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Highlights

mz-Compact Spaces

Wonderful Summation Formula

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On Certain Bicomplex Duals

Discovering Thoughts, Inventing Future

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Ψ-Eventual Stability of Differential Systems with Impulses

By Anju Sood & Sanjay K Srivastava
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Abstract- In our paper, we have established criterion of Ψ-Eventual Stability for impulsive differential systems with variable moments of impulses by using piecewise continuous auxiliary functions which are analogous to Lyapunov's functions. The work of Bainov, Kulev and Soliman has been extended. An example has been given to support the theoretical result. In the example, the zero solution is not stable in the sense of Lyapunov but it is uniformly eventually stable. Moreover a weight function Ψ is also associated with state vectors.

Keywords: Ψ -eventual stability, Ψ -uniform eventual stability, impulsive differential systems, lyapunov's function.

GJSFR-F Classification : MSC 2010: 34CXX; 34DXX; 34A37; 34K45

Ψ-Eventual Stability of Differential Systems with Impulses

Anju Sood [α] & Sanjay K Srivastava [σ]

Abstract- In our paper, we have established criterion of Ψ-Eventual Stability for impulsive differential systems with variable moments of impulses by using piecewise continuous auxiliary functions which are analogous to Lyapunov's functions. The work of Bainov, Kulev and Soliman has been extended. An example has been given to support the theoretical result. In the example, the zero solution is not stable in the sense of Lyapunov but it is uniformly eventually stable. Moreover a weight function Ψ is also associated with state vectors.

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AMS Mathematics Subject Classification: 34CXX; 34DXX; 34A37; 34K45

I. Introduction

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations of negligible duration in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is: in the form of impulses. It is known for example that many biological phenomenon involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Impulsive differential equations are adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jumps change of their states. For the description of the continuous change of such processes ordinary differential equations are used, while the moments and the magnitude of the change by jumps are given by the jump conditions. According to the way in which the moments of the change by jumps are determined, IDE are classified into two categories: Equations with fixed moments of impulsive effect and equations with unfixed (variable) moments of impulsive effect. The solutions of IDE with variable impulsive moments are piecewise continuous functions but unlike the solutions of the systems with fixed moments of impulse effect, different solutions of these IDE have different points of discontinuity. This leads to number of difficulties in the investigation of IDS with variable impulsive moments. That is why these systems have been an object of numerous investigations.

Moreover, when the trivial solution of the system does not exist, we may still have stability eventually, which generalizes Lyapunov Stability. For example, for the practical point of view, if a ship remains in an upright position, it is called stable. However, since the environmental forces acting on it as well as ship's disposition w.r.t. sea will change in time, the determination of a safe minimum amount of stability i.e.

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stability criterion becomes necessary. If one follows the intuitive concept of stability, it is natural to think that if the amplitude of the ship in every perceived combinations of ship- environment conditions, remain smaller than a pre-determined safe value, the ship should be considered stable, The mathematical counterparts of this definition are eventual stability and boundedness. For the problems arisen in this situations, to be solved a new notion of eventual stability is introduced by V. Lakshmikantham, S. Leela, A. A. Martynyuk.

The problem of Ψ - stability for the systems of ordinary differential equations has been studied by many authors e. g. J. Morchalo [4] introduced the notion of Ψ - stability, Ψ - uniform stability and Ψ - asymptotic stability of trivial solution of non linear system $x' = f(t, x)$. He has considered Ψ as scalar continuous function. Aurel Diamandescu [2] and [3] has introduced Ψ -stability by taking Ψ as diagonal matrix.

In [6], the criteria of eventual stability are established for impulsive differential systems with fixed moments of impulses by using piecewise Lyapunov functions by Zhang Yu. In [5], Soliman extended the notion of eventual stability to impulsive differential systems with variable moments.

In our paper we have established criterion of Ψ-Eventual Stability for impulsive differential systems with variable moments of impulses by using piecewise continuous auxiliary functions which are analogous to Lyapunov's functions. The paper is organized as follows: In section 2, some preliminary notes and definitions which will be used throughout the paper are introduced. In section 3, two theorems for Ψ - Eventual Stability and Ψ -Uniform Eventual Stability are proved. One example has been given in support of our theoretical results. Conclusion is given in Section 4.

II. Preliminary notes and Definitions

Let R^n denote n -dimensional Euclidean space with norm ‖. ‖

Let R_H^s be the s - dimensional Euclidean space with a suitable norm ‖. ‖. Let $R^n = [0, \infty)$ and $R_H^s = \left\{ x \in R^n : \|x\| < H \right\}$.

Consider the system of differential equations with impulses

$$\begin{aligned} x' &= f(t,x)+g(t,y), \quad t \ne \tau_i(x,y) \qquad \Delta x = A_t(x)+B_t(y), \quad t=\tau_i(x,y) \\ y' &= h(t,x,y), \qquad\qquad t \ne \tau_i(x,y) \qquad \Delta y = C_t(x,y), \quad t=\tau_i(x,y) \end{aligned} \tag{2.1}$$

Where $x \in R^n, y \in R^m,\ f: R^+ \times R_H^n \to R^n,\ g: R^+ \times R_H^m \to R^m,\ h: R^+ \times R_H^n \times R_H^m \to R^m$

$A_t : R_H^n \to R^n,\ B_t : R_H^m \to R^m,\ C_t : R_H^n \times R_H^m \to R^m,\ \tau_i : R_H^n \times R_H^m \to R^1$

$$\Delta x\Big|_{t=\tau(x,y)} = x(t+0)-x(t-0), \quad \Delta y\Big|_{t=\tau(x,y)} = y(t+0)-y(t-0).$$

Let $t_0 \in R^+$, $x_0 \in R_H^n$, $y_0 \in R_H^m$ Let $x(t, t_0, x_0, y_0),\ y(t, t_0, x_0, y_0)$ be the solution of the system (2.1), satisfying the initial conditions $x(t_0+0, t_0, x_0, y_0) = x_0$, $y(t_0+0, t_0, x_0, y_0) = y_0$ The solution $(x(t),\ y(t))$of the system (2.1) are piecewise continuous functions with points of discontinuities of the first type in which they are left continuous, i.e. at the moment t_i when the integral curve of the solution $(x(t),\ y(t))$ meets the hypersurface.

$$\sigma_t = \{(t,x,y) \in R^+ \times R_H^n \times R_H^m : t = \tau_i(x,y)\}$$

The following relations are satisfied:

$$x(t_i-0) = x(t_i),\ \ x(t_i+0) = x(t_i) + A_i(x(t_i)) + B_i(y(t_i))$$

$$y(t_i-0) = x(t_i),\ \ y(t_i+0) = y(t_i) + C_i(x(t_i), y(t_i))$$

Let $\tau_0(x,y) = 0$ for $(x,y) \in R_H^n \times R_H^m$.

Ref

[3]. Diamandescu, On the Ψ -asymptotic stability of nonlinear systems of differential equations, Analele Stntifice ale Universitatii, Mathematica, f2(2008), 293-306.

Following [1], we define the sets $G_i = \{(t,x,y) \in R^+ \times R_H^n \times R_H^m : \tau_{i-1}(x,y) < t < \tau_i(x,y)\}$.
Let $\Psi : R^+ (0, \infty)$ be a continuous finite function such that $\Psi(t_0) = \Psi_0$
Definition1. Let the sets K & K_1 be defined as

$$K = \{w \in C(R^+, R^+): \text{ strictly increasing and } w(0) = 0\},$$

$$K_1 = \{\phi \in C(R^+, R^+) : \text{ increasing and } \phi(s) < s \text{ for } s > 0\}$$

Ref

We use the classes ν_0 of piecewise continuous functions which are analogue to Lyapunov functions.
Definition2. We say that the function $V : R^+ \times R_H^n \times R_H^m \to R^+$ belongs to the class ν_0 if the following conditions hold:

1. The function V is continuous in $\cup_{i=1}^{\infty} G_i$ and is locally Lipschitzian with respect to x and y in each of the sets G_i
2. $V(t,0,0)=0$ for $t \in R^+$.
3. For each i =1,2,3................. and for any point $(t_0, x_0, y_0) \in \sigma_t$, there exist the finite limits

$$V(t_0 - 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0) \\ (t,x,y)\in G_i}} V(t,x,y)$$

$$V(t_0 + 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0) \\ (t,x,y)\in G_{i+1}}} V(t,x,y)$$

and the equality $V(t_0 - 0, x_0, y_0) = V(t_0, x_0, y_0)$ holds.

4. For any point $(t, x, y) \in \sigma_i$ the following inequality holds:

$$V(t+0, x + A_t(x) + B_t(y),\ y + C_t(x,y) \le V(t,x,y). \tag{2.2}$$

Let $V \in \nu_0$. For $(t,x,y) \in \cup_{i=1}^{\infty} G_i$ Following [1], we define

$$V'_{(2.1)}(t,x,y) = \lim_{s\to 0} \sup \frac{1}{s}[V(t+s, x+s(f(t,x)+g(t,y)),\ y + s\,h(t,x,y)) - V(t,x,y)]$$

Definition3 The set $\{(x,y) \in R_H^n \times R_H^m : x = 0 \text{ and } y = 0\}$ of the system (2.1) is said to be

1. Eventually stable if for all $\varepsilon > 0$ for all $t_0 \in R^+$ there exists $\tau = \tau(\in) > 0$ and $\delta = \delta(t_0, \varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|x_0\| + \|y_0\| < \delta$ implies $\|x(t,t_0,x_0,y_0)\| + \|y(t,t_0,x_0,y_0)\| < \varepsilon$, $t \ge t_0 \ge \tau(\in)$
2. Uniformly eventually stable if $\delta = \delta(\varepsilon)$ i.e. δ is independent of t_0.
3. Ψ - eventually stable if for all $\varepsilon > 0$, for all $t_0 \in R^+$, there exists $\tau = \tau(\in) > 0$ and $\delta = \delta(t_0, \varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon$ for $\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta$ and $t \ge t_0 \ge \tau(\in)$.
4. Ψ - uniformly eventually stable if for all $\varepsilon > 0$, for all $t_0 \in R^+$, there exists $\tau = \tau(\in) > 0$ and $\delta = \delta(\varepsilon) > 0$ for all $(x_0, y_0) \in R_H^n \times R_H^m$ such that $\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\| < \varepsilon$ for $\|\Psi_0 x_0\| + \|\Psi_0 y_0\| < \delta$ and $t \ge t_0 \ge \tau(\in)$.

Definition 4. We say that conditions (A) hold if the following are satisfied:

(A1) The functions $f(t,x)$, $g(t,y)$ and $h(t,x,y)$ are continuous in their definition domains, $f(t,0) = g(t,0) = 0$ and $h(t,0,0) = 0$ for $t \in R^+$.

(A2) The functions A_t, B_t and C_t are continuous in their definition domains and $A_{t(0)} = B_t(0) = C_t(0,0) = 0$

[1]. G. K. Kulev and D. D. Bainov, Asymptotic stability of systems with impulses by the direct method of Lyapunov, Bull Austral. Math. Soc., 38(1988), 113-123.

(A3) $x \in R_H^n$ *and* $y \in R_H^m$ then $\| xy + A_t(x) + B_t(y) \| \le \|x\|$ and $\| y + C_t(x,y) \| \le \|y\|$,

(A4) The functions τ_i (x, y) are continuous and for $(x, y) \in {}_H^n \times {}_H^m$, the following relations hold:

$$0 < \tau_1(x,y) < \tau_2(x,y) < .... < \lim_{t\to\infty} \tau_i(x,y) = \infty \text{ uniformly in } R_H^n \times R_H^m \text{ and}$$

$$\inf_{R_H^n \times R_H^m} \tau_{i+1}(x,y) - \sup_{R_H^n \times R_H^m} \tau_i(x,y) \ge \theta > 0,\ i = 1,2.............$$

(A5) For each point $(t_0, x_0, y_0) \in R^+ \times R_H^n \times R_H^m$ the solution $x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0)$ is unique and defined in (t_0,∞).
(A6) The integral curve of each of the solutions of system (2.1) meets each of the hyper surfaces$\{\sigma_i\}$ at most once.

III. Main Results

In this section, we extend the work of Kulev and Bainov [1] and A. A. Soliman [5] and established Ψ - eventually stability and Ψ - uniformly eventually stability for impulsive differential system with variable moments.
Theorem 1. Assume that
(H1) Conditions (A) holds.
(H2) There exists functions $V \in v_0,\ a \in K$ such that $a(\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\|) \le V(t,x,y)$, $(t,x,y) \in R^+ \times R_H^n \times R_H^m$ where Ψ is a function defined in section 2.
(H3) $V'_{(2.1)}(t,x,y) \le p(t)\, w(v(t,x,y))$ *for* $(t,x,y) \in G_i$ $i = 1,2,3,..............$, the functions $p, w: R^+ \to R^+$ are locally integrable.
(H4) There exists a no. L > 0 such that $|w((t,x,y))| \le L : (t,x,y) \in R^+ \times R_H^n \times R_H^m$ and $\int_{t_0}^{\infty} |p(s)|\, ds < \infty$
Then the set $\{(x,y) \in R_H^n \times R_H^m : x = 0 \text{ and } y = 0\}$ is Ψ - eventually stable set of the system (2.1).

Proof: Let $\varepsilon > 0$ be given and let the number $\tau = \tau(\varepsilon) > 0$ be chosen so that for $t \ge \tau(\varepsilon)$

$$\int_t^{\infty} |p(s)|\, ds < \frac{a(\varepsilon)}{2L} \quad \text{(This is possible because of condition (H4))} \tag{3.1}$$

Let $t_0 \ge \tau(\varepsilon)$ From property 4 of definition 2 , it follows that there exists a number $\delta(t_0, \in) > 0$ such that for

$$\|\Psi_0 x_0 + \Psi_0 y_0\| < \delta(t_0, \in),\ V(t_0 + 0, x_0, y_0) \le \frac{a(\varepsilon)}{2} \tag{3.2}$$

From (H3), (H4) and (3.2), we get

$$\int_{t_0}^{t} V'_{(2.1)}(s, x(s), y(s))\, ds \le \int_{t_0}^{t} p(s)\, w\big(v(s,x(s),y(s))\big)\, ds \le \int_{t_0}^{t} |p(s)| \big|w\big(v(s,x(s),y(s))\big)\big|\, ds$$

$$\le L \int_{t_0}^{t} |p(s)|\, ds \ \le L \int_{t_0}^{\infty} |p(s)|\, ds < L \frac{a(\varepsilon)}{2L} = \frac{a(\varepsilon)}{2} \text{ for } t \ge t_0 \tag{3.3}$$

Without loss of generality, Let $\tau_{k+l} < t < \tau_{k+l+1}$

Now $\int_{t_0}^{t} V'_{(2.1)}(s,x(s),y(s))\, ds \le \int_{t_0}^{\tau_1} V'_{(2.1)}(s,x(s),y(s))\, ds + \sum_{j=2}^{k+l} \int_{\tau_{j-1}}^{\tau_j} V'_{(2.1)}(s,x(s),y(s))\, ds + \int_{\tau_{k+l}}^{t} V'_{(2.1)}(s,x(s),y(s))\, ds$

$$= V(\tau_1, x(\tau_1), y(\tau_1)) - V(t_0 + 0,\ x(t_0), y(t_0)) + \sum_{j=2}^{k+l} \{V(\tau_j, x(\tau_j), y(\tau_j)) - V(\tau_{j-1} + 0, x(\tau_{j-1} + 0), y(\tau_{j-1} + 0))\}$$

Notes

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$$+V(t,x(t),y(t))-V(\tau_{k+l}+0,x(\tau_{k+l}+0),y(\tau_{k+l}+0))$$

$$=V(\tau_1,x(\tau_1),y(\tau_1))-V(t_0+0,x(t_0),y(t_0))+V(\tau_2,x(\tau_2),y(\tau_2))-V(\tau_1+0,x(\tau_1+0),y(\tau_1+0))$$
$$+V(\tau_3,x(\tau_3),y(\tau_3))-V(\tau_2+0,x(\tau_2+0),y(\tau_2+0))+\ldots$$
$$+V(\tau_{k+l},x(\tau_{k+l}),y(\tau_{k+l}))-V(\tau_{k+l-1}+0,x(\tau_{k+l-1}+0),y(\tau_{k+l-1}+0))$$
$$+V(t,x(t),y(t))-V(\tau_{k+l}+0,x(\tau_{k+l}+0),y(\tau_{k+l}+0))$$
$$\geq V(t,x(t),y(t))-V(t_0+0,x(t_0),y(t_0))$$

Notes

From (H2), (3.1), (3.2) and (3.3)

$$a(\|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|)\leq V(t,x,y)\leq V(t_0+0,x(t_0),y(t_0))+\int_{t_0}^{t}V'_{(2.1)}(s,x(s),y(s))\,ds<$$

$$\frac{a(\varepsilon)}{2}+\frac{a(\varepsilon)}{2}=a(\varepsilon)\text{ for } t\geq t_0\geq\tau(\varepsilon)$$

Thus for all $\varepsilon>0$, for all $t_0\in R^+$, there exists $\tau=\tau(\in)>0$ and $\delta=\delta(t_0,\varepsilon)>0$ for all $(x_0,y_0)\in R_H^n\times R_H^m$ such that $\|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|<\varepsilon$ for $\|\Psi_0x_0\|+\|\Psi_0y_0\|<\delta$ and $t\geq t_0\geq\tau(\in)$

Hence the set $\{(x,y)\in R_H^n\times R_H^m : x=0 \text{ and } y=0\}$ is Ψ - eventually stable set of the system (2.1).

Theorem 2. Assume that (H1) and (H3) of Theorem 1 holds. Moreover suppose that
(H5) Let There exists functions $V\in v_0$, $a,b\in K$, $\phi\in K_1$ such that

$$a(\|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|)\leq V(t,x,y)\leq b(\|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|):(t,x,y)\in R^+\times R_H^n\times R_H^m$$

(H6) For all $k\in N,(x,y)\in R_H^n\times R_H^m$, $V(\tau_k,x(\tau_k^-)+A_t(x)+B_t(y),y(\tau_k^-)+C_t(x,y))\leq\phi(V(\tau_k^-+0,x(\tau_k^-),y(\tau_k^-))$

(H7) There exists a constant $A>0$ such that $\int_{\tau_{k-1}}^{\tau_k}|p(s)|\,ds<A$ and $\int_{\mu}^{\phi^-(\mu)}\frac{ds}{w(s)}\geq A$

Then the set $\{(x,y)\in R_H^n\times R_H^m : x=0 \text{ and } y=0\}$ is Ψ - uniformly eventually stable set of the system (2.1).

Proof: Let $\varepsilon>0$ and choose $\delta(\in)>0$, $\tau=\tau(\varepsilon)>0$ such that $\delta<b^{-1}(\phi(a(\varepsilon)))$ for $t_0\geq\tau(\varepsilon)$.
In the following, we prove that for all $(x_0,y_0)(x_0,y_0)\in R_H^n\times R_H^m$,

$$\|\Psi_0x_0\|+\|\Psi_0y_0\|<\delta \text{ implies } \|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|<\varepsilon \text{ for } t\geq t_0\geq\tau(\in).$$

Let $t_0\in(\tau_{m-1},\tau_m)$ i.e. G_m for some $m\in N$
We first prove that $V(t,x,y)\leq\phi^{-1}(b(\delta))$ for $t_0\leq t<\tau_m$ (3.4)

Clearly $V(t_0,x_0,y_0)\leq b(\|\Psi_0x_0\|+\|\Psi_0y_0\|)<b(\delta)<\phi^{-1}(b(\delta))$

Now for $t\in(t_0,\tau_m)$ if (3.1) does not hold, then there exists $\hat{t}\in(t_0,\tau_m)$ such that

$$V(\hat{t},x(\hat{t}),y(\hat{t}))>\phi^{-1}(b(\delta))>b(\delta)\geq V(t_0,x_0,y_0)$$

From the continuity of $V(t,x,y)$ in (τ_{m-1},τ_m) and hence in $\left(t_0,\hat{t}\right)$ there is an $s_1\in\left(t_0,\hat{t}\right)$ such that

$$V(s_1,x(s_1),y(s_1))=\phi^{-1}(b(\delta)) \tag{3.5}$$

$$V(t,x(t),y(t))>\phi^{-1}(b(\delta)):s_1<t<\hat{t}$$
$$V(t,x(t),y(t))\leq\phi^{-1}(b(\delta)):t_0\leq t\leq s_1$$

and also there exists an $s_2\in(t_0,s_1)$ such that

$$V(s_2,x(s_2),y(s_2))=b(\delta) \tag{3.6}$$

$$V(t,x(t),y(t)) \geq b(\delta) : s_2 \leq t \leq s_1$$

Therefore we integrate (H3) between $\left[s_2, s_1\right]$

$$\int_{s_2}^{s_1} \frac{V'_{(2.1)}\left(t,x(t),y(t)\right)}{w\left(V\left(t,x(t),y(t)\right)\right)}dt \leq \int_{s_2}^{s_1} p(t)dt$$

$$\int_{V\left(s_2,x(s_2),y(s_2)\right)}^{V\left(s_1,x(s_1),y(s_1)\right)} \frac{du}{w(u)} \leq \int_{s_2}^{s_1} p(t)dt \leq \int_{\tau_{m-1}}^{\tau_m} p(t)dt < A \tag{3.7}$$

On the other hand from the inequalities (3.5), (3.6) and condition (H7)

$$\int_{V\left(s_2,x(s_2),y(s_2)\right)}^{V\left(s_1,x(s_1),y(s_1)\right)} \frac{du}{w(u)} = \int_{b(\delta)}^{\phi^{-1}(b(\delta))} \frac{du}{w(u)} \geq A$$

which contradicts the inequality (3.7). Therefore our assumption was wrong and hence (3.4) holds.

From condition (H6)

$$V(\tau_m, x(\tau_m), y(\tau_m)) = V(\tau_m, x(\tau_m^-) + A_t(x) + B_t(y), y(\tau_m^-) + C_t(x,y)) \leq \phi(V(\tau_m^-, x(\tau_m^-), y(\tau_m^-))$$

$$\leq \phi(\phi^{-1}(b(\delta)) < b(\delta) \tag{3.8}$$

Next we prove that $V(t,x,y) \leq \phi^{-1}\left(b(\delta)\right)$ for $\tau_m < t < \tau_{m+1}$

If inequality (3.9) does not hold good, there exists $\hat{r} \in (\tau_m, \tau_{m+1})$ such that

$$V(\hat{r}, x(\hat{r}), y(\hat{r})) > \phi^{-1}\left(b(\delta)\right) > b(\delta) \geq V(\tau_m, x(\tau_m), y(\tau_m)) \text{ [using (3.8)]}$$

From the continuity of $V(t,x,y)$ in (τ_m, τ_{m+1}) there is an $r_1 \in (\tau_m, \hat{t})$ such that

$$V\left(r_2, x\left(r_2\right), y\left(r_2\right)\right) = \phi^{-1}\left(b(\delta)\right)$$

$$V\left(t, x(t), y(t)\right) > \phi^{-1}\left(b(\delta)\right) : r_1 < t < \hat{r} \tag{3.10}$$

$$V\left(t, x(t), y(t)\right) \leq \phi^{-1}\left(b(\delta)\right) : \tau_m \leq t \leq r_1$$

and also there exists an $r_2 \in (\tau_m, r_1)$ such that

$$V\left(r_2, x\left(r_2\right), y\left(r_2\right)\right) = b(\delta) \tag{3.11}$$

$$V\left(t, x(t), y(t)\right) \geq b(\delta) : r_2 \leq t \leq r_1$$

Then we integrate the inequality (H3) in $\left[r_2, r_1\right]$ as done before and get contradiction. Hence (3.9) holds.

Thus we see that $V(t,x,y) \leq \phi^{-1}\left(b(\delta)\right)$ in G_i :1,2,3................... and hence in $\cup G_i$ (3.12)

Also as shown in (3.8), $V(\tau_m, x(\tau_m), y(\tau_m) < b(\delta) : m = 1,2,3$................... (3.13)

As $b(\delta) < \phi^{-1}(b(\delta))$, it follows by (3.4), (3.12) and (3.13) that

$$V\left(t, x(t), y(t)\right) < \phi^{-1}\left(b(\delta)\right) < a(\varepsilon)$$

therefore by condition (H5),

$$a(\|\Psi(t)x(t)\| + \|\Psi(t)y(t)\|) \leq V(t,x,y) < a(\varepsilon) \text{ for all } t \geq t_0 \geq \tau(\epsilon).$$

Thus we see that $\|\Psi(t)x(t)\|+\|\Psi(t)y(t)\|<\varepsilon$ whenever $\|\Psi_0 x_0\|+\|\Psi_0 y_0\|<\delta$ for $t\geq t_0\geq\tau(\epsilon)$.

Hence the set $\{(x,y)\in R_H^n\times R_H^m : x=0 \text{ and } y=0\}$ is Ψ - uniformly eventually stable set of the system (2.1).

Example

Consider the system

$$\begin{aligned} x' &= A(t)x(t-r(t))+B(t)y(t) \quad & t\neq\tau_i \quad & x(\tau_k^+)=\alpha' x(\tau_k)+\beta' y(\tau_k) \\ y' &= D(t)y(t), & t\neq\tau_i \quad & y(\tau_k^+)=\gamma' y(\tau_k) \end{aligned} \qquad (*)$$

Where $0<r(t)<r', x\in R, y\in R, A(t), B(t), D(t)\in C(R^+,R)$ and such that $|A(t)|<\alpha, |B(t)|<\beta, |D(t)|<\gamma$

Let us further assume that $x(s)\leq x(t)$ for $t-r'\leq s\leq t$.

Let (i) $\alpha'>0, \beta'>0, \gamma'>0$ and $\alpha+T>\gamma$ (ii) $3\alpha'^2+\beta'^2<2$ (iii) $\beta'^2+\gamma'^2<\alpha'^2$

(iv) $\tau_k-\tau_{k-1}<-\dfrac{\log(3\alpha'^2+\beta'^2)+\log 2}{(2\alpha+\beta)}$

Let us further define the following functions

$$\phi(s)=\frac{(3\alpha'^2+\beta'^2)}{2}s,\ w(s)=s,\quad p(t)=[2\alpha+\beta],\quad \Psi(t)=\underset{t\geq t_0}{\text{Max}}\, e^{-t},\ a,b\in K \text{ such that}$$

$$a(t)=\frac{t^2}{4e^{-t_0}},\ b(t)=\frac{t^2}{2e^{-t_0}} \text{ and } V(t,x,y)=\frac{1}{2}(x^2+y^2)e^{-t_0}$$

Therefore

$$V'(t,x,y)=e^{-t_0}\left(xx'+yy'\right)=e^{-t_0}\{A(t)x(t-r(t))+B(t)y(t)]+y(t)[D(t)y(t)]\}$$

$$\leq e^{-t_0}\{x(t)[\alpha x(t)+\beta y(t)]+y(t)[\gamma y(t)]\}=e^{-t_0}[\alpha x^2(t)+\beta x(t)y(t)+\gamma y^2(t)]$$

$$\leq e^{-t_0}[\alpha x^2(t)+\beta x(t)y(t)+\alpha y^2(t)]\leq e^{-t_0}\{\alpha[x^2(t)+y^2(t)]+\frac{\beta}{2}[x^2(t)+y^2(t)]\}$$

$$\frac{e^{-t_0}(2\alpha+\beta)}{2}[x^2(t)+y^2(t)]=p(t)\frac{[x^2(t)+y^2(t)]}{2}=p(t)w(V(t,x,y))$$

Thus (H3) holds. Let

$$\Psi(t)=\underset{t\geq t_0}{\text{Max}}\, e^{-t}=e^{-t_0}$$

$$a(|\Psi(t)x(t)|+|\Psi(t)y(t)|)=a(|e^{-t_0}x(t)|+|e^{-t_0}y(t)|)=a(|e^{-t_0}|(|x(t)|+|y(t)|))$$

$$=\frac{e^{-2t_0}(|x(t)|+|y(t)|)^2}{4e^{-t_0}}=\frac{e^{-t_0}(|x(t)|^2+|y(t)|^2+2|x(t)||y(t)|)}{4}\leq$$

$$\frac{e^{-t_0}(|x(t)|^2+|y(t)|^2+|x(t)|^2+|y(t)|^2)}{4}==\frac{e^{-t_0}(|x(t)|^2+|y(t)|^2)}{2}=V(t,x,y)$$

Also $b(|\Psi(t)x(t)|+|\Psi(t)y(t)|)=b((|e^{-t_0}||x(t)|+|e^{-t_0}||y(t)|))=b(e^{-t_0}(|x(t)|+|y(t)|))$

$$\frac{e^{-2t_0}(|x(t)|+|y(t)|)^2}{2e^{-t_0}}\geq\frac{e^{-t_0}(|x(t)|^2+|y(t)|^2)}{2e^{-t_0}}=V(t,x,y)$$

Thus (H5) holds. Now

$$V(\tau_k^+,x(\tau_k)+A_t(x)+B_t(y),y(\tau_k)+C_t(x,y))=V(\tau_k^+,x(\tau_k^+),y(\tau_k^+))$$

$$=V(\tau_k,\alpha' x(\tau_k)+\beta' y(\tau_k),\gamma' y(\tau_k))=\frac{e^{-t_0}}{2}[(\alpha' x(\tau_k)+\beta' y(\tau_k))^2+\gamma'^2 y^2(\tau_k)]$$

Notes

$$=\frac{e^{-t_0}}{2}[\alpha'^2 x^2(\tau_k)+\beta'^2 y^2(\tau_k)+2\alpha'\beta' x(\tau_k)y(\tau_k)+\gamma'^2 y^2(\tau_k)]$$
$$\le\frac{e^{-t_0}}{2}\{\alpha'^2 x^2(\tau_k)+\alpha'^2 y^2(\tau_k)+\alpha'\beta'[x^2(\tau_k)+y^2(\tau_k)]\}$$
$$=\frac{e^{-t_0}}{2}(\alpha'^2+\alpha'\beta')[x^2(\tau_k)+y^2(\tau_k)]\le\frac{e^{-t_0}}{2}(\frac{3\alpha'^2+\beta'^2}{2})[x^2(\tau_k)+y^2(\tau_k)]$$
$$=\phi(\frac{e^{-t_0}}{2}[x^2(\tau_k)+y^2(\tau_k)])=\phi(V(\tau_k,x(\tau_k),y(\tau_k)))$$

Thus (H6) holds.

Again, if we choose $A=-\log\frac{3\alpha'^2+\beta'^2}{2}>0$

$$\int_{\tau_{k-1}}^{\tau_k}|p(s)|\,ds=\int_{\tau_{k-1}}^{\tau_k}(2\alpha+\beta)\,ds=(2\alpha+\beta)(\tau_k-\tau_{k-1})$$
$$<[-\log(3\alpha'^2+\beta'^2)+\log 2]=-\log\frac{3\alpha'^2+\beta'^2}{2}=A$$

Also for any $\mu>0$,

$$\int_{\mu}^{\phi^{-1}(\mu)}\frac{ds}{w(s)}\,ds=\int_{\mu}^{\frac{2\mu}{(3\alpha'^2+\beta'^2)}}\frac{ds}{s}\,ds=[\log s]_{\mu}^{\frac{2\mu}{(3\alpha'^2+\beta'^2)}}=\log\frac{2\mu}{(3\alpha'^2+\beta'^2)}-\log\mu$$
$$=\log\frac{2}{(3\alpha'^2+\beta'^2)}=-\log\frac{(3\alpha'^2+\beta'^2)}{2}=A$$

Thus all the conditions of Theorem 2 hold and hence the set $\{(x,y)\in R_H^n\times R_H^m : x=0 \text{ and } y=0\}$ is $\Psi-$uniformly eventually stable set of the system (*).

IV. Conclusion

In [5], no example has been given in support of the results and in example given in [6], zero solution being equilibrium, implies Lyapunov stability and thus difference between Lyapunov stability and Eventual stability has not been shown. But in the example given above, the zero solution is not stable in the sense of Lyapunov as it is not equilibrium but it is uniformly eventually stable. Moreover a weight function Ψ is also associated with state vectors.

Thus our result shows that the system may not be stable in the sense of Lyapunov even then it can be eventually stable.

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Global Stability of Stifiers Impact on Meme Transmission Model

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Abstract- Spreaders and stiflers play a major role in the persistence or disappearance of memes. In this paper we study the impact of stiflers on the dynamics of memes transmission. We introduce and analyze qualitatively a mathematical model considering stiflers effect. Three equilibrium points of the model are examined: meme free equilibrium, meme cessation equilibrium and meme existence equilibrium. The reproduction number R_0 that generates new memes is found. Local and global stability of the equilibrium points are explored. Finally, we support our results using numerical simulations.

GJSFR-F Classification : MSC 2010: 00A69

GLOBALSTABILITYOFSTIFICRSIMPACTONMEMETRANSMISSIONMODEL

Strictly as per the compliance and regulations of :

Ref

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Global Stability of Stifiers Impact on Meme Transmission Model

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Abstract- Spreaders and stiflers play a major role in the persistence or disappearance of memes. In this paper we study the impact of stiflers on the dynamics of memes transmission. We introduce and analyze qualitatively a mathematical model considering stiflers effect. Three equilibrium points of the model are examined: meme free equilibrium, meme cessation equilibrium and meme existence equilibrium. The reproduction number R_0 that generates new memes is found. Local and global stability of the equilibrium points are explored. Finally, we support our results using numerical simulations.

I. Introduction

In recent times, spread of ideas is easily done through the use of fast tech communications. A unit of idea transmission is defined by Dawkins [1] as meme or a unit of imitation. Style of living, eating, clothing and thinking are some examples of memes. Individuals reaction towards memes is either acceptation or rejection. Those who imitate certain memes and is manifested in their life style help in spreading it, they are known as spreaders. On the other hand, those who refuse or not interested in adopting a meme are known as sti ers. Spreaders and sti ers play a significant role in the survival or extinction of memes.

There is a similarity between the propagation of memes and the spread of rumors. Both may be seen as a virus, that is transmitted to another individual by a certain mean. Researchers have applied epidemiological models to study the dynamics of social systems. In particular, the dynamics of rumors spread and transmission of ideas and thoughts. This was based on the fact that both biological diseases and social behavior are a result from interactions between individuals. Daley and kendall are among the earliest researchers to propose a rumor spread model that has some properties in common with epidemic model [2]. Also Cane [3] showed the similarity between the deterministic forms of models for the spread of an epidemic and of a rumor. At the beginning of this century, Thompson et al. [4] explored the dynamics of rumor spreading in chat rooms. Bettencourt et al. [5] applied models similar to epidemiology to the spread of ideas. Kawachi [6] and Kawachi et al. [7] proposed deterministic models for rumor transmission and explored the effects of various contact interactions. Al-Amoudi et al. [8, 9] analyzed qualitatively constant and variable meme propagation models. Piqueira [10] examined an equilibrium study of a rumor spreading model according to propagation parameters and initial conditions. Huang [11] studied the rumor spreading process with denial and skepticism. Wang and Wood [12] adopted an epidemiological approach to model viral meme propagation. Zhao et al. [13, 14] proposed rumor spreading models in social networks considering the forgetting mechanism of spreaders. Huo et al. [15] investigated the psychological effect with rumor spreading in emergency event. Zhao and Wang [16] established a dynamic rumor model considering the medium as a subclass. Recently, Afassinou [17] analyzed the impact of education rate on rumor final size. Finally, Zan et al. [18] examined the dynamics or rumor spread with counterattack mechanism and self-resistance parameter.

In this paper, we investigate the impact of sti ers on the transmission of memes in a population with constant immigration and emigration. We analyze the dynamics of the model qualitatively. The formulation of the model and its equilibria and basic reproduction number are described in section 2. Section 3 analyzes the stability of equilibria both locally and globally using linearization methods and Lyapunov method. Numerical simulations are illustrated in section 4. A brief conclusion is given in Section 5.

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II. Mathematical Model

Let $N(t)$ denote the total population. We assume that the population is divided into three disjoint classes of individuals: $S(t)$, the susceptible class, describing individuals who have not yet been exposed to a particular meme; $I(t)$, the spreader class, referring to individuals who have taken an active interest in the idea or concept that a meme represents, and therefore have a tendency to talk about the meme in social interactions; $Z(t)$, the sti er class, meaning individuals who reject the meme and have no interest to embrace it. Sti ers may play an in u ential role in memes cessation. We explore this assumption by introducing a new term in the dynamical model given by [8], where we consider that susceptibles become sti ers when contacting them at a certain rate.

The mathematical model is governed by the following nonlinear system of ordinary differential equations:

$$\begin{aligned} \frac{dS}{dt} &= B - \alpha SI - \delta SZ - \mu S, \\ \frac{dI}{dt} &= \alpha\theta SI - \beta I^2 - \gamma IZ - \mu I, \\ \frac{dZ}{dt} &= \alpha(1-\theta)SI + \beta I^2 + \gamma IZ + \delta SZ - \mu Z, \end{aligned} \tag{1}$$

where the positive parameters are defined as follows: B is the sum of the population's birth rate and immigration rate; μ is the sum of the population's death rate and emigration rate; α is the rate at which susceptibles change their meme class, where $\alpha = cq$ such that c is the average number of contact per unit time and q is the probability of transmitting the meme; β is the rate at which spreaders become stiflers by contacting with each other; γ is the rate at which spreaders become stiflers by contacting with stiflers; θ is the fraction of susceptibles who become spreaders (at rate α); $1-\theta$ is the fraction of susceptibles who become stiflers (at rate α), such that $\theta \in (0,1]$; and δ is the rate at which susceptibles become stiflers by contacting with stiflers.

Note that $N(t) = S(t) + I(t) + Z(t)$. It follows from system (1) that $N'(t) + \mu N(t) = B$, which has the solution $N(t) = N_0 \exp(-\mu t) + [1 - \exp(-\mu t)]B/\mu$, where $N_0 = N(0)$, and therefore, $\lim_{t\to\infty} N(t) = B/\mu$. Thus, the considered region for system (1) is

$$\Gamma = \{(S,I,Z) : S + I + Z \le \frac{B}{\mu}, S > 0, I \ge 0, Z \ge 0\}.$$

The vector field points into the interior of Γ on the part of its boundary when $S + I + Z = B/\mu$. Hence, Γ is positively invariant.

We find the equilibria of the model by equating to zero the right hand side of system (1):

$$\begin{aligned} B - \alpha SI - \delta SZ - \mu S &= 0, \\ \alpha\theta SI - \beta I^2 - \gamma IZ - \mu I &= 0, \\ \alpha(1-\theta)SI + \beta I^2 + \gamma IZ + \delta SZ - \mu Z &= 0. \end{aligned} \tag{2}$$

Solution to system (2) gives three equilibrium points: the meme free equilibrium $E_0 = (B/\mu, 0, 0)$; the meme cessation equilibrium $E_1 = (\frac{\mu}{\delta}, 0, \frac{B}{\mu} - \frac{\mu}{\delta})$, which exists if and only if $\delta > \mu^2/B$; and meme existence equilibrium $E^* = (S^*, I^*, Z^*)$, where

$$\begin{aligned} S^* &= \frac{B}{\alpha I^* + \delta Z^* + \mu}, \\ I^* &= \frac{\alpha\theta S^* - \gamma Z^* - \mu}{\beta} \\ Z^* &= \frac{-\alpha(1-\theta)S^* I^* - \beta {I^*}^2}{\delta S^* + \gamma I^* - \mu}. \end{aligned}$$

The basic reproduction number $\mathcal{R}_0$ may be calculated by the method of next generation matrix [19]. Let $X = (I, Z, S)^T$, then system (1) may be written as: $X' = \mathcal{F}(X) - \mathcal{V}(X)$ where

$$\mathcal{F}(X) = \begin{bmatrix} \alpha\theta SI \\ \alpha(1-\theta)SI \\ 0 \end{bmatrix}, \mathcal{V}(X) = \begin{bmatrix} \beta I^2 + \gamma IZ + \mu I \\ -\beta I^2 - \gamma IZ - \delta SZ + \mu Z \\ -B + \alpha SI + \delta SZ + \mu S \end{bmatrix}.$$

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The Jacobian matrices of $\mathcal{F}(X)$ and $\mathcal{V}(X)$ at the meme free equilibrium point E_0, are

$$D\mathcal{F}(E_0)=\begin{bmatrix} \alpha\theta S_0 & 0 & 0 \\ \alpha(1-\theta)S_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix},$$

$$D\mathcal{V}(E_0)=\begin{bmatrix} \mu & 0 & 0 \\ 0 & -\delta S_0 + \mu & 0 \\ \alpha S_0 & \delta S_0 & \delta S_0 + \mu \end{bmatrix} = \begin{bmatrix} V & 0 \\ J_1 & J_2 \end{bmatrix},$$

where $F = \begin{bmatrix} \alpha\theta S_0 & 0 \\ \alpha(1-\theta)S_0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} \mu & 0 \\ 0 & -\delta S_0 + \mu \end{bmatrix}$, $J_1 = [\ \alpha S_0 \ \ \delta S_0\ ]$, $J_2 = [\ \delta S_0 + \mu\ ]$.

Thus the next generation matrix is $FV^{-1} = \begin{bmatrix} \dfrac{\alpha\theta B}{\mu^2} & 0 \\ \dfrac{\alpha(1-\theta)B}{\mu^2} & 0 \end{bmatrix}$. Clearly, the spectral radius of matrix FV^{-1} is $\rho(\ FV^{-1}) = \alpha\theta B/\mu^2$. So, the basic reproduction number of the system is $\mathcal{R}_0 = \alpha\theta B/\mu^2$.

III. Stability of the Equilibria

3.1 Local stability

Here we investigate the local stability of E_0, E_1 and E^*. We state the following theorems:

Theorem 1 ***(Local stability of*** E_0***)*** *If* $\mathcal{R}_0 < 1$*and* $\delta < \mu^2/B$*, the meme free equilibrium point* E_0 *is locally asymptotically stable. If* $\mathcal{R}_0 = 1$ *and* $\delta < \mu^2/B$ *or* $\mathcal{R}_0 < 1$ *and* $\delta = \mu^2/B$*,* E_0 *is locally stable. If* $\mathcal{R}_0 > 1$ *or* $\delta > \mu^2/B$*,* E_0 *is unstable.*

Proof. Linearizing system (1) (by linearization method [20]) we obtain the Jacobian matrix evaluated at the equilibrium E_0:

$$J\,(E_0) = \begin{bmatrix} -\mu & \dfrac{-\alpha B}{\mu} & \dfrac{-\delta B}{\mu} \\ 0 & \dfrac{\alpha\theta B}{\mu} - \mu & 0 \\ 0 & \dfrac{\alpha\,(1-\theta)\,B}{\mu} & \dfrac{\delta B}{\mu} - \mu \end{bmatrix}.$$

Clearly the roots of the characteristic equation are: $\lambda_1 = -\mu < 0$; $\lambda_2 = \mu(\dfrac{\delta B}{\mu^2} - 1) < 0$, if $\delta < \mu^2/B$; and $\lambda_3 = \mu(\dfrac{\alpha\theta B}{\mu^2} - 1) = \mu\,(\mathcal{R}_0 - 1) < 0$, if $\mathcal{R}_0 < 1$. Hence, E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and $\delta < \mu^2/B$. If $\mathcal{R}_0 = 1$ and $\delta < \mu^2/B$ then the eigenvalues are λ_1, $\lambda_2 < 0$ and $\lambda_3 = 0$. Also if $\delta = \mu^2/B$ and $\mathcal{R}_0 < 1$ then the eigenvalues are $\lambda_1 < 0$, $\lambda_2 = 0$ and $\lambda_3 < 0$. So, E_0 is locally stable. If $\mathcal{R}_0 > 1$ or $\delta > \mu^2/B$, then the characteristic equation has a positive eigenvalue. So, E_0 is unstable. ■

Theorem 2 ***(Local stability of*** E_1***)*** *Let* $\mathcal{R}_1 = \dfrac{\delta(B\gamma + \mu^2)}{\mu^2(\gamma + \alpha\theta)}$*. If* $\mathcal{R}_1 > 1$*, the meme cessation equilibrium point* E_1 *is locally asymptotically stable. If* $\mathcal{R}_1 = 1$*,* E_1 *is locally stable. If* $\mathcal{R}_1 < 1$*,* E_1 *is unstable.*

Proof. The Jacobian matrix at the equilibrium E_1 gives:

$$J\,(E_1) = \begin{bmatrix} -\dfrac{\delta B}{\mu} & -\dfrac{\alpha\mu}{\delta} & -\mu \\ 0 & \dfrac{\alpha\theta\mu}{\delta} - \gamma(\dfrac{B}{\mu} - \dfrac{\mu}{\delta}) - \mu & 0 \\ \dfrac{\delta B}{\mu} - \mu & \dfrac{\alpha(1-\theta)\mu}{\delta} - \gamma(\dfrac{B}{\mu} - \dfrac{\mu}{\delta}) & 0 \end{bmatrix}.$$

The roots of the characteristic equation are: $\lambda_1 = -\mu < 0$; $\lambda_2 = \dfrac{1}{\mu\delta}\left(\mu^2(\gamma + \alpha\theta) - \delta(B\gamma + \mu^2)\right) < 0$, if $\mathcal{R}_1 > 1$; and $\lambda_3 = -\dfrac{1}{\mu}(B\delta - \mu^2) < 0$. Hence, E_1 is locally asymptotically stable if $\mathcal{R}_1 > 1$. If $\mathcal{R}_1 = 1$ then the eigenvalues are λ_1, $\lambda_3 < 0$ and $\lambda_2 = 0$. So, E_1 is locally stable. If $\mathcal{R}_1 < 1$ then the characteristic equation has a positive eigenvalue. So, E_1 is unstable. ■

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Theorem 3 ***(Local stability of*** E^****)*** *If* $a_1 > 0$, $a_3 > 0$ *and* $a_1a_2 > a_3$, *the meme existence equilibrium* E^* *is locally asymptotically stable.*

Proof. The Jacobian matrix at the equilibrium E^* gives:

$$J(E^*) = \begin{bmatrix} -\alpha I^* - \delta Z^* - \mu & -\alpha S^* & -\delta S^* \\ \alpha\theta I^* & \alpha\theta S^* - 2\beta I^* - \gamma Z^* - \mu & -\gamma I^* \\ \alpha(1-\theta) I^* + \delta Z^* & \alpha(1-\theta) S^* + 2\beta I^* + \gamma Z^* & \gamma I^* + \delta S^* - \mu \end{bmatrix}.$$

This may be simplified using system (2) to be

$$J(E^*) = \begin{bmatrix} -\dfrac{B}{S^*} & -\alpha S^* & -\delta S^* \\ \alpha\theta I^* & -\beta I^* & -\gamma I^* \\ \alpha(1-\theta)I^* + \delta Z^* & \beta I^* + \dfrac{\mu Z^*}{I^*} - \dfrac{\delta S^* Z^*}{I^*} & \delta S^* + \gamma I^* - \mu \end{bmatrix}.$$

The characteristic equation about E^* is $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where

$$\begin{aligned} a_1 &= \beta I^* + \mu + \frac{B}{S^*} - \delta S^* - \delta S^*, \\ a_2 &= \alpha\delta(1-\theta)S^*I^* + \delta^2 S^*Z^* + \gamma\mu Z^* - \gamma\delta S^*Z^* - \delta B - \delta\beta S^*I^* \\ &\quad -\frac{\gamma B I^*}{S^*} + \frac{B\mu}{S^*} + \mu\beta I^* + \frac{\beta B I^*}{S^*} + \alpha^2\theta S^* I^*, \\ a_3 &= \alpha\theta\delta\mu S^*Z^* - \delta^2\alpha\theta {S^*}^2 Z^* + \alpha\beta\delta S^* {I^*}^2 + \beta\delta^2 S^*I^*Z^* + \frac{\gamma B\mu Z^*}{S^*} \\ &\quad -B\gamma\delta Z^* - \alpha^2\gamma S^* {I^*}^2 - \alpha\gamma\delta S^*I^*Z^* - \delta B\beta I^* - \alpha^2\theta\delta {S^*}^2 I^* \\ &\quad +\frac{\beta B\mu I^*}{S^*} + \alpha^2\theta\mu S^* I^*. \end{aligned}$$

If $a_1 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$, then by using Routh-Herwitz Criteria [21] all eigenvalues of $J(E^*)$ have negative real parts. Thus, E^* is locally asymptotically stable. ∎

3.2 Global stability

First, we explore the global stability of E_0 Consider the Lyapunov function [22]:

$$L = I + Z.$$
$$\frac{dL}{dt} = (-\mu + \alpha S) I + (-\mu + \delta S)Z.$$

Since $E_0 \in \Gamma$ then $S \leq \dfrac{B}{\mu}$ and we have

$$\frac{dL}{dt} \leq (-\mu + \frac{\alpha B}{\mu})I + (-\mu + \frac{\delta B}{\mu})Z \leq 0 \text{ if } \alpha B < \mu^2 \text{ and } \delta < \mu^2/B.$$

It follows that $\dfrac{dL}{dt} < 0$ if $\alpha B \leq \mu^2$ and $\delta < \mu^2/B$; with $\dfrac{dL}{dt} = 0$ if and only if $I = Z = 0$. Hence, the only solution of system (1) in Γ on which $\dfrac{dL}{dt} = 0$ is E_0 . Therefore, by LaSalle's Invariance Principle [22], every solution of system (1), with initial conditions in Γ, approaches E_0 as $t \to \infty$. Thus, E_0 is globally asymptotically stable and we may state the following theorem.

Theorem 4 ***(Global stability of*** E_0***)*** *If* $\alpha B \leq \mu^2$ *and* $\delta < \mu^2/B$ *then* E_0 *is globally asymptotically stable in* Γ.

Next, we examine the global stability of E_1 and E^*. We may define the same Lyapunov function for both equilibriums. Consider the Lyapunov function:

$$L = \frac{1}{2}\left[(S - S^*) + (I - I^*) + (Z - Z^*)\right]^2.$$
$$\frac{dL}{dt} = \left[(S - S^*) + (I - I^*) + (Z - Z^*)\right]\left[B - \mu S - \mu I - \mu Z\right].$$

Using $B = \mu S^* + \mu I^* + \mu Z^*$, we have

Ref

[22] M.W.Hirsch, S.Smale, and R.L.Devaney, Differential equations, dynamical systems and an introduction to chaos, Elsevier Academic press (1974).

$$\frac{dL}{dt} = -\mu\left[(S - S^*) + (I - I^*) + (Z - Z^*)\right]^2 \leq 0.$$

Hence, E_1 and E^* are globally stable and we may state the following theorem.

Theorem 5 ***(Global stability of*** E_1 *and* E^****)*** *The equilibrium points* E_1 *and* E^* *are globally stable whenever they exist.*

Notes

We summarize the result of this section as follows:

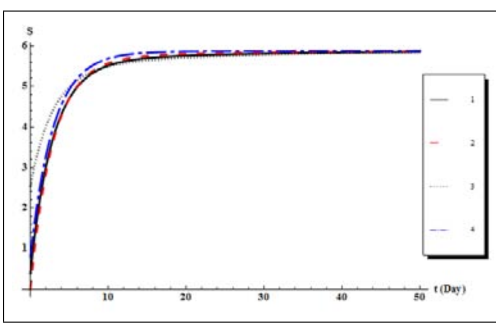
- If $\mathcal{R}_0 < 1$ and $\delta < \mu^2/B$, then E_0 is locally asymptotically stable. If $\alpha B \leq \mu^2$ and $\delta < \mu^2/B$ then E_0 is globally asymptotically stable.
- If $\mathcal{R}_1 > 1$ then E_1 is locally asymptotically stable. But E_1 is globally stable unconditionally whenever it exists.
- If $a_1 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$ then E^* is locally asymptotically stable. But E^* is globally stable unconditionally whenever it exists.

IV. Numerical Simulation

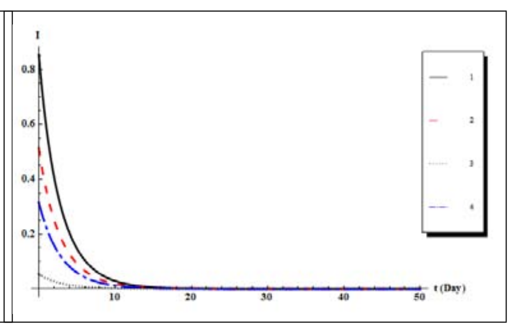
In this section, we illustrate numerical simulations of system (1) using different values of the parameters to support our results. Four different initial values are chosen such that $S + I + Z \leq B/\mu$:

1. S(0) =0.3890, I(0) =0.8540, Z(0) = 0.5360,
2. S(0) = 0.0010, I(0) = 0.5140, Z(0)= 0.3600,
3. S(0) = 2.5485, I(0) = 0.0540, Z(0) = 0.8675,
4. S(0) = 0.8000, I(0) = 0.3154, Z(0) = 0.0250.

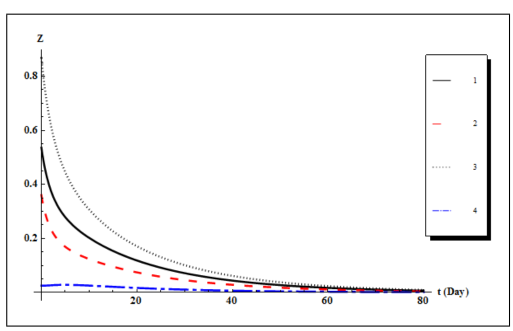
(a) Using the parameters: $\beta = 0.05, \mu = 0.34, \quad = 0.015, \alpha = 0.0125, \theta = 0.333, B = 2, \delta = 0.05$. Here $\mathcal{R}_0 = 0.0720156 < 1$. We see from Fig. 1(a) that the number of susceptibles to the meme increases as a function of time to approach the value of S_0 for the four sets of initial conditions. While Fig. 1(b,c) show that the number of spreaders and sti ers decreases as a function of time and approaches zero. Thus, for all sets of initial conditions the solution curves tend to the meme free equilibrium E_0. Hence, system (1) is locally asymptotically stable about E_0 for the above set of parameters.



(a)

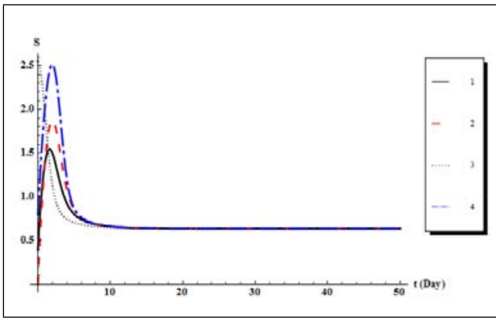


(b)

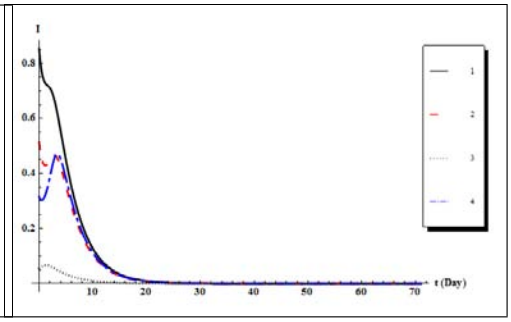


(c)

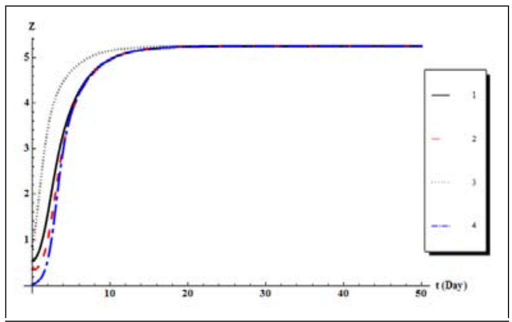
Figure 1 : Time plots of systems (1) with different initial conditions for $R_0 < 1$: (a) Susceptibles; (b) Spreaders; (c) Stiflers.



(a)



(b)

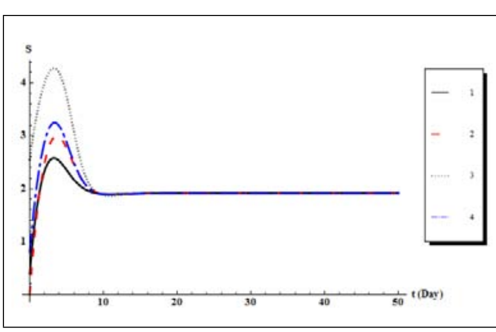


(c)

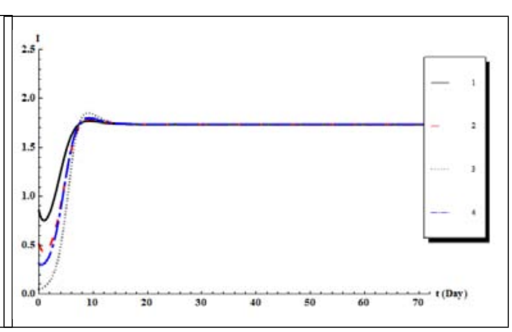
Figure 2 : Time plots of systems (1) with different initial conditions for R_0, $R_1 > 1$: (a) Susceptibles; (b) Spreaders; (c) Stiflers.

(b) Using the same initial conditions and parameters as in (a) except for: $\alpha = 0.4, \theta = 0.6, \delta = 0.535$. Here $\mathcal{R}_0 = 4.15225 > 1$ and $\mathcal{R}_1 = 2.6425 > 1$. We see from Fig. 2(a) that for the second value of parameters the number of susceptibles increases first then it starts to decrease and all solutions approach a certain value S_1. Fig. 2(b) shows that the number of spreaders at the beginning of the meme increases slightly, then it starts to decrease and finally approaches the value $I_1 = 0$. Fig. 2(c) shows a different behavior for the sti ers, they grow at first then they approach a certain value Z_1. Thus, for all sets of initial conditions the solution curves tend to the meme cessation equilibrium E_1. Hence, system (1) is locally asymptotically stable about E_1 for the above set of parameters.

(c) Using the same initial conditions and parameters as in (b) except for: $\delta = 0.0035$. Here $\mathcal{R}_0 = 4.15225 > 1$ and $\mathcal{R}_1 = 0.002 < 1$. We see from Fig. 3(a) that when the parameter δ is very small the number of susceptibles increases first then it starts to decrease and all solutions approach a certain value S^*. Fig. 3(b) shows that the number of spreaders at the beginning of the meme decreases slightly, then it starts to increase and finally approaches the value I^*. Fig. 3(c) shows a similar behavior for the sti ers, they decrease slightly at first then they increase to approach a certain value Z^*. Thus, for all sets of initial conditions the solution curves tend to the meme existence equilibrium E^*. Hence, system (1) is locally asymptotically stable about E^* for the above set of parameters.



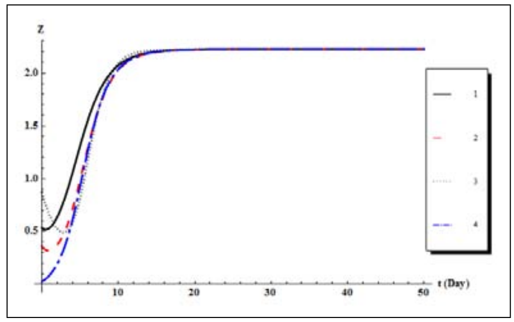
(a)



(b)

Notes

Notes



(c)

Figure 3 : Time plots of systems (1) with different initial conditions for $R_0, > 1$ and $R_1 < 1$: (a) Susceptibles; (b) Spreaders; (c) Stiflers.

V. Conclusions

In this paper, a nonlinear memes transmission model with sti ers effect on susceptibles in is analyzed. Sufficient conditions have been given ensuring local and global stability of the three equilibrium points. First, the meme free equilibrium point E_0 is shown to be locally asymptotically stable whenever the basic reproduction number R_0 of the model is less than unity and when the condition $\delta < \mu^2/B$ is satisfied. In addition, if $\alpha B \leq \mu^2$ is satisfied, then E_0 becomes globally asymptotically stable and the meme will disappear. Second, the meme cessation equilibrium point E_1, which exists only if $\delta > \mu^2/B$, is shown to be locally asymptotically stable if $\mathcal{R}_1 > 1$. Moreover, it is globally stable with no conditions. Thus, if E_1 exists the meme will eventually end. Lastly, the meme existence equilibrium E^*, if it exists, is shown to be locally asymptotically stable if the conditions: $a_1 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$ are satisfied and it is globally stable with no conditions also. Therefore, if E^* exists the meme will persists. Finally, some numerical simulations are used to support the qualitative results. In conclusion, sti ers in uence produces a new equilibrium point which eventually ceases the meme from spreading if the analytical conditions are satisfied.

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Notes

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On Certain Bicomplex Duals

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Abstract- We investigated certain bicomplex duals of the class of bicomplex sequences defined by Srivastava & Srivastava [S2] and of the subclasses defined by Nigam [N1] and Wagh [W2]. Köthe & Toeplitz duals for bicomplex sequence spaces, defined and studied in [W3], have been extended further. Two types of duals namely $\alpha\beta$- dual and $\beta\alpha$- dual have been defined and relations between these duals and the duals of classes defined by [S2] and subclasses defined by [W2] have been established. Relation between these duals and the i_2 – conjugate of a bicomplex number is also studied.

Keywords: bicomplex numbers, bicomplex sequences, i_2 - conjugate, köthe & töeplitz duals.

GJSFR-F Classification : MSC 2010: 46A45, 46B45

ONCERTAINBICOMPLEXDUALS

Strictly as per the compliance and regulations of :

Ref

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On Certain Bicomplex Duals

Mamta Amol Wagh [α] & Sanjeev Kumar [σ]

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AMS Subject Classification: 46A45, 46B45.

I. Introduction

Bicomplex Numbers were introduced by Corrado Segre (1860–1924) in 1892. In [S1], he defined an infinite set of algebras and gave the concept of multicomplex numbers. For the sake of brevity, we confine ourselves to the bicomplex version of his theory. The space of bicomplex numbers is the first in an infinite sequence of multicomplex spaces. The set of bicomplex numbers is denoted by C_2 and is defined as follows:

$$C_2 = \{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : x_1, x_2, x_3, x_4 \in C_0\} \text{ Or equivalently}$$

$$C_2 = \{z_1 + i_2 z_2 : z_1, z_2 \in C_1\}$$

Where $i_1^2 = i_2^2 = -1$, $i_1 i_2 = i_2 i_1$ and C_0, C_1 denote the space of real and complex numbers respectively. The binary compositions of addition and scalar multiplication on C_2 are defined coordinate wise and the multiplication in C_2 is defined term by term. With these binary compositions, C_2 becomes a commutative algebra with identity. Algebraic structure of C_2 differs from that of C_1 in many respects [P1]. Few of them, which pertain to our work, are mentioned below:

a) Idempotent Elements

Besides 0 and 1, there are exactly two nontrivial idempotent elements in C_2 defined as

$$e_1 = (1 + i_1 i_2)/2, \quad e_2 = (1 - i_1 i_2)/2.$$

Note that $e_1 + e_2 = 1$ and $e_1.e_2 = e_2.e_1 = 0$.

A bicomplex number $\xi = z_1 + i_2 z_2$ has a unique idempotent representation, [S3] as

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$$\xi = {}^1\xi\, e_1 + {}^2\xi\, e_2$$

$$\text{where } {}^1\xi = z_1 - i_1 z_2,\ {}^2\xi = z_1 + i_1 z_2 .$$

b) Two Principal Ideals

The Principal Ideals in C_2 generated by e_1 and e_2 are denoted by I_1 and I_2 respectively; thus

$$I_1 = \{\xi e_1 : \xi \in C_2\},$$

$$I_2 = \{\xi e_2 : \xi \in C_2\}.$$

Since $\xi = {}^1\xi e_1 + {}^2\xi e_2$, where ${}^1\xi$ and ${}^2\xi$ are the idempotent components of ξ, therefore these ideals can also be represented as

$$I_1 = \{(z_1 - i_1 z_2)e_1 : z_1, z_2 \in C_1\} = \{z\, e_1 : z \in C_1\}$$

$$I_2 = \{(z_1 + i_1 z_2)e_2 : z_1, z_2 \in C_1\} = \{z\, e_2 : z \in C_1\}$$

Note that $I_1 \cap I_2 = \{0\}$ and $I_1 \cup I_2 = O_2$, the set of all singular elements of C_2.

c) Zero Divisors

As we have seen, $e_1 . e_2 = e_2 . e_1 = 0$. Thus zero divisors exist in C_2.

In fact, two Bicomplex are divisors of zero if and only if one of them is a complex multiple of e_1 and the other is a complex multiple of e_2. In other words, two Bicomplex numbers are divisors of zero if and only if one of them is a member of $I_1 \sim$ –0″ and the other is a member of $I_2 \sim$ –0″.

d) Conjugates of a Bicomplex number

The i_2- conjugate of a bicomplex number is defined in [P1] as follows:

$$\xi^{\#} = z_1 - i_2\, z_2 = {}^2\xi\, e_1 + {}^1\xi\, e_2 \text{ where } \xi = z_1 + i_2 z_2$$

e) Norm of a Bicomplex number

The norm in C_2 is defined as

$$\|\xi\| = \left\{ |z_1|^2 + |z_2|^2 \right\}^{1/2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2} \right]^{1/2}$$

C_2 becomes a modified Banach algebra with respect to this norm in the sense that

$$\| \xi.\eta \| \le \sqrt{2} \| \xi \| . \| \eta \|$$

f) Holomorphic functions

Let X be a domain in C_2. A function $f : X \to C_2$ is said to be holomorphic function if $\forall \alpha \in X, \exists$ a discus $D(\alpha; r_1, r_2)$ with $r_1 > 0$, $r_2 > 0$ and a bicomplex power series representation in D such that

$$f(\xi) = \sum_{k \ge 1} \eta_k (\xi - \alpha)^k \ \forall \xi \in D(\alpha; r_1, r_2).$$

H(X) denotes the set of all holomorphic functions on C_2.

Ref

7. [P1] Price, G. Baley : "An Introduction to Multicomplex spaces and functions",Marcel Dekker, Inc., 1991.

g) Entire Bicomplex Sequence

If $f(\xi) = \sum_{k\geq 1} \alpha_k (\xi - \eta)^k$ represents an entire function, the series $\sum \alpha_k$ is called entire bicomplex series and the sequence $\{\alpha_k\}$ is called entire bicomplex sequence.

II. PRELIMINARIES

I. The Classes B, B', B" Of Entire Bicomplex Sequences

$$B = \left\{ \begin{array}{l} f : f = \{ \xi_k \} = \{ {}^1\xi_k e_1 + {}^2\xi_k e_2 \} : \\ \sup_{k\geq 1} k^k \left| {}^1\xi_k \right| < \infty, \sup_{k\geq 1} k^k \left| {}^2\xi_k \right| < \infty \end{array} \right\}$$

Srivastava & Srivastava [S2] defined the above class B. Every element of class B is the sequence of coefficients of an entire function and is, therefore, an entire bicomplex sequence.

Two subclasses of the class B have been defined in [N1],[W1] as follows:

$$B' = \left\{ f : f = \{ {}^1\xi_k e_1 \} : \sup_{k\geq 1} k^k \left| {}^1\xi_k \right| < \infty \right\}$$

$$B'' = \left\{ f : f = \{ {}^2\xi_k e_2 \} : \sup_{k\geq 1} k^k \left| {}^2\xi_k \right| < \infty \right\}$$

The elements of B′ and B′ ′ are the sequences with members in A_1 and A_2, respectively where A_1 and A_2 are the auxiliary space.

Note first that B′ is closed with respect to the binary compositions induced on B′ as a subset of B, owing to the consistency of idempotent representation and the algebraic structure of bicomplex numbers.

Norm in B' is defined as follows:

$$\| f \| = \sup_{k\geq 1} \left| {}^1\xi_k \right| , \quad f = \{ {}^1\xi_k . e_1 \} \in B'.$$

B′ is a Gelfand subalgebra of B (cf. [N1, W1]).

B′ is an algebra ideal of B which is not a maximal ideal [W2].

a) Duals Of Sequence Spaces

There are two types of dual of a sequence space, namely Algebraic dual and Topological dual. The set of all linear functionals, on a linear space V, with domain as V and range as K is denoted by L(V, K) = $V^{\#}$ and is called algebraic dual of V. If we consider the set of all continuous linear functional, then we get topological dual denoted by V *. From the point of view of the duality theory, the study of sequence spaces is much more profitable. Köthe & Toeplitz were the first to recognize the problem that it is difficult to find the topological duals of sequence spaces equipped with linear topologies. To resolve it, they introduced a kind of dual, α – dual, in quite many familiar and useful sequence spaces. In the same paper [K2], they also introduced another kind of dual namely β – dual which together with the given sequence space forms a nice dual system. A still more general notion of a dual, γ – dual was later introduced by Garling [G1]. For symmetric sequence spaces there is another notion of a dual, called a δ – dual due to Garling [G2] and Ruckle [R1].

Ref

13. [W2] Wagh, Mamta A. : "On a Class of Bicomplex Sequences ", Inter. J. Trends in Math. And Statistics, June 2014, pp. 158 – 171.

III. Bicomplex Köthe – Töeplitz Duals [W3]

If ω' is the family of all bicomplex sequences $\xi=(\xi_k)$ with $\xi_k \in C_2$, k ≥ 1 , where C_2 is the space of all bicomplex numbers. If λ be a bicomplex sequence space, then we denote α - , β - , γ- , and δ – duals of λ respectively by $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ *and* λ_δ.

These bicomplex duals are given as follows:

Definition II.1 $\underline{\alpha - dual}$: $\lambda_\alpha = \left\{ \xi : \xi \in \omega', \sum_{i\geq 1} \|\xi_i \eta_i\| < \infty, \forall \eta_i \in \lambda \right\}$

Definition II.2

$\underline{\beta - dual}$: $\lambda_\beta = \left\{ \xi : \xi \in \omega', \left\| \sum_{i\geq 1} \xi_i \eta_i \right\| < \infty, \forall \eta_i \in \lambda \right\}$ Definition II.3

$\underline{\gamma - dual}$: $\lambda_\gamma = \left\{ \xi : \xi \in \omega', \sup_n \left\| \sum_{i=1}^{n} \xi_i \eta_i \right\| < \infty, \forall \eta_i \in \lambda \right\}$

Definition II.4

$\underline{\delta - dual}$:

$$\lambda_\delta = \left\{ \xi : \xi \in \omega', \sum_{i\geq 1} \|\xi_i \eta_{\rho(i)}\| < \infty, \forall \eta_i \in \lambda \text{ and } \rho \in \pi \right\}$$

where π is the set of all permutations of N.

IV. Main Results

αβ - Dual and βα - Duals

Due to idempotent representation of the bicomplex numbers we have 16 types of duals, including the four types we have already defined, for bicomplex sequence Spaces.

These duals are denoted as

α α – dual, α β – dual, α γ – dual, α δ – dual
β α – dual, β β – dual, β γ – dual, β δ – dual
γ α – dual, γ β – dual, γ γ – dual, γ δ –dual
δ α – dual, δ β – dual , δ γ – dual, δ δ – dual

Note that α α – dual, β β –dual , γ γ –dual , δ δ – dual are same as α - , β - , γ- , and δ – duals defined above, respectively.

In particular, we give definition of αβ – dual and βα – dual of B.

Definition III.1 *αβ – dual of class B*

$$B_{\alpha\beta} = \left\{ (\eta_k) \in \omega' : \sum_{k\geq 1} \left| {}^1\xi_k \, {}^1\eta_k \right| < \infty, \left| \sum_{k\geq 1} {}^2\xi_k \, {}^2\eta_k \right| < \infty, \; \forall (\xi_k) \in B \right\}$$

Now we show how αβ – dual of class B is connected with the α – dual of its first idempotent sequence and β – dual of the second idempotent sequence.

Theorem III.1: A sequence (η_k) belongs to $\alpha\beta$ - dual of the class B if and only if its first idempotent sequence belongs to α - dual of the class B' and the second idempotent sequence belongs to β - dual of the class B'', that is

$$(\eta_k) \in B_{\alpha\beta} \Leftrightarrow ({}^1\eta_k e_1) \in (B')_\alpha \text{ and } ({}^2\eta_k e_2) \in (B'')_\beta$$

Notes

Notes

Proof: We know that B' and B'' are the idempotent spaces of the class B. Therefore every element of the class B can be written as a sum of elements from B' and B''.
That is

$$(\xi_k)=({}^1\xi_k e_1)+({}^2\xi_k e_2)$$

where

$$({}^1\xi_k e_1)\in B' \text{ and } ({}^2\xi_k e_2)\in B''$$

and From def. III.1,

$$(\eta_k)\in B_{\alpha\beta} \Leftrightarrow \sum_{k\geq1}\left|{}^1\xi_k\, {}^1\eta_k\right|<\infty, \left|\sum_{k\geq1} {}^2\xi_k\, {}^2\eta_k\right|<\infty,$$
$$\forall(\xi_k)=({}^1\xi_k e_1)+({}^2\xi_k e_2)\in B$$

Or

$$(\eta_k)\in B_{\alpha\beta} \Leftrightarrow \sum_{k\geq1}\left|{}^1\xi_k\, {}^1\eta_k\right|<\infty, \forall({}^1\xi_k e_1)\in B'$$

and

$$\left|\sum_{k\geq1} {}^2\xi_k\, {}^2\eta_k\right|<\infty, \forall({}^2\xi_k e_2)\in B''$$

Hence, $(\eta_k)\in B_{\alpha\beta} \Leftrightarrow ({}^1\eta_k e_1)\in(B')_\alpha \text{ and } ({}^2\eta_k e_2)\in(B'')_\beta$, from def. 4.6, 4.10.

Definition III.2 $\beta\alpha$ – *dual of class B*

$$B_{\beta\alpha}=\left\{\begin{array}{l}(\eta_k)\in\omega':\left|\sum_{k\geq1} {}^1\xi_k\, {}^1\eta_k\right|<\infty\ ,\\ \sum_{k\geq1}\left|{}^2\xi_k\, {}^2\eta_k\right|<\infty\ ,\ \forall(\xi_k)\in B\end{array}\right\}$$

We get a similar result as theorem III.1:

Theorem III.2: A sequence (η_k) belongs to $\beta\alpha$ - dual of the class B if and only if its first idempotent sequence belongs to β - dual of the class B' and the second idempotent sequence belongs to α - dual of the class B'', that is

$$(\eta_k)\in B_{\beta\alpha} \Leftrightarrow ({}^1\eta_k e_1)\in(B')_\beta \text{ and } ({}^2\eta_k e_2)\in(B'')_\alpha$$

Proof: similar as theorem III.1.

Theorem III.3: If a sequence (η_k) belongs to $\alpha\beta$ - dual of the class B, then the sequence made up of i_2 - conjugates of these sequences will be contained in $\beta\alpha$ - dual of B.
That is if $(\eta_k)\in B_{\alpha\beta}$, then $(\eta_k)^{\#}\in B_{\beta\alpha}$.

$$({}^1\eta_k e_1+{}^2\eta_k e_2)\in B_{\alpha\beta}\ ,\text{ then } ({}^2\eta_k e_1+{}^1\eta_k e_2)\in B_{\beta\alpha}$$

Proof: Given that $(\eta_k)\in B_{\alpha\beta}$

That is, $\left({}^{1}\eta_k\, e_1 + {}^{2}\eta_k\, e_2\right) \in B_{\alpha\beta}$

Then from def. III.1, $\sum_{k\geq 1}\left|{}^{1}\xi_k\ {}^{1}\eta_k\right| < \infty, \left|\sum_{k\geq 1} {}^{2}\xi_k\ {}^{2}\eta_k\right| < \infty, \forall\left(\xi_k\right)\in B$ (i)

As (i) holds for every sequence of B and if a sequence belongs to B then i_2 – conjugate of that sequence will also belong to the class B.

Thus (i) holds for $\left({}^{2}\xi_k\, e_1 + {}^{1}\xi_k\, e_2\right)$

That is $\sum_{k\geq 1}\left|{}^{2}\xi_k\ {}^{1}\eta_k\right| < \infty, \left|\sum_{k\geq 1} {}^{1}\xi_k\ {}^{2}\eta_k\right| < \infty$

Hence from def. III.2, $\left(\eta_k\right)^{\#} = \left({}^{2}\eta_k\, e_1 + {}^{1}\eta_k\, e_2\right) \in B_{\beta\alpha}$

Remark 1 : Converse of above statement is also true, that is, If a sequence $\left(\eta_k\right)$ belongs to $\beta\alpha$ - dual of the class B, then the sequence made up of i_2 - conjugates of these sequences will be contained in $\alpha\beta$ - dual of B.

That is if $\left({}^{1}\eta_k\, e_1 + {}^{2}\eta_k\, e_2\right) \in B_{\beta\alpha}$, then $\left({}^{2}\eta_k\, e_1 + {}^{1}\eta_k\, e_2\right) \in B_{\alpha\beta}$

Remark 2 : From the above two remarks we can say that $\left(\eta_k\right) \in B_{\beta\alpha} \Leftrightarrow \left(\eta_k\right)^{\#} \in B_{\alpha\beta}$, as i_2 - conjugate is symmetric.

Remark 3 : Thus $B_{\beta\alpha}$ can be treated as i_2 - conjugate of $B_{\beta\alpha}$.

Remark 4 : Due to the relation between convergence and absolute convergence of complex sequences we get $B_{\alpha\alpha} \subset B_{\alpha\beta}$ and $B_{\beta\alpha} \subset B_{\beta\beta}$.

Remark 5 : B is not a Köthe – space, or not a perfect sequence space, since $B_{\alpha\alpha} \neq B$. In fact, $B_{\alpha\alpha} = B_{\alpha}$.

Theorem III.4: $\phi \subset (B)_{\delta} \subset (B)_{\alpha} \subset (B)_{\beta} \subset (B)_{\gamma}$ (ii)

Proof: We know that the class B which is a class of bicomplex sequences, has two subclasses B' and B'' given by

$$B' = \left\{ f : f = \left\{{}^{1}\xi_k e_1\right\} : \sup_{k\geq 1} k^k \ \left|{}^{1}\xi_k\right| < \infty \right\}$$

$$B'' = \left\{ f : f = \left\{{}^{2}\xi_k e_2\right\} : \sup_{k\geq 1} k^k \left|{}^{2}\xi_k\right| < \infty \right\}$$

Let $B' = A_1 e_1$, where $A_1 = \left\{\left(x_k\right) : \left(x_k\, e_1\right) \in B'\right\}$

and $B'' = A_2\, e_2$, where $A_2 = \left\{\left(y_k\right) : \left(y_k\, e_2\right) \in B''\right\}$

that is, A_1 and A_2 are complex sequence spaces and are idempotent components of the class B. Therefore, B can be written as $B = A_1\, e_1 + A_2\, e_2$.

Define the mappings f_1 *and* f_2 as follows:

$$f_1 = B \to A_1 \text{ by } f_1(\xi_k) = (x_k) ,$$

$$f_2 = B \to A_2 \text{ by } f_2(\xi_k) = (y_k)$$

Such that $(\xi_k) = (x_k)e_1 + (y_k)e_2$. Here x_k works as $\left({}^{1}\xi_k\right)$ and y_k works as $\left({}^{2}\xi_k\right)$.

We know that the relation (ii) holds for a sequence space in C_0 and C_1 (c.f. [K2]).

Notes

Thus $\phi \subset (A_1)_\delta \subset (A_1)_\alpha \subset (A_1)_\beta \subset (A_1)_\gamma$

and $\phi \subset (A_2)_\delta \subset (A_2)_\alpha \subset (A_2)_\beta \subset (A_2)_\gamma$

Hence, we can say that $\phi \subset (B)_\delta \subset (B)_\alpha \subset (B)_\beta \subset (B)_\gamma$.

Remark 4.6:

$\phi \subset B_{\delta\delta} \subset B_{\alpha\alpha} \subset B_{\alpha\beta}$ or $B_{\beta\alpha} \subset B_{\beta\beta} \subset B_{\beta\gamma}$ or $B_{\gamma\beta} \subset B_{\gamma\gamma}$

Notes

V. Conclusion

Present investigation confirmed that first idempotent sequence of the members of $B_{\alpha\beta}$ is also the member of B'_α and the second idempotent sequence belongs to B''_β. Furthermore, i_2 – conjugate of $B_{\alpha\beta}$ is also associated with $B_{\beta\alpha}$.

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Generalized Methods for Generating Moments of Continuous Distribution

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Abstract- We propose a method of obtaining the moment of some continuous bi-variate distributions with parameters α_1, β_1 *and* α_2, β_2 in finding the nth moment of the variable $x^c y^d \left(c \geq 0, d \geq 0\right)$ where X and Y are continuous random variables having the joint pdf, f(x,y).Here we find t he so called $g_n(c,d)$ defined $g_n(c,d) = E\left(X^c Y^d + \lambda\right)^n$, the nth moment of expected value of the t distribution of the cth power of X and dth power of Y about the constant λ.These moments are obtained by the use of bi-variate moment generating functions, when they exist. The proposed $g_n(c,d)$ is illustrated with some continuous bi-variate distributions and is shown to be easy to use even when the powers of the random variables being considered are non-negative real numbers that need not be integers. The results obtained using $g_n(c,d)$ are the same as results obtained using other methods such as moment generating functions when they exist.

Keywords: moment generating functions, bivariate distributions, continuous random variables, joint pdf.

GJSFR-F Classification : MSC 2010, 46C50

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Generalized Methods for Generating Moments of Continuous Distribution

Oyeka ICA [α] & Okeh UM [σ]

Abstract- We propose a method of obtaining the moment of some continuous bi-variate distributions with parameters α_1, β_1 *and* α_2, β_2 in finding the nth moment of the variable $x^c y^d \left(c \geq 0, d \geq 0\right)$ where X and Y are continuous random variables having the joint pdf, f(x,y).Here we find the so called $g_n(c,d)$ defined $g_n(c,d) = E\left(X^c Y^d + \lambda\right)^n$, the nth moment of expected value of the t distribution of the cth power of X and dth power of Y about the constant λ.These moments are obtained by the use of bi-variate moment generating functions, when they exist. The proposed $g_n(c,d)$ is illustrated with some continuous bi-variate distributions and is shown to be easy to use even when the powers of the random variables being considered are non-negative real numbers that need not be integers. The results obtained using $g_n(c,d)$ are the same as results obtained using other methods such as moment generating functions when they exist.

Keywords: moment generating functions, bivariate distributions, continuous random variables, joint pdf.

I. Introduction

The purpose of this paper is to enable the reader and researcher learn some novel topics that may enable further studies in the areas of probability distributions and applied statistics. The paper presents more generalized methods of estimating moments of continuous random variables and of some continuous probability distributions including the beta and gamma families of distributions and their properties using more generalized methods. These moments are often tedious, cumbersome or impossible to obtain using some of the existing traditional methods.

Now some of the traditional methods often used to estimate moments of random variables and probability distributions are either based on the first principle definition of expected values of random variables, the concept of factorial moments and more generally on the theory of moment generating function. However moment generating functions of probability distributions do not always exist, and even if they exist they are sometimes difficult and tedious to evaluate in practical applications, especially if the estimation of higher moments is of interest. Furthermore if the powers or order of the random variables whose moments are to be determined are negative or fractional instead of merely whole numbers, then the method of moment generating functions may in general not be available.

The approach in estimation of moments of random variables and probability distributions has often been divided into methods of moments about the 'origin' or zero and moments about the mean of the distribution, an approach that is rather unnecessary and time consuming.

In this chapter we develop and present generalized methods for generating all conceivable moments of random variables and probability distributions about an

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arbitrarily chosen real valued constant. This approach homogenizes methods for moments about the 'origin' and moments about the mean treating them in essentially the same way. Moments about the 'origin' and moments about the mean are presented as only special cases of the more generalized methods. Methods are presented for both uni-variate and bi-variate random variables and probability distributions. Under specified conditions the methods may be used to determine moments of reciprocals or inverses and ratios of random variables. In the case of bi-variate distributions the method would enable the researcher estimate all conceivable moments of marginal probability distributions. The methods presented in this paper are available for use with only continuous random variables and probability distributions. Similar methods may also be developed as appropriate for discrete random variables and probability distributions.

Notes

The generating nth moment of X^c about λ

Generally $E\left(X^c+\lambda\right)^n$ is interpreted as the nth moment or expected value of the distribution of X^c about some real number λ where n and c are usually non-negative integers and λ is either 0 or $-\mu$ where μ is the mean of the random variable or probability distribution.

However for the present methods while 'n' may still be any non-negative integer, 'c' and λ may be any real numbers that are not necessarily integers or whole numbers. We will here still base the generalized method on the definition of expected values of random variables. To differentiate this method from the conventional moment generating function(mgf) of the random variable X usually designated by $M_X(t)=E\left(e^t X\right)$, we here refer to the present method as the generalized moment generating function (gmgf) designated by $g_n(c,\lambda)$ (which is read as g_n of c about λ) termed the nth moment of X^c about λ, for n=0,1,2,...; $-\infty \le c \le \infty$ and $-\infty \le \lambda \le \infty$. To develop the method, suppose X is a continuous random variable with probability density function (pdf), $f(x)$ for $-\infty \le x \le \infty$. Now

$$g_n(c;\lambda)=E\left(X^c+\lambda\right)^n \tag{1}$$

Expanding Equation 1 binomially and integrating we have

$$g_n(c;\lambda)=E\left(X^c+\lambda\right)^n=\int_{-\infty}^{\infty}\left(X^c+\lambda\right)^n f(x)\,dx$$

$$=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}(x^c)^r f(x)dx$$

$$g_n(c;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}x^{cr} f(x)dx=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr} f(x)dx$$

$$=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr} f(x)dx=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\mu_r(c)' \tag{2}$$

$$\mu_r(c)'=\int_{-\infty}^{\infty}x^{cr} f(x)dx \tag{3}$$

is the rth moment of X^c about zero, or the (c,r)th moment of X about zero(Oyeka et al, 2013).

Note that $g_n(c;\lambda)$ as defined in Equations 2 and 3 generates all conceivable moments of the distribution of X^c for all real values of 'c'.

Properties of $g_n(c;\lambda)$

Some of the properties of generalized moment generating function; $g_n(c;\lambda)$ from its definition include

Notes

$$g_0(c;\lambda) = 1 \quad (4)$$

$$g_1(c;\lambda) = E\left(X^c + \lambda\right) = \lambda + \mu_r(c)' \quad (5)$$

The first moment of the distribution of X^c *about* λ,

If $\lambda = 0$,that is if the nth moment of X^c is taken about the origin,zero,then

$$g_n(c;0) = E\left(X^c - 0\right)^n = \mu_r(c)' \quad (6)$$

If $\lambda = -\mu_1(c)' = -\mu$ the nth moment or mean value of X^c about the origin zero

$$g_n(c;-\mu) = E\left(X^c - \mu\right)^n = \mu_n(c), \quad (7)$$

the nth moment of X^c about its mean $\mu, \mu = \mu_1(c)'$.

Generalized moment generating functions, like the traditional moment generating functions when they exist, uniquely determine and define probability distributions. In particular, the moment generating function, $\mu_r(c)'$ of Equation 3 generates the same rth moment of the distribution of X^c about zero as would the corresponding traditional or regular moment generating function $M_X(t)$ when it exists. The later however is often more difficult to obtain and evaluate in practical applications.

Under specified conditions as illustrated later $g_n(c;\lambda)$ may be used to obtain all possible moments of the distribution of X^c ,where 'c' is some non-positive real number thereby enabling one obtain moments of random variables with negative and fractional values.

Note that the above properties of $g_n(c;\lambda)$ are quite consistent with existing theories of probability distributions. For example as can be seen from 4 the sum of all probability values over its range of definition is always 1.0.Equation 5 in particular also conforms with the known fact that first moments of distributions about then mean, $\lambda = -\mu_1(c)'$ is always zero. If in Equation 7, we let n=2,that is if the second moment of a distribution is taken about its mean, the resulting value is the variance of that distribution.

As noted above, generalized moment generating function $g_n(c;\lambda)$ may be used to obtain all conceivable moments of a continuous distribution. For example the variance, third and fourth moments of the distribution of $Y = X^c$ are obtained from Equation 2 by setting $\lambda = -\mu_1(c)' = -\mu$ where μ is the mean of X^c .Thus

$$\mu_2(c) = g_2(c;-\mu), \quad (8)$$

$$\textit{the variance of } X^c$$

$$\mu_3(c) = g_3(c;-\mu), \quad (9)$$

The third moment of X^c about its mean; and

$$\mu_4(c) = g_4(c;-\mu), \quad (10)$$

The fourth moment of the distribution of X^c about its mean. Hence the Skewness Sk(c) and Kurtosis, Ku(c) of the distribution of X^c are obtained as respectively

$$Sk(c) = \frac{\mu_3(c)}{(\mu_2(c))^{3/2}} = \frac{g_3(c;-\mu)}{(g_2(c;-\mu))^{3/2}} \quad (11)$$

And

$$Ku(c) = \frac{\mu_4(c)}{(\mu_2(c))^2} = \frac{g_3(c;-\mu)}{(g_2(c;-\mu))^2} \quad (12)$$

We now illustrate the use of $g_n(c;\lambda)$ with some examples.

Example 1

Suppose the random variable X has the probability density function (pdf),f(x)=2x,0¡x¡1.Interest is to find an expression for the estimation of all conceivable moments of the random variable Y=X. Note that conventionally the mean and variance of Y=X is by definition. $\mu = \mu_1' = \frac{2}{3}$ *and* $\sigma^2 = \frac{1}{18}$

Now to develop a more generalized expression for obtaining these moments and more, we have from Equation 2 that

$$g_n(c;-\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{2}{r+2}, where\ \mu_r(c)' = \frac{2}{cr+2} = \frac{2}{r+2},$$

Since in the present example c=1.
The first moment (n=1) of Y=X about λ is from the above expression

$$g_1(1;\lambda) = \frac{\lambda+2}{1+2} = \lambda + \frac{2}{3}$$

If now $\lambda = 0, then\ g_1(1;0) = 0 + \frac{2}{3} = \frac{2}{3} = \mu_1' = \mu$ the mean of Y=X as earlier obtained.

Hence if $\lambda = -\frac{2}{3}$ then as expected $g_1(1;-\frac{2}{3}) = 0$. If now we set n=2,that is if interest is in determining the second moment of Y=X about λ we have

$$g_2(1;-\lambda) = \lambda^2 + 2\lambda.\frac{2}{3} + \frac{2}{4} = \lambda^2 + \frac{4\lambda.}{3} + \frac{1}{2}.$$

If we now let $\lambda = -\mu_1(1)' = -\mu = -\frac{2}{3}$, then we would have that

$\left(-\frac{2}{3}\right)^2 + \frac{4}{3}\left(-\frac{2}{3}\right) + \frac{1}{2} = \frac{4}{9} - \frac{8}{9} + \frac{1}{2} = \frac{1}{1} = \frac{}{8}\sigma^2$, the variance of Y=X as obtained earlier. If we

Notes

had chosen $c = \frac{1}{2}$, that is if interest is in determining the moments of $Y = X^{\frac{1}{2}}$ *about* λ is $g_1(\frac{1}{2};\lambda) = \lambda + \frac{2}{\frac{1}{2}+2} = \lambda + \frac{4}{5}$ so that if $\lambda = 0$, then, $g_1(\frac{1}{2};0) = \frac{4}{5}$.

Notes

If $$\lambda = -\mu_1(\frac{1}{2})' = -\mu = -\frac{4}{5} \textit{ then } g_1(\frac{1}{2};-\frac{4}{5}) = 0.$$

The second moment of $Y = X^{\frac{1}{2}}$ *about* λ is $g_2(\frac{1}{2};\lambda) = \lambda^2 + 2\lambda(\frac{4}{5}) + \frac{2}{3}$. Hence, if

$$\lambda = -\mu_1(\frac{1}{2})' = -\mu = -\frac{4}{5},$$

$g_2(\frac{1}{2};-\mu) = g_2(\frac{1}{2};-\frac{4}{5}) = \left(-\frac{4}{5}\right)^2 + 2\left(-\frac{4}{5}\right)\left(\frac{4}{5}\right) + \frac{2}{3} = \frac{2}{3} - \frac{16}{25} = \frac{50-48}{75} = \frac{2}{75} = \sigma^2$, the variance of the distribution of $Y = X^{\frac{1}{2}}$ as obtained using the traditional method. If the researcher had wanted to estimate these moments using moment generating function we would have that the corresponding moment generating function for Y=X is $M_Y(t) = M_X(t) = \frac{te^t - (e^t - 1)}{t^2} = \frac{1 + te^t - e^t}{t^2}$, which is fairly cumbersome even if differential with respect to t, the resulting derivative do not exist at t=0.Hence the method of moment generating function cannot possible be used to obtain the moments of the distribution of the random variable Y=X and other similarly specified distributions.

Example 2.Beta family of Distributions

Suppose interest is in finding the gmgf of the distribution of the random variable $Y = X^c$,where X has the beta distribution with parameters α *and* β and pdf ,f(x) given as

$$f(x) = \frac{\overline{)\alpha+\beta}}{\overline{)\alpha.)\beta}}.x^{\alpha-1}.(1-x)^{\beta-1}, 0 < x < 1; \alpha > 0, \beta > 0.$$

To obtain the required gmgf we have from Equation 3 that

$$\mu_r(c)' = \frac{\overline{)\alpha+\beta}}{\overline{)\alpha.)\beta}} \int_0^1 x^{cr} x^{\alpha-1}.(1-x)^{\beta-1}\, dx = \frac{\overline{)\alpha+\beta}}{\overline{)\alpha.)\beta}} \int_0^1 x^{cr+\alpha-1}.(1-x)^{\beta-1}\, dx$$

Or

$$\mu_r(c)' = \frac{\overline{)\alpha+\beta}}{\overline{)\alpha.}}.\frac{\overline{)cr+\alpha}}{\overline{)cr+\alpha+\beta}} \tag{13}$$

Hence from Equation 2,we have that the gmgf of the beta family of distributions represented by the random variable $Y = X^c$ is

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\overline{)\alpha+\beta}}{\overline{)\alpha.}}.\frac{\overline{)cr+\alpha}}{\overline{)cr+\alpha+\beta}} \tag{14}$$

All desired moments of the beta family of distributions may be obtained using Equation 14.For instance, the first moment of X^c *about* λ is

$$g_1(c;\lambda) = \lambda + \mu_r(c)' = \lambda + \frac{\Gamma\alpha+\beta}{\Gamma\alpha.}.\frac{\Gamma cr+\alpha}{\Gamma cr+\alpha+\beta}$$

If c=1 that is if interest is in the first moment or mean of Y=X then we have $g_1(1;\lambda) = \lambda + \frac{\alpha}{\alpha+\beta}$ *so that if* $\lambda = 0$ *that is if the moment is taken about zero,*

then $g_1(1;0) = \mu_r(1)' = \frac{\alpha}{\alpha+\beta} = \mu,$

The mean of the beta distribution:If n=2 and c=1 then

$$g_2(1;\lambda) = \lambda^2 + 2\lambda\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Hence if $\lambda = -\mu_1(c)' = -\mu = \frac{-\alpha}{\alpha+\beta}$*, then we have*

$$g_2\left(1;\frac{\alpha}{\alpha+\beta}\right) = \left(-\frac{\alpha}{\alpha+\beta}\right)^2 + 2\left(-\frac{\alpha}{\alpha+\beta}\right)\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha.\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \sigma^2,$$

Which is the variance of the beta distribution. Other moments such as the skewness and kurtosis (Equation 11 and 12) of the various forms of the beta family of distributions can be similarly obtained. This is left as an exercise for the reader. If in Equation 14 we set $\alpha = \beta = 1$then we obtain the gmgf of the uniform distribution in a generalized form as

$$g_n(c;\lambda) = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}.\frac{\Gamma cr+1}{\Gamma cr+2} = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{1}{(cr+1)}. \quad (15)$$

Note that the moments of the beta family of distributions which are easily obtained using the present method namely $g_n(c;\lambda)$, cannot possibly be obtained using the method of moment generating function because the usual or traditional moment generating function do not exist for the better family of distributions.

Example 3.The Gamma family of distributions

Suppose the random variable X has the gamma distribution with parameters α *and* β and pdf,f(x) given as

$$f(x) = \frac{1}{\beta^\alpha \Gamma\alpha}x^{\alpha-1}.e^{-\frac{x}{\beta}}; x \geq 0; \alpha > 0; \beta > 0.$$

Interest is in determining the gmgf of the random variable $Y = X^c$ where X has the gamma distribution.To do this we have from Equation 3 that

Notes

$$\mu_r(c)' = \frac{1}{\beta^\alpha \overline{)\alpha}} \int_0^\infty x^{cr} x^{\alpha-1} e^{-\frac{x}{\beta}} .dx = \frac{1}{\beta^\alpha \overline{)\alpha}} \int_0^\infty x^{cr+\alpha-1} e^{-\frac{x}{\beta}} .dx$$

Letting $V = \frac{x}{\beta}$, integrating and simplifying we have that

Notes

$$\mu_r(c)' = \beta^{cr} . \frac{\overline{)cr+\alpha}}{\overline{)\alpha.}} \quad (16)$$

Hence using Equation 16 in Equation 2 yields the gmgf of the gamma family of distributions represented by the random variable $Y = X^c$,as

$$g_n(c;\lambda), = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} . \frac{\overline{)cr+\alpha}}{\overline{)cr}} . \quad (17)$$

As usual all conceivable moments of the gamma family of distributions are obtained using Equation 17.For example the variance of $Y = X^c$ is

$$g_2(c;\lambda), = \lambda^2 + 2\lambda . \frac{\overline{)c+\alpha}.\beta^c}{\overline{)\alpha}} + \frac{\overline{)2c+\alpha}.\beta^{2c}}{\overline{)\alpha}}$$

If c=1 then

$$g_2(1;\lambda), = \lambda^2 + 2\lambda.\alpha\beta + \alpha(\alpha+1).\beta^2$$

Hence if $\lambda = -\mu = -\alpha\beta$ *where* $\mu = \alpha\beta$ is the mean of the usual gamma distribution, then $g_2(1;\alpha\beta) = (-\alpha\beta)^2 + 2(-\alpha\beta)(\alpha\beta) + \alpha(\alpha+1).\beta^2 = \alpha\beta^2 = \sigma^2$, the variance of the usual gamma distribution. Note that the third moment of the gamma family of distributions about λ is obtained from Equation 9 as

$$g_n(n;\lambda) = \lambda^3 + 3\lambda^2.\beta^c . \frac{\overline{)c+\alpha}}{\overline{)\alpha}} + 3\lambda.\beta^{2c} \frac{\overline{)2c+\alpha}}{\overline{)\alpha}} + \beta^{3c} \frac{\overline{)3c+\alpha}}{\overline{)\alpha}}$$

If in particular $c = 1$ *and* $\lambda = -\alpha.\beta$ *where* $\alpha.\beta$ is the mean of the gamma distribution, then we have that

$$g_3(1;\alpha\beta) = (-\alpha\beta)^3 + 3(-\alpha\beta)^2.(\alpha\beta) + 3(-\alpha\beta)\alpha(\alpha+1).\beta^2 + \alpha(\alpha+1)(\alpha+2).\beta^3 = 2\alpha$$

Hence the Skewness of the beta distribution is easily obtained using Equation 11 as

$$Sk(1) = \frac{g_3(1;-\alpha\beta)}{g_2(1;-\alpha\beta)^{\frac{3}{2}}} = \frac{2\alpha\beta^3}{(\alpha\beta^2)^{\frac{3}{2}}} = \frac{2}{\alpha^{\frac{1}{2}}}$$

Similarly the fourth moment of the beta distribution about its mean is

$$g_4(1;\alpha\beta) = (-\alpha\beta)^4 + 4(-\alpha\beta)^3(\alpha\beta) + 6(-\alpha\beta)^2\alpha(\alpha+1).\beta^2 + \\ +4(-\alpha\beta)\alpha(\alpha+1)(\alpha+2).\beta^3 + \alpha(\alpha+1)(\alpha+2)(\alpha\beta)\beta^4 = 6\alpha\beta^4$$

Hence the corresponding Kurtosis is

$$Ku(1;-\alpha\beta) = \frac{g_4(1;\alpha\beta)}{(g_4(1;\alpha\beta))^2} = \frac{6\alpha\beta^4}{(\alpha\beta^2)^2} = \frac{6}{\alpha}$$

Notes

Setting $\alpha = 1$ in Equation 17 gives the gmgf of all forms of the exponential distribution as

$$g_n(n;\lambda) = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}.\beta^{cr}.\overline{)cr} + 1 = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}.\beta^{cr}(cr).\overline{)cr} \tag{18}$$

Similarly setting $\beta = 2 \; and \; \alpha = \frac{k}{2}$ where k=1,2,...gives the gmgf of the chi-square distribution with k degrees of freedom as

$$g_n(c;\lambda) = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}.2^{cr}.\frac{\overline{)cr + \frac{k}{2}}}{\overline{)\frac{k}{2}}} \tag{19}$$

Now as noted earlier gmgf s' can be used to obtain moments of powers of random variables with negative indeces.This is possible for example with the gamma distribution if in Equation 17, $cr + \alpha > 0$; that is if the real number c is such that $'c' \geq \frac{-\alpha}{r}$, for r=1,2,.. and some specified value of $\alpha > 0$. For example if in Equation 19,we choose $'c' = -\frac{3}{2} \; and \; k = 10$, and interest is in determining all possible moments of the random variable $Y = X^{-\frac{3}{2}}$, where X has the chi-square distribution with 10 degree of freedom. Then it is possible to generate moments up to the third moment of this random variable. Specifically the possible moments of $Y = X^{-\frac{3}{2}}$, are obtained from Equation 19 as

$$g_1(-\frac{3}{2};\lambda) = \lambda + 2^{-\frac{3}{2}}\frac{\overline{)-\frac{3}{2}+5}}{\overline{)5}} = \lambda + 2^{-\frac{3}{2}}\frac{\overline{)\frac{7}{2}}}{24} = \lambda + \frac{5\overline{)\pi}\,2^{-\frac{3}{2}}}{64}$$

Hence setting $\lambda = 0$ we have that the mean of the random variable $Y = X^{-\frac{3}{2}}$, where X has the chi-square distribution with degrees of freedom is $\mu = \frac{5\overline{)\pi}\,2^{-\frac{3}{2}}}{64} = \frac{5\overline{)2\pi}}{256}$

If n=2 the the variance of $Y = X^{-\frac{3}{2}}$, is obtained from the expression

$$g_2(-\frac{3}{2};\lambda)=\lambda^2+2\lambda\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)+2^{-3}\frac{\sqrt{-3}+8}{\sqrt{5}}=\lambda^2+2\lambda\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)+\frac{2^{-3}}{24}, \textit{ so that setting } \lambda=-\mu=\frac{-5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}=\frac{5\sqrt{2\pi}}{256}, \textit{gives}$$

$$g_2(-\frac{3}{2};-\mu)=\left(\frac{-5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)^2+2\left(\frac{-5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)+\frac{2^{-3}}{24}=\frac{1}{192}-\left(\frac{5\sqrt{\pi}\,2^{-\frac{3}{2}}}{64}\right)^2=\frac{1}{192}-\frac{78.575}{32768}=0.005-0.002=0.003$$

Although the usual or traditional moment generating function about the origin or zero and about the mean $\alpha\beta$ of the gamma family of distributions namely

$$M(t)=(1-\beta t)^{-\alpha} \textit{ and } M(t;\alpha\beta)=e^{-\alpha\beta t}(1-\beta t)^{-\alpha}$$

Respectively exist they are often relatively difficult to differentiate in practical applications and evaluate the required moments. Furthermore these moments cannot possibly be used if 'c' is not a whole number.

Example 4: The Normal Distribution

To obtain the gmgf of the random variable $Y=X^c$ where X has the normal distribution with parameters μ *and* σ^2 with pdf,f(x) given by

$$f(x)=\frac{1}{\sigma^2\sqrt{2\pi}}e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2},-\infty<x<\infty;-\infty<\mu<\infty;\sigma^2>0.$$

We have from Equation 3 that the rth moment of X^c about the origin or zero is

$$\mu_r(c)'=\frac{1}{\sigma^2\sqrt{2\pi}}\int_{-\infty}^{\infty}x^{cr}.e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2}dx$$

Now letting $'V'=\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2$, solving for x, expanding binomially and simplifying gives

$$\mu_r(c)'=\sum_{t=0}^{cr}\binom{cr}{t}\mu^{cr-t}\left(2\sigma^2\right)^{\frac{t}{2}}\frac{1}{\sqrt{\pi}}\int_0^{\infty}V^{\frac{t}{2}+\frac{1}{2}-1}e^{-V}dV$$

Or

$$\mu_r(c)'=\sum_{t=0}^{cr}\binom{cr}{t}\mu^{cr-t}\left(2\sigma^2\right)^{\frac{t}{2}}\frac{\sqrt{\frac{t}{2}+\frac{1}{2}}}{\sqrt{\pi}} \qquad (20)$$

For t=0,2,4;etc,that is for all even numbers. Hence the gmgf of the normal distribution represented by the random variable $Y=X^c$, where X is normally distributed with mean μ and variance σ^2 is from Equation 2

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} \left(2\sigma^2\right)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\Gamma\pi} \qquad (21)$$

For all even numbers t,that is for t=0,2,3.That is provided we set $\left(2\sigma^2\right)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\Gamma\pi} = 0$ for all odd values of t. Since with $V = \frac{x-\mu}{\sigma}$ we have that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V^t e^{-\frac{V^2}{2}} dV = 0$ for all odd values of t, that is for t=1,3,5…as may be easily verified. As with other gmgfs Equation21 may be used to generate all conceivable moments of all forms of the normal distribution represented by the random variable $Y = X^c$ for all real valued numbers ‘c’ provided $\mu_r(c)'$ is evaluated for all even values of ‘s’that is for all s=0,2,4,…etc.

For example the fourth moment of the random variable $Y = X^c$ about λ where X has the normal distribution with parameters μ *and* σ^2 is from Equation 21 with ‘c’=1

$$g_4(1;\lambda) = \lambda^4 + 4\lambda^3\mu + 6\lambda^2\left(\mu^2+\sigma^2\right) + 4\lambda\left(\mu^3 + 3\mu\sigma^2\right) + \left(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4\right)$$

If the moment is now taken about the mean, that is if we set $\lambda = -\mu$ we obtain the fourth moment of the normal distribution about its mean μ as

$$g_4(1;-\mu) = \mu^4 - 4\mu^4 + 6\mu^4 + 6\mu^2\sigma^2 - 4\mu^4 - 1\ \mu^2\sigma^2 + \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 = 3\sigma^4$$

Hence from Equation 12 we obtain the Kurtosis of the normal distributions

$$Ku(1) = \frac{g_4(1;-\mu)}{\left(g_4(1;-\mu)\right)^2} = \frac{3\sigma^4}{\left(\sigma^2\right)^2} = 3, \exp ected.$$

These results are relatively much more difficult to obtain using the usual conventional moment generating function of the normal distribution. Thus even if one uses the moment generating function of the normal distribution about its mean μ namely $e^{\frac{1}{2}}\sigma^2 t^2$, differentiation and evaluation of this function at t=0 up to four times is clearly tedious and cumbersome. The generalized moment generating functions (gmgf) of some other continuous probability distributions are similarly obtained using Equation 2.The results are summarized in Table 1 below.

Table 1 : Generalized Moment Generating Functions (gmgf) of Some Continuous probability Distributions

S/No	Distribution	Gmgf $g_n(c;\lambda)$	Mgf about mean $(\mu)\left(M^t_{X-\mu g}\right)$
1	Beta $\left(B(\alpha;\beta)\right)$	$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}.\frac{\Gamma(\alpha+\beta).\Gamma(cr++\alpha)}{\Gamma(\alpha)\Gamma(cr++\alpha+\beta)}$	-
2	Uniform $\left(\beta(1;1)\right)$	$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{\Gamma(cr+1)}{\Gamma(cr+2)}$	-

Notes

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3	Gamma $(G(\alpha;\beta))$	$\sum_{r=1}^{n}\binom{n}{r}\lambda^{n-r}\beta^{cr}\frac{\Gamma cr.\alpha}{\Gamma\alpha}$	$e^{-\alpha\beta t}.(1-\beta t)^{-\alpha}$
4	Exponential $(G(\alpha\beta))$	$\sum_{r=1}^{n}\binom{n}{r}\lambda^{n-r}\beta^{cr}\Gamma cr.+1$	$e^{-\beta t}.(1-\beta t)^{-1}$
5	Square (k) $\left((\frac{k}{2};2)\right)$	$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}2^{cr}\frac{\Gamma cr.+\frac{k}{2}}{\Gamma\frac{k}{2}}$	$e^{-kt}.(1-2t)t\frac{k}{2}$

Relationship between Generalized moment Generating Function and Factorial Moment generating Function about λ

The factorial moment generating function of the random variable X is usually defined as $F_X(t)=Et^X$ for some positive real value 't'. This expression is however only able to provide estimates of the moments of the distribution of the random variable X about zero or the origin. But just as is the case with generalized moment generating functions, a generalized approach to factorial moment generating functions would be more instructive and helpful in obtaining these moments about any desired real number, such as λ as used above. This would enable easier and quicker calculation of higher and more complicated moments of the distribution of the random variable X.

Thus, specifically suppose interest is in finding a generalized factorial moment generating function of the random variable $Y=X^C$ about an arbitrary chosen real number λ which is not necessarily zero or the mean of X^c ,where 'c' is any real number. Furthermore suppose that X is for the present purposes a continuous random variable with probability density function, f(x),for $-\infty<x<\infty$. Then the generalized factorial moment generating function for X^C *about* λ is given by

$$F_X(c,t;\lambda)=Et^{\left(X^C+\lambda\right)}=\int_{-\infty}^{\infty}t^{\left(X^C+\lambda\right)}.f(x)dx \qquad (22)$$

Now Equation 22 can be alternatively expressed as

$$F_X(c,t;\lambda)=\int_{-\infty}^{\infty}t^{l\,\text{int}^p}.f(x)dx=\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(l\,\text{int})^n}{n!\,p^n}f(x)dx \qquad (23)$$

where $p=x^c+\lambda.$

$$F_X(c,t;\lambda)=\sum_{n=0}^{\infty}\frac{(l\,\text{int})^n}{n!}\int_{-\infty}^{\infty}\left(x^c+\lambda\right)^n=\sum_{n=0}^{\infty}\frac{(l\,\text{int})^n}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr}.f(x)$$

That is

$$F_X(c,t;\lambda)=\sum_{n=0}^{\infty}\frac{(l\,\text{int})^n}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\mu_r(c)' \qquad (24)$$

Hence the nth factorial moment of the distribution of X^c about λ is the coefficient of $\frac{(l\,\text{int})^n}{n!}$ in the series expansion of Equation 23 or the nth derivative of this Equation, that is of $F_X(c,t;\lambda)$ with respect to t, evaluated at t=1.The result is seen to actually be the same as the generalized moment generating function, $g_n(c;\lambda)$ of the distribution of the random variable X^C about λ as already obtained above (Equation 2).Thus with the present approach, generalized moment generating functions are essentially the same yielding the same results, and need not be treated as different concepts.

Notes

II. Summary and Conclusion

We have presented in this paper method of obtaining the moment of some continuous bi-variate distributions with parameters α_1, β_1 *and* α_2, β_2 in finding the nth moment of the variable $x^c y^d\,(c \geq 0, d \geq 0)$ where X and Y are continuous random variables having the joint pdf, f(x,y).The proposed methods were the so called $g_n(c,d)$ defined $g_n(c,d) = E\left(X^c Y^d + \lambda\right)^n$, the nth moment of expected value of the t distribution of the cth power of X and dth power of Y about the constant λ.These moments are obtained by the use of bi-variate moment generating functions, when they exist. The proposed $g_n(c,d)$ exists for all continuous probability distributions unlike some of its competitors such as factorial moments of moment generating function which do not always exist. The results obtained using $g_n(c,d)$ are the same as results obtained using such other methods as moment generating functions of available. The proposed method is available and easy to use without the need for any modifications even when the powers of the random variable being considered are non-negative real numbers that do not need to be integers. The results obtained using $g_n(c,d)$ are the same as results obtained using other methods such as moment generating functions when they exist.

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mz-Compact Spaces

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Abstract- In this work we study mz-compact spaces and mz-Lindel of spaces, where m is an infinite cardinal number. Several new properties of them are given. It is proved that every mz-compact space is pseudocompact (a space on which every real valued continuous function is bounded). Characterizations of mz-compact and mz-Lindel of spaces by multifunctions are given.

Keywords: mz-compact space, mz-Lindel of space, m-compact space, pseudocompact space, realcompact space.

GJSFR-F Classification : AMS 2010: 54C, 54D

MZ-COMPACTSPACES

Strictly as per the compliance and regulations of :

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mz-Compact Spaces

A. T. Al-Ani

Abstract- In this work we study mz-compact spaces and mz-Lindel of spaces, where m is an infinite cardinal number. Several new properties of them are given. It is proved that every mz- compact space is pseudocompact (a space on which every real valued continuous function is bounded). Characterizations of mz-compact and mz-Lindelof spaces by multifunctions are given.

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I. Introduction

In this paper we study some properties of mz-compact spaces and mz-Lindelof spaces. Modifications of results about countably z-compact spaces [1] are proved. We relate mz-compact spaces to pseudocompact spaces (Theorem III b). Then we give some characterizations of mz-compact spaces. The collection of real valued continuous functions on a topological space X forms a ring denoted by $C(X)$ [2]. Characterizations of mz-compact spaces in terms of z-filter bases and z-ideals are given. No separation property is assumed unless otherwise is stated. Fordefinitions and notations not stated here see [2].

II. Preliminaries

a) Definition

A space X is called m-compact if every open cover of X of cardinality at most m, has a finite subcover. Recall that a *cozero set* in a space $X = (X, \tau)$ is an $f^{-1}[R \setminus \{0\}]$ with a continuous function $f: X \to R$. Cozero sets constitute a base of a topology $z\tau$ on X. (X, τ) is said to be mz-compact, (mz-Lindelof) if $zX = (X, z\tau)$ is m-compact (m-Lindelof). Filters and z -ideals here are modifications of their respective definitions [2] by taking z-closed set (closed in zX) instead of zero-set.

b) Definition

A multifunction α of a space X into a space Y is a set valued function on X into Y such that $\alpha(x) = \Phi$ for every $x \in X$. The class of all multifunctions on X into Y is denoted by $m(X, Y)$.

c) Definition

A multifunction α on X into Y is called closed graph if its graph $G(\alpha) = (x, y) \in X \times Y : y \in \alpha(x)$ is closed in $X \times Y$.

III. Some Properties of MZ -compact Spaces

We call a filter on X of cardinality at most m by m-filter. By mz-filter we mean a filter of z-closed sets, of cardinality at most m. The proof of the following theorem is straight forward.

a) Theorem

The following statements about a space X are equivalent.

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1. X is mz-compact.
2. Every family of subsets of X of cardinality at most m each is an intersection of zero sets, with the finite intersection property has a non empty intersection.
3. Every mz-filter on X is fixed.
4. Every mz-ideal in $C(X)$ is fixed.
5. zX is m-compact.

b) Theorem

Every mz-compact space is pseudo compact.

c) Example ([3], [4])

Let N be the set of positive integers. Topologize N by taking a subbase the collection $\beta = \{U_p(b) : b + np \in N, p$ prime $, b$ is not divisible by $p\}$ This space is a T_2 countably z-compact Lindel of space which is not countably compact. For $m = \chi_o$ in this example the space N is mz-compact but not m- compact.

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IV. A Characterization of MZ-compactness in Terms of Multifunctions

A space X is said to be of character m if every point of X has a local base of cardinality at most m. We give here a characterization of mz-compact space X in terms of multifunctions. Equivalently a characterization of m-compactness of zX. It is to be noted that a space is m-compact if every family of closed sets with cardinality at most m, satisfying the finite intersection property has a non-empty intersection. We shall use this fact in the proof of the second part of the following theorem.

a) Theorem

A space X is mz-compact if for every space Y with character m and closed graph multifunction $\alpha \in m(zX, Y)$, the image of every closed set in zX is closed in Y.

Proof

Let zX be m-compact space, Y be a space of character m, $\alpha \in m(zX, Y)$ with closed graph.

Let K be closed in zX and $y \in Y - \alpha(K)$.

Let $\{B_\lambda : \lambda \in \Lambda\}$ be a local base of cardinality at most m at y.

For each $x \in K$, there exist open set V_x in zX and B_λ in Y such that

$$(x, y) \in V_x \times B_\lambda$$

$$\text{and } (V_x \times B_\lambda) \cap G(\alpha) = \Phi.$$

For each $\lambda \in \Lambda$, let

$W_\lambda = \cup\{V_x : x \in K, (x, y) \in V_x \times B_\lambda\}$

Then $\{W_\lambda\}$ is an open cover of K of cardinality at most m. So, it has a finite

subcover $\{W_{\lambda_i} : i = 1, 2, \ldots, n\}$

Now, let W=È{Bl i :i=1, 2, …, n} .Then W is open in Y with $y \in W$ and $W \cap \alpha(K) = \Phi i$

So, $\alpha(K)$ is closed in Y.

To prove the converse, let $\{K_\lambda : \lambda \in \Lambda\}$ be a family of closed sets in zX of cardinality at most m, with the finite intersection property, let $y_0 \notin zX$. Topologize $zX \cup \{y_0\}$ by taking open sets all subsets of zX and sets containing $y_0 \cup \alpha(K_\lambda)$ for some $\lambda \in \Lambda$. Obviously, $zX \cup \{y_0\}$ has character m. Let β be the closure of the identity function of zX. Then β has a closed graph and so, by hypothesis, it maps closed sets in zX onto closed subsets in Y.

So, $\beta(K_\lambda)$ is closed in $zX \cup \{y_0\}$, for every $\lambda \in \Lambda$. So, $y_O \in \beta(K_\lambda)$ for every $\lambda \in \Lambda$.

Hence $\{K_\lambda : \lambda \in \Lambda\}$ has a non-empty intersection.

Therefore, zX is m-compact.

Notes

V. MZ-Lindel of Space

a) Definition

A space X is a P(m)-space if every intersection of at most m open sets in X is open.

The following result about mz-Lindelof space can be proved by the same technique of Theorems IV b.

b) Theorem

A space X is mz-Lindelof if for every P(m)-space Y and z-closed graph multi-function $\alpha \in m(X, Y)$ the image of every z-closed set in X is closed in Y.

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The proposed method, applying the extra sum of squares principle develops F ratio-test statistics for testing the significance of factor and interaction effects in analysis of variance models. The method also shows how using the extra sum of squares principle to build more parsimonious explanatory models for dependent or criterion variables of interest.

In addition, unlike the traditional approach with analysis of variance models the proposed method easily enables the simultaneous estimation of total or absolute and the so-called direct and indirect effects of independent or explanatory variables on given criterion variables. The proposed methods are illustrated with some sample data and shown to yield essentially the same results as would the two factor analysis of variance techniques when the later methods are equally applicable.

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I. Introduction

Analysis of variance and regression analysis whether single-factor or multi-factor, sometimes both in theory and applications have often been treated and presented as rather different concepts by various authors. In fact only limited attempts seem to have been made to present analysis of variance as a regression problem (Draper and Smith, 1966; Neter and Wasserman, 1974).

Nonetheless analysis of variance and regression analysis are actually similar concepts, especially when analysis of variance is presented from the perspective of dummy variable regression models. This is the focus of the present paper, which attempts to develop a method to use dummy variable multiple regression models and apply the "extra sum of squares principle" in the analysis of two-factor analysis of variance models with unequal replications per treatment combination as a multiple regression problem.

II. The Proposed Method

Regression techniques can be used for the analysis of data appropriate for two factor or two –way analysis of variance with replications and possible interactions. This approach is a more efficient method than the method of unweighted means discussed in Oyeka et al (2012).

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In a two factor analysis of variance involving factors A and B with interactions between these two factors, as discussed in Oyeka (2013), the resulting model is

$$y_{ilj} = \mu + \alpha_l + \beta_j + \lambda_{lj} + e_{ilj} \quad (1)$$

Where y_{ilj} is the i^{th} observation or response at the l^{th} level of factor A and j^{th} level of factor B; μ is the grand or overall mean, α_l is the effect of the l^{th} level of factor A, β_j is the effect of the j^{th} level of factor B; λ_{lj} is the interaction effect between the l^{th} level of factor A and j^{th} level of factor B; e_{ilj} are independent and normally distributed error terms with constant variance, for $i = 1,2..nij$, $l = 1,2\ldots a$, the 'a' levels of factor A; $j = 1,2\ldots b$, the 'b' levels of factor B, subject to the constraints

$$\sum_{l=1}^{a} \alpha_l = \sum_{j=1}^{b} \beta_j = \sum_{l=1}^{a} \lambda_{lj} = \sum_{j=1}^{b} \lambda_{lj} = 0 \quad (2)$$

Let $n = \sum_{l=1}^{a} \sum_{j=1}^{b} n_{lj}$ be the total sample size or observations for use in the analysis.

To obtain a dummy variable regression model of 1s and 0s equivalent to equation 1 and also subject to the constraints imposed on the parameters by equation 2, we would as usual use for each factor one dummy variable of 1s and 0s less than the number of levels, classes, or categories that factor has (Boyle 1974). Similarly the interaction effects will be factored in by taking the cross-products of the set of dummy variables representing one of the factors with the set of dummy variables representing the other factor. Thus factor A with 'a' levels will be represented by a-1 dummy variables of 1s and 0s, factor B with 'b' levels will be represented by b-1 dummy variables of 1s and 0s and the factors A by B interaction effects will be represented by (a-1)(b-1)dummy variables of 1s and 0s.Specifically to obtain the required dummy variables for factors A and B. we may define

$$x_{il;A} = \begin{cases} 1, \text{if the } i^{th} \text{observation or response, } y_{ilj} \text{ is at the } l^{th} \text{ level of factor A} \\ 0, \text{otherwise} \end{cases} \quad (3)$$

for $i = 1,2 \ldots n_{lj}$ $l = 1,2 \ldots a-1, for\ all\ j = 1,2 \ldots b$

For factor B define

$$x_{ij;B} = \begin{cases} 1, \text{if the } i^{th} \text{observation or response, } y_{ilj} \text{ is at the } J^{th} \text{ level of factor B} \\ 0, \text{otherwise} \end{cases} \quad (4)$$

for $i = 1,2 \ldots n_{lj}, j = 1,2 \ldots b-1, for\ all\ l = 1,2 \ldots a$

Using these specifications we have that the dummy variable multiple regression model equivalent to the two factor analysis of variance model of equation 1 is

$$y_{ilj} = \beta_0 + \beta_{1;A}.x_{i1;A} + \beta_{2;A.}x_{i2;A} + \ldots + \beta_{a-1;A}.x_{ia-1;A} + \beta_{1;B}.x_{i1;B} + \beta_2;_B.x_{i2;B} + \ldots + \beta_{b-1};_B.x_{ib-1};_B + \beta_1;I.x_{il};_I\ I$$
$$+ \beta_2;I.x_{i2};I + \ldots + \beta_{(a-1)(b-1)} : I.x_{i(a-1)(b-1)};I + e_{ilj}$$

OR when more compactly expressed

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Notes

$$y_{ilj} = \beta_0 + \sum_{l=1}^{a-1} \beta_{l;A}.x_{il;A} + \sum_{j=1}^{b-1} \beta_{j;B}.x_{ij;B} + \sum_{k=1}^{(a-1)(b-1)} \beta_{k;I}.x_{ik};I + e_{ilj} \quad (5)$$

Where the β_s are partial regression coefficients and e_{ij} are independent and normally distributed error terms with constant variance with $E(e_{ilj}) = 0$.
The expected value of y_{ilj} of Equation (5) is

Notes

$$E(y_{ilj}) = \beta_0 + \sum_{l=1}^{a-1} \beta_{l;A}.x_{il;A} + \sum_{j=1}^{b-1} \beta_{j;B}.x_{ij;B} + \sum_{k=1}^{(a-1)(b-1)} \beta_{k;}I.x_{ik};I \quad (6)$$

Note that the interaction terms may be more completely represented as

$$x_{ik;}I = x_{il:A}.x_{ij:B} = x_{il}.x_{ij};and\ \beta_{k};I = \beta_{ij}: AB = \beta_{lj}$$

For $l = 1,2 ... a-1 : j = 1,2, ... b-1$

Hence Equation 6 may alternatively be expressed as

$$E(y_{ilj}) = \beta_0 + \sum_{l=1}^{a-1} \beta_{l;A}.x_{il;A} + \sum_{j=1}^{b-1} \beta_{j;B}.x_{ij;B} + \sum_{l=1}^{(a-1)} \sum_{j=1}^{(b-1)} \beta_{lj}.x_{il}.x_{ij} \quad (7)$$

Now the mean value or mean response in the language of analysis of variance at the l^{th} level factor A and j^{th} level of factor B is obtained by setting $x_{il;A} = x_{ij:B} = 1$the $l^{th} and\ x_{ig} = 0$ for all 'g'not equal to $l.j$ in Equation (7) to obtain

$$E(y_{ilj}) = \mu_{lj} = \beta_0 + \beta_{l;A} + \beta_{j:B} + B_{lj} \quad \textbf{(8)}$$

For $l = 1,2, ... a-1; j = 1,2, ... b-1$

Similarly the mean response or mean of the criterion variable at the l^{th} level of factor A is obtained by setting x_{il};A=1 and all other $x_{igs} = 0$ $(g \neq l)$ while the mean response at the j^{th} level of factor B is obtained by setting $x_{ij}; B = 1$ and all other $x_{igs} = 0$ $(g \neq j)$ in Equation (6) giving

$$\mu_j = \beta_0 + \beta_l : A; and\ \mu_j = \beta_0 + \beta_j : B \quad (9)$$

For $l = 1,2, ... a-1; j = 1,2, ... b-1$

These are the same results that are obtained using conventional two factor analysis of variance methods. The partial regression parameter β_l: A is as usual interpreted as the change in the dependent variable 'Y' percent change in the l^{th} level of factor A compared with all its other levels holding the levels of all other independent variables in the model constant; β_j : B is similarly interpreted. The interaction effect β_{lj} is interpreted as the dependent variable Y per unit change at the l^{th} level of factor A cj^{th} level of the change at the l^{th} level of factor B confounded by or in the presence of the effect of the j^{th} level of factor B (l^{th} level of factor A).
Now Equation 5 can be more compactly expressed in matrix form as

$$\underline{y} = X\underline{\beta} + \underline{e} \quad (10)$$

Where $\underline{y}$ is an $nx1$ column vector of response outcomes or values of the criterion or dependent variable; X is an nxr design matrix of 1s and 0s; $\underline{\beta}$ is an $rx1$ column vector of partial regression parameters and $\underline{e}$ is an $nx1$ column vector of normally distributed error terms with constant variance with $E(\underline{e}) = \underline{0}$,where $r=a.b-1$ representing the number of dummy variables of 1s and 0s in the model.
The corresponding expected value of the criterion variable equivalent to Equation (6) is

$$E(\underline{y}) = X.\underline{\beta} \quad (11)$$

Note that use of Equations 3-5 or 10 makes it unnecessary, at least for the fixed effects model of primary interest here, to treat one observation per cell, equal and unequal observations per cell in two factor analysis of variance problems differently. The same dummy variable regression models can be used in all these cases except that in the case of one observation per cell where it is not possible to calculate the error sum of squares and hence the corresponding error mean square, the interaction mean square is instead used in all tests.

Use of the usual least squares methods with either Equations (5) or (10) yields unbiased estimates of the partial regression parameters which again expressed in matrix form is

$$\hat{\underline{\beta}} = \underline{b} = (X'X)^{-1}.X'\underline{y} \quad (12)$$

Where $(X'X)^{-1}$ is the matrix inverse of the non singular variance-covariance matrix $X'X$.
The resulting estimated or fitted value of the response or dependent variable is

$$\hat{\underline{y}} = X.\underline{b} \quad (13)$$

In the conventional two factor analysis of variance a null hypothesis that is usually of interest first is that treatment means are equal for all treatment combinations. In the dummy variable regression approach an equivalent null hypothesis would be that the specified model that is either Equations (5) or (10) fits. This null hypothesis when expressed in terms of the regression parameters would be

$$H_o : \underline{\beta} = \underline{0} \text{ versus } H_1 : \underline{\beta} \neq \underline{0} \quad (14)$$

This null hypothesis is tested using the usual F_{test} presented in the familiar analysis of variance table where the required sums of squares are obtained as follows:-
The total sum of squares is as usual calculated as

$$SS_{Total} = SS_{Tot} = \underline{y}'\underline{y} - n\bar{y}^2 \quad (15)$$

Which has the chi-square distribution with $n-1$ degrees of freedom where $\bar{y}$ is the mean of the criterion or dependent variable. The sum of squares regression or the so-called treatment sum of squares in analysis of variance parlance is

$$SSR = SST = \underline{b}'.X'\underline{y} - n\bar{y}^2 \quad (16)$$

Which has the chi-square distribution with $r = a.b-1$ degrees of freedom. Similarly the error sum of squares is

Notes

Notes

$$SSE = SS_{Total} - SSR = \underline{y}'\underline{y} - \underline{b}'.X'\underline{y} \qquad (17)$$

With $(n-1)$-$(a.b-1) = n - a.b$ degrees of freedom.
These results may be summarized in an analysis of variance Table (Table 1)

Table 1: Analysis of variance table for regression model of Equation (10)

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (DF)	Mean sum of Squares (MS)	F-Ratio
Regression (treatment)	$SSR = SST = \underline{b}'X'\underline{y} - n.\bar{y}^2$	$a.b-1$	MSR=$\frac{SSR}{a.b-1}$	F=$\frac{MSR}{MSE}$
Error	$SSE = \underline{y}'\underline{y} - \underline{b}'X'\underline{y}$	$n-a.b$	MSE=$\frac{SSE}{n-a.b}$	
Total	$SS_{Toat} = \underline{y}'\underline{y} - n.\bar{y}^2$	$n-1$		

The null hypothesis of Equation 4 is rejected at the α level of significance if the calculated $F-ratio$ of Table 1 is such that

$$F \geq F_{1-\alpha;a.b-1}, n-a.b \qquad (18)$$

Otherwise the null hypothesis is accepted.

If the model fits, that is if the null hypothesis of Equation (14) is rejected, in which case not all the regression parameters are equal to zero, then one can proceed to test other null hypothesis concerning factors A and B level effects as well as factors A by B interaction effects. Thus additional null hypothesis that may be tested are that factor A has no effects on the criterion variable; factor B has no effects on the criterion variable; and that there are no factors A by B interaction effects. Stated notation ally the null hypotheses are

$$H_{0A} : \underline{\beta}_A = \underline{0} \text{ Versus } H_{1A} : \underline{\beta}_A \neq \underline{0} \qquad (19)$$

$$H_{0B} : \underline{\beta}_B = \underline{0} \text{ Versus } H_{1B} : \underline{\beta}_B \neq \underline{0} \qquad (20)$$

$$H_{0AB} : \underline{\beta}_{AB} = \underline{0} \text{ Versus } H_{1AB} : \underline{\beta}_{AB} \neq \underline{0} \qquad (21)$$

To test these hypotheses one needs to calculate the contribution of each factor separately to the treatment or regression sum of squares. The treatment or regression sum of squares SST in analysis of variance parlance which is the regression sum of squares SSR in regression models distributed as chi-square with $a.b-1$ degrees of freedom is made up of three sums of squares each having the chi-square distribution, namely the sum of squares due to row or factor A, SSA with $a-1$ degrees of freedom, the sum of squares due to column or factor B, SSB with $b-1$ degrees of freedom, and the row by column of factors A by B interaction sum of squares, SSAB with $(a-1)(b-1)$ degrees of freedom. Thus notationally we have that

$$SST = SSR = SSA + SSB + SSAB \qquad (22)$$

To obtain these sums of squares we note that the design matrix X of Equation (10) with $ab-1$ dummy variables 0s and 1s because of 0s and 1s of dummy variables of 1s and 0s can be partitioned into three sub matrices namely an $n \times (a-1)$ matrix X_A of $a-1$ dummy variables representing the $(a-1)$. included levels of factor A, the

$n \times (b-1)$ matrix X_B comprising $b-1$ dummy variables of 1s and 0s representing the $b-1$ included levels of factor B; and the $n \times (a-1)(b-1)$ matrix X_{AB} of $(a-1)(b-1)$ dummy variable of 1s and 0s representing interaction between factors A and B .

The $(ab-1) \times 1$ column vector of estimated partial regression coefficients b can be similarly partitioned into the corresponding $(a-1) \times 1$ column vector b_A of effects due to factor A; the $(b-1) \times 1$ column vector of b_B of effects due to factor B and the $(a-1)(b-1) \times 1$ column vector $\underline{b}_{AB}$ of effects due to factors A by B interaction. Now the sum of squares $\underline{b}'.X'.\underline{y}$ of Equation (16) may hence equivalently be expressed as

$$\underline{b}' X' \underline{y} = (X.\underline{b})'.\underline{y} = \left(\begin{pmatrix} X_A & X_B & X_{AB} \end{pmatrix} \begin{bmatrix} b_A \\ b_B \\ b_{AB} \end{bmatrix} \right)' \underline{y}$$

OR

$$\underline{b}' X' \underline{y} = \underline{b}'_A.X'_A.\underline{y} + \underline{b}'_B.X'_B.\underline{y} + b'_{AB}.X'_{AB}.\underline{y} \tag{23}$$

The sum of squares regression or the treatment sum of squares, $SSR = SST$ of the full model of Equation 10 is

$$\begin{aligned} SSR = \underline{b}'.X'.\underline{y} - n.\bar{y}^2 &= \left(\underline{b}'_A.X'_A.\underline{y} - n.\bar{y}^2\right) + \left(\underline{b}'_B.X'_B.\underline{y} - n.\bar{y}^2\right) + \left(\underline{b}'_{AB}.X'_{AB}.\underline{y} - n.\bar{y}^2\right) + 2.n.\bar{y}^2 \\ SSR(SST) &= SSA + SSB + SSAB + mean\ (adjustment\ \ factor) \end{aligned} \tag{24}$$

Now to find the required sums of squares after fitting the full regression model of Equation (10) one then proceeds to fit, that is regress the dependent variable $\underline{y}$ separately as reduced models on X_A X_B *and* X_{AB} to obtain using the usual least square methods, the three terms of Equation (20) or (24). Now the sums of squares and the corresponding estimated regression coefficients on the right hand side of Equation (24) are obtained by fitting reduced regression models separately of X_A, X_B *and* X_{AB} as reduced design matrices. That is the dependent variable $\underline{y}$ is separately fitted, that is regressed on each of the reduced design matrices X_A, X_B *and* X_{AB}.

These regression models would yield estimates of the corresponding reduced partial regression parameters, $\underline{\beta}_A, \underline{\beta}_B$ *and* $\underline{\beta}_{AB}$ as respectively

$$\hat{\underline{\beta}}_A = \underline{b}_A = (X'_A.X_A)^{-1}.X'_A.\underline{y}; \hat{\underline{\beta}}_B = \underline{b}_B = (X'_B.X_B)^{-1}.X'_B.\underline{y}; \hat{\underline{\beta}}_{AB} = \underline{b}_{AB} = (X'_{AB}.X_{AB})^{-1}.X'_{AB}.\underline{y} \tag{25}$$

If the full model of Equation (10) fits, that is if the null hypothesis of Equation (14) is rejected, then the additional null hypothesis of Equations 19-21 may be tested using the extra sum of squares principle (Drapa and Smith, 1966). If we denote the sum of squares due to the full model of Equation (10) and the reduced models due to the fitting of the criterion variable $\underline{y}$ to any of the reduced design matrices by $SS(F)$ *and* $SS(R)$, respectively then following the extra sum of squares principle (Draper and Smith, 1966;Neter and Wasserman 1974), the extra sum of squares due to a given factor is calculated as $ESS = SS(F) - SS(R)$ Equation (26)with degrees of freedom obtained as the difference between the degrees of freedom of $SS(F)$ *and* $SS(R)$. That is as

Notes

$$Edf = df(F) - df(R) \tag{27}$$

Thus the extra sums of squares for factors A, B and A by B interaction are obtained as respectively

$$ESSA = SSR - SSA; ESSB = SSR - SSB; ESSAB = SSR - SSAB \tag{28}$$

Notes

With degrees of freedom of respectively

$$(ab-1)-(a-1) = a(b-1); (ab-1)-(b-1) = b(a-1); (ab-1)-(a-1)(b-1) = a+b-2 \tag{29}$$

Note that since each of the reduced models and the full model have the same total sum of squares, SS_{Tot} ,the extra sum of squares may alternatively be obtained as the difference between the error sum of squares of each reduced model and the error sum of squares of the full model. In other words the extra sum of squares is equivalently calculated as

$$ESS = SS(F) - SS(R) = (SS_{Tot} - SS(F)) - (SS_{Tot} - SS(R)) = SSE(R) - SSE(F) \tag{30}$$

With the degrees of freedom similarly obtained as

$$Edf = df\,SSE(R) - df\,SSE(F) \tag{31}$$

Thus the extra sums of squares due to factors A, B and A by B interaction are alternatively obtained as

$$ESSA = SSEA - SSE; ESSB = SSEA - SSE; ESSAB = SSEAB - SSE \tag{32}$$

Where SSR and SSE are respectively the regression sum of squares and the error sum of squares for the full model. The null hypothesis of Equations (19) - (21) are tested using the F ratios as summarized in Table 2 which for completeness also includes the values of Table 1 for the full model.

Table 2 : Two factor Analysis of Variance Table showing Extra Sums of Squares

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (DF)	Mean sum of Squares (MS)	F Ratio	Extra Sum of Squares (SS(F)-SS(R))	Degrees of Freedom (DF)	Extra Mean Sum of Squares (EMSR)	F Ratio
Regression	$SSR = \underline{b}'.X'.\underline{y} - n\bar{y}^2$	$ab-1$	$MSR = \frac{SSR}{ab-1}$	$F = \frac{MSR}{MSE}$	$SSR = \underline{b}'X'.\underline{y} - n\bar{y}^2$	$ab-1$	$MSR = \frac{SSR}{ab-1}$	$F = \frac{MSR}{MSE}$
Full Model								
Error	$SSE = \underline{y}'.\underline{y} - \underline{b}'.X'.\underline{y}$	$n-a.b$	$MSE = \frac{SSE}{n-a.b}$		$SSE = \underline{y}'\underline{y} - \underline{b}'X'.\underline{y}$	$n-a.b$	$MSE = \frac{SSE}{n-a.b}$	
Factor A								
Regression	$SSA = \underline{b}'_A.X'_A.\underline{y} - n\bar{y}^2$	$a-1$	$MSA = \frac{SSA}{a-1}$	$F = \frac{MSA}{MSEA}$	$ESSA = SSR - SSA$	$a(b-1)$	$EMSA = \frac{ESSA}{a(b-1)}$	$F = \frac{EMSA}{MSE}$
Error	$SSEA = \underline{y}'.\underline{y} - \underline{b}'_A.X'_A.\underline{y}$	$n-a$	$MSEA = \frac{SSEA}{n-a}$		$ESSEA = SSEA - SSE$ $= ESSA$	$a(b-1)$	$EMSSA = \frac{ESSEA}{a(b-1)}$	
Factor B								
Regression	$SSB = \underline{b}'_B.X'_B.\underline{y} - n\bar{y}^2$	$b-1$	$MSB = \frac{SSB}{b-1}$	$F = \frac{MSB}{MSEB}$	$ESSB = SSR - SSB$	$b(a-1)$	$EMSB = \frac{ESSB}{b(a-1)}$	$F = \frac{EMSB}{MSE}$
Error	$SSEB = \underline{y}'.\underline{y} - \underline{b}'_B.X'_B.\underline{y}$	$n-b$	$MSEB = \frac{SSEB}{n-a}$		$ESSEB = SSEB - SSE$ $= ESSB$	$b(a-1)$	$EMSEB = \frac{ESSEB}{b(a-1)}$	
Factor A by B interaction								
Regression	$SSAB = \underline{b}'_{AB}.X'_{AB}.\underline{y} - n\bar{y}^2$	$(a-1)(b-1)$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	$F = \frac{MSAB}{MSEAB}$	$ESSAB = SSR - SSAB$	$a+b-2$	$EMSAB = \frac{ESAB}{a+b-2}$	$F = \frac{EMSAB}{MSE}$
Error	$SSEB = \underline{y}'.\underline{y} - \underline{b}'_{AB}.X'_{AB}.\underline{y}$	$n-(a-1)(b-1)$	$MSEAB = \frac{SSEAB}{n-(a-1)(b-1)}$		$SSEAB - SSE = ESSAB$	$a+b-2$	$EMSEAB = \frac{ESSEAB}{a+b-2}$	
Total	$SS_{Tot} = \underline{y}'.\underline{y} - n.\bar{y}^2$	$n-1$			$SS_{Tot} = y'.y - n.\bar{y}^2$	$n-1$		

Notes

Notes

Note that the F ratios of Table 2 are each the ratio of the extra mean sum of squares of the corresponding reduced model to the mean sum of squares of the full model. Where SSR is the regression sum of Squares with $a.b-1$ degrees of freedom, SSE is the error sum of Squares with $n-a.b$ degrees of freedom and MSE is the mean error sum of Squares, all for the full model of Equation (10). The results of Table 2 are the same as would be obtained using the conventional two factor or two way analysis of variance with replications and interactions. Usually, the hypothesis of no interaction is tested first using the corresponding F ratio of Table 2. If the hypothesis of no interaction is accepted, then one may proceed to test the null hypotheses about factors A and B effects again using the corresponding F ratios of Table 2. If however the null hypothesis of no interaction is rejected, then one may use any of the familiar and appropriate methods of treating interactions and proceed with further analysis.

Thus if the model of Equation (10) fits, that is if the null hypothesis of Equation (14) is rejected then the null hypotheses of Equations 19-21 of no factors A, B and A by B interaction are respectively tested using the corresponding test statistics (see Table 2), namely

$$F_A = \frac{EMSA}{MSE}; F_B = \frac{EMSB}{MSE}; F_{AB} = \frac{EMSAB}{MSE} \quad (33)$$

With numerator degrees of freedom of $a(b-1), b(a-1)$, and $a+b-2$ respectively and common denominator degrees of freedom of $n-a.b$ for use to obtain the necessary critical $F-ratios$ for comparative purposes for rejection or acceptance of the corresponding null hypothesis.

Note that in general whether or not the independent or explanatory variables used in a regression model are dummy variables or numeric measurements, the extra sum of squares principle is most useful in determining the contribution of an independent variable or a subset of the independent variables among all the independent variables in the model in explaining the variation of a specified dependent on criterion variable. This would inform the inclusion or exclusion of the independent variable or the subset of the independent variables in the hypothesized model depending on the significance of the contribution.

Thus the extra sum of squares principle enables one select important variables and formulate a more parsimonious statistical model of explanatory variables for a dependent variable of interest. To do this, for example, for one independent variable $X_{j,}$ included in a regression model with say a total of $'r'$ independent variables, over fits the full model with all the independent variables and reduced model with only one independent variable X_{jA} .Suppose as discussed earlier that the regression sums of squares for the full model and the reduced model are respectively $SS(F)$ and $SS(R)$ which have degrees of freedom of $'r'$ and 1 respectively. Then from Equation (28) the extra sum of squares regression with respect to X_j is

$$ESS(X_j) = SS(F) - SS(R) \quad (34)$$

With r-1 degrees of freedom. The corresponding extra mean sum of squares is

$$EMS(X_j) = \frac{ESS(X_j)}{r-1} \quad (35)$$

The significance of β_j, the partial regression coefficient or effect X_j on the criterion variable y is determined using the test statistic.

$$FX_j = \frac{EMS(X_j)}{MSE} \qquad (36)$$

Which has $r-1$ and $n-r$ degrees of freedom for $j = 1,2,...r$; where MSE is the error mean square for the full model and 'n' is the total sample size.

Notes

An advantage of using dummy variable regression models in two factor and multi factor analysis of variance is that the method enables the estimation of other effects separately of several factors on a specified dependent or criterion variable. For example it enables the estimation of the total or absolute effect, the partial regression coefficient or the so-called direct effect of a given independent variable on the dependent variable through the effects of its representative dummy variables as well as the indirect effect of that parent independent variable through the meditation of other parent independent variables in the model (Wright, 1934).

The total or absolute effect of a parent independent variable on a dependent variable is estimated as the simple regression coefficient of that independent variable represented by codes assigned to its various categories, when regressed on the dependent variable. The direct effect of a parent independent variable on a dependent variable is the weighted sum of the partial regression coefficients or effects of the dummy variables representing that parent independent variable on the dependent variable, where the weights are the simple regression coefficients of each representative dummy variable regressing on the specified parent independent variable represented by codes. The indirect effect of a given parent independent variable on a dependent variable is then simply the difference between its total and direct effects (Wright 1973). Now the direct effect or partial regression coefficient of a given parent independent variable A say on a dependent variable Y is obtained by taking the partial derivative of the expected value of the corresponding regression model with respect to that parent independent variable. Thus the direct effect of the parent independent variable A on the dependent variable Y is obtained from Equation 7 as

$$\beta_A dir = \frac{dE(y_{ilj})}{d_A} = \sum_{l=1}^{a-1} \beta_l ; A. \frac{dE(x_{il} ; A)}{d_A} + \sum_g \beta_g ; Z \frac{dE(x_{ig} ; Z)}{dA} \qquad OR$$

$$\beta_A dir = \sum_{l=1}^{a-1} \beta_l ; A \frac{dE(x_{il;A})}{dA} \qquad (37)$$

Since $\sum_g \beta_g ; Z \frac{dE(x_{ig} : Z)}{dA} = 0$, for all other independent variables Z in the model different from A.

The weight, $\alpha_l ; A = \frac{dE(x_{il;A})}{dA}$ is estimated by fitting a simple regression line of the dummy variable $x_{il;A}$ regressing on its parent independent variable, A represented by codes and taking the derivative of its expected value with respect to A, Thus if the expected value of the dummy variable $x_{il;A}$ regressing on its parent independent variable A is expressed as

Notes

Then the derivative of $E(x_{il;A})$ with respect to A is

$$\frac{dE(x_{il;A})}{dA} = \alpha_{j;A} \quad (39)$$

Hence using Equation 39 in Equation 37 gives the direct effect of the parent independent variable A on the dependent variable Y as

$$\beta_{Adir} = \sum_{l=1}^{a-1} \alpha_{l;A_l} . \beta_{l;A} \quad (40)$$

Whose sample estimate is from Equation 12

$$\hat{\beta}_{Adir} = b_{Adir} = \sum_{l=1}^{a-1} \alpha_{l;A} . b_{l;A} \quad (41)$$

The total or absolute effect of A on Y is estimated as the simple regression coefficient or effect of the parent independent variable A represented by codes on the dependent variable Y as

$$\hat{\beta}_A = b_A \quad (42)$$

Where b_A is the estimated simple regression coefficient or effect of A on Y. The indirect effect of A on Y is estimated as the difference between b_A and $b_{Adir,}$ that is as

$$\hat{\beta}_{Aind} = b_{Aind} = b_A - b_{Adir} \quad (43)$$

The total, direct and indirect effects of factor B are similarly estimated. These results clearly give additional useful information on the effects of given factors on a specified dependent or criterion variable than would the traditional two factor analysis of variance model.

III. Illustrative Example

In a study of Encephalitic and Meningitic brain damage each of a random sample of 36 patients is given a battery of tests on mental acuity recording a composite score for each patient. Low scores on this composite measure indicate some degree of brain damage. The patients are divided into 3 groups according to the predisposing factor of initial infection and into 3crossed groups according to time to observed physical recovery from the illness. A control group of other mental patients are similarly studied with the following results. (Table 3)

Table 3 : Mental acuity of sample data of patients with diagnosed metal illness by factor and time to recovery.

	Time to Recovery (B)		
Predisposing factor (A)	**1 – 2 years (1)**	**3 – 5 years (2)**	**7 – 10 years (3)**
Encephalitis (1)	76 73 75 62	69 53 72	59, 43 41 57, 55
Meningitis (2)	81 89 83	82 70 91, 74 75	68 50 75 47
Other (Control (3)	75 79 84 65 63	85 76 87	98 100 82 79

Do there seem to be significant differences in performance among the encephalitic, meningitic and other (control) patients? Among patients according to time to recovery? Is there any interaction between predisposing factor of illness and time to recovery of patients?

To answer these questions using dummy variable multiple regression analysis or model, we represent the predisposing factor here called factor A which has three levels with two dummy variables of Is and Os namely $x_{i1.A}$ for (1) Encephalitis and $x_{12;A}$ for (2) Meningitis. Time to recovery here called factor B also with three levels is represented by two dummy variables of Is and 0s namely $x_{i1;B}$ for (1) 1 – 2 years and $x_{i2;B}$ for (2) 3 – 5 years. The interaction terms are represented by the cross products of these dummy variables namely $x_{i3} = x_{i1;A}.x_{i2;B}; x_{i4};= x_{i1;A}.x_{i2;B}; x_{i5} = x_{i2;B}.x_{i1;B}$ *and* $x_{i6} = x_{i2;A.}.x_{i2;B}$ *for* $i = 1, 2...36$ yielding the design matrix of Table 4.

Table 4 : Design Matrix X for the Data of Table 3

S/N	y_{ilj}	x_{i0}	$x_{i1;A}$	$x_{i2;A}$	$x_{i1;B}$	$x_{i2;B}$	x_{i3} $(x_{i1;A}.x_{i1;B})$	x_{i4} $(x_{i1;A}.x_{i2;B})$	x_{i5} $(x_{i2;A}.x_{i1;B})$	x_{i6} $(x_{i2;A}.x_{i2;B})$
1.	76	1	1	0	1	0	1	0	0	0
2.	73	1	1	0	1	0	1	0	0	0
3.	75	1	1	0	1	0	1	0	0	0
4.	62	1	1	0	1	0	1	0	0	0
5.	69	1	1	0	0	1	0	1	0	0
6.	53	1	1	0	0	1	0	1	0	0
7.	72	1	1	0	0	1	0	1	0	0
8.	59	1	1	0	0	0	0	0	0	0
9.	43	1	1	0	0	0	0	0	0	0
10.	41	1	1	0	0	0	0	0	0	0
11.	57	1	1	0	0	0	0	0	0	0
12.	55	1	1	0	0	0	0	0	0	0
13.	81	1	0	1	1	0	0	0	1	0
14.	89	1	0	1	1	0	0	0	1	0
15.	83	1	0	1	1	0	0	0	1	0
16.	82	1	0	1	0	1	0	0	0	1
17.	70	1	0	1	0	1	0	0	0	1
18.	91	1	0	1	0	1	0	0	0	1
19.	74	1	0	1	0	1	0	0	0	1
20.	75	1	0	1	0	1	0	0	0	1
21.	68	1	0	1	0	0	0	0	0	0
22.	50	1	0	1	0	0	0	0	0	0
23.	75	1	0	1	0	0	0	0	0	0
24.	47	1	0	1	0	0	0	0	0	0
25.	75	1	0	0	1	0	0	0	0	0
26.	79	1	0	0	0	0	0	0	0	0
27.	84	1	0	0	0	0	0	0	0	0
28.	65	1	0	0	0	0	0	0	0	0
29.	63	1	0	0	0	0	0	0	0	0
30.	85	1	0	0	1	1	0	0	0	0
31.	76	1	0	0	1	1	0	0	0	0
32.	87	1	0	0	1	1	0	0	0	0
33.	98	1	0	0	1	0	0	0	0	0
34.	100	1	0	0	1	0	0	0	0	0
35.	82	1	0	0	1	0	0	0	0	0
36.	79	1	0	0	1	0	0	0	0	0

Fitting the full model of Eqn 10 using the design matrix X of table 4, we obtain the fitted regression equation

$$\hat{y}_{ilj} = 83.642 - 32.642x_{i1;A} - 18.726x_{i2A} - 10.442x_{i1;B} + 7.168x_{i2;B} - 30.942x_{i3} + 6.499x_{i4} + 29.860x_{i5} + 2.383x_{i6} \quad (Pvalue = 0.0000) \tag{44}$$

Notes

Notes

A P-value of 0.0000 clearly shows that the model fits.

The expected score by patients on the mental acuity test by predisposing factor (factor A), is obtained by setting $x_{i1;A} = x_{i2;A} = 1$, and all other $x_{ijs} = 0$ in equation (44) giving

$$\hat{y}_{ilj} = 83.642 - 32.642 - 18.726 = 32.274$$

The estimated response or score on the mental acuity test by length of time to observed physical recovery is similarly estimated by setting $x_{i1;B} = x_{i2;B} = 1$ and all other $x_{ils} = 0$ in Equation (44) yielding

$$\hat{y}_{ilj} = 83.642 - 10.442 + 7.168 = 80.368$$

The corresponding analysis of variance table for the full model is presented in Table 5.

Table 5 : Anova Table for the Full Model of Equation (44)

Source of Variation	**Sum of Squares (SS)**	**Degrees of freedom (Df)**	**Mean Sum of Squares (MS)**	**F-Ratio**	**P-Value**
Regression (treatment)	4597.321	8	574.665	5.468	0.0000
Error	2837.652	27	105.098		
Total	7434.972	35			

Having fitted the full model which is here seen to fit, we now proceed to fit the dependent variably y separately on each of the sub matrices X_A and X_B each with two dummy variables of Is and 0s and X_{AB} with four dummy variables of Is and 0s as reduced models to obtain the corresponding sum of squares due to each of these factors. The sums of squares due to factor A, B and A by B interaction are calculated following Equation (24). The results are summarized in a two factor analysis of variance Table with extra sums of squares (Table 6)

Table 6 : Two factor Analysis of Variance Table with Extra Sums of Squares for the Sample Data of Table 3

Source of Variation	Sum of Squares (SS)	Degrees of freedom (Df)	Mean sum of squares MS	F-Ratio	Extra Sum of Squares (ESS)	Degrees of freedom (Df)	Extra mean sum of squares (EMS)	F-Ratio	Critical F value P-value
Full Model									
Regression	4597.321	8	574.665	5.468	4597.321	8	574.665	5.468	3.030
Error	2837.652	27	105.098		2837.652	27	105.098		
Factor A									
Regression	2413.556	2	1206.778	7.931	2183.765	6	363.963	3.463	2.46
Error	5021.417	33	152.164		2837.652	6	472.942		
Factor B									
Regression	817.650	2	408.825	2.039	3779.671	6	629.945	1.096	2.46
Error	6617.322	33	200.525		-3779.67	6	-629.95		
Factor A by B Interaction									
Regression	624.201	4	156.050	0.710	3973.12	4	993.28	1.728	2.73
Error	6810.771	31	219.702		-3973.12	4	-993.23		
Total	7434.972	35			7434.972	35			

Note: * indicates statistical significance at the 5 percent level

These analyses indicate that the hypothesized model fits, that is that not all the factor level effects are zero. Furthermore, there does not seem to exist any significant interaction between predisposing factor of illness A and time to observed physical recovery B. However only the predisposing factor of illness A is seen to have significant effect on the criterion variable Y namely patient composite score on mental acuity.

Finally to estimate the direct effect or partial regression coefficient of A, say, represented by the dummy variables $x_{i1;A}$ and $x_{i2;A}$, we first estimate the simple regression coefficient resulting when theses dummy variables are each regressed on A using Equation 39, yielding.

Notes

$$\alpha_{1;A} = -\frac{1}{2} = -0.5;\ \alpha_{2;A} = 0$$

Using these results with Equation (44) in (41), we obtain an estimate of the direct effect of A on 'y' as

$$b_A dir = (-0.5)(-32.642) + (0)(-10.442) = 16.321$$

The estimated simple regression coefficient or effect of A on y is $b_A = 9.917$
Hence the estimated indirect effect of A on'y' is from Equation (43)

$$b_A ind = 9.917 - 16.321 = -6.404$$

The absolute, direct and indirect effects of B on 'y' are similarly estimated.

IV. Summary and Conclusion

We have in this paper proposed and developed a method that enabled the use of dummy variable multiple regression techniques for the analysis of data appropriate for use with two factor analysis of variance models with unequal observations per treatment combination and with interactions. The proposed model and method employed the extra sum of squares principle to develop appropriate test statistics of F ratios to test for the significance of factor and interaction effects.

The method which was illustrated with some sample data was shown to yield essentially the same results as would the traditional two factor analysis of variance model with unequal observations per cell and interaction. However the proposed method is more generalized in its use than the traditional method since it can easily be used in the analysis of two-factor models with one observation, equal, and unequal observations per cell as a rather unified analysis of variance problem.

Furthermore unlike the traditional analysis of variance models the proposed method is able to enable one using the extra sum of squares principle, to determine the relative contributions of independent variables or some combinations of these variables in explaining variations in a given dependent variable and hence build a more parsimonious explanatory model for any variable of interest. In addition, the method enables the simultaneous estimation of the total or absolute, direct and indirect effects of a given independent variable on a dependent variable, which provide additional useful information.

Notes

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An Wonderful Summation Formula Convoluted with Contiguous Relation

By Salahuddin, M. P. Chaudhary & Diriba Kejela Geleta
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Keywords: gauss second summation theorem, recurrence relation, prudnikov.

GJSFR-F Classification : MSC 2010: 33C05, 33C20, 33D15, 33D50, 33D60

ANWONDERFULSUMMATIONFORMULACONVOLUTEDWITHCONTIGUOUSRELATION

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Notes

An Wonderful Summation Formula Convoluted with Contiguous Relation

Salahuddin [α], M. P. Chaudhary [σ] & Diriba Kejela Geleta [ρ]

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I. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A & ; \\ & & z \\ b_1, b_2, \cdots, b_B & ; \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers and $| z |= 1$

Contiguous Relation is defined by

[Andrews p.363(9.16), E. D. p.51(10)]

$$ (a-b)\ {}_2F_1\left[\begin{matrix} a,\ b\ ; & \\ c\ \ ; & z \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b & ; & \\ c & ; & z \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1 & ; & \\ c\ ; & & z \end{matrix}\right] \tag{2} $$

Gauss second summation theorem is defined by [Prudnikov., 491(7.3.7.5)]

$$ {}_2F_1\left[\begin{matrix} a,\ b & ; & \\ \frac{a+b+1}{2} & ; & \frac{1}{2} \end{matrix}\right] = \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \tag{3} $$

$$ = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})} \tag{4} $$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$$ {}_2F_1\left[\begin{matrix} a,\ b & ; & \\ \frac{a+b-1}{2} & ; & \frac{1}{2} \end{matrix}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\ \Gamma(b)}\right] \tag{5} $$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

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$$ {}_2F_1\left[\begin{matrix} a, \ b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & ; & \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{6} $$

Recurrence relation is defined by

$$ \Gamma(\zeta+1) = \zeta\ \Gamma(\zeta) \tag{7} $$

II. Main Summation Formula

$$ {}_2F_1\left[\begin{matrix} a, \ b & ; & \frac{1}{2} \\ \frac{a+b+51}{2} & ; & \end{matrix}\right] = \frac{2^b\ \Gamma(\frac{a+b+51}{2})}{(a-b)\ \Gamma(b)\left[\prod_{\varrho=1}^{25}\{a-b-(2\varrho-1)\}\right]\left[\prod_{\varsigma=1}^{25}\{a-b+(2\varsigma-1)\}\right]} $$

$$ \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})}\Big\{16777216(a^{26}+a^{25}(-625+1224b)+25a^{24}(7396-9775b+9163b^2) \right. $$

$$ +100a^{23}(-344425+752114b-262395b^2+156604b^3)+230a^{22}(19709459-33907350b+33726980b^2 $$

$$ -5313350b^3+2265165b^4)+17710a^{21}(-25320175+58459396b-30894950b^2+19064140b^3-1697875b^4 $$

$$ +549712b^5)+230a^{20}(150530498030-292095255025b+300323641639b^2-76706596750b^3 $$

$$ +33242331080b^4-1893929625b^5+483162491b^6)+2300a^{19}(-929947187675+2221532039250b $$

$$ -1430366003815b^2+898110785264b^3-134850725625b^4+43814588818b^5-1725580325b^6) $$

$$ +354976524b^7+2185a^{18}(49213453442791-102599357455300b+107537450188000b^2 $$

$$ -34831583707900b^3+15171737968030b^4-1494100037500b^5+378247207240b^6-10846504900b^7 $$

$$ +182418491 5b^8)+2185a^{17}(-203704868006 1175+4974851454538832b-3608737921763100b^2 $$

$$ +2282171335907400b^3-445423671139650b^4+143879155669032b^5-9955177409100b^6 $$

$$ +2009515142040b^7-43458522975b^8+60090797 20b^9)+1955a^{16}(78196640069934680 $$

$$ -170970567062484075b+180998915556065411b^2-67807865563445700b^3+29427554850189300b^4 $$

$$ -3827247742614450b^5+952626139804962b^6-48449597307300b^7+7889492452380b^8 $$

$$ -131448618875b^9+14872472307b^{10})+109480a^{15}(-3999070285605812 5+9903668635739917 0b $$

$$ -77792742051082475b^2+49149496844294628b^3-11276126774591250b^4+3589210801875180b^5 $$

$$ -330978283892550b^6+64822304809584b^7-2491392044625b^8+327096283330b^9-4215553375b^{10} $$

$$ +380166268b^{11})+340a^{14}(308246102285629047091-696416403959283989450b $$

$$ +739141133551046804900b^2-305205676801283739250b^3+130862315493685966715b^4 $$

$$ -20189297381974528500b^5+4886797542957632400b^6-331816588273247700b^7 $$

$$ +51414459002198985b^8-1505891473581250b^9+155745474282860b^{10}-1496904680250b^{11} $$

$$ +96467190505b^{12})-340a^{13}(616880343249751686 7075-1539090800459834740695 6b $$

$$ +1277232167086191465065 0b^2-799781334941830654602 0b^3+204195075593483602498 5b^4 $$

$$ -633657121834647845368b^5+69430174491959238300b^6-12968488981389722760b^7 $$

$$ +658096757009317605b^8-79151871484967260b^9+1708599193535850b^{10}-125088007991380b^{11} $$

$$ +6752703335 35b^{12})-340a^{12}(-1030178237495060404752 10+2379106017872272801356 25b $$

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$$-2512856661714001729250 11b^2+110831320079452337254890b^3-464659644318469856201 40b^4$$
$$+79921660067640129252 15b^5-18495329291190086654 41b^6+147559784213610898220b^7$$
$$-2099324861691459857 0b^8+773406749919025495b^9-65078685048990613b^{10}+781590483290890b^{11}$$
$$-6752703335 35b^{13}+964671905 05b^{14})-680a^{11}(71491457735594668993512 5$$
$$-178556846901254034742639 0b+1536661412180534230322244 5b^2-944266221187015176370984b^3$$
$$+25811706291459790694310 5b^4-7679544303557898329649 0b^5+927716137299884330898 5b^6$$
$$-15949113105370119111444b^7+9058999166670274321 5b^8-888467336318330769 0b^9$$
$$+17995652657792309 5b^{10}-39079524164544 5b^{12}+62544003995690b^{13}-748452340125b^{14}$$
$$+612067691 48b^{15})-5a^{10}(-1114416507876347373186164291+260525163250631984626500660 0b$$
$$-271323653056815525998794928 0b^2+124538926846370632829993788 0b^3$$
$$-50224716889889719356281449 2b^4+9148768058933389501718740 0b^5$$
$$-1953908608042226785251848 0b^6+163727225039124348165564 0b^7-19039436677003398214959 4b^8$$
$$+5856587190172219901400b^9-244740876145975409 20b^{11}+44253505833313616 84b^{12}$$
$$-11618474516043780 0b^{13}+105906922512344 80b^{14}-923037566990 00b^{15}+58151366720 37b^{16})$$
$$-35a^9(149313615080134446367938792 5-3703909632762109598425147640b$$
$$+3246559298435769978327965800b^2-1926953531613304593979387120b^3$$
$$+5425076960982796694640578 20b^4-149417805588085678869647632b^5$$
$$+182131370915097153559646 00b^6-2565922911004141451388080b^7$$
$$+114192589173533471911310b^8-836655312881745700200b^{10}+17261651105613283512 0b^{11}$$
$$-75130941420705333 80b^{12}+76890389442539624 0b^{13}-14628660029075000b^{14}$$
$$+10231571742562 40b^{15}-73423442828 75b^{16}+37513826252 0b^{17})$$
$$-5a^8(-79203796430606208986686969380+18528875873668454397762440792 5b$$
$$-187454251356830104937440572625b^2+86812396275445363314471791160b^3$$
$$-32530060756440164285483566920b^4+5816403994485737376716026860b^5$$
$$-1020905487646592853629193884b^6+64402163268671537755435080b^7$$
$$-7993481242147343033791 70b^9+19039436677003398214959 4b^{10}-12320238866671573077240b^{11}$$
$$+142754090595019270276 0b^{12}-44750579476633597140b^{13}+349618321214953098 0b^{14}$$
$$-5455152020910900 0b^{15}+30847915488805 80b^{16}-18991374540075b^{17}+7971688078 55b^{18})$$
$$-20a^7(119162173363594180659738941625-2898652383209820419758759218 90b$$
$$+252318392792718541698360880915b^2-139829590206394802306612896492b^3$$
$$+37872700780492240721071991460b^4-8601790875678594877022106360b^5$$
$$+768050259516698029488037820b^6-1610054081716788443885877 0b^8$$
$$+449036509425724753992914 0b^9-409318062597810870413910b^{10}+54226984558258404989096b^{11}$$
$$-250851633163138526974 0b^{12}+2204643126836252869 20b^{13}-564088200064521090 0b^{14}$$
$$+354837296527662816b^{15}-4735948136788575b^{16}+219539529267870b^{17}-11849806603 25b^{18}$$

Notes

$$+40822300260b^{19}) - 2a^{6}(-5561476346804512129684103946537$$
$$+1277058733162600297821865148025 0b - 1215050620501969520067226174110 0b^{2}$$
$$+533236851546392079166462649785 0b^{3} - 165485613821717093329982731247 5b^{4}$$
$$+212504641903276935470565517000b^{5} - 76805025951669802948803782 00b^{7}$$
$$+255226371911648213407298471 0b^{8} - 318729899101420018729380500b^{9}$$
$$+48847715201055669631296200b^{10} - 3154234866819606725054900b^{11}$$
$$+314420597950231473124970b^{12} - 11803129663633070511000b^{13} + 830755582302797508000b^{14}$$
$$-18117751260278187000b^{15} + 931192051659350355b^{16} - 10876031319441750b^{17}$$
$$+413235073909700b^{18} - 1984417373750b^{19} + 555636864465b^{20})$$
$$-2a^{5}(1946737181928490708624750112 9625 - 4513753924827938583441615238762 8b$$
$$+3683225654502972913203972747525 0b^{2} - 170079393341983782553260397593 00b^{3}$$
$$+326046607622060573774506693982 5b^{4} - 212504641903276935470565517000b^{6}$$
$$+86017908756785948770221063600b^{7} - 14541009986214343441790067150b^{8}$$
$$+2614811597791499380218833560b^{9} - 228719201473334737542968500b^{10}$$
$$+26110450632096854320806600b^{11} - 1358668221149882197286550b^{12}$$
$$+107721710711890133712560b^{13} - 3432180554935669845000b^{14} + 196473399294647353200b^{15}$$
$$-3741134668405624875b^{16} + 157187977568417460b^{17} - 1632304290968750b^{18} + 50386777140700b^{19}$$
$$-217801906875b^{20} + 4867699760b^{21}) - 10a^{4}(-97289856165422209495685970550 50$$
$$+2086053242070434000130711241717 5b - 1670653850805727909383994949492 1b^{2}$$
$$+513634242186040713256004842881 0b^{3} - 652093215244121147549013387965b^{5}$$
$$+330971227643434186659965462495b^{6} - 75745401560984481442143982920b^{7}$$
$$+16265030378220082142741783460b^{8} - 1898776936343978843124202370b^{9}$$
$$+251123584449448596781407246b^{10} - 17551960278192657672131140b^{11}$$
$$+1579842790682797511084760b^{12} - 69426325701784424849490b^{13} + 4449318726785322868310b^{14}$$
$$-123451035928225005000b^{15} + 5753086973212008150b^{16} - 97325072144013525b^{17}$$
$$+3315024746014555b^{18} - 31015666893750b^{19} + 764573614840b^{20} - 3006936625b^{21} + 52098795b^{22})$$
$$-140a^{3}(114591674826841350084919611187 5 - 2288297987564913630461499692850b$$
$$+1302380715384648078117984831015b^{2} - 366881601561457652325717744915b^{4}$$
$$+242970561917119689361800567990b^{5} - 76176693078056011309494664255b^{6}$$
$$+19975655743770686043801842356b^{7} - 3100442724123048689802563970b^{8}$$
$$+481738382903326148494846780b^{9} - 44478188159418083153569210b^{10}$$
$$+4586435931479787999516208b^{11} - 269161777335812819047590b^{12} + 19423260991444458754620b^{13}$$
$$-741213786517403366750b^{14} + 38434906532238399096b^{15} - 946888408403831025b^{16}$$
$$+35618174063983350b^{17} - 543621502869725b^{18} + 14754677186480b^{19} - 126017980375b^{20}$$
$$+24116137710b^{21} - 8729075b^{22} + 111860b^{23}) - 35a^{2}(-4326299607531226662947437702875$$

Notes

Notes

$$+65572957809681708252947317335 00b - 520952286153859231247193932406 0b^3$$

$$+477329671658779402681141414 06b^4 - 21047003740016988075451272843 00b^5$$

$$+694314640286839725752700670920b^6 - 144181938738696309541920503380b^7$$

$$+26779178765261443562491510375b^8 - 32465592984357699783279658 00b^9$$

$$+3876052186525936085697070 40b^{10} - 2985513600807895076055036 0b^{11}$$

$$+24410607570936016798429 64b^{12} - 124073981945515742320600b^{13} + 7180228154495883247600b^{14}$$

$$-243335697135785981800b^{15} + 1011008228320308224 3b^{16} - 225288353115782100b^{17}$$

$$+6713409390308000b^{18} - 93995480250700b^{19} + 1973555359342b^{20} - 15632844700b^{21} + 221634440b^{22}$$

$$-749700b^{23} + 6545b^{24}) + a(-584358414459472720534554743906 25$$

$$+229505352333885978885315610672500b^2 - 320361718259087908264609956999000b^3$$

$$+20860532420704340001307112417 1750b^4 - 9027507849655877166883230477525 6b^5$$

$$+25541174663252005956437302960500b^6 - 5797304766419640839517518437800b^7$$

$$+926443793683422719888122039625b^8 - 129636837146673835944880167400b^9$$

$$+13026258162531599231325033000b^{10} - 1214186558928527436249945200b^{11}$$

$$+80889604607657275246112500b^{12} - 5232908721563438118365040b^{13}$$

$$+236781577346156556413000b^{14} - 10842536422408061131600b^{15} + 334247458607156366625b^{16}$$

$$-10870050428167347920b^{17} + 224179596039830500b^{18} - 5109523690275000b^{19} + 67181908655750b^{20}$$

$$-1035315903160b^{21} + 77986905 00b^{22} - 75211400b^{23} + 244375b^{24} - 1224b^{25})$$

$$-b(-584358414459472720534554743906 25 + 15142048626359293320316031960062 5b$$

$$-160428344757577890118887455662500b^2 + 97289856165422209495685970550500b^3$$

$$-38934743638569814172495002259250b^4 + 11122952693609024259368207893074b^5$$

$$-2383243467271883613194778832500b^6 + 396018982153031044933434846900b^7$$

$$-52259765278047056228778577375b^8 + 5572082539381736865930821455b^9$$

$$-486141912602043749155885000b^{10} + 35026060074832053761571400b^{11}$$

$$-2097393167049155734805500b^{12} + 104803674777113876010940b^{13} - 4378182148681243525000b^{14}$$

$$+152874431336722299400b^{15} - 4450951365933667375b^{16} + 107531395772498335b^{17}$$

$$-2138878531652500b^{18} + 34622014546900b^{19} - 448420299250b^{20} + 4533175570b^{21} - 34442500b^{22}$$

$$+184900b^{23} - 625b^{24} + b^{25}))\Big\} - \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\Big\{67108864(25a^{25} + 25a^{24}(-208 + 391b) + 20a^{23}(101177$$

$$-58344b + 52479b^2) + 460a^{22}(-455884 + 892959b - 185640b^2 + 106267b^3) + 2530a^{21}(13346809$$

$$-117041 60b + 10766882b^2 - 1163344b^3 + 475405b^4) + 1610a^{20}(-1353942200 + 2704589143b$$

$$-912494856b^2 + 527319226b^3 - 34900320b^4 + 108224 55b^5) + 460a^{19}(426815174659 - 459263696248b$$

$$+426282394643b^2 - 777006406 40b^3 + 317269916 25b^4 - 1414134120b^5 + 345116065b^6)$$

$$+8740a^{18}(-984715225276 + 1984309333429b - 852036675616b^2 + 492261535759b^3 - 56362562620b^4$$

$$+17304624547b^5 - 5521857704b^6 + 1084650 49b^7) + 2185a^{17}(222718961858047 - 272078729517312b$$

$$+252683899863948b^2-600219604931520b^3+242934064133460b^4-19049883914880b^5+4560891507480b^6$$

$$-108581054400b^7+17383409190b^8)+1955a^{16}(-7708080913032800+15557493409997523b$$

$$-7762351183361712b^2+4451249998293588b^3-6747586418116320b^4+2035390612512180b^5$$

$$-115481519613600b^6+22018358544840b^7-4017499012800b^8+525794475500b^9)$$

$$+15640a^{15}(36085356535274431-48105532096658072b+44411429571288773b^2$$

$$-12447420898782912b^3+4957788393308982b^4-519828215919600b^5+121009816113354b^6$$

$$-51452942311168b^7+7879835002830b^8-1115971948000b^9+118035494500b^{10})$$

$$+4760a^{14}(-2634493563069450044+5295166795741208983b-2927845137628855608b^2$$

$$+1655174300820813575b^3-298864444151010600b^4+87812255490403086b^5-6688314593762592b^6$$

$$+1222111405424742b^7-396512804352600b^8+484060497087500b^9-52610106120b^{10}+4276870515b^{11})$$

$$+68a^{13}(467895671183131654401500-664211804312945679056000b+605745723918476854018600b^2$$

$$-189910158086799121563200b^3+73820498865313394747700b^4-9253320058537152192000b^5$$

$$+2068269630695661010800b^6-117497931172729161600b^7+16817367098205284100b^8$$

$$-413181917915312000b^9+38348174094522600b^{10}-290599095259200b^{11}+13505406670700b^{12})$$

$$-68a^{12}(7453455831561136614620000-14833615758189486852900500b+8818248061484284778045600b^2$$

$$-4875550359055230530730600b^3+988192525136718144152000b^4-279349067814604592832300b^5$$

$$+25309581683326575252800b^6-4330728616047077498800b^7+183106044650014668000b^8$$

$$-19654472755109273900b^9+333115617728493600b^{10}-17520703785002600b^{11}+13505406670700b^{13})$$

$$-136a^{11}(-641411470422114063568765+952391263870199916976200b$$

$$-851530880817831240234369b^2+288041151657164042271680b^3-107958722843046786837217b^4$$

$$+15112252054344345746664b^5-3169716881851024760437b^6+208882234126009254720b^7$$

$$-26340965887457983671b^8+757967917191378712b^9-45609432306336899b^{10}+876035189250130b^{12}$$

$$-14529954762960b^{13}+149690468025b^{14})-136a^{10}(720758456722307762556694000$$

$$-1410402685661213589796162500b+880354323721928908588008000b^2-470473545063530596863655100b^3$$

$$+102352434142240386262813200b^4-27211170476712261843685700b^5+2684523857819735023092000b^6$$

$$-405565744913865730494300b^7+17874998553495527043600b^8-1246553930163278837100b^9$$

$$+45609432306336899b^{11}-1665578088642468b^{12}+191740870472613b^{13}-1841353714200b^{14}$$

$$+135740818675b^{15})+a^9(11235300352609743619544070295-17171398091754762959354650880b$$

$$+14882192077639511485086884504b^2-5258867741091186554073629888b^3$$

$$+1858219828829485034803945748b^4-272222373360422257212406848000b^5$$

$$+50524473238981550170383112b^6-32577350013182071698240640b^7+2674718055280627866049540b^8$$

$$-1695313345022059218456b^{10}+103083636738027504832b^{11}-1336504147347430625200b^{12}$$

$$+28096370418241216000b^{13}-230412796613650000b^{14}+174538012667200b^{15}-10279281996025b^{16})$$

$$-7a^8(1228564811375702148246572440000-233867765745928381485427142150b$$

$$+14953240991029729138762688352b^2-75548565974360910525754061520b^3$$

Notes

$$+1674125532113613106842481440b^4 - 3947941718217439868675390 76b^5$$

$$+3662572723990770898850505 6b^6 - 3610588219953128377714296b^7 + 382102579325803980864 22b^9$$

$$-3472856861821988111328b^{10} + 5117673372420408256 08b^{11} - 177874443374299963 20b^{12}$$

$$+163368708953994188 4b^{13} - 26962870695976800b^{14} + 1760580277775160b^{15} - 112203008143 20b^{16}$$

$$+5426107011 45b^{17}) - 4a^7(-15712233652488463119205654378 5$$

$$+2424172071633021784484783203 60b - 1992988130339918704148524685 03b^2$$

$$+7034958803762113032227084736 0b^3 - 2210966988815633449444990304 4b^4$$

$$+296454973456199577448088284 8b^5 - 3598877513916596981875885 24b^6$$

$$+6318529384917974661000018b^8 - 8144337503295517924560 16b^9 + 137892353270714348368062b^{10}$$

$$-7101995960284314660480b^{11} + 736223864728003174796b^{12} - 19974648299363957472b^{13}$$

$$+145431257245544298 0b^{14} - 20118100443866880b^{15} + 1076147273879055b^{16} - 5931240096600b^{17}$$

$$+2369961320 65b^{18}) - 4a^6(7679418765227492517434309934 60 - 139286917291503525054856188250 5b$$

$$+8769029498190944172406133134 00b^2 - 395331281111828674826897085587 7b^3$$

$$+786391137384008165111099062 24b^4 - 1216196580962694219142238776 4b^5$$

$$+359887751391659698187588524b^7 - 640950226698384907298838 48b^8$$

$$+12631118309745387542595778b^9 - 912738111658709907851280b^{10} + 10777037398293484185485 8b^{11}$$

$$-430262888616551779297 6b^{12} + 35160583721826237183 6b^{13} - 795909436657748448 0b^{14}$$

$$+473148381003214140b^{15} - 5644159271114700b^{16} + 249138698596095b^{17} - 1206525763240b^{18}$$

$$+396883474 75b^{19}) - 2a^5(-65661054840081273864142856957 85$$

$$+9871915481562597552130646417760b - 7272009082608192934087529171754b^2$$

$$+2271587750081918842847295429008b^3 - 469652076662064339904368905593b^4$$

$$+24323931619253884382844775528b^6 - 592909946912399154896176569 6b^7$$

$$+138177960137610395403638676 6b^8 - 1361118668021112860620342 40b^9$$

$$+1850359592416433805370627 6b^{10} - 1027633139695415510773152b^{11} + 94978683056965561562982b^{12}$$

$$-3146128819902631745280b^{13} + 208993168067159344680b^{14} - 40650566484912720 00b^{15}$$

$$+198959432373065595b^{16} - 20811998177006 40b^{17} + 756212092703 90b^{18} - 3252508476 00b^{19}$$

$$+87120762 75b^{20}) - 2a^4(1807737016418235427869101483820 0$$

$$-2955202084317163683899023976515 5b + 1629666266495228542412923046810 4b^2$$

$$-485198523392127447313064675351 4b^3 + 469652076662064339904368905593b^5$$

$$-157278227476801633022219812444 8b^6 + 44219339776312668988899806088b^7$$

$$-5859439362397645873948685040b^8 + 929109914414742517401972874b^9$$

$$-69599655216723462658712976b^{10} + 7341193153327181504930756b^{11}$$

$$-335985458546484169011680b^{12} + 25098969614206554214218b^{13} - 711297377079405228000b^{14}$$

$$+3876990523567623924 0b^{15} - 659576572370870280b^{16} + 26540546506580505b^{17}$$

$$-246304398649400b^{18} + 7297208073750b^{19} - 28094757600b^{20} + 601387325b^{21})$$

$$
\begin{aligned}
&-4a^3(-1876765205895500933173193686132 5+24566189373961227625674141811080b\\
&-12015540473174435277887487066981b^2+24259926169606372365653233767757b^4\\
&-11357938750409594214236477145 04b^5+3953312811182867482689708558 77b^6\\
&-7034958803762113032227084736 0b^7+132209990455131593420069607 66b^8\\
&-131471693527279663851840747 2b^9+159961005321600402933642734b^{10}\\
&-9793399156343577437237120b^{11}+828843561039389190224202b^{12}-32284726874755850665744b^{13}\\
&+1969657417976768154250b^{14}-48669415714241185920b^{15}+21755484366659911 35b^{16}\\
&-3278699591938428 0b^{17}+1075591455633415b^{18}-8935573673600b^{19}+212245988465b^{20}\\
&-7358150 80b^{21}+12220705b^{22})-28a^2(3057116769612200345160349480500\\
&-33473565454857880466475405879 75b+17165057818820621825553552952 83b^3\\
&-11640473332108775302949450334 36b^4+51942922018629949529196636941 1b^5\\
&-12527184997415634532008761620 0b^6+2847125900485598148783606692 9b^7\\
&-373831024775743228469067208 8b^8+53150685991569683875310301 8b^9\\
&-42760067152207975559988960b^{10}+4136007135400894595424078b^{11}\\
&-214157452921761201752536b^{12}+14710967580877295026166b^{13}-497733673396905453360b^{14}\\
&+24806955660534157490b^{15}-541978448695433820b^{16}+19718368614383085b^{17}\\
&-265957162317280b^{18}+7003210769135b^{19}-52468454220b^{20}+972864695b^{21}-3049800b^{22}+37485b^{23})\\
&-5b(929590553691019718130054434512 5-171198539098283219328979570908 00b\\
&+15014121647164007465385549489060b^2-7230948065672941711476405935280b^3\\
&+26264421936032509545657142783 14b^4-614353501218199401394744794768b^5\\
&+125697869219907704953645235028b^6-17199907359259830075452014160b^7\\
&+2247060070521948723908814059b^8-196046300228467711415420768b^9\\
&+17446391995481502529070408b^{10}-1013669993092314579588320b^{11}\\
&+63633811280905904998604b^{12}-2508037872042116441888 8b^{13}+112874995242338420168b^{14}\\
&-3013859636995824800b^{15}+97328186331966539b^{16}-1721282213782448b^{17}+39266996068628b^{18}\\
&-435969388400b^{19}+6753485354b^{20}-41941328b^{21}+404708b^{22}-1040b^{23}+5b^{24})\\
&-5a(-929590553691019718130054434512 5+187451966547204130612262272926 60b^2\\
&-1965295149916898210053931344886 4b^3+11820808337268654735596095906 062b^4\\
&-39487661926250390208522585671 04b^5+11142953383320282004388495060 04b^6\\
&-1939337657306417427587826562 88b^7+327414872044299734079597999 01b^8\\
&-34342796183509525918709301 76b^9+383629530499850096424556200b^{10}\\
&-25905042377269437741752640b^{11}+2017371743113770211994468b^{12}\\
&-90332805386560612351616b^{13}+5040998789545630951816b^{14}-150474104398346449216b^{15}\\
&+6082979923309031493b^{16}-118898404799065344b^{17}+3468572714833892b^{18}-42252260054816b^{19}\\
&+870877704046b^{20}-5922304960b^{21}+82152228b^{22}-233376b^{23}+1955b^{24}))\Big\}\Big] \qquad (8)
\end{aligned}
$$

Notes

III. Derivation of the Summation Formula

Substituting $c=\frac{a+b+51}{2}$ and $z=\frac{1}{2}$ in equation (2), we get

$$(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b & ; & \frac{1}{2} \\ \frac{a+b+51}{2} & ; & \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b & ; & \frac{1}{2} \\ \frac{a+b+51}{2} & ; & \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1 & ; & \frac{1}{2} \\ \frac{a+b+51}{2} & ; & \end{matrix}\right]$$

Notes

Now involving the derived formula [Salahuddin et. al. p.12-41(8)], the summation formula is obtained.

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Solution of Complex Differential Equation System by using Differential Transform Method

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Abstract- In this study, using differential transform method second order complex differential equation system was solved. Firstly we seperated real and imaginer parts these equations system. Thus from two unknown equation system four equality was obtained. Later using two dimensional differential transform we obtained real and imaginer parts of solutions.

Keywords: differential equation system, differential transform method.

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Solution of Complex Differential Equation System by using Differential Transform Method

Murat DÜZ [α] & Kübra HEREDAĞ [σ]

Abstract- In this study, using differential transform method second order complex differential equation system was solved. Firstly we seperated real and imaginer parts these equations system. Thus from two unknown equation system four equality was obtained. Later using two dimensional differential transform we obtained real and imaginer parts of solutions.

Keywords: differential equation system, differential transform method.

I. Introduction

The concept of differential transform (one dimension) was first proposed and applied to solve linear and non linear initial value problems in electric circuit analysis by Zhou [1] :Solving partial differential equations by two dimensional differential transform method was proposed by Chao Kuang Chen and Shing Huei Ho by [2] : Partial differential equations was solved by using two dimensional DTM in [2]-[3]. System of differential equation was solved using two dimensional DTM in [4]. The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations was compared in [5]. By using differential transform method was solved that integral equations, fractional differential equations, difference equations, integral and integro differential equations in [6], [7], [8],[9],[10] .

In this paper using [1] complex partial differential equations was solved. Let w = w (z; z) be a complex function. Here $z = x + iy$, $w(z,\bar{z}) = u(x,y) + iv(x,y)$. Derivative according to z and $\bar{z}$ of $w(z,\bar{z})$ is defined as follows:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \quad (1)$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \quad (2)$$

Here

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (3)$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \quad (4)$$

Similarly second order derivatives according to z and $\bar{z}$ of $w(z,\bar{z})$ is defined as follows:

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$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right) \quad (5)$$

$$\frac{\partial^2 w}{\partial \bar{z}^2} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} + 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right) \quad (6)$$

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{1}{4}\Delta w \quad (7)$$

Notes

II. Two Dimensional Differential Transform

Definition 1: Two dimensional differential transform of function f (x,y) is defined as follows

$$F(k,h) = \frac{1}{k!.h!}\left[\frac{\partial^{k+h} f(x,y)}{\partial x^k \partial y^h}\right]_{x=0,y=0} \quad (8)$$

In Equation(8) , $f(x,y)$ is original function and $F(k,h)$ is transformed function, which is called T-function is brief.

Definition 2: Differential inverse transform of $F(k,h)F(k,\ h)$ is defined as follows

$$f(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} F(k,h) x^k y^h$$

$$f(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!.h!}\left[\frac{\partial^{k+h} f(x,y)}{\partial x^k \partial y^h}\right]_{x=0,y=0} x^k y^h \quad (9)$$

Equation 9 implies that the concept of two dimensional differential transform is derived from two dimensional Taylor series expansion.

Theorem 3 [2] : If $w(x,y) = u(x,y) \pm v(x,y)$ then $W(k,h) = \lambda U(k,h)$

Theorem 4 [2] : If $w(x,y) = \lambda u(x,y)$ then $W(k,h) = \lambda U(k,h)$

Theorem 5 [2] **:** If $w(x,y) = \frac{\partial u(x,y)}{\partial x}$ then $W(k,h) = (k+1)U(k+1,h)$

Theorem 6 [2] **:** If $w(x,y) = \frac{\partial u(x,y)}{\partial y}$ then $W(k,h) = (h+1)U(k,h+1)$

Theorem 7 [2] **:** If $w(x,y) = \frac{\partial^{r+s} f(x,y)}{\partial x^r \partial y^s}$ then

$$W(k,h) = (k+1)(k+2)\ldots(k+r)(h+1)(h+2)\ldots(h+s)U(k+r,h+s)$$

Theorem 8 [2] : If $w(x,y) = u(x,y).v(x,y)$ then $W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,k-s)V(k-r,s)$

Theorem 9 [2] : If $w(x,y) = x^m y^n$ then $W(k,h) = \delta(k-m,h-n)$

Example 1 : Solve the following complex differential equation system

$$\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial \bar{z}} = 2z + 3 \quad (10)$$

$$\frac{\partial w_1}{\partial \bar{z}} + \frac{\partial w_2}{\partial z} = 7 \quad (11)$$

with initial conditions

$$w_1(x,0) = x^2 + 2x \quad (12)$$

$$w_2(x,0) = 8x \quad (13)$$

Since $w_1 = u_1 + iv_1$, $w_2 = u_2 + iv_2$ and from (1),(2), system (10)-(11) is equivalent that :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} - \frac{\partial v_2}{\partial y} = 4x + 6 \quad (14)$$

$$\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}=4y \quad (15)$$

$$\frac{\partial u_1}{\partial x}-\frac{\partial v_1}{\partial y}+\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}=14 \quad (16)$$

$$\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y}+\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}=0 \quad (17)$$

Notes

From (12) and (13) initial conditions we have that:

$$U_1(0,0)=0,\ U_1(1,0)=2,\quad U_1(2,0)=1,\quad U_1(i,0)=0\ (i>2),\ \ V_1(i,0)=0\quad (i\epsilon N)$$

$$U_2(0,0)=0,\quad U_2\ (1,0)=8,\quad U_2(i,0)=0\ \ (i\geq 2)\quad V_2(i,0)=0\quad (i\epsilon N) \quad (18)$$

From differential transform of (14)- (17) is get that:

$$(k+1)U_1(k+1,h)+(h+1)V_1(k,h+1)+(k+1)U_2(k+1,h)-(h+1)V_2(k,h+1) = 4\delta(k-1,h)+6\delta(k,h) \quad (19)$$

$$(k+1)V_1(k+1,h)-(h+1)U_1(k,h+1)+(h+1)(k+1)U_2(k,h+1)+(k+1)V_2(k+1,h) = 4\delta(k,h-1) \quad (20)$$

$$(k+1)U_1(k+1,h)-(h+1)V_1(k,h+1)+(k+1)U_2(k+1,h)+(h+1)V_2(k,h+1) = 14\delta(k,h) \quad (21)$$

$$(k+1)V_1(k+1,h)+(h+1)U_1(k,h+1)+(k+1)V_2(k+1,h)-(h+1)U_2(k,h+1) = 0 \quad (22)$$

In (19) equality if we write $k=0, h=0$ than by using (18) is obtained

$$V_1(0,1)-V_2(0,1)=-4,\ V_1(0,1)=a,\ V_2(0,1)=a+4\ \ (a\in R) \quad (23)$$

By sum of (19) with (21) we get that:

$$U_1(k+1,h)+U_2(k+1,h)=\frac{4\delta(k,h-1)+20\delta(k,h)}{2(k+1)} \quad (24)$$

By mines of (20) with (22) we get that:

$$U_2(k,h+1)-U_1(k,h+1)=\frac{4\delta(k,h-1)}{2(h+1)} \quad (25)$$

By mines of (19) with (21) we get that:

$$V_1(k,h+1)-V_2(k,h+1)=\frac{4\delta(k-1,h)-8\delta(k,h)}{2(h+1)} \quad (26)$$

By sum of (20) with (22) we get that:

$$V_1(k+1,h)+V_2(k+1,h)=\frac{4\delta(k,h-1)}{2(k+1)} \quad (27)$$

If we write in place of $h, h+1$ in (24) than we get

$$U_1(k+1,h+1)+U_2(k+1,h+1)=\frac{4\delta(k,h)+20\delta(k,h+1)}{2(k+1)} \quad (28)$$

If we write in place of $k, k+1$ in (25) than we get

$$U_2(k+1,h+1)-U_1(k+1,h+1)=\frac{4\delta(k+1,h-1)}{2(h+1)} \quad (29)$$

If we write in place of $k, k+1$ in (26) than we get

$$V_1(k+1,h+1)-V_2(k+1,h+1)=\frac{4\delta(k,h)-8\delta(k+1,h)}{2(h+1)} \quad (30)$$

If we write in place of h , $h+1$ in (27) than we get

$$V_1(k+1,h+1)+V_2(k+1,h+1)=\frac{4\delta(k,h)}{2(k+1)} \tag{31}$$

By sum of (28) with (29) we get that:

$$U_2(k+1,h+1)=\frac{\delta(k,h)+5\delta(k,h+1)}{k+1}+\frac{\delta(k+1,h-1)}{h+1} \tag{32}$$

By mines of (29) from (28) we get that:

$$U_1(k+1,h+1)=\frac{\delta(k,h)+5\delta(k,h+1)}{k+1}-\frac{\delta(k+1,h-1)}{h+1} \tag{33}$$

By sum of (30) with (31) we get that:

$$V_1(k+1,h+1)=\frac{\delta(k,h)-2\delta(k+1,h)}{h+1}+\frac{\delta(k,h)}{k+1} \tag{34}$$

By mines of (30) from (31) we get that

$$V_2(k+1,h+1)=\frac{\delta(k,h)}{k+1}-\frac{\delta(k,h)-2\delta(k+1,h)}{h+1} \tag{35}$$

In (34) equality if we write $k=0,h=0$ than is obtained

$$V_1(1,1)=2 \tag{36}$$

In (20) equality if we write $k=0,h=1$ than by using (31) is obtained

$$-U_1(0,2)+U_2(0,2)=1, U_1(0,2)=b, U_2(0,2)=b\ , b\in R \tag{37}$$

If we write $k=0,h=0$ in (25) than we get

$$U_1(0,1)=U_2(0,1)=\text{c}, \text{c}\epsilon\text{R} \tag{38}$$

By using (32),(33),(34) and (35) it is seen that all the other components of u_1, v_1, u_2 and v_2 is zero. Thus it is obtained

$$u_1(x,y)=\sum_{k=0}\sum_{h=0}U_1(k,h)x^ky^h$$

$$=x^2+by^2+2x+cy \tag{39}$$

$$v_1(x,y)=\sum_{k=0}\sum_{h=0}V_1(k,h)x^ky^h$$

$$=2xy+ay \tag{40}$$

$$u_2(x,y)=\sum_{k=0}\sum_{h=0}U_2(k,h)x^ky^h$$

$$=8x+(b+1)y^2+cy \tag{41}$$

$$v_2(x,y)=\sum_{k=0}\sum_{h=0}V_2(k,h)x^ky^h$$

$$=(a+4)y \tag{42}$$

From (39)-(42) we get

$$w_1(x,y)=u_1(x,y)+v_1(x,y)$$

$$=x^2+by^2+2x+cy+i(2xy+ay)$$

$$w_2(x,y)=u_2(x,y)+v_2(x,y)$$

Notes

Notes

Example 2 : Solve the following complex differential equation system

$$2\frac{\partial w_1}{\partial z}-\frac{\partial w_2}{\partial \bar{z}}=6z^2-4z \tag{43}$$

$$\frac{\partial w_1}{\partial \bar{z}}+3\frac{\partial w_2}{\partial z}=12\bar{z}-2 \tag{44}$$

with initial conditions

$$w_1(x,0)=x^3-2x \tag{45}$$

$$w_2(x,0)=4x^2 \tag{46}$$

If we write system in (43)-(44) system $w_1=u_1+iv_1\,, w_2=u_2+iv_2$ □□□□ we get following equations.

$$2\left(\frac{\partial u_1}{\partial x}+i\frac{\partial v_1}{\partial x}-i\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial y}\right)-\left(\frac{\partial u_2}{\partial x}+i\frac{\partial v_2}{\partial x}+i\frac{\partial u_2}{\partial y}-\frac{\partial v_2}{\partial y}\right)=12(x+iy)^2-8(x+iy)$$

$$\frac{\partial u_1}{\partial x}-\frac{\partial v_1}{\partial y}+i\frac{\partial u_1}{\partial x}+i\frac{\partial v_1}{\partial x}+3\frac{\partial u_2}{\partial x}+3i\frac{\partial v_2}{\partial x}-3i\frac{\partial u_2}{\partial y}+3\frac{\partial v_2}{\partial y} \tag{47}$$

$$=24(\mathrm{x}-\mathrm{iy})-4$$

If (47) equalities is seperated into real and imaginary parts then it is get following egualities.

$$2\frac{\partial u_1}{\partial x}+2\frac{\partial v_1}{\partial y}-\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}=12(x^2-y^2)-8x \tag{48}$$

$$2\frac{\partial v_1}{\partial x}-2\frac{\partial u_1}{\partial y}-\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}=24xy-8y \tag{49}$$

$$\frac{\partial u_1}{\partial x}-\frac{\partial v_1}{\partial y}+3\frac{\partial u_2}{\partial x}+3\frac{\partial v_2}{\partial y}=24x-4 \tag{50}$$

$$\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}+3\frac{\partial v_2}{\partial x}-3\frac{\partial u_2}{\partial y}=-24y \tag{51}$$

From (45)-(46) initial conditions we have that:

$U_1(0,0)=0,\ U_1(1,0)=-2,\quad U_1(2,0)=0,\ \ U_1(3,0)=1,\ U_1(i,0)=0(i>3),\ \ V_1(i,0)=0\ \ (i\epsilon N)$

$$U_2(0,0)=0,\ U_2\ (1,0)=0,\ \ U_2(2,0)=4,\ U_2(i,0)=4(i\geq 3)\ ,\ \ V_2(i,0)=0\quad (i\epsilon N) \tag{52}$$

From differential transform of (48)- (51) is get following equalities:

$$2(k+1)U_1(k+1,h)+2(h+1)V_1(k,h+1)-(k+1)U_2(k+1,h)+(h+1)V_2(k,h+1)$$
$$=12\delta(k-2,h)-12\delta(k,h-2)$$
$$-8\delta(k-1,h) \tag{53}$$

$$2(k+1)V_1(k+1,h)-2(h+1)U_1(k,h+1)-(k+1)V_2(k+1,h)-(h+1)U_2(k,h+1)$$
$$=24\delta(k-1,h-1)-\delta(k,h-1) \tag{54}$$

$$(k+1)U_1(k+1,h)-(h+1)V_1(k,h+1)+3(k+1)U_2(k+1,h)+3(h+1)V_2(k,h+1)$$
$$=24\delta(k-1,h)-4\delta(k,h) \tag{55}$$

$$(h+1)U_1(k,h+1)+(k+1)V_1(k+1,h)+3(k+1)V_2(k+1,h)-3(h+1)U_2(k,h+1)$$

$$=-24\,\delta(k,h-1) \tag{56}$$

If we write $h=0$ in (54) and (56) from (52) we get that:

$$U_1(k,1)=U_2(k,1)=0 \tag{57}$$

If we write $k = 0, h = 0$ in (53) and (55) from (52) we get that:

$$\mathrm{V}_1(0,1) = 2, \mathrm{V}_2(0,1) = 0 \tag{58}$$

If we write $h = 0$ in (54) and (56) from (52) we get that:

$$\mathrm{U}_1(\mathrm{k},1) = \mathrm{U}_2(\mathrm{k},1) = 0 \tag{59}$$

If we write $k = 1, h = 0$ in (53) and (55) from (52) we get that:

$$V_1(1,1) = 0, V_2(1,1) = 0 \tag{60}$$

If we write $k = 0, h = 1$ in (54) and (56) from (52) we get that:

$$U_1(0,2) = 0, U_2(0,2) = 4 \tag{61}$$

If we write $h = 1$ in (53) and (55) from (52) and (57) we get that:

$$V_1(k,2) = V_2(k,2) = 0 \tag{62}$$

If we write $h = 2$ in (54) and (56) from (62) we get that:

$$U_1(k,3) = U_2(k,3) = 0 \tag{63}$$

If we write k=2,h=0 1 in (53) and (55) we get that

$$V_1(2,1) = 3, V_2(2,1) = 0 \tag{64}$$

If we write k=1,h=1 1 in (54) and (56) we get that

$$U_1(1,2) = -3, U_2(1,2) = 0 \tag{65}$$

By using equalities (57)-(65) we see other components of U_1, U_2, V_1 and V_2 are equal zero.

Thus it is obtained

$$u_1(x,y) = \sum_{k=0}\sum_{h=0} U_1(k,h)x^k y^h$$
$$= x^3 - 3xy^2 - 2x$$

$$v_1(x,y) = \sum_{k=0}\sum_{h=0} V_1(k,h)x^k y^h$$
$$= 3x^2y - y^3 + 2y$$

$$u_2(x,y) = \sum_{k=0}\sum_{h=0} U_2(k,h)x^k y^h$$
$$= 4x^2 + 4y^2$$

$$v_2(x,y) = \sum_{k=0}\sum_{h=0} V_2(k,h)x^k y^h$$
$$= 0$$

Therefore

$$w_1(x,y) = u_1(x,y) + v_1(x,y)$$
$$= x^3 - 3xy^2 - 2x + i(3x^2y - y^3 + 2y)$$
$$= z^3 - 2\bar{z}$$

$$w_2(x,y) = u_2(x,y) + v_2(x,y)$$
$$= 4x^2 + 4y^2$$
$$= 4z\bar{z}$$

Example:

Solve the following complex diferential equation system

$$\frac{\partial^2 w_1}{\partial z^2} + \frac{\partial^2 w_2}{\partial z \partial \bar{z}} = 3 \tag{66}$$

$$\frac{\partial^2 w_2}{\partial z^2}+\frac{\partial w_1}{\partial z}-\frac{\partial w_2}{\partial \bar{z}}=z \tag{67}$$

with initial conditions

$$w_1(x,0)=x^2 \tag{68}$$

$$\frac{\partial w_1}{\partial y}(x,0)=2ix \tag{69}$$

$$w_2(x,0)=x^2 \tag{70}$$

$$\frac{\partial w_2}{\partial y}(x,0)=0 \tag{71}$$

Notes

Since $w_1=u_1+iv_1$, $w_2=u_2+iv_2$ and from (2),(5) and (7) system (66)-(67) is equivalent that :

$$\frac{1}{4}\left(\frac{\partial^2 w_1}{\partial x^2}-2i\frac{\partial^2 w_1}{\partial x\partial y}-\frac{\partial^2 w_1}{\partial y^2}\right)+\frac{1}{4}\left(\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_2}{\partial y^2}\right)=3 \tag{72}$$

$$\frac{1}{4}\left(\frac{\partial^2 w_2}{\partial x^2}-2i\frac{\partial^2 w_2}{\partial x\partial y}-\frac{\partial^2 w_2}{\partial y^2}\right)+\frac{1}{2}\left(\frac{\partial w_1}{\partial x}-i\frac{\partial w_1}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial w_2}{\partial x}+i\frac{\partial w_2}{\partial y}\right)=z \tag{73}$$

$$\left(\frac{\partial^2 u_1}{\partial x^2}+i\frac{\partial^2 v_1}{\partial x^2}\right)-2i\left(\frac{\partial^2 u_1}{\partial x\partial y}+i\frac{\partial^2 v_1}{\partial x\partial y}\right)-\left(\frac{\partial^2 u_1}{\partial y^2}+i\frac{\partial^2 v_1}{\partial y^2}\right)+\left(\frac{\partial^2 u_2}{\partial x^2}+i\frac{\partial^2 v_2}{\partial x^2}\right)+\left(\frac{\partial^2 u_2}{\partial y^2}+i\frac{\partial^2 v_2}{\partial y^2}\right)=12 \tag{74}$$

$$\left(\frac{\partial^2 u_2}{\partial x^2}+i\frac{\partial^2 v_2}{\partial x^2}\right)-2i\left(\frac{\partial^2 u_2}{\partial x\partial y}+i\frac{\partial^2 v_2}{\partial x\partial y}\right)-\left(\frac{\partial^2 u_2}{\partial y^2}+i\frac{\partial^2 v_2}{\partial y^2}\right)+2\left(\frac{\partial u_1}{\partial x}+i\frac{\partial v_1}{\partial x}\right)-2i\left(\frac{\partial u_1}{\partial y}+i\frac{\partial v_1}{\partial y}\right)$$

$$-2\left(\frac{\partial u_2}{\partial x}+i\frac{\partial v_2}{\partial x}\right)-2i\left(\frac{\partial u_2}{\partial y}+i\frac{\partial v_2}{\partial y}\right)=4z \tag{75}$$

$$\frac{\partial^2 u_1}{\partial x^2}+2\frac{\partial^2 v_1}{\partial x\partial y}-\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=12 \tag{76}$$

$$\frac{\partial^2 v_1}{\partial x^2}-2\frac{\partial^2 u_1}{\partial x\partial y}-\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}=0 \tag{77}$$

$$\frac{\partial^2 u_2}{\partial x^2}+2\frac{\partial^2 v_2}{\partial x\partial y}-\frac{\partial^2 u_2}{\partial y^2}+2\frac{\partial u_1}{\partial x}+2\frac{\partial v_1}{\partial y}-2\frac{\partial u_2}{\partial x}+2\frac{\partial v_2}{\partial y}=4x \tag{78}$$

$$\frac{\partial^2 v_2}{\partial x^2}-2\frac{\partial^2 u_2}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial y^2}+2\frac{\partial v_1}{\partial x}-2\frac{\partial u_1}{\partial y}-2\frac{\partial v_2}{\partial x}-2\frac{\partial u_2}{\partial y}=4y \tag{79}$$

From (68)-(71) initial conditions we have that:

$$U_1(0,0)=0,\ U_1(1,0)=0,\ U_1(2,0)=1,\ U_1(i,0)=0(i>2,i\in N),V_1(i,0)=0(i\epsilon N)$$
$$U_1(i,1)=0(i\in N),\quad V_1(0,1)=0,\quad V_1(1,1)=2,\quad V_1(i,1)=0,(i>1)$$
$$U_2(0,0)=0,\quad U_2(1,0)=0,\quad U_2(2,0)=1\quad V_2(i,0)=0=U_2(i,1)=V_2(i,1)=0(i\epsilon N) \tag{80}$$

From diferential transform of (76)- (79) is get that:

$$(k+2)(k+1)U_1(k+2,h)+2(k+1)(h+1)V_1(k+1,h+1)-(h+2)(h+1)U_1(k,h+2)$$
$$+(k+2)(k+1)U_2(k+2,h)+(h+2)(h+1)U_2(k,h+2)=12\delta(k,h) \tag{81}$$

$$(k+2)(k+1)V_1(k+2,h)-2(k+1)(h+1)U_1(k+1,h+1)-(h+2)(h+1)V_1(k,h+2)$$
$$+(k+2)(k+1)V_2(k+2,h)+(h+2)(h+1)V_2(k,h+2)=0 \tag{82}$$

$$(k+2)(k+1)U_2(k+2,h)+2(k+1)(h+1)V_2(k+1,h+1)-(h+2)(h+1)U_2(k,h+2)+$$
$$2(k+1)U_1(k+1,h)+2(h+1)V_1(k,h+1)-2(k+1)U_2(k+1,h)+2(h+1)V_2(k,h+1)=$$
$$4\delta(k-1,h) \tag{83}$$

$(k+2)(k+1)V_2(k+2,h) - 2(k+1)(h+1)V_2(k+1,h+1) - (h+2)(h+1)V_2(k,h+2) + 2(k+1)V_1(k+1,h) - 2(h+1)U_1(k,h+1) - 2(k+1)V_2(k+1,h) - 2(h+1)U_2(k,h+1) = 4\delta(k,h-1)$ (84)

When $h = 0$ is written in the equality (80),(81),(82), (83)

$U_1(0,2) = -1, U_2(0,2) = 1,\ U_2(i,2) = 0,\ \ U_1(i,2) = 0(i \geq 1),\ V_1(i,2) = V_2(i,2) = 0(i \geq 0)$ (85)

are obtained.

It is clear that all of the other components U_i and V_i are zero. Thus

$$u_1(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_1(k,h)x^k y^h$$
$$= x^2 - y^2$$
$$v_1(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V_1(k,h)x^k y^h$$
$$= 2xy$$
$$u_2(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_2(k,h)x^k y^h$$
$$= x^2 + y^2$$
$$v_2(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V_2(k,h)x^k y^h$$
$$= 0$$

are obtained.

Therefore

$$w_1(z) = u_1(x,y) + iv_1(x,y)$$
$$w_1(z) = x^2 - y^2 + i2xy = z^2$$

and

$$w_2(z) = u_2(x,y) + iv_2(x,y)$$
$$w_2(z) = x^2 + y^2 = (x+iy)(x-iy) = z.\bar{z}$$

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An Wonderful Summation Formula of Half Argument Involving Contiguous Relation

By Salahuddin, M. P. Chaudhary & Gedefa Negassa Feyissa
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GJSFR-F Classification : MSC 2010: 40A25

ANWONDERFULSUMMATIONFORMULAOFHALFARGUMENTINVOLVINGCONTIGUOUSRELATION

Notes

An Wonderful Summation Formula of Half Argument Involving Contiguous Relation

Salahuddin [α], M. P. Chaudhary [σ] & Gedefa Negassa Feyissa [ρ]

Abstract- In this paper we have developed a summation formula of half argument involving recurrence relation of Gamma function and contiguous relation.

2010 MSC NO : 33C05 , 33C20 , 33D15 , 33D50 , 33D60

Keywords: gauss second summation theorem, recurrence relation, contiguous relation.

I. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1}$$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers and $\mid z \mid = 1$

Contiguous Relation is defined by

[Andrews p.363(9.16), E. D. p.51(10)]

$$ (a-b)\ {}_2F_1\left[\begin{matrix} a, & b\ ; & \\ & c\ ; & z \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1, & b & ; & \\ & c & ; & z \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a, & b+1 & ; & \\ & c\ ; & & z \end{matrix}\right] \tag{2}$$

Gauss second summation theorem is defined by [Prudnikov., 491(7.3.7.8)]

$$ {}_2F_1\left[\begin{matrix} a, \ b & ; & \\ \frac{a+b+1}{2} & ; & \frac{1}{2} \end{matrix}\right] = \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \tag{3}$$

$$ = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})} \tag{4}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

$$ {}_2F_1\left[\begin{matrix} a, \ b & ; & \\ \frac{a+b-1}{2} & ; & \frac{1}{2} \end{matrix}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\ \Gamma(b)}\right] \tag{5}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

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$$ {}_2F_1\left[\begin{matrix} a,\ b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & ; & \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{6} $$

Recurrence relation is defined by

$$ \Gamma(\zeta+1) = \zeta\ \Gamma(\zeta) \tag{7} $$

II. Main Summation Formula

$$ {}_2F_1\left[\begin{matrix} a,\ b\ ; & \frac{1}{2} \\ \frac{a+b+52}{2}\ ; & \end{matrix}\right] = \frac{2^b\ \Gamma(\frac{a+b+52}{2})}{(a-b)\ \Gamma(b)\Big[\prod_{\tau=1}^{25}\{a-b-2\tau\}\Big]\Big[\prod_{\upsilon=1}^{25}\{a-b+2\upsilon\}\Big]} $$

$$ \Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\Big\{33554432(-520469842636666622693081088000000 + a^{25} $$

$$ +1364009155068468067713291386880000b - 1422743599044869675233377797712000b^2 $$

$$ +897138976290180907385572923801600b^3 - 346869302630338134364227895296000b^4 $$

$$ +105376534094283873384116219019264b^5 - 21454722011693905387175458897920b^6 $$

$$ +3874217504689761765837326254080b^7 - 478037833101498222002097356800b^8 $$

$$ +56654588151086887435417681920b^9 - 4546070710610585366480158720b^{10} $$

$$ +373181829933472199021690880b^{11} - 20194930448705937178009600b^{12} $$

$$ +1182641581542335852359680b^{13} - 43795287670944443576320b^{14} + 1853638447567267092480b^{15} $$

$$ -46802276105796851200b^{16} + 1430111919273504000b^{17} - 24034443398763520b^{18} $$

$$ +521282486734080b^{19} - 5521231861600b^{20} + 81628506960b^{21} - 485121520b^{22} + 4481880b^{23} $$

$$ -11050b^{24} + 51b^{25} + 25a^{24}(-26+51b) + 100a^{23}(2002-2652b+2499b^2) + 460a^{22}(-84500+185198b $$

$$ -64974b^2 + 39151b^3) + 2530a^{21}(2110264 - 3645616b + 3657108b^2 - 581672b^3 + 251685b^4) $$

$$ +17710a^{20}(-31163600 + 72380152b - 38575992b^2 + 24139388b^3 - 2181270b^4 + 721497b^5) $$

$$ +2300a^{19}(19390040384 - 37869421408b + 39415220936b^2 - 10203690224b^3 + 4514501810b^4 $$

$$ -262624908b^5 + 69023213b^6) + 8740a^{18}(-330455840000 + 796316947392b - 518916043856b^2 $$

$$ +332176816104b^3 - 50876724500b^4 + 17020114230b^5 - 690232130b^6 + 147906885b^7) $$

$$ +2185a^{17}(69789853509376 - 146863394961408b + 156544599068928b^2 - 51655821883008b^3 $$

$$ +23139727485168b^4 - 2344230332640b^5 + 617895802776b^6 - 18458779248b^7 + 3283532847b^8) $$

$$ +37145a^{16}(-178814098777600 + 442041296102656b - 326043316343808b^2 + 211636160789760b^3 $$

$$ -42441185113824b^4 + 14259524785296b^5 - 1026870520368b^6 + 219152761464b^7 - 5021873766b^8 $$

$$ +751134965b^9) + 15640a^{15}(15383673087527680 - 34078827668715520b + 36908462651058432b^2 $$

$$ -14183694144313344b^3 + 6392060676118656b^4 - 864220759200960b^5 + 227240073284784b^6 $$

$$ -12234189336096b^7 + 2154383845614b^8 - 39059018180b^9 + 4957490769b^{10}) $$

$$ +109480a^{14}(-66546397551296000 + 167579244365076736b - 134688825529456128b^2 $$

$$ +88134640433903872b^3 - 20988432423037440b^4 + 7047385998698880b^5 - 687149634511200b^6 $$

Notes

$$+1454306204738\,40b^7 - 60764672568\,60b^8 + 8947703654\,50b^9 - 131525265\,30b^{10} + 1425623505b^{11})$$

$$+340a^{13}(5438758782734394\,05056 - 12513406208678195\,17952b + 136963463787553875\,4048b^2$$

$$-585790021790811299840b^3 + 264369205161369776384b^4 - 43063456581573765120b^5$$

$$+11250117922841403648b^6 - 829489361267409024b^7 + 14406876671953365\,6b^8$$

$$-4808165137958000b^9 + 597642037172180b^{10} - 72649773814\,80b^{11} + 675270333535b^{12})$$

$$+68a^{12}(-57891485665006958848000+147793035892802393226240b-126616282707392263152640b^2$$

$$+83166467194741279078912b^3 - 22375541602754700878336b^4 + 7479867859429706168064b^5$$

$$-888687861460172067840b^6 + 185865876600461680896b^7 - 107349545276614848\,48b^8$$

$$+1551008942251917512b^9 - 423041195284307760b^{10} + 44672345918720\,52b^{11} - 456482745469\,66b^{12}$$

$$+3646459801089b^{13}) + 136a^{11}(516122352488702370611200 - 1222665035084907763814400b$$

$$+1347127778676226885524480b^2-624384680472188982900736b^3+281174089484815331587328b^4$$

$$-52343935058412888963072b^5 + 13540741049183460286720b^6 - 1227811867433180510208b^7$$

$$+209649585286086114624b^8 - 9747684004211325536b^9 + 1182716835244945000b^{10}$$

$$-268600743748078\,56b^{11} + 241935855098\,9198b^{12} - 210684344062\,92b^{13} + 14470078575\,75b^{14})$$

$$+1496a^{10}(-698335459257875445760000 + 1800988475513078748282880b$$

$$-1621553919233981940480000b^2+1065534475796573110279168b^3-314457416256610890658304b^4$$

$$+104318352623346207347456b^5 - 14296859989592591756800b^6 + 294559999197144423142\,4b^7$$

$$-210849048023648116992b^8 + 29796435316570523328b^9 - 1139854171756540240b^{10}$$

$$+11687847178142480\,8b^{11} - 22446358449646\,04b^{12} + 172785378722866b^{13} - 1297317389550b^{14}$$

$$+765084614\,35b^{15}) + 5a^9(2581175998232799743087476736 - 6258043140275689339905376256b$$

$$+6919213805404635084180160512b^2 - 3416233934301369756896362496b^3$$

$$+1530382721637759977964662784b^4 - 315508062566630172274008064b^5$$

$$+80574190195760192772212736b^6 - 8493409725734008691359744b^7$$

$$+1421343108547670498910720b^8 - 82452450586173649293312b^9 + 9733792240058435028480b^{10}$$

$$-311885299252958342912b^{11} + 27127086010213084896b^{12} - 446352585997782464b^{13}$$

$$+29362226326633200b^{14} - 191991813933920b^{15} + 9691894453395b^{16})$$

$$+7a^8(-1875021106872959208390656\,0000 + 4871521431211746242043314\,1760b$$

$$-4565236873600306051540254\,7200b^2 + 2992473576924649942984320\,6144b^3$$

$$-950355538984328496378266\,4192b^4 + 311955328010581966119519\,4368b^5$$

$$-47722913716694790617831\,4240b^6 + 96565086727541027651\,008512b^7$$

$$-8089542881324122750811136b^8+1114515430601677475131904b^9-53509897670705946961920b^{10}$$

$$+5308977943764198173184b^{11} - 144618581171963825472b^{12} + 10685144436609403488b^{13}$$

$$-152327779533540000b^{14} + 8546129120240400b^{15} - 49088816062650b^{16} + 2110152726675b^{17})$$

$$+4a^7(27161118659360298611748372\,4800 - 67100953574301314561217658\,8800b$$

$$+742537262505646916824150835200b^2 - 385962449734386495153439506432b^3$$

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$$+171518199003453816671172427776b^4 - 383668134151354875488977059 84b^5$$

$$+964395233319582493938632704 0b^6 - 114248022242588338059635097 6b^7$$

$$+186774351185946338965882368b^8 - 127558956146156273047429 12b^9$$

$$+14599589837019929360819 20b^{10} - 590263389520699184209 92b^{11} + 49377352701047552796 16b^{12}$$

$$-11580898074356995238 4b^{13} + 726314821417947216 0b^{14} - 90585228574276800b^{15}$$

$$+4316543225775690b^{16} - 219455883357420b^{17} + 7971688078855b^{18})$$

$$+4a^6(-18017770822654921249495449 60000 + 47052952251733732863310508851 20b$$

$$-45577502434075177624792255692 80b^2 + 29723594874460098538212016783 36b^3$$

$$-10026012034945279526333548462 08b^4 + 32469655522212633720590381875 2b^5$$

$$-54273684471851519667616440320b^6 + 10750868281998240831232831488b^7$$

$$-10183112848954039612344330 24b^8 + 13628432126760216117441689 6b^9$$

$$-7745409165666979057771520b^{10} + 740516808004401899767296b^{11} - 255808110789357670369 28b^{12}$$

$$+18055754541382548856 32b^{13} - 3686224405669574496 0b^{14} + 1956546120608308320b^{15}$$

$$-21529912483492140b^{16} + 865664492098770b^{17} - 39212087305 30b^{18} + 119065042425b^{19})$$

$$+2a^5(18753968220881607367780007411712 - 47053273644557609628716803031040b$$

$$+51988342777586963198472565555200b^2 - 28212003541558220003379112837120b^3$$

$$+12401919064860226460358520340480b^4 - 29682663112712973386928999628 80b^5$$

$$+73201585661176445088191873024 0b^6 - 95343588407789300565345730560b^7$$

$$+15171159746147710152216606720b^8 - 1178042776563322567881052160b^9$$

$$+130170212925783829505674240b^{10} - 6261749868483139430400000b^{11}$$

$$+501333835550888494103040b^{12} - 14983086238859523133440b^{13} + 890751084074763932160b^{14}$$

$$-15982376875606176000b^{15} + 71396638299896 7000b^{16} - 69840808386703 20b^{17} + 2346692609501 80b^{18}$$

$$-952520339400b^{19} + 23813008485b^{20}) + 2a^4(-74316927802894107465164311756800$$

$$+194687654859464119954736897064960b - 193933948498446449994109865164800b^2$$

$$+125495553471106303674735340290048b^3 - 44547016316848225192524239601664b^4$$

$$+14188032961507814336191755452416b^5 - 2553999485189716495501831045120b^6$$

$$+493409092826910829114626146304b^7 - 51679565064449610215572549632b^8$$

$$+6689358860110427498742499328b^9 - 434548470425455105249187840b^{10}$$

$$+39830233126021756366527488b^{11} - 1645331321028515241698304b^{12}$$

$$+110272324699694369046016b^{13} - 2882608118845349529600b^{14} + 143701546123284441600b^{15}$$

$$-2285514984990274800b^{16} + 85196618367004200b^{17} - 745838913885800b^{18} + 20657300856500b^{19}$$

$$-755046610 50b^{20} + 1521156175b^{21}) + 4a^3(1073107756402550250380470 51776000$$

$$-27272864866763569892098572288 0000b + 30013943835081059356480955744256 0b^2$$

$$-169021755790313032483730677039104b^3 + 73268126616610426499718092685312b^4$$

$$-18582054310213755452736057376768b^5 + 4478731498377030241256147517440b^6$$

Notes

$$-6320180137369413006519679057926b^{7}+9743478861044483692526901657 6b^{8}$$

$$-841293137686916819017414246 4b^{9}+89256967406725730278001664 0b^{10}$$

$$-4933826918121795147939430 4b^{11}+3756115885754254624189952b^{12}$$

$$-1349302395401127295098888b^{13}+7544883035012338122240b^{14}-1742133105684636672 00b^{15}$$

$$+72270984408513609 60b^{16}-102686829707390880b^{17}+3154219002555320b^{18}-248620595300000b^{19}$$

$$+5566273849 50b^{20}-1839537700b^{21}+28943775b^{22})+28a^{2}(-301031703889953907349127168 00000$$

$$+789593862143801362731837161472 00b-805847570029289506984944677683 20b^{2}$$

$$+5158486694641724282461058865561 6b^{3}-1914670919130635432112583134412 8b^{4}$$

$$+5973414332894411456012491948032b^{5}-1146763745373159912248725995520b^{6}$$

$$+215009405133117108367464071168b^{7}-245438288964984926786500362 24b^{8}$$

$$+305439846182531108180957593 6b^{9}-22189045217867783827912704 0b^{10}$$

$$+193595650180635509708943 36b^{11}-924105012556794821229568b^{12}+583064647537596639134 72b^{13}$$

$$-1842268763764152652800b^{14}+853636617610608102 40b^{15}-175698814220972928 0b^{16}$$

$$+5994552273927840 0b^{17}-766522310449840b^{18}+19051214640920b^{19}-135958559100b^{20}$$

$$+2391399010b^{21}-7167030b^{22}+83895b^{23})+35a(2837273074690669772132686233 6000$$

$$-728941711965287714774600122368 00b+796791005924426195370502835404 80b^{2}$$

$$-4637439284488715656570591405670 4b^{3}+1974152067591897650840512285900 8b^{4}$$

$$-5272051559468802725848012555424 0b^{5}+1235283313014379920045983662080b^{6}$$

$$-187103240593140852984746344448b^{7}+277616560609072452300754124 80b^{8}$$

$$-262878613997995871863177216 0b^{9}+26563632937497319230904729 6b^{10}$$

$$-1652300848553745699618816 0b^{11}+1184461743479967024246784b^{12}$$

$$-49479148909541030371328b^{13}+2571557404807467722752b^{14}-72237271723083759616b^{15}$$

$$+2742555462044058888 0b^{16}-5076687713692262 4b^{17}+1399684890082048b^{18}-16224631558016b^{19}$$

$$+317606253328b^{20}-2062966048b^{21}+27285544b^{22}-74256b^{23}+595b^{24}))\Big\}$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Big\{33554432(-520469842636666622693081088000000+51a^{25}$$

$$+993045576141734420246440181760000b-842888770891870940577556070400000b^{2}$$

$$+429243102561020100152188207104000b^{3}-148633855605788214930328623513600b^{4}$$

$$+37507936441763214735560014823424b^{5}-7207108329061968499798179840000b^{6}$$

$$+1086444746374411944469934899200b^{7}-131251477481107144587345920000b^{8}$$

$$+12905879991163998715437383680b^{9}-1044709847049781666856960000b^{10}$$

$$+70192639938463522403123200b^{11}-3936621025220473201664000b^{12}$$

$$+184917798612969397719040b^{13}-7285499603915886080000b^{14}+240600647088932915200b^{15}$$

$$-6642049699093952000b^{16}+152490829917986560b^{17}-2888184041600000b^{18}+44597092883200b^{19}$$

$$-551907356000b^{20}+5338967920b^{21}-38870000b^{22}+200200b^{23}-650b^{24}+b^{25}+425a^{24}(-26+49b)$$

Notes

$$+1020a^{23}(4394-2548b+2303b^2)+7820a^{22}(-62036+122122b-25662b^2+14805b^3)$$
$$+301070a^{21}(271128-239824b+222404b^2-24440b^3+10105b^4)+27370a^{20}(-201725680+406146104b$$
$$-139088040b^2+81348540b^3-5517330b^4+1740081b^5)+7820a^{19}(66660164544-72616637408b$$
$$+68214067768b^2-12717166000b^3+5283197150b^4-243611340b^5+609028835b^6)$$
$$+148580a^{18}(-161760959744+3297144437696b-1444516401440b^2+849163818160b^3-100395600200b^4$$
$$+31588270420b^5-1055649140b^6+214609990b^7)+6555a^{17}(218171154732800-2710664683130880b$$
$$+2560602039206400b^2-626616809808640b^3+259943915688800b^4-21309171132480b^5+5282468296560b^6$$
$$-133916633760b^7+22534048950b^8)+1955a^{16}(-23939783174320640+49099458399765760b$$
$$-25164024543157248b^2+14786902180770048b^3-2338122746793120b^4+730400391814800b^5$$
$$-44050971833232b^6+8831801996472b^7-175765581810b^8+24787453845b^9)$$
$$+109480a^{15}(16931297475038976-23093756944719872b+21832138557816064b^2$$
$$-6365119129282560b^3+2625165256179840b^4-291968887022400b^5+71485061037936b^6$$
$$-3309653948640b^7+546427693110b^8-8768351020b^9+1045457237b^{10})$$
$$+4760a^{14}(-920069068717320243 2+18908510329466674432b-10836875080965603840b^2$$
$$+6340237844548183296b^3-1211179881867793920b^4+374265161375951232b^5-30976675677895584b^6$$
$$+6103485894268464b^7-224011440490500b^8+30842674712850b^9-407728322430b^{10}$$
$$+41343081645b^{11})+68a^{13}(1739178796385788 0181760-2546720899755788 3279360b$$
$$+2400854431037162631 7312b^2-793707291412427820 6464b^3+3243303667638069677824b^4$$
$$-440679007025280092160b^5+106210320831662052096b^6-6812292984915879552b^7$$
$$+1099941339062732712b^8-32820043088072240b^9+3801278331903052b^{10}-42136868812584b^{11}$$
$$+3646459801089b^{12})+68a^{12}(-29698427130449907614 7200+6096494267911594977 4080b$$
$$-38051382869985669109 4528b^2+22094799327966203671 7056b^3-48392097677309271814 656b^4$$
$$+14745112810320249826560b^5-1504753592878574531584b^6+290455015888515016448b^7$$
$$-14887206885349217328b^8+1994638677221550360b^9-4938198858922128 8b^{10}$$
$$+4838717101978396b^{11}-45648274546966b^{12}+3376351667675b^{13})$$
$$+136a^{11}(2743984043628472051630080-425224483083684555048 9600b$$
$$+3985792797836613435184128b^2-1451125564153469161158656b^3+585738722441496417154816b^4$$
$$-92084556889457932800000b^5+21779906117776526463744b^6-1736068792707938777088b^7$$
$$+273256217693745494208b^8-11466371296064644960b^9+1285663189595672888b^{10}$$
$$-26860074374807856b^{11}+2233617295936026b^{12}-18162443453700b^{13}+1147626921525b^{14})$$
$$+680a^{10}(-668539810383909612717 6704+13672458129594208427671552b$$
$$-9136665677945558046787584b^2+5250409847454454722235392b^3-1278083736545456191909376b^4$$
$$+382853567428775969134336b^5-45561230386276347398656b^6+8587994021776429035776b^7$$
$$-550837181904325924608b^8+71572001765135551680b^9-2507679177864388528b^{10}$$
$$+236543367048989000b^{11}-4230411952843076b^{12}+298821018586090b^{13}-2117556771330b^{14}$$

Notes

$$+1140222876876b^{15}) + 119a^9(476088976059553675927879680 - 773172394111752564303462400b$$
$$+718681991017720254543429632b^2 - 282787609306526661854593024b^3$$
$$+112426199329587016785588224b^4 - 19799038261568446519009280b^5$$
$$+4580985588826963400820736b^6 - 428769600491281590075392b^7 + 65559731211863380890112b^8$$
$$-3464388680091329802240b^9 + 374583758265458007552b^{10} - 11140210290527229184b^{11}$$
$$+886290824143952864b^{12} - 137376146798800000b^{13} + 823188736214000b^{14} - 5133470960800b^{15}$$
$$+234461414075b^{16}) + 119a^8(-4017124647911749764723507200 + 8165192959090366244139827200b$$
$$-5775018563881998277329420288b^2 + 3275118944888902081521647616b^3$$
$$-868564118730245549841555456b^4 + 254977474725171599196917760b^5$$
$$-34228950752786687725859840b^6 + 6278129451628448368601088b^7 - 475855463607301338283008b^8$$
$$+59720298678473550374400b^9 - 2650673746583004899328b^{10} + 239599526041241273856b^{11}$$
$$-6134259730092277056b^{12} + 411625047770096160b^{13} - 5590349876311200b^{14} + 283147591137840b^{15}$$
$$-1567542025530b^{16} + 60290077905b^{17}) + 68a^7(5697378683367296714466656256 0$$
$$-9630313854058720374214885376 0b + 8853328446657763285719108812 8b^2$$
$$-3717753021982007650893928857 6b^3 + 1451203214196796556219488665 6b^4$$
$$-2804223188464391193098403840b^5 + 632404016588131813601931264b^6$$
$$-67204718966228434152726528b^7 + 9940523633717458728780288b^8$$
$$-624515421009853580247040b^9 + 64803199823371773091328b^{10} - 2455623734866361020416b^{11}$$
$$+185865876600461680896b^{12} - 4147446806337045120b^{13} + 234143298962882400b^{14}$$
$$-2813863547302080b^{15} + 119712195949710b^{16} - 593124009660b^{17} + 19010384925b^{18})$$
$$+340a^6(-6310212356380560407992782028 8 + 1271615175161861682400277299 20b$$
$$-9443936726602493394989508198 4b^2 + 5269095880443564989713111472 64b^3$$
$$-1502352638346892056177547673 6b^4 + 4305975627128026181658345472b^5$$
$$-638513934962959054913134592b^6 + 113458262743480293404545024b^7$$
$$-9825305765201868656612352b^8 + 1184914561702355776061952b^9 - 62906183954207403729920b^{10}$$
$$+5416296419673384114688b^{11} - 177737572292034413568b^{12} + 11250117922841403648b^{13}$$
$$-221262182312606400b^{14} + 10453043371100064b^{15} - 1121856043502041b^{16} + 3970889203134b^{17}$$
$$-17743025930b^{18} + 466921735b^{19}) + 34a^5(3099309826302466864238712324096$$
$$-5427111899453179276608248217600b + 4919282391795397669657346310144b^2$$
$$-2186124036495735935616006750208b^3 + 834590174206342019775985614848b^4$$
$$-174603900663017490511347056640b^5 + 38199594732014863200694566912b^6$$
$$-4513742754721822064576200704b^7 + 642260969433551106716657664b^8$$
$$-46398244495092672393236480b^9 + 4590007515427233123288064b^{10}$$
$$-209375740233651555852288b^{11} + 14959735718859412336128b^{12} - 430634565815737651200b^{13}$$
$$+22692582915810393600b^{14} - 397541549232441600b^{15} + 15578530827935880b^{16}$$

$$
\begin{gathered}
-150651272847600b^{17}+4375170540300b^{18}-17765802600b^{19}+375815055b^{20}) \\
+34a^{4}(-1020203831265700395188905574400+2032215363697541699394645000192 0b \\
-1576787815754640944092715522457 6b^{2}+8619779601954167823496246198272b^{3} \\
-2620412724520483834854367035392b^{4}+7295246508741309682563835549440b^{5} \\
-1179530827640621120745123348 48b^{6}+201786116474651549024908738 56b^{7} \\
-195661434496773513960231 3216b^{8}+225056282593788232053626880b^{9} \\
-138361263152908791889653 76b^{10}+1124696357939261326349312b^{11}-447510832055094017566 72b^{12} \\
+26436920516136977638 40b^{13}-675827524021805568 00b^{14}+29403479110145817 60b^{15} \\
-4636699473685272 0b^{16}+148706778103212 0b^{17}-130783109450 00b^{18}+305392769500b^{19} \\
-1136185050b^{20}+18728325b^{21})+476a^{3}(1884745748508783418877254041600 \\
-3409881826829937982772493680640b+3034403938024543695565328744448b^{2} \\
-1420350888994227163728829218816b^{3}+527292241475236570061913194496b^{4} \\
-118537830006547142871340810240b^{5}+24977810818874032385052114944b^{6} \\
-3243381930541062984482684928b^{7}+440069643665389697497694208b^{8} \\
-35884810234258085681684480b^{9}+3348822638217801203734528b^{10} \\
-178395622992053995114496b^{11}+11880923884963039868416b^{12}-418421444136293785600b^{13} \\
+20270967299797890560b^{14}-466035664741724160b^{15}+16515178975915200b^{16} \\
-237117585744480b^{17}+6099212967960b^{18}-49303545200b^{19}+898127230b^{20}-3091660b^{21}+37835b^{22}) \\
+2380a^{2}(-5977914281701133089215873024 00+1171751479300626757897798287360b \\
-948055964740340596452876091392b^{2}+504436030841698476579511861248b^{3} \\
-162969704620543235289167953920b^{4}+43687683006375599326447534080b^{5} \\
-7660084442701710525175169024b^{6}+1247961785723776330796892160b^{7} \\
-134271672752950177986478080b^{8}+14536163456732426647437312b^{9} \\
-1019262463518502934016000b^{10}+76978730210070107744256b^{11}-3617608077354064661504b^{12} \\
+195662091125076964864b^{13}-6195685974354981888b^{14}+242541325992669696b^{15} \\
-5088604615794432b^{16}+143718465951936b^{17}-1905599253488b^{18}+38090339560b^{19}-287050764b^{20} \\
+3887598b^{21}-12558b^{22}+105b^{23})+5a(2728018310136936135426582773760 00 \\
-5102591983757014003422200856576 00b+4421725628005287631298288104243 20b^{2} \\
-2181829189341085591367885783040 00b^{3}+7787506194378564798189475882598 4b^{4} \\
-1882130945782304385148672121241 6b^{5}+3764236180138698629064840708096b^{6} \\
-536807628594410516489741271040b^{7}+68201300036964447388606398464b^{8} \\
-6258043140275689339905376256b^{9}+538855751873513161486237696b^{10} \\
-33256488954309491175751680b^{11}+2009985288142112547876864b^{12} \\
-85091162219011727220736b^{13}+3669315134617720211456b^{14}-106598572947742146560b^{15} \\
+3283924788746631424b^{16}-64179303598135296b^{17}+1391962024041216b^{18}-17419933847680b^{19} \\
+256370498384b^{20}-1844681696b^{21}+17038216b^{22}-53040b^{23}+255b^{24}))\Big\}\Bigg] \qquad (8)
\end{gathered}
$$

Notes

III. Derivation of the Summation Formula

Substituting $c = \frac{a+b+52}{2}$ and $z = \frac{1}{2}$ in equation (2), we get

$$(a-b)\ {}_2F_1\left[\begin{array}{cc} a,\ b & ; \\ \frac{a+b+52}{2} & ; \end{array} \frac{1}{2}\right] = a\ {}_2F_1\left[\begin{array}{cc} a+1,\ b & ; \\ \frac{a+b+52}{2} & ; \end{array} \frac{1}{2}\right] - b\ {}_2F_1\left[\begin{array}{cc} a,\ b+1 & ; \\ \frac{a+b+52}{2} & ; \end{array} \frac{1}{2}\right]$$

Now involving the derived formula [Salahuddin et. al.], the summation formula is obtained.

Notes

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Notes

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Modification of Earth's Gravity Sphere

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Abstract- The standard radius of the Earth's gravity sphere is 917 000 km. Here we present that the radius is 1400000 km.

Keywords: celestial mechanics, gravitation, earth, sun.

GJSFR-F Classification : MSC 2010: 76B15

Modification of Earth's Gravity Sphere

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Abstract- The standard radius of the Earth's gravity sphere is 917 000 km. Here we present that the radius is 1400000 km.

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I. Introduction

Earth's gravity sphere is a space around the Earth (as a material point) where in the Earth's force of attraction is stronger than the gravitational forces of other bodies, including the gravitational force of Sun. The formula that determines the radius ρ of the so called sphere of influences (gravy sphere) of the Earth's gravity in this case is ([1], p. 196),

$$\rho = r \sqrt[5]{(m_1/M)^2}, \tag{1}$$

where r is the distance between the Earth and Sun, $m_1 = M_\oplus$ is the mass of the Earth, and $M_\odot \approx 333000\, m_1$ is the mass of the Sun. The size of this radius of the Earth's sphere amounts approximately to $\rho = 917\,000\, km$ or ([2], p.108) 923 000km.

$$F = \kappa \frac{m_1 m_2}{\rho^2}. \tag{3}$$

Verification of the formula (1) with the use of the Newton's formula of "universal gravitational force"

$$F = \kappa \frac{m_1 m_2}{\rho^2}. \tag{2}$$

led to a paradoxical result. According to formula (2), at the boundary of the Earth's gravity sphere, it should be $F_\oplus = F_\odot$. However, the calculation shows the opposite. And indeed, let us show this with some more details.

Let it be assumed that: $m_1 = M_\oplus$ is the mass of the Earth, $m_2 = M_\odot$ is the mass of the Sun, and m is the mass of any body at the boundary $\rho_\oplus = x = 917\,000$ km. For the above mentioned assertions of the book the mass of the Sun is $M_\odot \approx 333\,000 M_\oplus$, whereas a tabulated distance of the Earth from the Sun is $\rho_\odot = a = 149\,600\,000$ km.

First. The Sun and the Earth act at the same time on a body having the mass m in a critical boundary point at the distance $\rho_\oplus = x$ with the forces according to Newton's formula (2):

$$F_\oplus = \kappa \frac{M_\oplus m}{x^2}, \qquad F_\odot = \kappa \frac{M_\odot\, m}{(r-x)^2}. \tag{3}$$

Therefore, in a critical point $\rho_\oplus = x = 917\,000$, it should be

$$F_\odot = \kappa \frac{M_\odot\, m}{(149\,600\,000 - x)^2} = 1,5063 \cdot 10^{-11} \kappa M_\odot m. \tag{4}$$

and

$$F_\oplus = \kappa \frac{M_\oplus\, m}{x^2} = 0,11892 \cdot 10^{-11} \kappa M_\oplus m. \tag{5}$$

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Ref

[1] T. Andjelić and R. Stojanović, Racionalna mehanika, Zavod za udžbenike, Beograd, 1966.

This shows that, according to Newton's formula, the gravitational force of the Sun at the distance of 917 000 km from the center of the Earth is more than 12 times greater than the value of the Earth's gravitational force, i.e.

$$F_\odot = 12,666\,611\,F_\oplus \Longleftrightarrow F_\oplus = 0,0789478\,F_\odot.$$

However, this is not in compliance either with the definition of gravity sphere, or with the phenomena in the nature. The Moon moves around the Earth at an average distance of 384 400 km, under the dominant attraction of the Earth, not the Sun.[7]

The second. Let's determine the boundary of the Earth's gravity sphere with the use of a strict procedure, by means of the universal gravity formula (2). According to the Newton's gravity theory (2) would follow, so that it should be:

$$\frac{M_\oplus m}{x^2} = \frac{M_\odot m}{(r-x)^2},$$

or for $M_\odot = 333\,000 M_\oplus$ follow $(\rho - x)/x)^2 = M_\odot / M_\oplus = 333\,000$.

Further calculation gives: $(\rho - x)^2 = (577,6152\,x)^2$, i.e. $\rho - x = 577,6152\,x$, or $\rho = 578,0652\,x$, and from there, for $\rho = 149\,600\,000\,\mathrm{km}$, it follows that

$$x = 258\,795,993\,\mathrm{km}.$$

This is contradictory to the fundamental laws of dynamics, as well as the actual state of the motion of the Moon around the Earth at an average distance of 384 400 km, and particularly the formula (1), which demonstrates the radius of the sphere of the Earth's gravity. Doubt about the validity of the Newton's formula is increased by a fact from the above mentioned book. According to the Newton's formula (1) it follows that the acceleration of gravity depends not only on the distance, but it is asserted that at the first cosmic velocity of 7, 91 km/s, a body will escape from the Earth's attraction and will rotate around the planet Earth under an assumption that the resistance of the medium is ignored. At the second cosmic velocity $v_{or} = 11,19$ km/s, a missile will leave the area of the Earth's gravity sphere.

II. One Modification Theory of Gravity

In the papers [1, 2, 3, 4] author is demonstrated that our formula of mutual action of two bodies has the form

$$F_\rho = \frac{\dot\rho^2 + \rho\ddot\rho - v_{or}^2}{m_1 + m_2}\frac{m_1 m_2}{\rho} = M^* \frac{\dot\rho^2 + \rho\ddot\rho - v_{or}^2}{\rho} = F^* + F^{**}. \quad (8)$$

where we introduced notations:

$$M^* = \frac{m_1 m_2}{m_1 + m_2}, \quad F^* = M^* \frac{\dot\rho^2 + \rho\ddot\rho}{\rho}, \quad F^{**} = M^* \frac{v_{or}^2}{\rho}.$$

For the escaping boundary of the attraction of a body having a mass of m and the body having a mass of M, it will be

$$M^* \frac{\dot\rho^2 + \rho\ddot\rho - v_{or}^2}{\rho} = 0,$$

or in Simić's form

$$\frac{d}{dt}(\rho\dot\rho) - v_{or}^2 = 0.$$

For the purpose of clearer and more straightforward comprehension of this assertion, let us mention that formula (6), in relation to the natural coordinate system, can be reduced to a simpler form. It is sufficient to observe that it is $v^2 = \dot\rho^2 + \rho^2\dot\theta^2$ so as to reduce the formula (6) to a form

$$F_\rho = M^*(\ddot\rho - \rho\dot\theta^2).$$

In the state of motion where $F_\rho = 0$, the known formula for normal acceleration follows

$$\ddot\rho = \rho\dot\theta^2 = \frac{v^2}{\rho},$$

Ref

[7] V. A. Vujičić, *Modification of the characteristic gravitation constants*, Astronomical and Astrophysical Transactions, 25(4), (2006), 317–325.

as well as formula for the force of mutual attraction

$$F^{**} = M^{*}\frac{v^2}{\rho}, \tag{7}$$

where $\rho = R = const.$

Table. It has been shown what the radial accelerations of the satellites are at different altitudes H above the Earth according to the standard formula $\gamma = gR^2/\rho^2$, as well as the formula $\gamma^* = v^2/\rho$, which follows from the formula (6).

Altitude	Velocity	Acceleration	Acceleration
H km	v km/s	γ	γ^*
0	7, 91	981, 0	982, 3
100	7, 84	948, 9	950, 0
1000	7, 35	732, 1	733, 0
10000	4, 93	148, 4	148, 4
100000	1, 94	3, 5	3, 5
384400	1, 02	0, 002693	0.002706

Let's note that the last type of table refers to the average speed of the Moon's motion around the Earth and its average distance from the center of the Earth.

By the application of formula (8) to the motion of the Moon in relation to the Sun and in relation to the Earth, it has been proven that the gravitational force of the Earth, which acts on the Moon, is greater than the corresponding force of the Sun. In this way, dynamical paradox in the theory of the Moon's motion has been removed, [8]. It is logical that it is possible to determine the boundary of the Earth's gravity sphere in the same way.

Using this procedure, we obtain a significant modification of the Earth's gravitation sphere . Starting from the aforementioned definitional of the gravity sphere of two bodies, let us find the boundary x of the gravity sphere of the Earth in relation to the gravitational force of the Sun for that same body. By the very nature of things and by mathematical logics, initial relation of that task is that the gravitational force of the Earth is greater than, and at the boundary of the sphere $\rho = x$ is equal to, the Sun's gravitational force, i.e.,

$$F_{\oplus} = F_{\odot},$$

where :

$$F_{\oplus} = \frac{M_{\oplus} m}{M_{\oplus} + m}\frac{v^2_{or\oplus}}{x}, \quad v_{or\oplus} < 1\text{km/s},$$

$$F_{\odot} = \frac{M_{\odot}\, m}{M_{\odot} + m}\frac{v^2_{or\odot}}{a - x}, \quad v_{or\odot} = 29,8 - (19,5 + 0,3) = 10\,\text{km/s}.$$

Ratio of the gravitational forces $F_{\oplus}$ and $F_{\odot}$ at the boundary of the Earth's gravity sphere is:

$$\frac{F_{\oplus}}{F_{\odot}} \equiv \frac{v^2_{or\oplus}}{x} : \frac{v^2_{or\odot}}{a - x} = 1.$$

From here, it follows that

$$x = \frac{a}{1 + (v_{or\odot}/v_{or\oplus})^2}. \tag{8}$$

Value of the fraction which is derived, depends, as we can see, on the ratio of the orbital speeds of bodies in relation to the Sun and the Earth at the boundary x of the Earth's gravity sphere. Let us analyze that for our needs.

First: $v_{or\odot} \neq v_{or\oplus}$, because it is $v_{or\odot} = v_{\oplus} \pm v_{or\oplus} - v_{\odot}$; $\quad v_{or\oplus} \neq v_{\odot}$.

The second: For $v_{\oplus} = 1$ is $x = a/(1 + v^2_{or\odot})$.

The third: for $v_{or} > 1$ the value of the fraction is decreased, and already for $v_{or} > 1$ the fraction (10) is decreased, and for $v_{or} < 1$ it is increased. In view of the

Ref

[8] Veljko A. Vujičić, *Dynamical Paradox in Theory of Lunar Motion*, International Journal of Nonlinear Sciences and Numerical Simulation, Freund Publishing House Ltd, 10(11-12): 1539-1543, (2009).

sphere depends on the ratio of the speeds of two bodies in relation to the Earth $v_{or\oplus}$ and in relation to the Sun $v_{or\odot}$. Usually the velocity $v_{or\oplus}$ is not known, so that we are left only with a hypothetical analysis on the basis of the average standard data. The velocity of the Sun $v_{or\odot}$is even less known. Speeds of the Sun in relation to various groups of stars [4]. The standard velocity of the Sun is usually taken to be $v_{\odot} = 20\,000$ km/s. Since the mean velocity of the Earth's motion around the Sun is $v_{\oplus} \approx 30\,000$ km/s. In this state of motion, it is

$$v_{or\odot} \approx v_{\oplus} - v_{\odot} = 10\,\text{km/s}.$$

For this logical choice and numerical values of the standard quantities (see for example [3]):

$$\frac{M_{\oplus}\, m}{M_{\oplus} + m} = 0,987; \quad \frac{M_{\odot}\, m}{M_{\odot} + m} = 0,999;$$
$$a = 149\,600\,000 km, \quad M_{\odot} = 333\,000\, M_{\oplus},$$

it is obtained that the radius of the gravitation sphere of the Earth is $x = 1\,481\,188$ km, or

$$x \approx 1\,481\,000\,\text{km}. \tag{9}$$

Therefore, for the standard data which are taken, the radius of the gravitation sphere of the Earth is significantly greater than the radius $x = 917\,000$km, and expressly than $x = 258\,795$km.

III. Conclusion

In the first part of this paper it is proven that the formula of the gravitational sphere of the Earth (1) has not been derived on the basis of the Newton's formula (3). By direct calculation with the use of the formula (3) it is shown that the formula leads to the results, which are not in accordance with the nature of the motion between the Sun and the Earth. Convincing example is the motion of the Moon, for which the formula (3) leads to paradoxical dynamic result of the Newton's gravity theory.

With the use of the formula (6) for the mutual attraction of two bodies, the above mentioned paradox in the theory of the Moon's motion is removed and one solution to the problem of three bodies (Sun-Earth-Moon) is obtained. That was a reason to consider the boundary (2) of the gravity sphere of the Earth in this paper. Approximately correct result for the radius of the Earth's gravity sphere on the basis of the formula (11) amounts to **1 400 000** km, which is considerably different from the value (2).

IV. Acknowledgment

The author is grateful Milan Dimitrijević for his suggestions.

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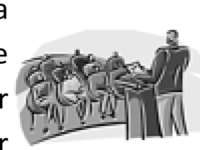

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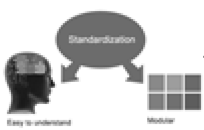

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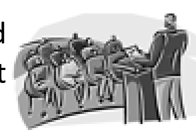

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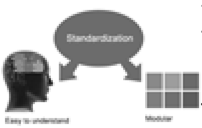

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