-336812449654932675107340 80a^{15}c^{4}\\
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+96771949538749896798656000258948937925263360a^{2}c^{5}\\
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+17500649307492985336645896877813256159232a^{5}c^{5}\\
-1342913503427012602902038507668300431360a^{6}c^{5}\\
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+20146859117162372335682859790630912a^{9}c^{5}\\
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-18109354408937955271680a^{16}c^{5}-893900850334015 1808a^{17}c^{5}+107629731469702 0416a^{18}c^{5}\\
+7012520037146624a^{19}c^{5}+7807184985600a^{20}c^{5}-68281461248a^{21}c^{5}-220706304a^{22}c^{5}\\
-139776a^{23}c^{5}+1219452440746708186504120196094670817848197 12c^{6}\\
-73262589227235648928947694717825943435476992ac^{6}\\
+15804566062521407536243800589322586553319424a^{2}c^{6}\\
-1418413680398920787753723297190036231946240a^{3}c^{6}\\
+28847132612377292540622016932544310935552a^{4}c^{6}\\
+2905172584429293571275607213323087052800a^{5}c^{6}\\
-110942999852391914486916318472837939200a^{6}c^{6}\\
-3446863471917812820141304100276101120a^{7}c^{6}
\end{gathered}
$$

$$+12268469637363436826537507697147904 0a^{8}c^{6}$$

$$+335788591259770961710812060726886 4a^{9}c^{6}$$

$$-52571581303890717505978219222016a^{10}c^{6}-2032001315070727524982798179840a^{11}c^{6}$$

$$-910841492102055143172920832a^{12}c^{6}+52240691771457827555562444 8a^{13}c^{6}$$

$$+51480286461058156254192 64a^{14}c^{6}-2628117991930866626048 0a^{15}c^{6}$$

$$-717409565359998843904a^{16}c^{6}-34128180646683002 88a^{17}c^{6}+10298713056952320a^{18}c^{6}$$

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$$+6551130579757610085613174488095947489 28a^{4}c^{7}$$

$$+3179623657729477005038035182758928056 32a^{5}c^{7}$$

$$-545994982334205597413156781099532288 0a^{6}c^{7}$$

$$-371811172494493784286861359574859776a^{7}c^{7}$$

$$+44540318736861578852575748357652 48a^{8}c^{7}$$

$$+2788051413510239810041131529256 96a^{9}c^{7}-19582300215009021734626956288 0a^{10}c^{7}$$

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$$-166807531038720a^{18}c^{7}+1105505223680a^{19}c^{7}+3928184832a^{20}c^{7}+2489344a^{21}c^{7}$$

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$$-1357159339491002745191232161282238929960960ac^{8}$$

$$+2045448793115856439599970062538505904455 68a^{2}c^{8}$$

$$-1086100573871317287377201692978012094464 0a^{3}c^{8}$$

$$-1273561213711467634924830311575148298 24a^{4}c^{8}$$

$$+2488473803881860640534757437959659520 0a^{5}c^{8}$$

$$-570445682182050350015994912039485 44a^{6}c^{8}$$

$$-2594249772059345895997313257854566 4a^{7}c^{8}-3496045767970256623765595950694 4a^{8}c^{8}$$

$$+1449576021953783071740949759488 0a^{9}c^{8}+126111171470049320851084478976a^{10}c^{8}$$

$$-3372654774871565025846322176a^{11}c^{8}-572853862817892384546554 88a^{12}c^{8}$$

$$+558283125817994551603 20a^{13}c^{8}+6801429694776171586560a^{14}c^{8}$$

$$+431597097155331993 60a^{15}c^{8}-7377840085392691 2a^{16}c^{8}-163669079388672 0a^{17}c^{8}$$

$$-4857338703360a^{18}c^8+112020480a^{19}c^8+11202048a^{20}c^8$$
$$+3580681795012108766673181582522169950208000c^9$$
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$$-8514852498772950753280a^{13}c^9+9106572374054098944 0a^{14}c^9$$
$$+978854002949222400a^{15}c^9+1864917249761280a^{16}c^9-9949509672960a^{17}c^9$$
$$-39256954880a^{18}c^9-24893440a^{19}c^9+34335917493562665015210536243285133885440c^{10}$$
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$$+1119022732382276855739722073732947116032a^2c^{10}$$
$$-27113940876700284373713963616491274240a^3c^{10}$$
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$$-302744340976836734976a^{13}c^{10}+19176305976885657 6a^{14}c^{10}+110567544502272 00a^{15}c^{10}$$
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$$-720619864788519425470474123689852928a^3c^{11}$$
$$-94421547101751318296316588549734400a^4c^{11}$$
$$+169359887115733371868551910411468 8a^5c^{11}+7704144112355377002971076624384 0a^6c^{11}$$
$$-75934795006743891117887825510400a^7c^{11}-363020664447826532505935216640a^8c^{11}$$
$$-2180926426512110789191680 0a^9c^{11}+72882989589371879679590 40a^{10}c^{11}$$

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$$+541480026871867328593 92a^{11}c^{11} - 3312660889925338398 72a^{12}c^{11}$$

$$-5124375210119553024a^{13}c^{11} - 12317896466104320a^{14}c^{11} + 54993447075840a^{15}c^{11}$$

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$$-2937793198538188339648 9216a^{2}c^{18} + 25181305173728125313351680a^{3}c^{18}$$

$$+363836969115629020774 40a^{4}c^{18} - 547542200447729270784 0a^{5}c^{18}$$

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$$-44104417280a^{9}c^{18} - 8820883456a^{10}c^{18} + 1200209792427319199199657984 0c^{19}$$

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$$+30336414566559839682 56a^{3}c^{20}+26335287904781205504a^{4}c^{20}-2170588032820838 40a^{5}c^{20}$$

$$-1932594283282432a^{6}c^{20} + 44568674304a^{7}c^{20} + 11142168576a^{8}c^{20}$$

$$+324665907950852599971840 0c^{21} - 86773650318458492026880ac^{21}$$

$$-13934970090459307704 32a^{2}c^{21}+20788815106731409408a^{3}c^{21}+2300105024811827 20a^{4}c^{21}$$

$$-565491584204800a^{5}c^{21} - 6668322603008a^{6}c^{21} - 4244635648a^{7}c^{21}$$

$$+400570592519651957145 60c^{22}-71444889622330474496 0ac^{22}-14006605481099919360a^{2}c^{22}$$

$$+93964840258764800a^{3}c^{22}+1254936176230400a^{4}c^{22}-28940697600a^{5}c^{22}-9646899200a^{6}c^{22}$$

$$+39910783637287600128 0c^{23} - 4335073295272509440ac^{23} - 9908277655633920 0a^{2}c^{23}$$

$$+223513621299200a^{3}c^{23}+3951034368000a^{4}c^{23}+2516582400a^{5}c^{23}+312992021046362112 0c^{24}$$

$$-17703518810931200ac^{24} - 472876318720000a^{2}c^{24} + 10905190400a^{3}c^{24} + 5452595200a^{4}c^{24}$$

$$+1859040086851584 0c^{25} - 38742215622656ac^{25} - 1368819499008a^{2}c^{25} - 872415232a^{3}c^{25}$$

$$+78570521100288c^{26} - 1811939328ac^{26} - 1811939328a^{2}c^{26} + 210453397504c^{27}$$

Notes

$$+134217728ac^{27}+268435456c^{28}\Big\}\Big] \tag{8}$$

Notes

Derivation of the result (8):

Substituting $b=-a-56, z=\frac{1}{2}$ in given result (2),we get

$$(2a+56)\ {}_2F_1\left[\begin{array}{ccc} a & , & -a-56 & ; & \frac{1}{2} \\ & c & & ; & \end{array}\right]$$

$$=a\ {}_2F_1\left[\begin{array}{ccc} a+1 & , & -a-56 & ; & \frac{1}{2} \\ c & & & ; & \end{array}\right]+(a+56)\ {}_2F_1\left[\begin{array}{ccc} a & , & -a-55 & ; & \frac{1}{2} \\ & c & & ; & \end{array}\right]$$

Now using the result of salahuddin et al [Salahuddin et al, p.76-90(8)], we can prove the main result.

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Parallel Surfaces Satisfying the Properties of Ruled Surfaces in Minkowski 3-Space

By Yasin Ünlütürk & Cumali Ekici
Kirklareli University & Eskisehir Osmangazi University, Turkey

Abstract- In this study, some properties of timelike parallel surfaces have been investigated in Minkowski 3-space. The main motivation of this work is to study the conditions under which parallel surfaces of timelike ruled surfaces with timelike ruling become timelike ruled surfaces. Also some characterizations of ruled surfaces such as distribution parameter, striction curve and orthogonal trajectory have been given for timelike parallel ruled surfaces.

Keywords: Developable timelike surface, striction curve, orthogonal trajectory, timelike parallel surface, timelike parallel ruled surface with timelike ruling, timelike ruled surface with timelike ruling.

GJSFR-F Classification : *MSC 2010: 53A05, 53B25, 53B30*

PARALLELSURFACESSATISFYINGTHEPROPERTIESORULEDSURFACESINMINKOWSKI3-SPAC

Strictly as per the compliance and regulations of :

Notes

[2] Y.J. Chen, B. Ravani, Offset surface generation and contouring in computer-aided design, J. Mech. Trans. and Auto. in Design: ASME Transactions 109 (3) (1987) 133–142.

Parallel Surfaces Satisfying the Properties of Ruled Surfaces in Minkowski 3-Space

Yasin Ünlütürk [α] & Cumali Ekici [σ]

Abstract- In this study, some properties of timelike parallel surfaces have been investigated in Minkowski 3-space. The main motivation of this work is to study the conditions under which parallel surfaces of timelike ruled surfaces with timelike ruling become timelike ruled surfaces. Also some characterizations of ruled surfaces such as distribution parameter, striction curve and orthogonal trajectory have been given for timelike parallel ruled surfaces.

Keywords: Developable timelike surface, striction curve, orthogonal trajectory, timelike parallel surface, timelike parallel ruled surface with timelike ruling, timelike ruled surface with timelike ruling.

I. Introduction

Parallel surfaces and ruled surfaces are some of the main topics in both the classical and the modern differential geometry. These surfaces have many applications especially in physics and engineering [2,8,17,18,19]. It is possible to see many interesting papers which study on these two fields in terms of differential geometry such as [3,4,6,7,9,10,11,12,15,17,21,23,24].

If we mention briefly these studies: Craig studied to find the parallel of ellipsoid [4]. Çöken et al. studied parallel timelike ruled surfaces with timelike rulings in Dual space D_1^3 [3]. Görgülü and Çöken gave the dupin indicatrices for parallel pseudo Euclidean hypersurfaces in semi Euclidean space R_1^n [9]. Eisenhart studied parallel surfaces within a chapter of his book [6]. Nizamoğlu investigated a parallel ruled surface as a one-parameter curve using E. Study theorem and obtained some geometric characterizations of such a surface [15]. Güneş studied the relations among curves under parallel map preserving the connection [11]. Park examined offsets of ruled surfaces in Euclidean space [17]. Küçük and Gürsoy researched Bertrand offsets of trajectory ruled surfaces in view of their integral invariants [12]. Tarakçı and Hacısalihoğlu dealt with parallel surfaces as surfaces at a constant distance from the edge of regression on a surface in the general sense, [21]. Ekici and Çöken gave the parallel timelike ruled surface with a timelike ruling and its geometric invariants in terms of the main surface in Dual space D_1^3 [7]. Ünlütürk studied parallel ruled surfaces in Minkowski 3-space in detail [23, 24].

In this study, we have given some properties of timelike parallel surfaces in Minkowski 3-space. We have also studied the conditions under which parallel

Author α: Department of Mathematics Kirklareli University 39060, Kirklareli -Turkey. e-mail: yasinunluturk@klu.edu.tr
Author σ: Eskisehir Osmangazi University Department of Mathematics-Computer 26480, Eskisehir – Turkey. e-mail: cekici@ogu.edu.tr

surfaces of timelike ruled surfaces with timelike ruling become timelike ruled surfaces. Furthermore we obtained some characterizations of ruled surfaces such as distribution parameter, striction curve and orthogonal trajectory have been given for timelike parallel ruled surfaces.

II. Preliminaries

Let E_1^3 be the three-dimensional Minkowski space, that is, the three-dimensional real vector space E^3 with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2$$

where $\mathbf{x} = (x_1, x_2, x_3)$ denotes the canonical coordinates in E^3. A vector $\mathbf{x}$ of E_1^3 is called to be spacelike, timelike and lightlike, respectively, if it satisfies $< \mathbf{x}, \mathbf{x} >> 0$ or $\mathbf{x} = \mathbf{0}$, $< \mathbf{x}, \mathbf{x} >< 0$, $< \mathbf{x}, \mathbf{x} >= 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or null vector in E_1^3 is said to be causal. The norm of $\mathbf{x} \in E_1^3$ is defined by $\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, then the vector $\mathbf{x}$ is called a spacelike or timelike unit vector if it satisfies $< \mathbf{x}, \mathbf{x} >= 1$ or $< \mathbf{x}, \mathbf{x} >= -1$, respectively. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [16]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of E_1^3, the inner product is the real number $< \mathbf{x}, \mathbf{y} >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [14].

A one-parameter family of lines $\{\alpha(u), X(u)\}$, the parameterized surface

$$\varphi(u, v) = \alpha(u) + vX(u), \qquad u \in I, \qquad v \in R \tag{1}$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$ where $\alpha(u)$ is a point in E_1^3 and a vector $X(u) \in E_1^3$. The normal vector of the surface is denoted by N. Let us take timelike ruled surface φ with a spacelike directrix and timelike ruling. So the system $\{T, X, N\}$ establishes an orthonormal frame such that $T = \alpha'(u)$. Therefore

$$< T, T >=< X, X >= -1, \; < N, N >= 1 \text{ and } < D_T N, N >= 0. \tag{2}$$

Derivative equations of the frame $\{T, X, N\}$ are

$$D_T T = aX + bN, \quad D_T X = aT + cN, \quad D_T N = -bT + cX. \tag{3}$$

Also the cross products of these vectors are as follows:

$$T \wedge X = N, \; T \wedge N = X, \; X \wedge N = T. \tag{4}$$

The distribution parameter is expressed as

$$\lambda = \frac{\det(\alpha', X, X')}{|X'|^2} \tag{5}$$

where, as usual, (α', X, X') is a short for $\langle \alpha' \wedge X, X' \rangle$ [22].

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Theorem 2.1. A surface in Minkowski 3-space is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal vector on the surface is a spacelike vector [1].

The coefficients which belong to the parametric equation of the surface given in (1) are as follows:

$$E = \langle X_u, X_u \rangle, \qquad F = \langle X_u, X_v \rangle, \qquad G = \langle X_v, X_v \rangle,$$

where the differentiable functions $E, F, G : U \to R$ are called the coefficients of the first fundamental form I. So the first fundamental form is

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

The following differentiable functions

$$\begin{aligned} l &= -\langle X_u, N_u \rangle = \langle N, X_{uu} \rangle \\ m &= -\langle X_u, N_v \rangle = -\langle X_v, N_u \rangle = \langle N, X_{uv} \rangle \\ n &= -\langle X_v, N_v \rangle = \langle N, X_{vv} \rangle \end{aligned}$$

are called the coefficients of the second fundamental form II. So the second fundamental form is

$$II = ldu^2 + 2mdudv + ndv^2,$$

[14].

Theorem 2.2. Up to Lorentzian motions, a ruled surface is uniquely determined by the quantities

$$Q = \langle \alpha', X \wedge X' \rangle, \quad J = \langle X, X'' \wedge X' \rangle, \quad F = \langle \alpha', X \rangle \tag{6}$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [13].

Theorem 2.3. The Gauss K and mean H curvatures of a timelike ruled surface φ with timelike ruling in terms of the parameters Q, J, F, D in E_1^3 are obtained as

$$K = -\frac{Q^2}{D^4} \quad \text{and} \quad H = \frac{1}{2D^3}(-QF - Q^2J - Jv^2 + vQ'), \tag{7}$$

where $D = \sqrt{-\varepsilon Q^2 + \varepsilon v^2}$, respectively [5].

Theorem 2.4. The parameter curves are lines of curvature if and only if $F = m = 0$, where the coefficients F and m belong, respectively, to the first and second fundamental forms in E_1^3 [14].

Definition 2.5. Let M and M^r be two surfaces in E_1^3. The function $f : M \to M^r$, $f(p) = p + r\mathbf{N}_p$ is said to be the parallelization function between M and M^r and furthermore M^r is said to be a parallel surface to M in E_1^3 where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [9].

Theorem 2.6. Let M be a surface and M^r be a parallel surface of M in E_1^3. Let $f : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$, we have the following relations:

1. $f_*(X) = X + rS(X)$
2. $S^r(f_*(X)) = S(X)$
3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X) \tag{8}$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction of X [9].

Definition 2.7. Let M be a hypersurface of $\overline{M}$- manifold and M^r be a parallel surface of M in E_1^3. If σ is a curve passing through p on M and T is the tangent vector field of σ on M, then $\sigma^r = f \circ \sigma$ is a curve passing through a point $f(p)$ on M^r and $f_*(T) \in T_{f(p)}M^r$ is a tangent of σ^r at $f(p)$. The connection D^r belongs to the parallel surface M^r of M and the vector $\mathbf{N}^r$ is the unit normal vector of M^r, where $\langle \mathbf{N}^r, \mathbf{N}^r \rangle = \varepsilon = \pm 1$. Therefore the Gauss equation is as follows:

$$\overline{D}_{f_*(T)} f_*(T) = D^r_{f_*(T)} f_*(T) - \varepsilon \langle S^r(f_*(T)), f_*(T) \rangle \mathbf{N}^r \tag{9}$$

[11, 16].

Theorem 2.8. Let $\varphi(u,v)$ be a surface in E_1^3 with the normal vector N. Then the shape operator S of φ is given in terms of the basis $\{\varphi_u, \varphi_v\}$ by

$$\begin{aligned} -S(\varphi_u) &= N_u = \frac{mF - lG}{EG - F^2}\varphi_u + \frac{lF - mE}{EG - F^2}\varphi_v \\ -S(\varphi_v) &= N_v = \frac{nF - mG}{EG - F^2}\varphi_u + \frac{mF - nE}{EG - F^2}\varphi_v \end{aligned} \tag{10}$$

[20].

III. Timelike Parallel Surfaces

The representation of points are obtained on M^r by using the representations of points on M. Let φ be the position vector of a point P on M and φ^r be the position vector of a point $f(P)$ on the parallel surface M^r. Then $f(P)$ is at a constant distance r from P along the normal to the surface M. Therefore the parameterization for M^r is given by

$$\varphi^r(u,v) = \varphi(u,v) + r\mathbf{N}(u,v) \tag{11}$$

where r is a constant scalar and $\mathbf{N}$ is the unit normal vector field on M. In E_1^3, let the parallel surface of a timelike surface $\varphi(u,v)$ be as given in (11), is defined in E_1^3 as where $\mathbf{N}$ is the unit normal vector on the surface $\varphi(u,v)$ such that $\langle \mathbf{N}, \mathbf{N} \rangle = 1$ and $r \in R$. Fundamental forms' coefficients of timelike parallel surfaces can be given relative to ones of timelike surface as follows:

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$$\begin{array}{ll} E^r = E - 2rl + r^2 \langle \mathbf{N}_u, \mathbf{N}_u \rangle, & l^r = l - r \langle \mathbf{N}_u, \mathbf{N}_u \rangle \\ F^r = F - 2rm + r^2 \langle \mathbf{N}_u, \mathbf{N}_v \rangle, & m^r = m - r \langle \mathbf{N}_u, \mathbf{N}_v \rangle \\ G^r = G - 2rn + r^2 \langle \mathbf{N}_v, \mathbf{N}_v \rangle, & n^r = n - r \langle \mathbf{N}_v, \mathbf{N}_v \rangle, \end{array} \tag{12}$$

where E, F, G, l, m, n, are fundamental forms' coefficients of the surface φ, and E^r, F^r, G^r, l^r, m^r, n^r, are fundamental forms' coefficients of the parallel surface φ^r[23].

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To get explicit formulas for H^r and K^r, we work in a parallel surface M^r. Let $\varphi^r(u, v)$ be a timelike parallel surface with first and second fundamental forms

$$E^r du^2 + 2F^r dudv + G^r dv^2 \quad \text{and} \quad l^r du^2 + 2m^r dudv + n^r dv^2,$$

respectively. Define 2×2 matrices $\mathcal{F}_{I^r}$ and $\mathcal{F}_{II^r}$ by

$$\mathcal{F}_{I^r} = \begin{bmatrix} E^r & F^r \\ F^r & G^r \end{bmatrix}, \quad \mathcal{F}_{II^r} = \begin{bmatrix} e^r & f^r \\ f^r & g^r \end{bmatrix},$$

also the matrix of S_p^r, with respect to the basis $\{\varphi_u^r, \varphi_v^r\}$ of T_pM^r, is

$$\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}. \tag{13}$$

Definition 3.1. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Let $\mathbf{N}^r$ and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian and mean curvature functions are defined as $K^r : M^r \to R, K^r(f(P)) = \det S_{f(P)}^r$ and $H^r : M^r \to R, H^r(f(P)) = \dfrac{1}{2} \operatorname{tr} S_{f(P)}^r$ where $P \in M$, $f(P) \in M^r$ and $\langle \mathbf{N}, \mathbf{N} \rangle = 1$, respectively.

Theorem 3.2. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Let $\mathbf{N}^r$ and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian K^r and mean H^r curvatures are given in terms of the coefficients of the fundamental forms I^r and II^r as follows:

$$K^r = \frac{l^r n^r - m^{r2}}{E^r G^r - F^{r2}} \quad \text{and} \quad H^r = \frac{l^r G^r - 2m^r F^r + n^r E^r}{2(E^r G^r - F^{r2})}, \tag{14}$$

respectively.

Proof. Since $S_p^r = \mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}$ for the matrices $\mathcal{F}_{I^r}$ and $\mathcal{F}_{II^r}$, using the Definition 3.1 and the equation (13) we get

$$K^r = \det(\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}) = \frac{\det \mathcal{F}_{II^r}}{\det \mathcal{F}_{I^r}} = \frac{l^r n^r - m^{r2}}{E^r G^r - F^{r2}}.$$

For the mean curvature H^r, first the matrix $\mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r}$ is obtained as

$$\begin{aligned} \mathcal{F}_{I^r}^{-1} \mathcal{F}_{II^r} &= \frac{1}{E^r G^r - F^{r2}} \begin{bmatrix} G^r & -F^r \\ -F^r & E^r \end{bmatrix} \begin{bmatrix} l^r & m^r \\ m^r & n^r \end{bmatrix} \\ &= \frac{1}{E^r G^r - F^{r2}} \begin{bmatrix} l^r G^r - m^r F^r & m^r G^r - n^r F^r \\ m^r E^r - l^r F^r & n^r E^r - m^r F^r \end{bmatrix}. \end{aligned}$$

From Definition 3.1, the mean curvature is found as

$$H^r = \tfrac{1}{2}\mathrm{tr}(\mathcal{F}_{I^r}^{-1}\mathcal{F}_{II^r}) = \frac{l^r G^r - 2m^r F^r + n^r E^r}{2(E^r G^r - F^{r2})}.$$

Lemma 3.3. Let M be a timelike surface and M^r be its parallel surface in E_1^3. The surface M is a timelike one if and only if the surface M^r is a timelike parallel surface.

Proof. ($\Rightarrow$): If M is a timelike surface, then by Theorem 2.1, the unit normal vector $\mathbf{N}$ of M has to be as follows

$$\langle \mathbf{N}_P, \mathbf{N}_P \rangle > 0. \tag{15}$$

Between the unit normal vectors of the surfaces M and M^r,there is the following relation:

$$\mathbf{N}_P = \mathbf{N}^r_{f(P)}. \tag{16}$$

By substituting (16) into (15), we get

$$\left\langle \mathbf{N}^r_{f(P)}, \mathbf{N}^r_{f(P)} \right\rangle > 0. \tag{17}$$

The inequality (17) means that M^r is a timelike surface in accordance with Theorem 2.1.

($\Leftarrow$): If M^r is a timelike parallel surface, in accordance with Theorem 2.1, we get

$$\left\langle \mathbf{N}^r_{f(P)}, \mathbf{N}^r_{f(P)} \right\rangle > 0. \tag{18}$$

By substituting (16) into (18), we have

$$\langle \mathbf{N}_P, \mathbf{N}_P \rangle > 0. \tag{19}$$

The inequality (19) means that M^r is a timelike surface in accordance with Theorem 2.1.

Theorem 3.4. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Then we have

$$K^r = \frac{K}{1 + 2rH + r^2 K} \quad \text{and} \quad H^r = \frac{H + rK}{1 + 2rH + r^2 K} \tag{20}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [23].

Corollary 3.5. Let M be a timelike surface and M^r be a parallel surface of M in E_1^3. Then we have

$$K = \frac{K^r}{1 - 2rH^r + r^2 K^r} \quad \text{and} \quad H = \frac{H^r - rK^r}{1 - 2rH^r + r^2 K^r} \tag{21}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [23].

Theorem 3.6. Let M be a timelike surface and M^r be a its parallel surface in E_1^3. The curves on the timelike parallel surface M^r which correspond to the lines of curvature on the timelike surface M are also the lines of curvature.

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Proof. If the lines of curvature on M are chosen as parameter curves, then in accordance with Theorem 2.4, we have

$$F = m = 0. \tag{22}$$

It suffices to see $F^r = m^r = 0$ such that the curves on M^r which correspond to the lines of curvature on M are the lines of curvature. The parametric representation of M^r is as in (11). From Weingarten equations given in (10), by using the equation (22), we get

$$N_u = -\frac{l}{E}\varphi_u \quad \text{and} \quad N_v = -\frac{n}{G}\varphi_v. \tag{23}$$

If the values of F^r and m^r are used in the equations (22) and (23), then we have

$$F^r = F - 2rm + r^2 \langle N_u, N_v \rangle = r^2 \langle N_u, N_v \rangle = r^2 \frac{ln}{EG} \langle \varphi_u, \varphi_v \rangle = 0 \tag{24}$$

and

$$m^r = m - r \langle N_u, N_v \rangle = -r \langle N_u, N_v \rangle = -r \frac{ln}{EG} F = 0. \tag{25}$$

In (24) and (25), it is seen that the coefficients F^r and m^r vanish. So the curves on M^r are the lines of curvature.

IV. Timelike Parallel Ruled Surfaces

Theorem 4.1. Let M be a timelike ruled surface with a timelike ruling and M^r be a parallel surface of M in E_1^3. A parallel surface of a timelike developable ruled surface is again a timelike ruled surface.

Proof. Let the timelike ruled surface M with a timelike ruling be given as

$$\varphi(u,v) = \alpha(u) + vX(u), \;\; \langle \alpha', \alpha' \rangle = 1, \;\; \langle X, X \rangle = -1, \;\; \langle X', X' \rangle = 1. \tag{26}$$

We get its normal vector as follows:

$$N = \alpha' \wedge X + vX' \wedge X. \tag{27}$$

For the normal vector of a timelike developable ruled surface which is constant along its ruling and is independent from the parameter v, we infer that the expressions $\alpha' \wedge X$ and $X' \wedge X$ in (27) are linearly dependent, that is

$$\alpha' \wedge X = \lambda X' \wedge X,$$

for $\lambda \in \mathbb{R}$. Also, from (26), we obtain the normal vector of the surface M as

$$N = (\lambda + v) X' \wedge X. \tag{28}$$

In the end, the unit normal vector of the surface M is as follows

$$\mathbf{N} = X' \wedge X. \tag{29}$$

We get the parallel surface of the ruled surface $\varphi(u,v) = \alpha(u) + vX(u)$ as

$$\varphi^r(u,v) = \alpha(u) + rX'(u) \wedge X(u) + vX(u). \tag{30}$$

We call this surface obtained in (30) as timelike parallel ruled surface. The ruling of timelike parallel ruled surface is

$$f_*(X) = f_*(T) \wedge N^r = (T - rbT) \wedge N = (1 - rb)X. \tag{31}$$

And also we find

$$f \circ \alpha(u) = \alpha(u) + rN(u) = \alpha(u) + rX'(u) \wedge X(u). \tag{32}$$

The coefficient n^r of the second fundamental form of the surface M^r is calculated as

$$n^r = -\langle \varphi_v^r, \mathbf{N}_v \rangle = -\langle X, 0 \rangle = 0.$$

The drall of timelike parallel ruled surface is obtained as

$$P^r =< \frac{df \circ \alpha}{du}, f_*'(X) \wedge f_*(X) > . \tag{33}$$

From (33), we find

$$P^r \quad = \quad \langle \alpha' + rX'' \wedge X, (1 - rb)^2 X' \wedge X \rangle = 0. \tag{34}$$

Finally, timelike parallel ruled surface given in (30) is a developable ruled surface.

The coefficients of the first I^r and second II^r fundamental forms for timelike parallel ruled surface M^r parameterized in (30) are given by

$$\begin{aligned} E^r \quad = \quad & \langle \alpha', \alpha' \rangle + 2r \langle \alpha', X'' \wedge X \rangle + 2v \langle \alpha', X' \rangle + r^2 \langle X'' \wedge X, X'' \wedge X \rangle \\ & + 2rv \langle X', X'' \wedge X \rangle + v^2 \langle X', X' \rangle . \end{aligned} \tag{35}$$

Since $\langle X', X' \rangle = 1$ and $\langle X'', X' \rangle = 0$, X'' lies in the plane spanned by the vectors X and $X' \wedge X$. Therefore

$$X'' = wX + yX' \wedge X, \tag{36}$$

where $w, y \in \mathbb{R}$. By using (36), we have

$$X'' \wedge X = (wX + yX' \wedge X) \wedge X = (yX' \wedge X) \wedge X = -yX'. \tag{37}$$

Substituting (37) into (35), the coefficients E^r, F^r and G^r are found as

$$\begin{aligned} & E^r = 1 + (ry - v)^2 \\ & F^r = \langle \varphi_u^r, \varphi_v^r \rangle = \langle \alpha' + rX'' \wedge X + vX', X \rangle = \langle \alpha', X \rangle \\ & G^r = \langle \varphi_v^r, \varphi_v^r \rangle = \langle X, X \rangle = -1. \end{aligned}$$

Also, the normal vector of the surface is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

Additionally, the coefficients l^r, m^r and n^r of the second fundamental form are computed as follows:

$$\begin{aligned} & l^r = -\langle \varphi_u^r, N_u^r \rangle = -\langle \alpha', \alpha'' \wedge X \rangle + \langle X', \alpha'' \wedge X \rangle (ry - v) + v^2 y - rvy^2 \\ & m^r = \langle -\varphi_u^r, N_v^r \rangle = -\langle \alpha', X' \wedge X \rangle \\ & n^r = -\langle \varphi_v^r, N_v^r \rangle = -\langle X, X' \wedge X \rangle = 0. \end{aligned}$$

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Corollary 4.2. Let M^r be a timelike parallel ruled surface. Then the causal characters of the directrix and the ruling of M^r are a spacelike curve and a timelike vector, respectively.

Proof. The ruling of timelike parallel ruled surface M^r given in (30) is timelike since $\langle X, X\rangle = -1$. The causal character of the directrix is seen by the following computation:

$$\left\langle \frac{df\circ\alpha(u)}{du}, \frac{df\circ\alpha(u)}{du}\right\rangle = \langle \alpha' + rX'' \wedge X, \alpha' + rX'' \wedge X\rangle . \qquad (38)$$

By using $\langle X', X'\rangle = 1$ and $\langle X', X''\rangle = 0$, from (38), the following result is obtained that

$$\left\langle \frac{df\circ\alpha(u)}{du}, \frac{df\circ\alpha(u)}{du}\right\rangle = 1 + r^2y^2 > 0 \qquad (39)$$

which means that the causal character of the directrix is spacelike.

Theorem 4.3. Let M^r be a timelike parallel ruled surface with timelike ruling and $f_*(T)$, $f_*(X)$ and N^r be the tangent vector field of the directrix, tangent vector field of the ruling and the normal vector field of the surface M^r, respectively. Hence we have

$$f_*(T) \wedge \mathbf{N}^r{=}f_*(X), \quad f_*(T) \wedge f_*(X){=}(1\text{-}rb)\mathbf{N}^r, \quad f_*(X) \wedge \mathbf{N}^r{=}f_*(T). \qquad (40)$$

Proof. Frenet equations for timelike developable ruled surface M are obtained in (3) by taking $c = 0$. And also the cross products of the unit vectors T, X, $\mathbf{N}$ for timelike developable ruled surface M are as in (4). By means of these information, we have the following results:

$$\begin{aligned} &f_*(T) \wedge \mathbf{N}^r = (T - rbT) \wedge \mathbf{N} = (1 - rb)X = f_*(X) \\ &f_*(T) \wedge f_*(X){=}(T\text{-}rbT) \wedge (X\text{-}rbX) = (1\text{-}rb)^2\mathbf{N} = (1\text{-}rb)^2\mathbf{N}^r \\ &f_*(X) \wedge \mathbf{N}^r = (1 - rb)X \wedge \mathbf{N} = (1 - rb)T = f_*(T). \end{aligned} \qquad (41)$$

Theorem 4.4. The vectors $f_*(T)$, $f_*(X)$ and $\mathbf{N}^r$ for timelike parallel ruled surface M^r are spacelike, timelike and spacelike vectors, respectively, while the unit vectors T, X, $\mathbf{N}$ for timelike developable ruled surface M with timelike ruling are spacelike, timelike and spacelike vectors, respectively.

Proof. The unit normal vector $\mathbf{N}^r$ of the timelike parallel ruled surface M^r is a spacelike vector because

$$\langle \mathbf{N}^r, \mathbf{N}^r\rangle = \langle \mathbf{N}, \mathbf{N}\rangle = 1.$$

The tangent vector field of the directrix is a spacelike vector since

$$\langle f_*(T), f_*(T)\rangle = (1 - rb)^2 > 0.$$

From (31), the vector $f_*(X)$ is a timelike vector because of

$$\langle f_*(X), f_*(X)\rangle = \langle (1 - rb)X, (1 - rb)X\rangle = -(1 - rb)^2 = -1 < 0.$$

We get the position vector of the striction curve on the timelike parallel ruled surface M^r as

$$\overrightarrow{O\gamma} = \overrightarrow{Of \circ \alpha} + \overrightarrow{\theta f_*(X)}. \tag{42}$$

Using $f_*(X) = X^r$ in (42), we have the striction curve as

$$(u) = f \circ \alpha(u) + \theta X^r(u) \text{ and } \theta = \theta(u). \tag{43}$$

By (43), we obtain the value θ as follows:

$$\theta = -\frac{\left\langle \dfrac{df \circ \alpha}{du}, \dfrac{dX^r}{du} \right\rangle}{\left\langle \dfrac{dX^r}{du}, \dfrac{dX^r}{du} \right\rangle} = -\frac{\langle \alpha', X' \rangle + r \langle X'' \wedge X, X' \rangle}{(1 - rb) \langle X', X' \rangle}. \tag{44}$$

Hence using (43) and (44), we find the striction curve as

$$(u) = \alpha(u) + rX'(u) \wedge X(u) - \frac{\langle \alpha', X' \rangle + r \langle X'' \wedge X, X' \rangle}{\langle X', X' \rangle} X. \tag{45}$$

After some arrangements in (45), it becomes

$$(u) = \alpha(u) + rX'(u) \wedge X(u) - \frac{(1 - rya)}{a} X. \tag{46}$$

Corollary 4.5. The striction curve of timelike parallel ruled surface with a timelike ruling is also the directrix provided that $\langle \alpha', X' \rangle = 0$ and $\langle X'' \wedge X, X' \rangle = 0$.

Proof. Straightforward calculation by using (45).

Corollary 4.6. The striction curve of timelike parallel ruled surface with a timelike ruling is also the directrix provided that $1 - rya = 0$.

Proof. Straightforward calculation by using (46).

Theorem 4.7. The striction curve of timelike parallel ruled surface M^r with a timelike ruling is a timelike curve.

Proof. The normal vector N^r of timelike parallel ruled surface M^r with a timelike ruling is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

For $v = 0$, we have

$$N^r(u, 0) = \alpha'(u) \wedge X(u). \tag{47}$$

From (47), we get

$$\begin{aligned} \langle N^r(u,0), N^r(u,0) \rangle &= \langle \alpha'(u) \wedge X(u), \alpha'(u) \wedge X(u) \rangle \\ &= F^2 + 1 > 0. \end{aligned} \tag{48}$$

The result (48) means that the striction curve is a timelike one because the vector, which is normal to it, is a spacelike vector.

Theorem 4.8. The striction curve of timelike parallel ruled surface M^r with a timelike ruling does not depend on the choice of the base curve $f \circ \alpha$.

Proof. Let $f \circ \alpha$ and ρ be two different directrices of timelike parallel ruled surface with a timelike ruling. Then timelike parallel ruled surface is given as

Notes

Notes

$$\varphi^r(u,v) = f \circ \alpha(u) + vX^r(u) = \rho(u) + sX^r(u) \tag{49}$$

for some function $s = s(v)$. Assume that the curves $\gamma(u)$ and $\overline{\gamma}(u)$ are the striction curves of the surfaces in (49). Then as analogous to (45) from (49) we obtain

$$\gamma(u) - \overline{\gamma}(u) \;\; = (v-s)X^r - \frac{\langle (v-s)X^{r\prime}, X'\rangle}{\langle X', X'\rangle} X(u) = 0. \tag{50}$$

The proof is completed by the result obtained in (50).

Theorem 4.9. Given timelike parallel ruled surface M^r which is parallel to timelike developable ruled surface M with a timelike ruling. There exists a unique orthogonal trajectory of M^r through each point of M. This orthogonal trajectory in terms of the magnitudes of timelike ruled surface M with timelike ruling is as follows:

$$\beta(s) = \alpha(s) + rX'(s) \wedge X(s) + g(s)X(s).$$

Here, the function $g(s)$ has been taken instead of $v(1-rb)$.

Proof. Let

$$\begin{aligned} \varphi^r : \;\; & I \times J \longrightarrow \;\; E_1^3 \\ & (u,v) \longrightarrow \;\; \varphi^r(u,v) \;\; = f \circ \alpha(u) + vX^r(u) \\ & \qquad\qquad\qquad\qquad\quad = \alpha(u) + rX'(u) \wedge X(u) + v(1-rb)X. \end{aligned}$$

An orthogonal trajectory of M^r is given by

$$\begin{aligned} \beta : \;\; & \widetilde{I} \longrightarrow M^r \\ & s \longrightarrow \beta(s) = f \circ \alpha(s) + g(s)X^r(s). \end{aligned} \tag{51}$$

We may assume $\widetilde{I} \subset I$. Since

$$\langle \beta'(s), X^r(s)\rangle \;\; = \;\; \langle \alpha'(s), X(s)\rangle - g'(s) = 0, \tag{52}$$

we have

$$g(s) = \int \langle \alpha'(s), X(s)\rangle \, ds + h,$$

where h is a real constant. So $h = g(s_0) - F(s_0)$, where

$$-\int \langle \alpha'(s), X(s)\rangle \, ds = F(s).$$

Therefore we find that the orthogonal trajectory of the surface M^r through the point P_0 is unique. Thus, we have $\widetilde{I} = I$ since the orthogonal trajectory of M^r meets each one of the rulings of the surface M^r.

Corollary 4.10. Let M^r be a timelike parallel ruled surface with timelike ruling. The Gaussian and mean curvatures K^r and H^r of the surface M^r are as follows:

$$K^r = \frac{-Q^2}{D^4 - rQFD - rQ^2JD + rvQ'D - rv^2JD - r^2Q^2} \tag{53}$$

$$H^r = \frac{-QFD - Q^2JD - v^2JD + vQ'D - 2rQ^2}{2D^4 - 2rQFD - 2rQ^2JD + 2rvQ'D - 2rv^2JD - 2r^2Q^2},$$

respectively, in terms of the parameters Q, J, F, D.

Proof. Using (7) in (20), the values of Gauss curvature K^r and mean curvature H^r are obtained as in (53).

Theorem 4.11. Let $\varphi(u, v)$ be a timelike ruled surface in E_1^3 with $F = m = 0$. Then the parallel surface

$$\varphi^r(u, v) = \varphi(u, v) + rN(u, v)$$

is a timelike developable ruled surface while one of the parameters of parallel surface is constant and the other is variable.

Proof. Every surface u=u_0 (a constant) is a ruled one as it is the union of the straight lines given by v =constant. This surface is developable provided that the curve $(u) = \varphi(u, v_0)$ is a line of curvature of M, i.e., if φ_v is a principal vector. This is true since the matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal. Similarly for the surfaces v =constant. If $F = m = 0$, then $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal. So $\mathcal{F}_I^{-1}\mathcal{F}_{II}$, the matrix of Weingarten map, depends on the basis $\{\varphi_u, \varphi_v\}$. It means that the principal vectors φ_u and φ_v are lines of curvature. Hence, the ruled surface M is a developable ruled surface. From Theorem 4.1., $\varphi^r(u, v)$ is a developable timelike ruled surface.

Theorem 4.12. Let M^r be a timelike parallel ruled surface with timelike ruling. The rulings of M^r are both an asymptotic and a geodesic line in M^r.

Proof. Let $f_*(X) \in \chi(M^r)$ be a tangent vector field for a ruling of M^r while $\overline{D} \in \chi(E_1^3)$, $D \in \chi(M)$ and $D^r \in \chi(M^r)$. Each one of the rulings is geodesic since each one of the rulings is a straight line in E_1^3. Thus we have

$$\overline{D}_{f_*(X)} f_*(X) = 0. \tag{54}$$

The Gauss equation for M^r is

$$\overline{D}_{f_*(X)} f_*(X) = {D^r}_{f_*(X)} f_*(X) - S^r(f_*(X)), f_*(X)\rangle\, N^r, \tag{55}$$

where $\overline{D}$ is Levi-Civita connection on M^r. By using (54), the equation (55) becomes

$${D^r}_{f_*(X)} f_*(X) = \langle S^r(f_*(X)), f_*(X)\rangle\, N^r.$$

Furthermore, since

$${D^r}_{f_*(X)} f_*(X) \in \chi(M^r) \text{ and } \langle S^r(f_*(X)), f_*(X)\rangle\, N^r \in \chi^{\perp}(M^r),$$

we get the following results:

$${D^r}_{f_*(X)} f_*(X) = 0 \text{ or } \langle S^r(f_*(X)), f_*(X)\rangle\, N^r = 0.$$

Also, for the normal vector N^r of the surface M^r, we write

$$\chi(E_1^3) = \chi(M^r) \oplus \chi^{\perp}(M^r) \text{ and } \chi(M^r) \cap \chi^{\perp}(M^r) = \{0\}.$$

Then, we obtain

Notes

$$D^r{}_{f_*(X)} f_*(X) = 0 \text{ and } \langle S^r(f_*(X)), f_*(X) \rangle = 0.$$

The last equations completes the proof.

Theorem 4.13. Let M^r be a timelike parallel ruled surface. Then the Gaussian curvature $K^r(f(P))$ of M^r satisfies

Notes

$$K^r \geq 0$$

at each point $f(P) \in M^r$.

Proof. Let $f_*(X)$ be the timelike vector field of the rulings through the point $f(P) \in M^r$. We get an orthogonal base $\{f_*(X), f_*(Y)\}$ of $\chi(M^r)$ in which $f_*(Y)$ is a spacelike vector field. We obtain the matrix corresponding to the shape operator of M^r derived from N^r as follows:

$$S^r = \begin{bmatrix} -\langle S(X), X \rangle & \langle S^r(f_*(X)), f_*(Y) \rangle \\ -\langle S^r(f_*(Y)), f_*(X) \rangle & \langle S(Y), f_*(Y) \rangle \end{bmatrix}.$$

Using $\langle S^r(f_*(X)), f_*(X) \rangle = \langle S(X), X \rangle = 0$ by means of Theorem 4.12 and Definition 3.1, we have the Gaussian curvature K^r as follows:

$$\begin{aligned} K^r &= -\det S^r \\ &= \langle S^r(f_*(Y)), f_*(X) \rangle^2 = \langle S(Y), X \rangle^2 \geq 0. \end{aligned}$$

Example 4.14. A hyperbolic cylinder has the parameterization

$$\varphi(u, v) = (\cosh u, \sinh u, v). \tag{56}$$

It is easily seen that its base curve is a spacelike curve and its ruling is a timelike vector. A hyperbolic cylinder is a developable ruled surface since its drall λ vanishes. The unit normal vector of hyperbolic cylinder is found as

$$\mathbf{N} = (-\frac{5\cosh u}{\sqrt{2\cosh^2 u - 1}}, \frac{5\sinh u}{\sqrt{2\cosh^2 u - 1}}, 0).$$

By using the expression $\varphi^r = \varphi + r\mathbf{N}$ for $r = 5$, parallel surface of hyperbolic cylinder can be parameterized as

$$\varphi^r(u, v) = (-\frac{5\cosh u}{\sqrt{2\cosh^2 u - 1}} + \cosh u, \frac{5\sinh u}{\sqrt{2\cosh^2 u - 1}} + \sinh u, v), \tag{57}$$

where the base curve C^r is

$$C^r = (-\frac{5\cosh u}{\sqrt{2\cosh^2 u - 1}} + \cosh u, \frac{5\sinh u}{\sqrt{2\cosh^2 u - 1}} + \sinh u, 0)$$

and the ruling X^r is $(0, 0, 1)$. The surface in (57) is a ruled surface because it can be written in the form $\varphi^r = C^r + vX^r$. Also, parallel surface of a hyperbolic cylinder is a developable ruled surface since its drall λ^r vanishes. It means that the surface given with the parametrization in (57) is a timelike parallel ruled surface with timelike ruling, so the red and blue surfaces in (Fig. 1) show timelike hyperbolic cylinder and its timelike parallel ruled surface, respectively.

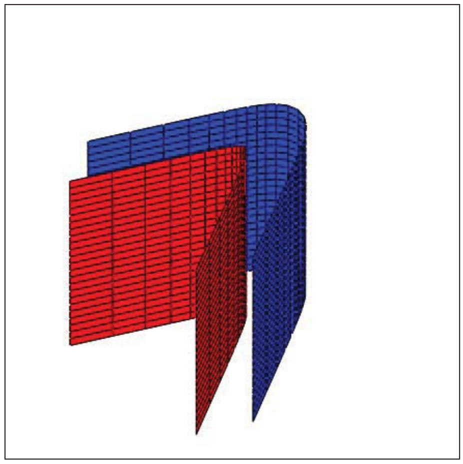

Figure 1 : Hyperbolic cylinder and its parallel surface

Also by using the equation (46), the striction curve of timelike parallel ruled surface with timelike ruling is found as $(u) = (\cosh u, \sinh u, -1)$ by taking $r = -5$, $y = 1$ and $a = 0$. Its orthogonal trajectory is calculated as

$$\beta(u) = (\cosh u, \sinh u, v(1 + 5b))$$

by means of Theorem 4.9. The Gaussian curvature K^r of timelike parallel ruled surface vanishes because the main surface is developable, therefore timelike parallel ruled surface is also developable from Theorem 4.1. Nevertheless, the vanishing of the Gaussian curvature can be seen by computating the coefficients of the first and second fundamental forms of the surface given in in (56) or by calculating the values of Q, J, F, D in (6) and then putting them into (53) in Corollary 4.10. For instance, the values of Q, J, F, D are as follows:

$$Q = 0, \; J = 0, \; F = 0, \; D = \sqrt{\varepsilon}\,|v|$$

for the surface given in (56). As a result, the accuracy of Theorem 4.13 is seen.

Example 4.15. The helicoid of the 3 rd kind has the parametrization

$$\varphi(u, v) = (v \cosh u, v \sinh u, u). \tag{58}$$

It is easily seen that its base curve is a spacelike curve and its ruling is a timelike vector. The helicoid of the 3 rd kind is not a developable ruled surface since its drall λ doesn't vanish. Hence, the parallel surface of the helicoid of the 3 rd kind can not become a ruled surface because of Theorem 4.1. The unit normal vector for the helicoid of the 3 rd kind is found as

$$\overrightarrow{\mathbf{N}} = \left(\frac{5 \sinh u}{\sqrt{2\cosh^2 u - 1 - v^2}}, -\frac{5 \cosh u}{\sqrt{2\cosh^2 u - 1 - v^2}}, -\frac{v}{\sqrt{2\cosh^2 u - 1 - v^2}}\right).$$

The helicoid of the 3 rd kind in Minkowski 3-space is seen in (Fig. 2):

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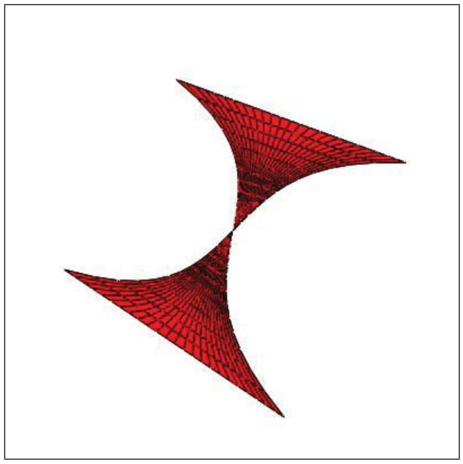

Figure 2 : The Helicoid of the 3rd kind

By using the expression $\varphi^r = \varphi + r\,\mathbf{N}$ for $r = -1$, parallel surface of the helicoid of the 3 rd kind can be parameterized as

$$\begin{aligned}\varphi^r(u,v) \;=\; & \Big(\frac{5\sinh u}{\sqrt{2\cosh^2 u-1-v^2}} + v\cosh u, -\frac{5\cosh u}{\sqrt{2\cosh^2 u-1-v^2}} + v\sinh u, \\ & -\frac{v}{\sqrt{2\cosh^2 u-1-v^2}} + u\Big). \qquad (59)\end{aligned}$$

Again we can state that the surface in (58) is not a ruled surface because both it can not be written in the form $\varphi^r = C^r + vX^r$ and the main surface is, as stated previously, not developable one. The parallel surface of the helicoid of the 3 rd kind and the two surfaces together are seen in (Fig. 3) and (Fig. 4), respectively.

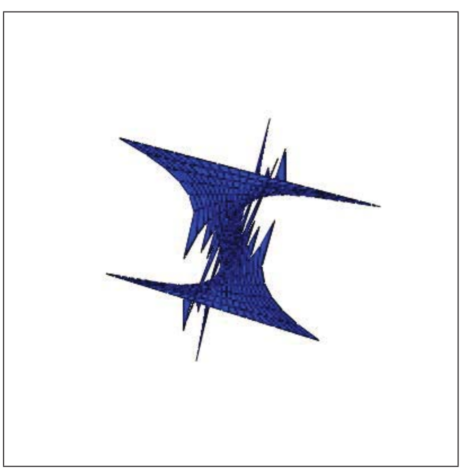

Figure 3 : The parallel surface of the helicoid of the 3rd kind

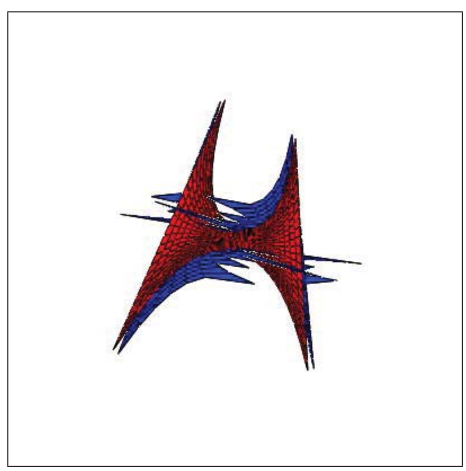

Figure 4 : The two surfaces together

V. Conclusion

In this paper, we have constructed timelike parallel ruled surfaces by using the elements of differential geometry in Minkowski 3-space. Furthermore, we have presented some characterizations of timelike parallel ruled surfaces whose original surfaces are timelike ruled surfaces with timelike ruling. Researchers can try to see the results we obtained in this work, in Euclidean and

Lorentzian n-spaces. The results have a number of applications in computer-aided design and manufacturing of sculptured surfaces.

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Construction of a Mixed Quadrature Rule using Three Different Well-Known Quadrature Rules

By Debasish Das, Rajani B. Dash & parthasarathi Das
Ravenshaw University, India

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GJSFR-F Classification : *MSC 2010: 65D30, 65D32*

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Strictly as per the compliance and regulations of :

Construction of a Mixed Quadrature Rule using Three Different Well-Known Quadrature Rules

Debasish Das [α], Rajani B. Dash [σ] & parthasarathi Das [ρ]

Abstract- This paper deals with construction of a mixed quadrature rule of precision nine by using Gauss-Legendre 3-point rule, Lobatto 4-point rule and Clenshaw-Curtis 5-point rule, each having precision five. This mixed rule is successfully tested on different real definite integrals.

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I. Introduction

Real definite integral of the form

$$I(f)=\int_a^b f(x)dx \tag{1.1}$$

can be approximated by using (i) Gauss-Legendre quadrature rule,(ii) Lobatto quadrature rule,(iii)Clenshaw-Curtis quadrature rule. Among these three quadrature rules, Lobatto and Clenshaw-Curtis quadrature rules are of closed type where as Gauss-Legendre quadrature rule[2] is of open type. An n-point Clenshaw-Curtis rule[4] is of degree of precision n, while an n-point Gauss-Legendre rule and an n-point Lobatto rule are of degree of precision $2n$-1 and $2n$-3 respectively. That means Gauss-Legendre rule needs less number of nodes to maintain a particular precision than Lobatto and Clenshaw-Curtis rule. Usually by increasing the value of 'n' in these above three rules, we can optimize the accuracy of approximation of the real definite integral (1.1).The nodes of n-point Gauss-Legendre rule and n-point Lobatto rule are zeros of $P_n(x),\ n\geq 2$ and $P'_{n-1}(x),\ n\geq 3$ respectively. The nodes of n-point Clenshaw-Curtis rule are obtained from the following equation.

$$x_i=\cos\frac{i\pi}{n} \qquad i=0,1,2,\ldots n \tag{1.2}$$

In the Gauss-Legendre and Lobatto rules, the computational complexity for the evaluation of the zeros of $P_n(x)$ and $P'_{n-1}(x)$ increases for large n. Also the computational complexity may arise to find the nodes of Clenshaw-Curtis rule for large n. In these three rules, as we move from lower order rule to higher order rule, almost all the information obtained in computing the former gets discarded because the nodes and weights are different for different values of n.

Authors α σ: Department of Mathematics. e-mail: debasisdas100@gmail.com
Author ρ: Department of Physics Ravenshaw University Cuttack-753003, Odisha (India).

Keeping these facts in view, we desire to construct a mixed quadrature rule[3] of precision nine which is a linear combination of Gauss-Legendre 3-point rule, Lobatto 4-point rule and Clenshaw-Curtis 5-point rule each having precision five. The construction of the mixed quadrature rule is out-lined in the following section.

II. Construction of the Mixed Quadrature Rule of Precision Nine

Notes

We choose the Gauss-Legendre 3-point rule $\left(R_{GL_3}(f)\right)$:

$$I(f)=\int_a^b f(t)dt=\int_{-1}^{1} f(x)dt \approx R_{Gl_3}(f)$$

$$=\frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{2.1}$$

the Clenshaw-Curtis 5-point rule $\left(R_{CC_5}(f)\right)$:

$$I(f)=\int_a^b f(t)dt=\int_{-1}^{1} f(x)dt \approx R_{cc_5}(f)$$

$$=\frac{1}{15}\left[f(-1)+8f\left(\frac{-1}{\sqrt{2}}\right)+12f(0)+8f\left(\frac{1}{\sqrt{2}}\right)+f(1)\right] \tag{2.2}$$

and the Lobatto 4-point rule $\left(R_{L_4}(f)\right)$:

$$I(f)=\int_a^b f(t)dt=\int_{-1}^{1} f(x)dt \approx R_{L_4}(f)$$

$$=\frac{1}{6}\left[f(-1)+5f\left(\frac{-1}{\sqrt{5}}\right)+5f\left(\frac{1}{\sqrt{5}}\right)+f(1)\right] \tag{2.3}$$

Each of these rules (2.1), (2.2) and (2.3) is of precision 5. Let $E_{GL_3}(f), E_{CC_5}(f)$ and $E_{L_4}(f)$ denote the errors in approximating the integral $I(f)$ by the rules (2.1), (2.2) and (2.3) respectively .Then

$$I(f)=R_{GL_3}(f)+E_{GL_3}(f) \tag{2.4}$$

$$I(f)=R_{CC_5}(f)+E_{CC_5}(f) \tag{2.5}$$

and $$I(f)=R_{L_4}(f)+E_{L_4}(f) \tag{2.6}$$

Assuming $f(x)$ to be sufficiently differentiable in $-1\le x\le 1$, using Maclaurin's expansion of the function $f(x)$ we can express the errors associated with the quadrature rules under reference as

$$E_{GL_3}(f)=\frac{8}{7!\times 25}f^{(vi)}(0)+\frac{88}{9!\times 125}f^{(viii)}(0)+\frac{656}{11!\times 625}f^{(x)}(0)+\dots$$

$$E_{CC_5}(f)=\frac{2}{7!\times 15}f^{(vi)}(0)+\frac{1}{9!\times 5}f^{(viii)}(0)+\frac{1}{11!\times 6}f^{(x)}(0)+\dots$$

and $E_{L_4}(f)=\frac{-32}{7!\times 75}f^{(vi)}(0)-\frac{128}{9!\times 125}f^{(viii)}(0)-\frac{3136}{11!\times 1875}f^{(x)}(0)-\ldots$

Now multiplying the Eqs(2.4),(2.5) and (2.6)by60,-32 and 35respectively, then adding the results we obtain,

$$I(f)=\frac{1}{63}\left[60R_{GL_3}(f)+35R_{L_4}(f)-32R_{CC_5}(f)\right]+$$

$$\frac{1}{63}\left[60E_{GL_3}(f)+35E_{L_4}(f)-32E_{CC_5}(f)\right]$$

or $I(f)=R_{GL_3L_4CC_5}(f)+E_{GL_3L_4CC_5}(f)$ (2.7)

where $R_{GL_3L_4CC_5}(f)=\frac{1}{63}\left[60R_{GL_3}(f)+35R_{L_4}(f)-32R_{CC_5}(f)\right]$ (2.8)

and $E_{GL_3L_4CC_5}(f)=\frac{1}{63}\left[60E_{GL_3}(f)+35E_{L_4}(f)-32E_{CC_5}(f)\right]$ (2.9)

Eq (2.8), expresses the desired mixed quadrature rule for the approximate evaluation of $I(f)$ and Eq (2.9) expresses the error generated in this approximation.

so, $E_{GL_3L_4CC_5}(f)=\frac{-16}{11!\times 1125}f^{(x)}(0)-\ldots$ (2.10)

As the first term of $E_{GL_3L_4CC_5}(f)$ contains 10th order derivative of the integrand, so the degree of precision of the mixed quadrature rule is 9. It is called a mixed type rule as it is constructed from three different types of rules of equal precision.

III. Error Analysis of the Mixed Quadrature Rule

An asymptotic error estimate and an error bound of the mixed quadrature rule (2.8) is given in theorems 3.1 and 3.2 respectively.

Theorem-3.1

Let *f(x)* be sufficiently differentiable function in the closed interval [-1,1].Then the error $E_{GL_3L_4CC_5}(f)$ associated with the mixed quadrature rule $R_{GL_3L_4CC_5}(f)$ is given by

$$\left|E_{GL_3L_4CC_5}(f)\right|\cong\frac{16}{11!\times 1125}\left|f^{(x)}(0)\right|$$

Proof:

The proof Follows from Eq(2.10).

Theorem-3.2

The bound for the truncation error

$$E_{GL_3L_4CC_5}(f)=I(f)-R_{GL_3L_4CC_5}(f)$$

is given by

$$E_{GL_3L_4CC_5}(f) \leq \frac{4M}{33075}$$

where, $M = \max_{-1\leq x\leq 1}\left|f^{(vii)}(x)\right|$

Proof:

We have

$$E_{GL_3}(f) = \frac{8}{7!\times 25} f^{(vi)}(\eta_1) \qquad \eta_1 \in [-1,1]$$

$$E_{L_4}(f) = \frac{-32}{7!\times 75} f^{(vi)}(\eta_2) \qquad \eta_2 \in [-1,1]$$

$$E_{CC_5}(f) = \frac{2}{7!\times 15} f^{(vi)}(\eta_3) \qquad \eta_3 \in [-1,1]$$

(Refer to Conte and Boor [1])

$$\text{so, } E_{GL_3L_4CC_5}(f) = \frac{1}{63}\left[60E_{GL_3}(f) + 35E_{L_4}(f) - 32E_{CC_5}(f)\right]$$

$$= \frac{96}{7!\times 315} f^{(vi)}(\eta_1) - \frac{224}{7!\times 945} f^{(vi)}(\eta_2) - \frac{64}{7!\times 945} f^{(vi)}(\eta_3)$$

Let $K = \max_{x\in[-1,1]}\left|f^{(vi)}(x)\right|$ and $k = \min_{x\in[-1,1]}\left|f^{(vi)}(x)\right|$. As $f^{(vi)}(x)$ is continuous and [-1,1] is compact, hence there exist points b and a in the interval [-1,1] such that $K = f^{(vi)}(b)$ and $k = f^{(vi)}(a)$. Thus

$$E_{GL_3L_4CC_5}(f) \leq \frac{96}{7!\times 315} f^{(vi)}(b) - \frac{224}{7!\times 945} f^{(vi)}(a) - \frac{64}{7!\times 945} f^{(vi)}(a)$$

$$= \frac{2}{33075}\left[f^{(vi)}(b) - f^{(vi)}(a)\right]$$

$$= \frac{2}{33075}\int_a^b f^{(vii)}(x)\,dx$$

$$= \frac{2}{33075}(b-a) f^{(vii)}(\xi) \qquad \text{for some } \xi \in [-1,1]$$

by Mean value theorem[1]

Hence by choosing $\left|(b-a)\right| \leq 2$

we have, $E_{GL_3L_4CC_5}(f) \le \frac{2}{33075}|(b-a)|\left|f^{(vii)}(\xi)\right| \le \frac{4M}{33075}$

where, $M = \max_{x\in[-1,1]}\left|f^{(vii)}(x)\right|$

Notes

IV. Numerical Verification

For the Numerical verification of the mixed quadrature rule $\left(R_{GL_3L_4CC_5}(f)\right)$, the following integrals are considered.

Table :1

Exact value of $I_1(f)=\int_0^{\frac{\pi}{4}}\cos^2(x)dx=0.6426990818$

Quadrature/Mixed quadrature rule	Aproximate value of $I_1(f)$
$R_{GL_3}(f)$	**0.6427**011120
$R_{L_4}(f)$	**0.6426**963791
$R_{CC_5}(f)$	**0.64269**993278
$R_{GL_3L_4CC_5}(f)$	**0.64269908**169

Table:2

Exact value of $I_2(f)=\int_0^{\frac{\pi}{4}}e^{\cos x}dx=1.9397348506$

Quadrature/Mixed quadrature rule	Aproximate value of $I_2(f)$
$R_{GL_3}(f)$	**1.9397**36725
$R_{L_4}(f)$	**1.9397**323524
$R_{CC_5}(f)$	**1.9397**356328
$R_{GL_3L_4CC_5}(f)$	**1.9397348506**67

Table : 3

Exact value of $I_3(f)=\int_0^1 e^{-x^2}dx=0.7468241328$

Quadrature/Mixed quadrature rules	Aproximate value of $I_3(f)$
$R_{GL_3}(f)$	**0.7468**145841
$R_{L_4}(f)$	**0.7468**365980
$R_{CC_5}(f)$	**0.7468**198579
$R_{GL_3L_4CC_5}(f)$	**0.74682413**53

Table: 4

Exact value of $I_4(f)=\int_0^1\frac{1}{1+e^x}dx=0.3798854936$

Quadrature/Mixed quadrature rules	Aproximate value of $I_4(f)$
$R_{GL_3}(f)$	**0.379885**308
$R_{L_4}(f)$	**0.37988**57384
$R_{CC_5}(f)$	**0.37988549**149
$R_{GL_3L_4CC_5}(f)$	**0.379885493**04

Table : 5

Exact value of $I_5(f)=\int_0^1(1+x^2)^{\frac{3}{2}}dx=1.5679519622$

Quadrature/Mixed quadrature rules	Aproximate value of $I_5(f)$
$R_{GL_3}(f)$	**1.5679**493894
$R_{L_4}(f)$	**1.5679**552310
$R_{CC_5}(f)$	**1.5679**507093
$R_{GL_3L_4CC_5}(f)$	**1.56795196**43

Table: 6

Exact value of $I_6(f)=\int_0^1 x^2e^{-x}dx=0.1606027942$

Quadrature/Mixed quadrature rules	Aproximate value of $I_6(f)$
$R_{GL_3}(f)$	**0.1605**953868
$R_{L_4}(f)$	**0.1606**126841
$R_{CC_5}(f)$	**0.1605**997226
$R_{GL_3L_4CC_5}(f)$	**0.16060279**415

Table:7

Exact value of $I_7(f)=\int_0^1 \frac{1}{1+x^4}dx=0.8669729870$

Quadrature/Mixed quadrature rules	Aproximate value of $I_7(f)$
$R_{GL_3}(f)$	**0.867**51846
$R_{L_4}(f)$	**0.866**2609
$R_{CC_5}(f)$	**0.867**21664
$R_{GL_3L_4CC_5}(f)$	**0.86697**313

Table:8

Exact value of $I_8(f)=\int_0^{\frac{\pi}{4}} e^{3x}\sin(2x)dx=2.5886286324$

Quadrature/Mixed quadrature rules	Aproximate value of $I_8(f)$
$R_{GL_3}(f)$	**2.58**9258
$R_{L_4}(f)$	**2.58**778613
$R_{CC_5}(f)$	**2.588**8872
$R_{GL_3L_4CC_5}(f)$	**2.58862863**8

Table: 9

Exact value of $I_9(f)=\int_1^{1.5} x^2\ln(x)dx=0.1922593577$

Quadrature/Mixed quadrature rules	Aproximate value of $I_9(f)$
$R_{GL_3}(f)$	**0.19225937**
$R_{L_4}(f)$	**0.1922593**316
$R_{CC_5}(f)$	**0.1922593**658
$R_{GL_3L_4CC_5}(f)$	**0.1922593577**326

Table:10

Exact value of $I_{10}(f)=\int_0^1 \frac{4}{1+x^2}dx=\pi\cong 3.141592654$

Quadrature/Mixed quadrature rules	Aproximate value of $I_{10}(f)$
$R_{GL_3}(f)$	**3.141**068
$R_{L_4}(f)$	**3.14**2276
$R_{CC_5}(f)$	**3.141**35
$R_{GL_3L_4CC_5}(f)$	**3.141592**72

Table:11

Exact value of $I_{11}(f)=\int_3^{3.5} \frac{x}{\sqrt{x^2-4}}dx=0.6362133458$

Quadrature/Mixed quadrature rules	Aproximate value of $I_{11}(f)$
$R_{GL_3}(f)$	**0.636213**1959
$R_{L_4}(f)$	**0.636213**5467
$R_{CC_5}(f)$	**0.636213**2845
$R_{GL_3L_4CC_5}(f)$	**0.63621334**577

V. Conclusion

In this article, we construct a new mixed quadrature rule of precision nine for evaluating the real definite integrals. An asymptotic error estimate and an error bound of the mixed quadrature rule are given. Finally we give some numerical examples to show the superiority of the mixed quadrature rule with respect to the usual Gauss-Legendre 3-point rule, Clenshaw-Curtis 5-point rule and Lobatto 4-point rule.

Notes

VI. Acknowledgement

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Notes

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Notes

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Noiseless Coding Theo rems Connected with Tuteja and Bhaker's 'useful' Inaccuracy Measure

By Rayees Ahmad Dar

Abstract- A new measure, $L_{\alpha}^{\beta}(U)$ called average code word length of order and type has been defined and its relationship with a result of Tuteja and Bhaker 'useful' inaccuracy measure has been discussed. Using $L_{\alpha}^{\beta}(U)$, some noiseless coding theorems for discrete noiseless channel has been proved. The results obtained in this paper generalizes some well known results available in the literature.

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GJSFR-F Classification : *AMS: 94A17, 94A24*

NOISELESSCODINGTHEOREMSCONNECTEDWITHTUTEJAANDBHAKERSUSEFULINACCURACYMEASURE

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Noiseless Coding Theorems Connected with Tuteja and Bhaker's 'useful' Inaccuracy Measure

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Abstract- A new measure, $L_\alpha^\beta(U)$ called average code word length of order **α** and type **β** has been defined and its relationship with a result of Tuteja and Bhaker 'useful' inaccuracy measure has been discussed. Using $L_\alpha^\beta(U)$ some noiseless coding theorems for discrete noiseless channel has been proved. The results obtained in this paper generalizes some well known results available in the literature.

Keywords: inaccuracy measure, average code length, holder's inequality.

I. Introduction

Consider the model given below for a finite scheme random experiment having $(x_1, x_2, ..., x_n)$ as the complete system of events, happening with respective probabilities $P = (p_1, p_2, ..., p_n)$ and credited with utilities $U = (u_1, u_2, ..., u_n)$, $u_i > 0$, $i = 1,2,...,n$. Denote

$$(1.1) \qquad X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \\ u_1 & u_2 & \dots & u_n \end{bmatrix}$$

We call scheme (1.1) as utility information scheme.

Now let us suppose that experimenter asserts that the ith outcome x_i has the probability q_i whereas the true probability is p_i with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. Thus we have two utility information schemes, (1.1) of a set of n events after an experiment and

$$(1.2) \qquad X^* = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ q_1 & q_2 & \dots & q_n \\ u_1 & u_2 & \dots & u_n \end{bmatrix}$$

of the same set of n events before the experiment. In both the schemes (1.1) and (1.2), the utility distribution is the same because we assume that the utility u_i of an event x_i is independent of its probability of occurrence p_i or predicted probability q_i, u_i is only a "utility" or value of the outcome for an observer relative to some specified goal [14].

Author : Department of Bio statistics, SKIMS Srinagar, Kashmir (J&K) India. e-mail: rayees_stats@yahoo.com

Ref

[14] Longo, G, "Quantitative@qualitative measure of information", Springer, New York (1972).

The quantitative@qualitative measure of inaccuracy [24] associated with the statement of an experimenter is given by

(1.3) $$I(P/Q;U) = -\sum_{i=1}^{n} u_i p_i \log q_i$$

Let $X = [x_1, x_2, ..., x_n]$ the finite set of n input symbols which are to be encoded using alphabet of D symbols. It has been shown Feinestein [5] that there is a unique decipherable code with lengths $l_1, l_2 ..., l_n$ iff

(1.4) $$\sum_{i=1}^{n} D^{-l_i} \leq 1$$

Noiseless coding theorem for Shannon entropy with ordinary code mean length

(1.5) $$L = \sum_{i=1}^{n} l_i p_i$$

under the condition of unique decipherability, has played an important role in ordinary communication theory.

A successful attempt in this direction has been made by Autar and Soni [2], who established noiseless coding theorem for inaccuracy of order α given by sharma [21] with the average code length of order t given by Campbell [4] under the condition

(1.6) $$\sum_{i-1}^{n} p_i q_i^{-1} D^{-l_i} \leq 1$$

Inequality (1.6) is in a way generalization of (1.4).
Guiasu and Picard [6] defined the following quantity

(1.7) $$L(U) = \frac{\sum_{i=1}^{n} u_i l_i p_i}{\sum_{i=1}^{n} u_i p_i}$$

and call it 'useful' mean length of the code. They also derived a lower bound for it. Also (1.7) is generalization of (1.5).

Jain and Tuteja [9] studied the generalization of (1.7) as

(1.8) $$L_1^{\beta}(U) = \frac{1}{2^{1-\beta}-1}\left[\left\{\frac{\sum_{i=1}^{n} u_i p_i D^{l_i \frac{1-\beta}{\beta}}}{\sum_{i=1}^{n} u_i pi}\right\}^{\beta} - 1\right] \quad , \beta > 0, \beta \neq 1$$

and found the bounds for the generalization in terms of 'useful' entropy of type which is given by

(1.9) $$H^{\beta}(P,U) = \frac{1}{(2^{1-\beta}-1)\sum_{i=1}^{n} p_i}\left[\sum_{i=1}^{n} u_i\, p_i\left(p_i^{\beta-1} - 1\right)\right] \quad , \ \beta > 0, \ \beta \neq 1$$

under the condition

(1.10) $$\sum_{i=1}^{n} u_i D^{-l_i} \leq \sum_{i=1}^{n} u_i p_i$$

which is generalization of (1.4).

Ref

[24] Taneja, H.C and Tuteja, R.K, "Characterization of Quantitative@ Qualitative measure of inaccuracy", Kybernetika, Vol 22, 393@402(1985).

It may be seen that the mean code word length (1.5) had been generalized parametrically and their bounds had been studied in terms of generalized measure of information. Here we give another generalization of (1.5) and study its bounds in terms of generalized 'useful' inaccuracy measure of order α ant type β given by Tuteja and Bhaker [25].

Longo [14]. Gurdial and Pessoa [7], Singh, Kumar and Tuteja [23], Praskash and Sharam [17], Hooda and Bhaker [8], Khan , Bhat and Pirzada [11], Arndt [1], Baig and Rayees [3], Rayees and Baig [18], Kerridge [10], Satish and Rajesh [12], Mc@Millan [15], Pirzada and Bhat [16], Roy [19] and Satish Kumar [13] have studied generalized coding theorems by considering different generalized measure of Shannon's entropy [20] under the condition (1.4) of unique decipherability.

In this communication, generalization of (1.7) have been studied and the bounds for this generalization are obtained interms of 'useful' inaccuracy of order α and type β given by Tuteja and Bhaker [25] which is given by

$$I_{\alpha}^{\beta}(P/Q;U)=\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta}\left(q_i^{\alpha-1}-1\right)}{\left(2^{1-\alpha}-1\right)\sum_{i=1}^{n}{}^{\beta}},\quad \alpha>0(\neq 1), \beta>0 \tag{1.11}$$

under the condition

$$\sum_{i=1}^{n}(u_i p_i)^{\beta} q_i^{-1} D^{-l_i} \leq \sum_{i=1}^{n}(u_i p_i)^{\beta} \tag{1.12}$$

which is generalization of (1.6).

Remarks:

(i) For $\beta=1$ and $\alpha\to 1$, (1.11) reduces to $I(P/Q;U)$ given by Taneja and Tuteja [24].

(ii) For $\beta=1$, $\alpha\to 1$ and $u_i=1\ \forall\, i=1,2,...,n$, (1.11) reduces to the measure given by Kerridge [10]. Further if $p_i=q_i\ \ \forall\, i=1,2,...,n$, it reduces to Shannon's entropy [20].

(iii) For $\beta=1$ and $p_i=q_i\ \ \forall\, i=1,2,...,n$, (1.11) reduces to measure given by Jain and Tuteja [9].

II. Generalization of 'Useful' mean Length and the Coding Theorems

$$L_{\alpha}^{\beta}(U)=\frac{1}{\left(2^{1-\alpha}-1\right)\sum_{i=1}^{n}p_i^{\beta}}\left[\left\{\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right\}^{\alpha}-1\right],\ \alpha>0(\neq 1),\ \beta>0 \tag{2.1}$$

Ref

[7] Gurdial and Pessoa, F, "On useful information of order α", journal of comb. Information and System Sciences, Vol 2, 30@35 (1977).

Remarks:

(i) For $\beta=1$ and $\alpha\to 1$, (2.1) reduces to the mean length of the code given by Guiasu and Picard [6].

(ii) For $\beta=1$, $\alpha\to 1$ and $u_i=1\ \forall\ i=1,2,\ldots,n$, (2.1) reduces to the mean length of the code given by Shannon [20].

(iii) For $\beta=1$, (2.1) reduces to the mean length given by Jain and Tuteja [9].

In the following theorem we obtain lower bound for (2.1) in terms of $I_\alpha^\beta(P/Q;U)$

Theorem 1: If $l_1,l_2,\ldots,l_n$ denote the lengths of a code satisfying (1.12). Then

(2.2) $$L_\alpha^\beta(U)\geq \frac{I_\alpha^\beta(P/Q,U)}{\bar{U}}\ ,\ \alpha>0(\neq 1),\beta>0$$

where $\bar{U}=\sum_{i=1}^{n}(u_ip_i)^\beta$

with equality iff

(2.3) $$l_i=-\log q_i^{\alpha}+\log\frac{\sum_{i=1}^{n}(u_ip_i)^\beta q_i^{\alpha-1}}{\sum_{i=1}^{n}(u_ip_i)^\beta}$$

Proof: By Holders inequality [22]

(2.4) $$\sum_{i=1}^{n}x_iy_i\geq\left[\sum_{i=1}^{n}x_i^{\,p}\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n}y_i^{\,q}\right]^{\frac{1}{q}}$$

for all $x_i\geq 0, y_i\geq 0$, $i=1,2,\ldots,n$ when $p<1(\neq 0), q<0$ or $q<1(\neq 0), p<0$ and $\frac{1}{p}+\frac{1}{q}=1$, with equality iff there exists a positive number c such that $x_i^{\,p}=cy_i^{\,q}$.

Put $$x_i=\left[\frac{(u_ip_i)^\beta}{\sum_{i=1}^{n}(u_ip_i)^\beta}\right]^{\frac{\alpha}{\alpha-1}}D^{-l_i}\qquad ,y_i=\left[\frac{(u_ip_i)^\beta}{\sum_{i=1}^{n}(u_ip_i)^\beta}\right]^{\frac{1}{1-\alpha}}q_i^{-1}$$

$$p=\frac{\alpha-1}{\alpha}\quad ,q=1-\alpha$$

in (2.4), we get

$$\sum_{i=1}^{n}\left[\frac{(u_ip_i)^\beta}{\sum_{i=1}^{n}(u_ip_i)^\beta}\right]^{\frac{\alpha}{\alpha-1}}D^{-l_i}\left[\frac{(u_ip_i)^\beta}{\sum_{i=1}^{n}(u_ip_i)^\beta}\right]^{\frac{1}{1-\alpha}}q_i^{-1}$$

Ref

[9] Jain, P and Tuteja, R.K, "On coding theorem connected with 'useful' entropy of order@ β ", International Journal of Math. and Math. Sci.,Vol. 12 No.1,193@ 198(1989).

Notes

$$\geq \left\{\sum_{i=1}^{n}\left[\left(\frac{(u_i p_i)^{\beta}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right)^{\frac{\alpha}{\alpha-1}} D^{-l_i}\right]^{\frac{\alpha-1}{\alpha}}\right\}^{\frac{\alpha}{\alpha-1}} \left\{\sum_{i=1}^{n}\left[\left(\frac{(u_i p_i)^{\beta}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right)^{\frac{1}{1-\alpha}} q_i^{-1}\right]^{1-\alpha}\right\}^{\frac{1}{1-\alpha}}$$

$$\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} D^{-l_i} q_i^{-1}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}} \geq \left[\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right]^{\frac{\alpha}{\alpha-1}} \left[\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right]^{\frac{1}{1-\alpha}}$$

using inequality (1.12), we get

(2.5)
$$\left[\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right]^{\frac{\alpha}{1-\alpha}} \geq \left[\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right]^{\frac{1}{1-\alpha}}$$

Let $0<\alpha<1$. Raising both sides of (2.5) to the power $(1-\alpha)$, we get

(2.6)
$$\left[\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right]^{\alpha} \geq \left[\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right]$$

since $2^{1-\alpha}-1>0$ for $0<\alpha<1$, a simple manipulation proves (2.2) for $0<\alpha<1$.The proof for $1<\alpha<\infty$ follows on the same lines.

Theorem 2: For every code with lengths $l_1, l_2, ..., l_n$ and satisfying (1.12), $L_{\alpha}^{\beta}(U)$ can be made to satisfy the inequality

(2.7)
$$L_{\alpha}^{\beta}(U) < \frac{I_{\alpha}^{\beta}(P/Q;U)D^{1-\alpha}}{\bar{U}} + \frac{D^{1-\alpha}-1}{\left(2^{1-\alpha}-1\right)\sum_{i=1}^{n} p_i^{\beta}} \quad , \ \alpha>0(\neq 1), \beta>0$$

Proof: Let l_i be the positive integer satisfying the inequality

(2.8)
$$-\log_D q_i^{\alpha} + \log_D\left(\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right) \leq l_i < -\log_D q_i^{\alpha} + \log_D\left(\frac{\sum_{i=1}^{n}(u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n}(u_i p_i)^{\beta}}\right) + 1$$

Consider the intervals

(2.9) $$\delta_i = \left[-\log_D q_i^{\alpha} + \log_D \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right), -\log_D q_i^{\alpha} + \log_D \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right) + 1 \right]$$

of length 1. In every δ_i, there lies exactly one positive integer l_i such that

(2.10) $$0 < -\log_D q_i^{\alpha} + \log_D \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right) \le l_i < -\log_D q_i^{\alpha} + \log_D \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right) + 1$$

we will first show that sequence $\{l_1, l_2, \ldots, l_n\}$, thus defined satisfies (1.12), from (2.10) we have

$$-\log_D q_i^{\alpha} + \log_D \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right) \le l_i$$

$$-\log_D q_i^{\alpha} + \log_D \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right) \le -\log_D D^{-l_i}$$

or

(2.11) $$\frac{q_i^{\alpha}}{\dfrac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}}} \ge D^{-l_i}$$

Multiplying both sides of (2.11) by $(u_i p_i)^{\beta} q_i^{-1}$ and summing over $i = 1,2,\ldots,n$, we get (1.12).

The last inequality of (2.10) gives

(2.12) $$D^{l_i} < D q_i^{-\alpha} \frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}}$$

Let $0 < \alpha < 1$. Raising both sides of (2.12) to the power $\left(\frac{1-\alpha}{\alpha} \right)$, we get

Notes

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(2.13) $$D^{l_i\left(\frac{1-\alpha}{\alpha}\right)} < q_i^{\alpha-1} D^{\left(\frac{1-\alpha}{\alpha}\right)} \left(\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right)^{\frac{1-\alpha}{\alpha}}$$

Multilply both sides of (2.13) by $\frac{(u_i p_i)^{\beta}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}}$, summing over $i = 1,2,...,n$ and after then raising both sides to the power α, we get

(2.14) $$\left[\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right]^{\alpha} < D^{(1-\alpha)} \left[\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} q_i^{\alpha-1}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right]$$

since for $0 < \alpha < 1$, $2^{1-\alpha} - 1 > 0$, a simple manipulation in (2.14) proves the Theorem 2 for $0 < \alpha < 1$. Let $1 < \alpha < \infty$, the proof follows on the same lines.

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Analysis of Pulsatile Flow in Elastic Artery

By Anamol Kumar Lal & Dr. Smita Dey
Ranchi University, India

Abstract- In this paper, we have considered a Pulsatile flow in an elastic arterial tube and witnessed the efforts on the flow due to elasticity of the tube. The expression for "Volumetric flow rate" and "impedance of n^{th} harmonic" have been found. Conclusions have been drawn with the aid of graphs. MATLAB software has been used for sketching graphs.

Keywords: elastic artery, shear stress, radial stress.

GJSFR-F Classification : MSC 2010: 00A69, 30C30

ANALYSISOFPULSATILEFLOWINELASTICARTERY

Strictly as per the compliance and regulations of :

Analysis of Pulsatile Flow in Elastic Artery

Anamol Kumar Lal [α] & Dr. Smita Dey [σ]

Abstract- In this paper, we have considered a Pulsatile flow in an elastic arterial tube and witnessed the efforts on the flow due to elasticity of the tube. The expression for "Volumetric flow rate" and "impedance of n^{th} harmonic" have been found. Conclusions have been drawn with the aid of graphs. MATLAB software has been used for sketching graphs.

Keywords: elastic artery, shear stress, radial stress.

Nomenclature :-

Q_n	=	Volumetric flow rate
Z_n	=	Impedance
Q_r	=	Radial Component of Velocity
Q_θ	=	Transverse Component of Velocity
Q_z	=	Axial Component of Velocity
P	=	Pressure

I. Introduction

In a Pulsatile flow in an elastic arterial tube, the following effects on the flow due to the elasticity of the tube take place :-

i. As the wall of the tube is elastic, therefore due to the deformation of the wall, the flow will be radial together with axial.

ii. There is an axial variation of pressure and the shape of the curve between pressure and time will vary with z. Also, the pressure gradient will have a radial component.

iii. The boundary conditions for continuity of shear and radial stresses in the fluid and the elastic material at the common boundary give a coupling between fluid flow and elastic deformation.

Thurston [1] attempted to study all of the rheological properties of blood with a model including non-Newtonian viscosity, viscoelasticity, and thixotropy, Liepsch 'and Moravec [2] investigated the flow of a shear thinning blood, analog fluid in pulsatile flow through arterial branch model and observed large differences in velocity profiles relative to those measured with Newtonian fluids having the high shear rate viscosity of the analog fluid/ Rindt *et al.* [3] considered both experimentally and numerically the two-dimensional steady and pulsatile flow, Nazemi *et al.* [4] made important contributions to the identification of atherogenic sites. Rodkiewicz *et al.* [5] used several different non-Newtonian models for blood for simulation of blood flow in large arteries and they' observed that there is no effect of the yield stress of blood on either the

Author α: Department of Mathematics Marwari College, Ranchi. e-mail: myid.anmol@gmail.com
Author σ: Department of Mathematics Ranchi University, Ranchi.

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1. G.8. Thurston, Rheological parameters for the viscosity, visco-elasticity and thixotropy of blood, *Biorheology* 16, 149-155, (1979).

velocity profiles or the wall shear stress, Boesiger *et al,* [6] used magnetic resonance imaging (MRI) to study arterial homodynamics. Perktold *et al* [7] modeled the flow in stenotic vessels as that of an incompressible Newtonian fluid in the rigid vessels. Sharma and Kapur [8] made a mathematical analysis of blood flow through arteries using finite element method, Dutta and Tarbell [9] studied the two different rheological models of blood displaying shearing thinning viscosity and oscillatory flow visco-elasticity. Lee and Libby [10] made a study of vulnerable atherosclerotic plaque containing a large necrotic core, and covered by this fibrous cap,

Korenga *et al* [11] considered biochemical factors such as gene expression and albumin transport in atherogenessis and in plaque rupture, which were shown to .be activated by hemodynarnic factors in wall shear stress. Rachev *et al* [12] considered a model for geometric and mechanical adaptation of arteries. Rees and Thompson [13] studied a simple model derived from laminar boundary layer theory to investigate the flow of blood in arteria″ stenoses up to Reynolds numbers of 1000. Tang *et al* [14] analysed triggering events, which are believed to be primarily homo-dynamic including cap tension, bending of torsion of the artery. Zendehbudi and Moayery [15] made a comparison of physiological and simple pulsatile flows through stenosed arteries.

Berger- and Jou [16] measured wall shear stress downstream of axi-symmetric stenoses in the presence of hernodynamic forces acting on the plaque, which may be responsible for plaque rupture. Botnar *et al* [17] based on the correspondence between MRI velocity measurements and numerical simulations used two approaches to study in detail the role of different flow patterns for the initiation and amplification of atherosclerotic plaque sedimentation. Stroud *et al* [18] found differences in flow fields and in quantities such as wall shear stress among stenotic vessels with same degree of stenosis. Sharma *et al* [19] made a mathematical analysis of blood flow through arteries using finite element Galerkin approaches.

In the current study, we are interested in the analysis of blood flow in elastic arteries.

Basic equation are Mathematical Information :-

Let (q_r, q_θ, q_z) be the components of velocity in radial, transverse and axial directions respectively.

Due to the assumptions, the velocity profile is given by

$$q_r = q_r(r,z,t),\ q_\theta = 0,\ q_z = q_z(r,z,t)$$

and

$$p = p(r,z,t)$$

The equation of continuity gives

$$\frac{1}{r}\frac{\partial}{\partial r}(r.\, q_r) + \frac{\partial q_z}{\partial z} = 0 \qquad \text{.............(1)}$$

And the equations of motion on neglecting inertial term are given by

$$\rho\frac{\partial q_r}{\partial t} = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 q_r}{\partial r^2} + \frac{\partial^2 q_z}{\partial z^2} + \frac{1\partial\, q_r}{r\ \partial t} - \frac{q_r}{r^2}\right) \qquad \text{................(2)}$$

and

$$\rho\frac{\partial q_z}{\partial t} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 q_z}{\partial r^2} + \frac{\partial^2 q_z}{\partial r^2} + \frac{1\partial\, q_r}{r\ \partial r}\right) \qquad \text{................(3)}$$

Ref

6. P. Boesiger, S.E. Maier, L. Kecheng, M.B. Scheidegger and D. Meier, Visualisation and quantification of the human blood flow by magnetic resonance imaging, *J. of Biomechanics* 25, 55-67, (1992).

Notes

Let (u_r, u_θ, u_z) be the components of deformation vector of the material of the wall of the tube and $\tau_{rr}, \tau_{r\theta}, \tau_{rz}, \tau_{\theta\theta}, \tau_{\theta z}$ and τ_{zz} are the components of the symmetric stress tensor.

Here, $u_\theta = 0$ and ρ_ω is the density of the material of the wall, G the shear modulus and W the negative mean of normal stress. Then the equation of elasticity are

$$\rho_w \frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} \quad \text{.................................. (4)}$$

$$\rho_w \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} \quad \text{.................................. (5)}$$

$$\tau_{ij} = 2G \in_{ij} - \Omega\delta_{ij} \quad \text{.................................. (6)}$$

$$\delta_{ij} = 1 \text{ If } i = j \text{ and } \delta_{ij} = 0 \; i \neq j \quad \text{.................................. (7)}$$

$$\in_{rr} = \frac{\partial u_r}{\partial r}, \; \in_{r\theta} = 0 = \in_{\theta r}, \in_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \in_{zr} \quad \text{.................................. (8)}$$

$$\in_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta} = 0, \; \in_{\theta r} = 0 = \in_{r\theta} \quad \text{.................................. (9)}$$

$$\in_{zz} = \frac{\partial u_z}{\partial z}, \; \in_{z\theta} = 0 = \in_{\theta z} \quad \text{.................................. (10)}$$

Above equations for $(u_r, 0, u_z)$ become :

$$\rho_w \frac{\partial^2 u_r}{\partial t^2} = \mathrm{G}\left(\frac{\partial^2 u_r}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right) - \frac{\partial \Omega}{\partial r} \quad \text{.................................. (11)}$$

$$\rho_w \frac{\partial^2 u_z}{\partial t^2} = \mathrm{G}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right) - \frac{\partial \Omega}{\partial z} \quad \text{.................................. (12)}$$

And the equation of continuity becomes :

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \quad \text{.................................. (13)}$$

The partial differential equations for q_r, q_z and p are the same as those for $\frac{\partial u_r}{\partial t}, \frac{\partial u_z}{\partial t}$ and Ω and both sets are independent Due to the coupling between fluid flow and elastic deformation, we have following boundary conditions :

(i) From the symmetry of velocity field

$$q_r = 0, \frac{\partial q_z}{\partial r} = 0 \qquad \text{at } r = 0$$

(ii) From the continuity of motion at the interface of wall of the tube, we have

$$q_r = \frac{\partial u_r}{\partial t}, q_z = \frac{\partial u_z}{\partial t} \qquad \text{at } r = a \text{ (inner radius of tube)}$$

(iii) From the continuity of the shear stress and radial stress at the inner wall, we have

$$\mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right) = G\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \qquad \text{at } r = a$$

$$\text{and } -p + 2\mu\frac{\partial q_z}{\partial r} = -\Omega + 2\mathrm{G}\frac{\partial u_r}{\partial r} \qquad \text{at } r = a$$

(iv) It is assumed that the outer wall is constrained radially and axially, then we have

$$G = \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0 \qquad \text{at } r = b \text{ (outer radius)}$$

If the inner wall is perturbed and the perturbations are small, then the boundary condition can be taken the same as that at the undisturbed inner wall. Since the outer wall is constrained radially and axially, therefore we can replace the boundary conditions by some others.

Suppose the solutions of equations (1), (2) and (3) are of the form

$$q_r = \mathrm{U}_1(r)e^{-iy_n z}e^{in\omega t} \qquad \text{.................................. (14)}$$

$$q_z = \mathrm{U}_2(r)e^{-iy_n z}e^{in\omega t} \qquad \text{.................................. (15)}$$

and $$P = P(r)e^{-iy_n z}e^{in\omega t} \qquad \text{.................................. (16)}$$

Using (14), (15) and (16), equations (1), (2) and (3) becomes :

$$\frac{d^2U_1}{dr^2} + \frac{1}{r}\frac{dU_1}{dr} - \frac{U_1}{r^2} - y_n^2 U_1 - \frac{\rho}{\mu} in\omega U_1 = \frac{1}{\mu}\frac{dp}{dr} \qquad \text{............................ (17)}$$

$$\frac{d^2U_2}{dr^2} + \frac{1}{r}\frac{dU_2}{dr} - y_n^2 U_2 - \frac{\rho}{\mu} in\omega U_2 = \frac{1}{\mu}(-iy_r)P \qquad \text{......................... (18)}$$

and $$-iU_2 y_n + \frac{dU_1}{dr} + \frac{U_1}{r} = 0 \qquad \text{.................................. (19)}$$

Let us take $\frac{in\omega}{v} + y_n^2 = K_n^2 (v = \frac{\mu}{\rho})$, So that the equations (17), (18) and (19) reduce to

Notes

$$\frac{d^2U_1}{dr^2}+\frac{1}{r}\frac{dU_1}{dr}-\left(K_n^2+\frac{1}{r}\right)U_1=\frac{1}{\mu}\frac{dp}{dr} \quad \text{.................................. (20)}$$

$$\frac{d^2U_2}{dr^2}+\frac{1}{r}\frac{dU_2}{dr}-K_n^2U_2=-\frac{iy_n}{\mu}p \quad \text{.................................. (21)}$$

and $\frac{d}{dr}(rU_1)=iy_nrU_2$ .... (22)

Notes

Since the expressions

$$\left.\begin{aligned} X &= A_1J_1\left(iy_nr\right)+A_2J_1\left(ik_nr\right) \\ \text{and} \quad Y &= B_1J_0\left(iy_nr\right)+B_2J_0\left(ik_nr\right)\end{aligned}\right] \quad \text{.................................. (23)}$$

are the solutions of the respective equations

$$\left.\begin{aligned} &\frac{d^2X}{dr^2}+\frac{1}{r}\frac{dX}{dr}-\left(K_n^2+\frac{1}{r^2}\right)X=-\frac{in\omega}{v}A_1J_1\left(iy_nr\right) \\ \text{and} \quad &\frac{d^2Y}{dr^2}+\frac{1}{r}\frac{dY}{dr}-K_n^2Y=-\frac{in\omega}{v}B_1J_1\left(iy_nr\right)\end{aligned}\right] \quad \text{........................ (24)}$$

with the help of the equations (17) – (24), we get

$$\left.\begin{aligned} U_1(r) &= -i\left[C_1Y_nJ_1\left(iy_nr\right)+C_2Y_nJ_1(ik_nr)\right] \\ U_2(r) &= -i\left[C_1Y_nJ_0\left(iy_nr\right)+C_2Y_nJ_0(ik_nr)\right] \\ \text{and} \quad P(r) &= -C_1in\omega\rho J_0\left(iy_nr\right)\end{aligned}\right) \quad \text{.............................. (25)}$$

Where C_1 and C_2 are arbitrary constants

Putting these values in (14), (15) and (16), we get general solutions as

$$\left.\begin{aligned} q_r &= -\sum_n i[C_1Y_nJ_1(iy_nr)+C_2Y_nJ_1(ik_nr)]e^{in\omega t-iy_nz} \\ q_z &= -\sum_n i[C_1Y_nJ_0(iy_nr)+C_2Y_nJ_0(ik_nr)]e^{in\omega t-iy_nz} \\ \text{and} \quad P &= -\sum_n C_1in\omega\rho\, J_0(iy_nr)e^{in\omega t-iy_nz}\end{aligned}\right] \quad \text{.......... (26)}$$

Let Q_n be the volumetric flow rate for the n-th harmonic, then

$$Q_n=\int_0^a 2\pi rq_zdr$$

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$$= -2\pi a[C_1 J_1(iy_n a) + C_2 J_2(ik_n a)]\, e^{in\omega t - iy_n z} \quad \text{.................................. (27)}$$

$$[\therefore \int_0^a rJ_0(r)dr = aJ_1(a)]$$

If Z_n be the impedance of *n*-th harmonic, then

$$Z_n = \frac{-in\omega\rho\, [iy_n a J_n(iy_n a)] C_1}{2\pi a^2 [C_1 J_1(iy_n a) C_2 J_1(ik_n a)]} \quad \text{.................................. (28)}$$

Notes

The solution for $\frac{\partial u_r}{\partial t}, \frac{\partial u_z}{\partial t}$ and W are similar to (26), but these solutions will have four more arbitrary constants. The boundary conditions (i) is trivially satisfied by (26). The other six boundary conditions give six equations to determine the six constants. These six equations is equivalent to an equation to find y_n of the form

$$ay_n = f\left(\frac{a^2\omega\rho}{\mu}, \frac{b}{a}, \frac{\rho_w v^2}{Ga^2}, \frac{\rho}{\rho_w}\right)$$

Where, $\frac{a^2\omega\rho}{\mu}$, $\frac{b}{a}$, $\frac{\rho_\omega v^2}{Ga^2}$ and $\frac{\rho}{\rho_\omega}$ are all dimensionless parameters.

II. Numerical Results and Discussion

In order to see the effects of various parameters on volumetric flow rate, impedance etc., the following values of the parameters are taken:

a = 1.0, 0.2, 0.3, 0.4, 0.5 (in cm)

ρ = 1.05 gm/cm^3

μ = 0.04gm/cm^{-sec}

ω = 8 rad./sec

1st set for J_1 $(iy_n a)$ and J_2 $(ik_n a)$ are respectively

–.7, .5, .4, .1, .2″, –.6, .6, .3, .2, .3″

IInd set of values for J_1 $(iy_n a)$ and J_2 $(ik_n a)$ are respectively

–.3, .1, .2, .3, .4″ –0, .6, .7, .5, .1″

IIIrd set of values for J_1 $(iy_n a)$ and J_2 $(ik_n a)$ are

J_1 $(iy_n a)$	J_2 $(ik_n a)$
i. .68, .48, .43, .15, .21 ii. .61, .45, .42, .17, .25 iii. .7, .5, .4, .1, .2	i. .59, .61, .32, .25, .33 ii. .58, .63, .31, .30, .35 iii. .6, .6, .3 .2, .3

It has been observed that:

On changing the set of values for J_1 $(iy_n a)$ and J_2 $(ik_n a)$ arbitrarily within the range -.7 to +.7, the graphs show maximum deflections in the value of $|Q_n|$ when artery radius

= .4 but if 'a' lies between 0.2cm to 0.3 cm, the value of $|Q_n|$ becomes constant for 1st set of values but for 2nd set of values, the value of $|Q_n|$ in creases uniformly upto a =0.3cm (fig, (i)).

In figure (ii), it is observed that for value of 'a' between $a = .3cm$ to $a = .5\ cm$, the value of $|Q_n|$ increase as the value of 'n' increases from $n = 3$ to $n = 4$ i.e. Z_4 has more value of impedance that Z_3.

In fig (iii), if the difference in the value of $J_1\ (iy_n a)$ or $J_2\ (ik_n a)$ for three sets of values are small, then it has been observed that $|Q_n|$ is directly proportional to the value of 'a'.

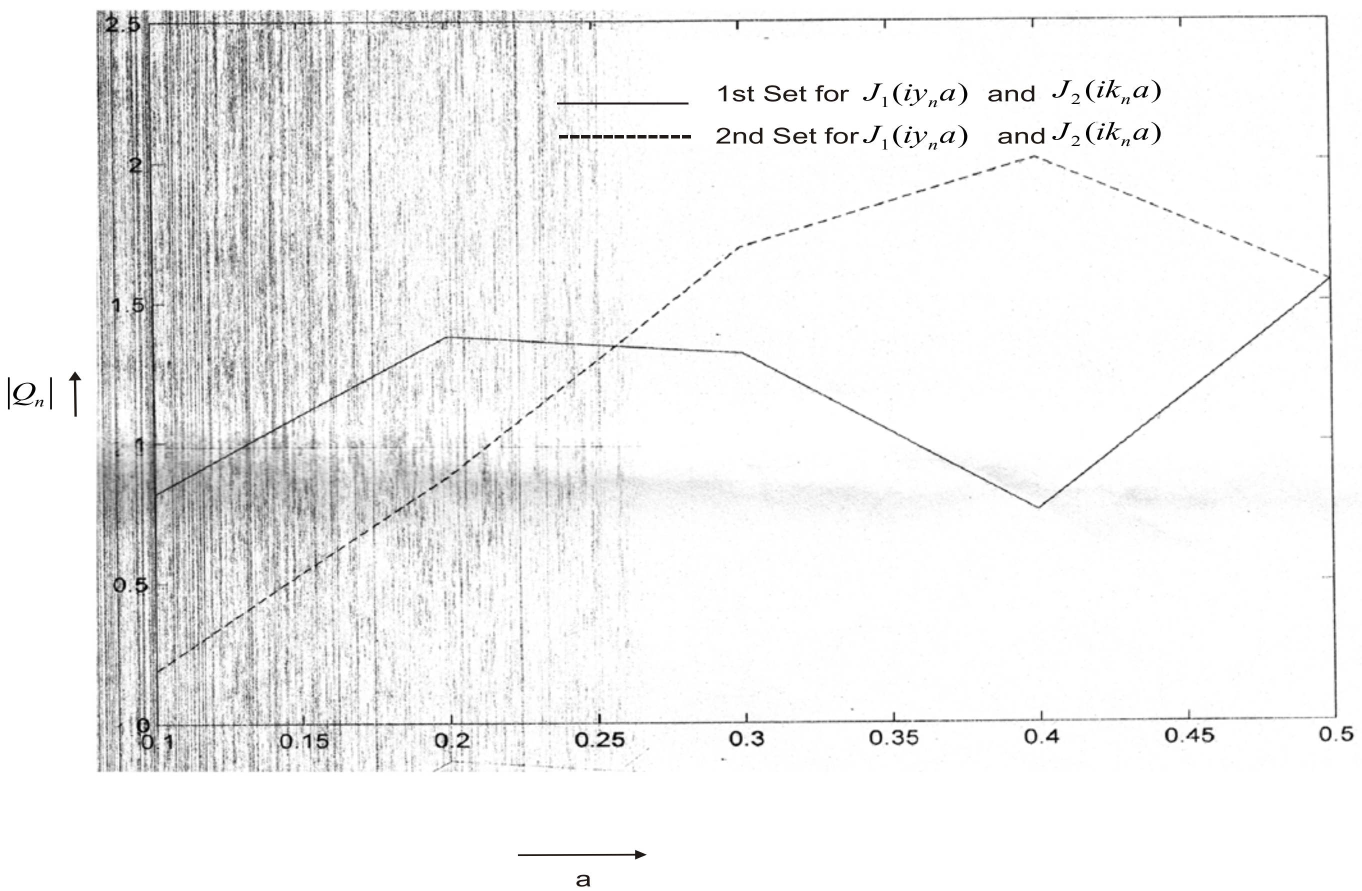

Figure : 1

Fig : Variation of $|Q_n|$ with 'a' for two different sets of values of $J_1(iy_na)$ and $J_2(ik_na)$

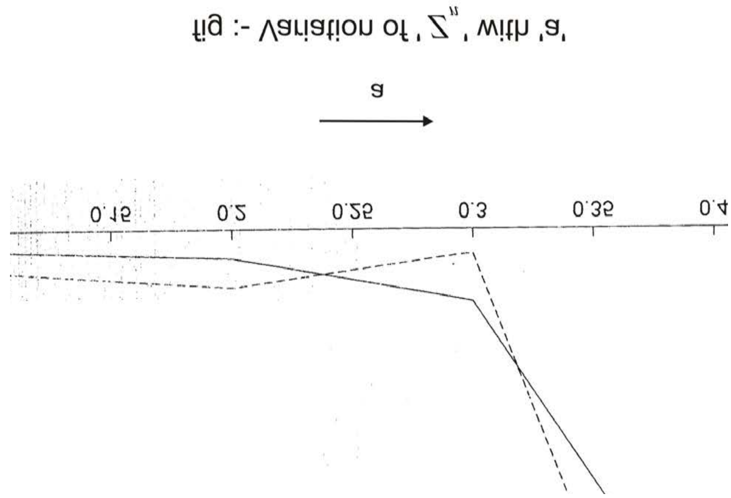

Figure : 2

Notes

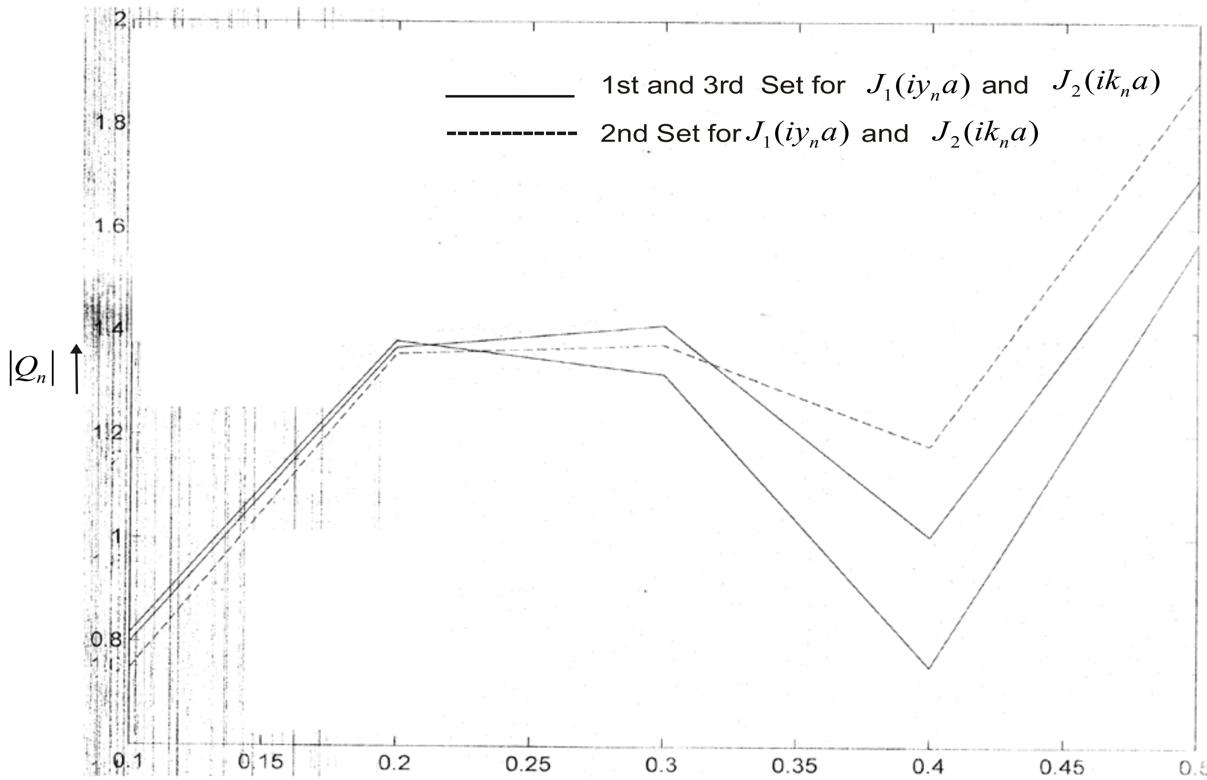

Figure : 3

Fig : Variation of $|Q_n|$ with 'a' for two different sets of values of $J_1(iy_na)$ and $J_2(ik_na)$

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Notes

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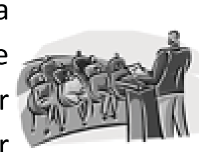

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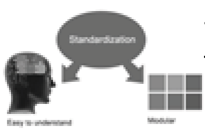

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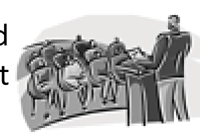

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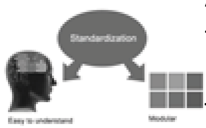

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TECHNIQUES FOR WRITING A GOOD QUALITY RESEARCH PAPER:

1. Choosing the topic: In most cases, the topic is searched by the interest of author but it can be also suggested by the guides. You can have several topics and then you can judge that in which topic or subject you are finding yourself most comfortable. This can be done by asking several questions to yourself, like Will I be able to carry our search in this area? Will I find all necessary recourses to accomplish the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

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15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

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General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

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The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

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- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

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- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

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- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

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- In spite of position, each table must be titled, numbered one after the other and complete with heading
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The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

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- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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