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Certain Sequences and its Integral Representations in Terms of Laguerre Polynomials

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Certain Sequences and its Integral Representations in Terms of Laguerre Polynomials

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I. Introduction and Main Results

By using some special functions and some particular integrals, we recall some integral representations for certain integer (or real) sequences.

a) Some special functions

The Gamma function is defined by the definite integral

$$\Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x}\,\mathrm{d}x, \quad \Re e(z) > 0.$$

We can see directly, that $\Gamma(1) = 1$, and using integration by parts, that $\Gamma(z+1) = z\Gamma(z)$. Notice that, for $z = n \in \mathbb{N}\setminus\{0\}$, the following formulas hold

$$\begin{aligned} n! &= \Gamma(n+1) \\ &= \int_0^{+\infty} x^n e^{-x}\,\mathrm{d}x, \qquad (1) \\ \frac{(2n)!}{2^{2n}n!}\sqrt{\pi} &= \Gamma\big(n+\frac{1}{2}\big) \\ &= \int_0^{+\infty} x^n \frac{e^{-x}}{\sqrt{x}}\,\mathrm{d}x. \qquad (2) \end{aligned}$$

The Bêta function is given in terms of the integral

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\,\mathrm{d}x, \quad \Re e(s),\ \Re e(t) > 0.$$

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which is symmetric in s and t, i.e., $B(s,t)=B(t,s)$. Notice that, after a change of variable $x=\frac{1}{1+y}$, we get

$$B(s,t)=\int_0^{+\infty}\frac{x^{s-1}}{(1+x)^{s+t}}\,\mathrm{d}x.$$

This function also admits the following representation in terms of the Gamma function [3]

$$B(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

In particular, if s and t are non-zero integers, then we have

$$\begin{aligned}\frac{n!p!}{(n+p+1)!} &= B(n+1,p+1)\\ &= \int_0^1 x^n(1-x)^p\,\mathrm{d}x, &(3)\\ &= \int_0^{+\infty}\frac{x^n}{(1+x)^{n+p+2}}\,\mathrm{d}x,\quad n,\ p\geq 0. &(4)\end{aligned}$$

The monic Hermite polynomials $H_n(x)$ are orthogonal in the interval $(-\infty,+\infty)$ with respect to the weight function e^{-x^2} and fulfil the following orthogonality relation [2]

$$\int_{-\infty}^{+\infty}e^{-x^2}H_n(x)H_m(x)\,\mathrm{d}x=\frac{\sqrt{\pi}}{2^m}n!\delta_{n,m},\quad n,\ m\geq 0,$$

where $\delta_{n,m}$ is the Kronecker delta.

The canonical moments, $(\mathcal{H})_{n\geq 0}$, of the Hermite polynomials have the representation [3]

$$\begin{aligned}\frac{\big(1+(-1)^n\big)n!}{2^{n+1}\Gamma(\frac{n}{2}+1)} &= (\mathcal{H})_n\\ &= \int_{-\infty}^{+\infty}x^ne^{-x^2}\,\mathrm{d}x,\quad n\geq 0. &(5)\end{aligned}$$

b) Some other integrals

The Wallis integral is given by

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x,\quad n\geq 0.$$

By a simple integration by parts, we can obtain

$$I_{2n}=\frac{(2n)!\pi}{2^{2n+1}(n!)^2},\quad I_{2n+1}=\frac{2^{2n}(n!)^2}{(2n+1)!},\quad n\geq 0.$$

By the change of variable $t=\sin x$, this gives the following formulas

$$\frac{(2n)!\pi}{2^{2n+1}(n!)^2}=\int_0^1\frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x,\quad n\geq 0, \tag{6}$$

$$\frac{2^{2n}(n!)^2}{(2n+1)!}=\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}\,\mathrm{d}x,\quad n\geq 0. \tag{7}$$

Now, let consider the following integral

Ref

3. P. Maroni, *Fonctions Eulériennes, Polynômes Orthogonaux Classiques.* Techniques de l'Ingénieur, Traité Généralités (Sciences Fondamentales) A 154 Paris, 1994. 1-30.

$$T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, \mathrm{d}x, \quad n \geq 0.$$

It is easy to see that

$$T_{n+2} + T_n = \frac{1}{n+1}, \quad n \geq 0.$$

We get by iteration the two following formulas

Notes

$$T_{2n+1} = \sum_{k=1}^{n} \frac{(-1)^{n+k}}{2k} + (-1)^n T_1, \quad n \geq 0.$$

$$T_{2n+2} = \sum_{k=0}^{n} \frac{(-1)^{n+k}}{2k+1} + (-1)^{n+1} T_0, \quad n \geq 0.$$

Then, by the change of variable $t = \tan x$, we get

$$\frac{(-1)^n}{2}\left(\ln 2 + \sum_{k=1}^{n} \frac{(-1)^k}{k}\right) = \int_0^1 \frac{x^{2n+1}}{1+x^2} \, \mathrm{d}x, \quad n \geq 0, \tag{8}$$

$$(-1)^n\left(\frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2k+1}\right) = \int_0^1 \frac{x^{2n}}{1+x^2} \, \mathrm{d}x, \quad n \geq 0, \tag{9}$$

with the convention $\sum_{k=1}^{0} = \sum_{k=0}^{-1} = 0$.

We also consider the following integral

$$R_n = \int_0^1 \frac{x^n}{1+x} \, \mathrm{d}x, \quad n \geq 0.$$

It is easy to see that $R_n + R_{n+1} = \frac{1}{n}$, $n \geq 1$, and hence the following formula

$$(-1)^n\left(\ln 2 + \sum_{k=1}^{n} \frac{(-1)^k}{k}\right) = \int_0^1 \frac{x^n}{1+x} \, \mathrm{d}x, \quad n \geq 0. \tag{10}$$

Finally, we consider the integral

$$B_n = \frac{1}{n!} \int_0^1 (1-x)^n e^x \, \mathrm{d}x, \quad n \geq 0.$$

For $n \geq 1$, integration by parts yields $B_n = B_{n-1} - \frac{1}{n!}$, and we obtain the formula

$$e - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{n!} \int_0^1 (1-x)^n e^x \, \mathrm{d}x, \quad n \geq 0.$$

This gives, after a change of variable $t = 1 - x$, the following relation

$$n!\left(1 - \frac{1}{e} \sum_{k=0}^{n} \frac{1}{k!}\right) = \int_0^1 x^n e^{-x} \, \mathrm{d}x, \quad n \geq 0. \tag{11}$$

In this paper, we introduce the following connection formula, between the monomial $\{x^n\}_{n\geq 0}$ and the shifted Laguerre polynomials,

$$x^n = \frac{1}{(n+m)!}\int_0^{+\infty} t^m e^{-t} L_n^{(m)}\big(t(x+1)\big)\ \mathrm{d}t, \quad n \geq 0,\ m \in \mathbb{N}\setminus\{0\}.$$

As an application of our formula, we give the integral representations in terms of Laguerre polynomials for the sequences given by the equations (1)−(11).

II. Integral Representations in Terms of Laguerre Polynomials

Let $\{L_n^{(m)}\}_{n\geq 0}$ be the monic Laguerre polynomial sequence, with parameter $m \in \mathbb{N}\setminus\{0\}$, [4]

$$L_n^{(m)}(x) = \sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\frac{(n+m)!}{(\nu+m)!}\,x^\nu, \quad n \geq 0. \tag{12}$$

For any $c \in \mathbb{C}$ and any polynomial p, let introduce in the space of polynomials the linear isomorphism $\mathfrak{S}_c$, called intertwining operator, and given by [1]:

$$\mathfrak{S}_c(p)(x) = \int_0^{+\infty} te^{-t} p\big(t(x-c)+c\big)\ \mathrm{d}t.$$

The operator $\mathfrak{S}_c$ can be characterized taking into account its linearity as well as the fact

$$\mathfrak{S}_c\big((x-c)^n\big) = (n+1)!(x-c)^n, \quad n \geq 0. \tag{13}$$

By (13) and (12), it is easy to prove that

$$\mathfrak{S}_0\big(x^{m-1}L_n^{(m)}(x)\big) = (n+m)!x^{m-1}(x-1)^n, \quad n \geq 0.$$

Hence, we can obtain the following result.

Theorem 2.1 For every integer $m \in \mathbb{N}\setminus\{0\}$, the following formula holds

$$x^n = \frac{1}{(n+m)!}\int_0^{+\infty} t^m e^{-t} L_n^{(m)}\big(t(x+1)\big)\ \mathrm{d}t, \quad n \geq 0. \tag{14}$$

Now, as an application of the above formula, we can express the sequences given by the equations (1)−(11) by integral representations in terms of Laguerre polynomials. Indeed, substituting expression (14) into (1)−(11), we can state the following theorem.

Theorem 2.2 For every integers $m \geq 1$, and $n,\ p \geq 0$, the following formulas hold

$$n!(n+m)! = \int_0^{+\infty}\int_0^{+\infty} t^m e^{-(x+t)} L_n^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(2n)!m!\sqrt{\pi}}{4^n}\binom{n+m}{n} = \int_0^{+\infty}\int_0^{+\infty} \frac{t^m}{\sqrt{x}}\ e^{-(x+t)} L_n^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{n!p!(n+m)!}{(n+p+1)!} = \int_0^{1}\int_0^{+\infty} (1-x)^p t^m e^{-t} L_n^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{n!p!(n+m)!}{(n+p+1)!} = \int_0^{+\infty}\int_0^{+\infty} \frac{t^m e^{-t}}{(1+x)^{n+p+2}}\ L_n^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

Ref

1. H. Hochstadt, *The Functions of Mathematical Physics*. Dover Publications Inc. New York, 1971.

Notes

$$\frac{\left(1+(-1)^{n}\right)n!(n+m)!}{2^{n+1}\Gamma\left(\frac{n}{2}+1\right)}=\int_{-\infty}^{+\infty}\int_{0}^{+\infty}t^{m}e^{-(x^{2}+t)}L_{n}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(2n)!(2n+m)!\pi}{(n!)^{2}2^{2n+1}}=\int_{0}^{1}\int_{0}^{+\infty}\frac{t^{m}e^{-t}}{\sqrt{1-x^{2}}}\ L_{2n}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(n!)^{2}(2n+m+1)!2^{2n}}{(2n+1)!}=\int_{0}^{1}\int_{0}^{+\infty}\frac{t^{m}e^{-t}}{\sqrt{1-x^{2}}}\ L_{2n+1}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(2n+m+1)!(-1)^{n}}{2}\Big(\ln 2+\sum_{k=1}^{n}\frac{(-1)^{k}}{k}\Big)=\int_{0}^{1}\int_{0}^{+\infty}\frac{t^{m}e^{-t}}{1+x^{2}}\ L_{2n+1}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$(2n+m)!(-1)^{n}\Big(\frac{\pi}{4}+\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{2k+1}\Big)=\int_{0}^{1}\int_{0}^{+\infty}\frac{t^{m}e^{-t}}{1+x^{2}}\ L_{2n}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$(n+m)!(-1)^{n}\Big(\ln 2+\sum_{k=1}^{n}\frac{(-1)^{k}}{k}\Big)=\int_{0}^{1}\int_{0}^{+\infty}\frac{t^{m}e^{-t}}{1+x}\ L_{n}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$n!(n+m)!\Big(1-\frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\Big)=\int_{0}^{1}\int_{0}^{+\infty}t^{m}e^{-(t+x)}L_{n}^{(m)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

with the convention $\sum_{k=1}^{0}=\sum_{k=0}^{-1}=0$.

Corollary 2.3 For $m=1$ and $p=0$ we have, for every integer $n\geq 0$, the following special cases

$$n!(n+1)!=\int_{0}^{+\infty}\int_{0}^{+\infty}te^{-(x+t)}L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(2n)!(n+1)\sqrt{\pi}}{2^{2n}}=\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{t}{\sqrt{x}}\ e^{-(x+t)}L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$n!=\int_{0}^{1}\int_{0}^{+\infty}te^{-t}L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$n!=\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{te^{-t}}{(1+x)^{n+2}}\ L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{\left(1+(-1)^{n}\right)n!(n+1)!}{2^{n+1}\Gamma\left(\frac{n}{2}+1\right)}=\int_{-\infty}^{+\infty}\int_{0}^{+\infty}te^{-(x^{2}+t)}L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(2n+1)!\pi}{2^{2n+1}}\binom{2n}{n}=\int_{0}^{1}\int_{0}^{+\infty}\frac{te^{-t}}{\sqrt{1-x^{2}}}\ L_{2n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$n!(n+1)2^{2n+1} = \int_0^1 \int_0^{+\infty} \frac{te^{-t}}{\sqrt{1-x^2}}\, L_{2n+1}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$\frac{(2n+2)!(-1)^n}{2}\Big(\ln 2 + \sum_{k=1}^{n} \frac{(-1)^k}{k}\Big) = \int_0^1 \int_0^{+\infty} \frac{te^{-t}}{1+x^2}\, L_{2n+1}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$(2n+1)!(-1)^n\Big(\frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2k+1}\Big) = \int_0^1 \int_0^{+\infty} \frac{te^{-t}}{1+x^2}\, L_{2n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$(n+1)!(-1)^n\Big(\ln 2 + \sum_{k=1}^{n} \frac{(-1)^k}{k}\Big) = \int_0^1 \int_0^{+\infty} \frac{te^{-t}}{1+x}\, L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

$$n!(n+1)!\Big(1 - \frac{1}{e}\sum_{k=0}^{n} \frac{1}{k!}\Big) = \int_0^1 \int_0^{+\infty} te^{-(t+x)} L_{n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x$$

with the convention $\sum_{k=1}^{0} = \sum_{k=0}^{-1} = 0.$

Remark 2.1 Note that, if we take n even in the fifth formula, we obtain

$$\frac{(2n)!(2n+1)!}{2^{2n}n!} = \int_{-\infty}^{+\infty} \int_0^{+\infty} te^{-(x^2+t)} L_{2n}^{(1)}\big(t(x+1)\big)\ \mathrm{d}t\mathrm{d}x.$$

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References Références Referencias

1. H. Hochstadt, *The Functions of Mathematical Physics*. Dover Publications Inc. New York, 1971.
2. N. N. Lebedev, *Special Functions and their Applications*. Revised English Edition, Dover Publications, New York, 1972.
3. P. Maroni, *Fonctions Eulériennes, Polynômes Orthogonaux Classiques*. Techniques de l'Ingénieur, Traité Généralités (Sciences Fondamentales) A 154 Paris, 1994. 1-30.
4. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.

Notes