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New Oscillation Results for Class of Third Order Neutral Delay Differential Equations with Distributed Deviating Arguments

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New Oscillation Results for Class of Third Order Neutral Delay Differential Equations with Distributed Deviating Arguments

E. M. Elabbasy [α] & O. Moaaz [σ]

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I. Introduction

In this scientific work we consider new class of third order neutral delay differential equations with distributed deviating arguments of the form

$$\left(r_2(t)\left(\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2}\right)' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\sigma(\xi) = 0, t \geq t_0, \quad (1.1)$$

where $z(t) = x(t) + p(t)x(\tau(t))$ and we consider the following conditions

(A_1): $p, \tau \in C(I, \mathbb{R}), 0 < p(t) \leq p < 1, \tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty, \alpha_1$ and α_2 are a quotient of odd positive integers, $\alpha_1\alpha_2 = \beta$ and $I = [t_0, \infty)$,

(A_2): $r_i \in C(I, (0, \infty)), \int_{t_0}^{\infty} (r_i(t))^{-1/\alpha_i} dt = \infty, i = 1, 2$,

(A_3): $f \in C(\mathbb{R}, \mathbb{R}), xf(x) > 0$ for $t \geq t_0$,

(A_4): $q \in C(I \times [a, b], [0, \infty)), q(t, \xi)$ is not zero on any half line $[t_\mu, \infty) \times [a, b], t_\mu \geq t_0$,

(A_5): $g \in C(I \times [a, b], \mathbb{R}),\ g(t, \xi) \leq t$ for $t \geq t_0$ and $\xi \in [a, b]$, $g(t, \xi)$ is continuous, has positive partial derivative on $I \times [a, b]$ with respect to t, nondecreasing with respect to ξ and $\lim_{t\to\infty} g(t, \xi) = \infty$,

(A_6): $\sigma \in C([a, b], \mathbb{R})$, σ is nondecreasing and the integral of Eq. (1.1) is in the sense Riemann-stieltijes.

We intend to a solution of Eq. (1.1) a function $x(t) : [t_x, \infty) \to \mathbb{R}, t_x \geq t_0$ such that $x(t), r_1(t)(z'(t))^{\alpha_1}$ and $r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}$ are continuously differentiable for all $t \in [t_x, \infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. A solution of Eq. (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

In the last decades, there have been many research activity with regard to the oscillation of solutions of neutral delay differential equations. Significantly, this is due

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to recognition of the importance of differential equations in different applications, see [14].

Recently, there has been an growing interest in getting sufficient conditions for the oscillation of solutions of second/third order nonlinear neutral delay differential equations (see, for example [1]-[11], [13] and the references quoted therein). The oscillation problem for delay equation such as

$$(r_2(t)z''(t))' + f(t, z(t), z'(t)) = 0$$

and the half linear delay differential equation

$$(r_2(t)(z''(t))^{\alpha_2})' + q(t)x^{\alpha_2}(g(t)) = 0$$

have been discussed by many authors by different methods. Some results can be found in [11], [17] and also references therein. As well, [3] obtained some sufficient conditions of oscillation for neutral delay differential equation

$$(r_2(t)(z''(t))^{\alpha_2})' + q(t)f(x(g(t))) = 0$$

The aim of this paper is to discuss asymptotic behavior of solutions of class of third order neutral delay differential equation. By using Riccati transformation technique and new comparison principles, we established sufficient conditions which insure that solution of class of third order neutral delay differential equation is oscillatory or tended to zero. The results of this study basically generalize and improve the previous results. Examples given in the study to clarify the new results.

Let's recall the two sets of conditions that are commonly used, and we rely on:

(S_1): $\frac{f(x)}{x^\beta} \geq k > 0$ for $x \neq 0$ and $t \geq t_0$.
(S_2): $f'(x) > 0$ for $x \neq 0$ and $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$.

To discuss our main results, we review the following Theorem:
Consider the differential equation of the form

$$(-1)^n x^{(n)}(t) = F(t, x(v_1(t)), x(v_2(t)), ..., x(v_m(t))) \text{ for } t \geq t_0, \qquad \text{(E)}$$

where $F \in C([t_0,\infty) \times [0,\infty)^m)$ and $v_j \in C([t_0,\infty))$ such that $\lim_{t\to\infty} v_j(t) = \infty$ for $j = 1, 2, ..., m$. The function $F = F(t, u_1, u_2, ..., u_m)$ is supposed to be increasing in each of $u_1, u_2, ..., u_m$. Furthermore, it is assumed that F is positive on $[t_0,\infty) \times [0,\infty)^m$ and $v_j(t) < t$ for every $t \geq t_0$ and $j = 1, 2, ..., m$.

Theorem 1.1. [16] *if y is a positive and strictly decreasing solution of the integral inequality*

$$y(t) \geq \int_{t_0}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} F(t, y(v_1(s)), y(v_2(s)), ..., y(v_m(s)))ds,$$

then there exists a positive solution $x(t)$ of the differential Equation (E) being such that $x(t) \leq y(t)$ for all large t and satisfying $\lim_{t\to\infty} x^{(i)}(t) = 0$ monotonically ($i = 1, 2, ..., n-1$).

Lemma 1.1. If X is nonnegative, $U \geq 0, V > 0$ and $\eta > 0$ then

$$UX - VX^{\frac{\eta+1}{\eta}} \leq \frac{\eta^\eta}{(\eta+1)^{\eta+1}} U^{\eta+1} V^{-\eta}.$$

Proof. Let

$$K(X) = UX - VX^{\frac{\eta+1}{\eta}}, X > 0.$$

Ref

3. B. Baculikova and J. Dzurina, On the asymptotic behavior of a class of third order nonlinear neutral differential equations, Cent. Eur. J. Math. 8(6) .2010, 1091-1103.

$K(X)$ obtains its maximum at $X=\left(\frac{\eta}{\eta+1}\right)^{\eta}U^{\eta+1}V^{-\eta}$ and

$$K(X)\le \max_{X>0} K(X)=\frac{\eta^{\eta}}{(\eta+1)^{\eta+1}}U^{\eta+1}V^{-\eta}$$

The proof is complete. □

II. Main Results

In this section, we will establish new oscillation criteria for solutions of the Eq. (1.1). Assume that there exists a positive function $\rho(t)$. For the sake of convenience, we insert the next notation:

$$E_0(z(t))=z(t), E_i(z(t))=r_i(t)\left(\frac{d}{dt}E_{i-1}(z(t))\right)^{\alpha_i}, i=1,2,$$

$$R(t,t_0)=\left(\frac{1}{r_1(t)}\int_{t_0}^{t}\frac{1}{r_2^{1/\alpha_2}(s)}ds\right)^{1/\alpha_1}, \overline{R}(t,t_0)=\int_{t_0}^{t}R(u,t_0)du,$$

$$Q(t)=\int_a^b q(t,\xi)d\sigma(\xi), l(t)=\rho(t)\left(\beta R\left(g(t,a),t_0\right)g'(t,a)\right)^{-\beta}$$

$$\Theta_1(t)=k\rho(t)(1-p)^{\beta}Q(t), \Theta_2(t)=\int_a^b q(t,\xi)f(1-p(g(t,\xi)))d\sigma(\xi)$$

and

$$\mu=\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}.$$

Lemma 2.1. Let $x(t)$ be a positive solution of Eq. (1.1). Then $z(t)$ has only one of the following two properties eventually:

(P_1)**:** $z(t)>0, z'(t)>0$ and $\frac{d}{dt}E_1(z(t))>0$,

(P_2)**:** $z(t)>0, z'(t)<0$ and $\frac{d}{dt}E_1(z(t))>0$.

Proof. Let $x(t)$ be a positive solution of Eq. (1.1). From (A_1) and (A_5), there exists a $t_1\ge t_0$ such that $x(t)>0, x(\tau(t))>0$ and $x(g(t,\xi))>0$ for $t\ge t_1$. Then $z(t)>0$ and Eq. (1.1) implies that $\frac{d}{dt}E_2(z(t))\le 0$. Hence, $E_2(z(t))$ is a non-increasing function and of one sign. We claim that $E_2(z(t))>0$ for $t\ge t_1$. Suppose that $E_2(z(t))<0$ for $t\ge t_2\ge t_1$, then there exists a $t_3\ge t_2$ and constant $K_1>0$ such that

$$\frac{d}{dt}E_1(z(t))<-K_1(r_2(t))^{-1/\alpha_2},$$

for $t\ge t_3$. By integrating the last inequality from t_3 to t, we get

$$E_1(z(t))<E_1(z(t_3))-K_1\int_{t_3}^{t}(r_2(s))^{-1/\alpha_2}ds.$$

Letting $t\to\infty$, from (A_2), we have $\lim_{t\to\infty}E_1(z(t))=-\infty$. Then there exists a $t_4\ge t_3$ and constant $K_2>0$ such that

$$z'(t)<-K_2(r_1(t))^{-1/\alpha_1},$$

for $t\ge t_4$. By integrating this inequality from t_4 to t and using (A_2), we get $\lim_{t\to\infty}z(t)=-\infty$, which contradicts $z(t)>0$. Now we have $E_2(z(t))>0$ for $t\ge t_1$. Therefore, $E_1(z(t))$ is increasing function. Thus (P_1) or (P_2) holds for $z(t)$, eventually. □

a) Oscillation results for f(x) without monotonicity. The purpose of this section is to study criteria of oscillation for solutions of the Eq. (1.1) by using a Riccati transformation technique.

Lemma 2.2. *Let (S_1) holds, $x(t)$ be a positive solution of Eq. (1.1), and $z(t)$has the property (P_2). Assume that*

$$\int_{t_0}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}Q(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1}dv=\infty. \tag{2.1}$$

Then the solution $x(t)$ is converges to zero as $t\to\infty$.

Proof. Let $x(t)$ be a positive solution of Eq. (1.1). Since $z(t)$ satisfies the property (P_2), we get

$$\lim_{t\to\infty} z(t)=\gamma.$$

Now. We shall prove that $\gamma=0$. Let $\gamma>0$, then we have $\gamma<z(t)<\gamma+\varepsilon$ for all $\varepsilon>0$ and t enough large. Choosing $\varepsilon<\frac{1-p}{p}\gamma$, we obtain

$$\begin{aligned} x(t) &= z(t)-p(t)x(\tau(t)) \\ &> \gamma-pz(\tau(t)) \\ &> L(\gamma+\varepsilon)>Lz(t), \end{aligned} \tag{2.2}$$

where $L=\frac{\gamma-p(\gamma+\varepsilon)}{\gamma+\varepsilon}>0$. Hence, from (1), (S_1) and (A_5), we have

$$\begin{aligned} \frac{d}{dt}E_2(z(t)) &< -kL^{\beta}\int_a^b q(t,\xi)z^{\beta}(g(t,\xi))d\sigma(\xi) \\ &< -kL^{\beta}z^{\beta}(t)Q(t) \\ &< -kL^{\beta}\gamma^{\beta}Q(t). \end{aligned}$$

By integrating two times from t to ∞, we get

$$-z'(t)>C\left(\frac{1}{r_1(t)}\int_t^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}Q(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1},$$

where $C=k^{1/\beta}L\gamma>0$. Integrating the last inequality from t_1 to ∞, we have

$$z(t_1)>C\int_{t_1}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}Q(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1}dv.$$

This contradicts to the condition (2.1), then $\lim_{t\to\infty} z(t)=0$, which implies that $\lim_{t\to\infty} x(t)=0$. ☐

Theorem 2.1. *Let (S_1) and (2.1) hold. Assume that there exists a positive function $\rho(t)$such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\Theta_1(s)-\mu\left(\frac{\rho'(s)}{\rho(s)}\right)^{\beta+1}l(s)\right)ds=\infty. \tag{2.3}$$

Then every solution of Eq. (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Notes

Global Journal of Science Frontier Research (F) Volume XV Issue IX Version I Year 2015 4

Notes

Proof. Let x be a non-oscillatory solution of Eq. (1.1) on the interval I. Without loss of generality we may assume that $x(t) > 0$. Then there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t,\xi)) > 0$ for $t \geq t_1$. By Lemma **??**, we have that $z(t)$ has the property (P_1) or the property (P_2). If $z(t)$ has the property (P_2). From Lemma 2.2, we obtain $\lim_{t\to\infty} x(t) = 0$. Now, for $t \geq t_2 \geq t_1$, let $z(t)$ satisfies the property (P_1), then we get

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1 - p(t))z(t) \geq (1 - p)z(t). \tag{2.4}$$

Thus, from (1), (S_1) and (A_5), we have

$$\frac{d}{dt}E_2(z(t)) \leq -k(1-p)^{\beta} z^{\beta}(g(t,a))Q(t). \tag{2.5}$$

We define

$$\omega(t) = \rho(t)\frac{E_2(z(t))}{z^{\beta}(g(t,a))}.$$

By differentiating and using (2.5), we get

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \Theta_1(t) - \beta\rho(t)\frac{E_2(z(t))}{z^{\beta+1}(g(t,a))}z'(g(t,a))g'(t,a) \tag{2.6}$$

From (P_1), we have

$$\begin{aligned} E_1(z(t)) &= E_1(z(t_2)) + \int_{t_2}^{t} \frac{E_2^{1/\alpha_2}(z(s))}{r_2^{1/\alpha_2}(s)}ds \\ &\geq E_2^{1/\alpha_2}(z(t)) \int_{t_2}^{t} \frac{1}{r_2^{1/\alpha_2}(s)}ds, \end{aligned} \tag{2.7}$$

for $t \geq t_2$. Since $\frac{d}{dt}E_2(z(t)) \leq 0$, we obtain

$$z'(g(t,a)) \geq E_2^{1/\beta}(z(t))R(g(t,a),t_2).$$

Hence, (2.6) implies

$$\omega'(t) \leq -\Theta_1(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - l^{-1/\beta}(t)\omega^{\frac{\beta+1}{\beta}}(t).$$

If $\eta = \beta, U = \frac{\rho'}{\rho}, V = l^{-1/\beta}$ and $X = \omega$, then from Lemma 1.1, we obtain

$$\frac{\rho'}{\rho}\omega - l^{-1/\beta}\omega^{\frac{\beta+1}{\beta}} \leq \mu\left(\frac{\rho'}{\rho}\right)^{\beta+1} l.$$

Therefore, we get

$$\omega'(t) \leq -\Theta_1(t) + \mu\left(\frac{\rho'(t)}{\rho(t)}\right)^{\beta+1} l(t).$$

By integrating the above inequality from t_2 to t we have

$$\omega(t) \leq \omega(t_2) - \int_{t_2}^{t}\left(\Theta_1(s) - \mu\left(\frac{\rho'(s)}{\rho(s)}\right)^{\beta+1} l(s)\right)ds.$$

Taking the superior limit as $t \to \infty$ and using (2.3), we get $\omega(t) \to -\infty$, which contradicts that $\omega(t) > 0$. This completes the proof of Theorem 2.1. □

Example 2.1. *Consider the third order neutral delay differential equation*

$$\left(t\left(\left(\frac{1}{t}\left(z'(t)\right)^{1/3}\right)'\right)^5\right)' + \int_1^2 \frac{t^2 e^{t^2(\xi-1)}}{e^{t^2}-1} x^{5/3}(\xi(t-1))\left(x^2(\xi(t-1))+2\right)d\xi = 0, \tag{2.8}$$

where $z(t) = x(t) + \frac{1}{2}x(t-1)$ *and* $t > 1$. *Choose* $\rho(t) = 1$ *and* $k = 2$. *It is easy to see that the conditions (2.1) and (2.3) are hold. Then, from Theorem 2.1, every nonoscillatory solution of Eq. (2.8) tends to zero as* $t \to \infty$.

Remark 2.1. *If* $\alpha_1 = \alpha_2 = 1$, $\tau(t) = t - \tau$ *and* $f(x) = x$, *then Theorem 2.1 extend and improve Theorem 2.1 in* [5].

b) Oscillation results for f(x) with monotonicity. In this section, we will establish some new criteria of oscillation for solutions of the Eq. (1.1) by using new comparison principles.

Lemma 2.3. *Let* (S_2) *holds,* $x(t)$*be a positive solution of Eq. (1.1) and* $z(t)$ *has the property* (P_2). *If the condition (2.1) holds, then the solution* $x(t)$ *is converges to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Lemma 2.2, we see that (2.2) holds. Hence, from (1), (S_2) and (A_5), we have

$$\begin{aligned} \frac{d}{dt}E_2(z(t)) &\leq -\int_a^b q(t,\xi)f(Lz(g(t,\xi)))d\sigma(\xi) \\ &< -f(L)f(\gamma)Q(t). \end{aligned}$$

The rest of the proof runs as in Lemma 2.2. □

Theorem 2.2. *Let the condition* (S_2) *and (2.1) hold. If the first order delay differential equation*

$$y'(t) + \Theta_2(t)f\left(y^{1/\beta}(g(t,a))\right)f\left(\overline{R}(g(t,a),t_0)\right) = 0 \tag{2.9}$$

is oscillatory, then every solution of Eq. (1.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof. Let x be a non-oscillatory solution of Eq. (1.1) on the interval I. Without loss of generality we may assume that $x(t) > 0$. As in the proof of Theorem 2.1, by Lemma **??**, we have that $z(t)$ has the property (P_1) or (P_2) for $t \geq t_2$. If $z(t)$ has the property (P_2). Then, from Lemma **??**, we obtain $\lim_{t\to\infty} x(t) = 0$. On the other hand, when $z(t)$ satisfies the property (P_1), we get that (2.4) and (2.7) hold. Thus, from (2.7), we obtain

$$z'(t) \geq E_2^{1/\beta}(z(t))R(t,t_2),$$

for $t \geq t_2 \geq t_1$. By integrating this inequality from t_2 to t, we get

$$z(t) \geq z(t_2) + \int_{t_2}^t E_2^{1/\beta}(z(s))R(s,t_2)ds.$$

Since $\frac{d}{dt}E_2(z(t)) < 0$, we obtain

$$z(g(t,a)) \geq E_2^{1/\beta}(z(g(t,a)))\overline{R}(g(t,a),t_2). \tag{2.10}$$

From (1), (S_2) and (2.4), we have

Notes

$$
\begin{aligned}
-\frac{d}{dt}E_2\left(z\left(t\right)\right) &\geq \int_a^b q\left(t,\xi\right) f\left(z\left(g\left(t,\xi\right)\right)\right) f\left(1-p\left(g\left(t,\xi\right)\right)\right) d\sigma\left(\xi\right) \\
&\geq f\left(z\left(g\left(t,a\right)\right)\right)\Theta_2\left(t\right),
\end{aligned}
$$

where $z'\left(t\right)>0$. Hence, from (2.10), we get

$$
-\frac{d}{dt}E_2\left(z\left(t\right)\right) \geq \Theta_2\left(t\right) f\left(E_2^{1/\beta}\left(z\left(g\left(t,a\right)\right)\right)\right) f\left(\overline{R}\left(g\left(t,a\right),t_2\right)\right).
$$

By integrating above inequality from t to ∞ and Let $y\left(t\right)=E_2\left(z\left(t\right)\right)$, we obtain

$$
y\left(t\right) \geq \int_t^{\infty} \Theta_2\left(s\right) f\left(y^{1/\beta}\left(g\left(s,a\right)\right)\right) f\left(\overline{R}\left(g\left(s,a\right),t_2\right)\right) ds,
$$

The function $y(t)$ is obviously strictly decreasing. Hence, by Theorem 1.1 there exists a positive solution of equation (2.9) with $\lim_{t\to\infty} y(t)=0$ which contradicts that Equation (2.9) is oscillatory. This completes the proof of Theorem 2.2. □

Example 2.2. Consider the third order delay differential equation

$$
\left(t\left(\left(\frac{1}{t}\left(z'\left(t\right)\right)^{1/3}\right)'\right)^3\right)' + \int_0^1 \frac{6\xi^2}{t} x\left(\frac{1}{2}\left(\xi+1\right)t\right) d\xi = 0, \tag{2.11}
$$

where $z(t)=x(t)+\frac{1}{2}x\left(\frac{t}{3}\right)$ *and* $t\geq 1$, *It easy to see that the Condition (2.1) holds and Eq. (2.9), reduces to*

$$
y'\left(t\right)+\widehat{q}(t)y\left(\frac{t}{2}\right)=0, \tag{2.12}
$$

where

$$
\widehat{q}\left(t\right)=\frac{9}{896}+\frac{9}{1024}t^5-\frac{243}{8192}2^{2/3}t^{13/3}+\frac{243}{3584}2^{1/3}t^{11/3}-\frac{27}{512}t^3.
$$

On the other hand, Theorem 2.1.1 in [15] *guarantees the oscillation of (2.12). Since*

$$
\liminf_{t\to\infty}\int_{t/2}^{t}\widehat{q}\left(t\right)ds > \frac{1}{e}.
$$

Then, from Theorem 2.2, every nonoscillatory solution of Equation (2.11) tends to zero as $t\to\infty$.

In the following Theorem, we are concerned with the oscillation of solutions of Eq. (1.1) when $\tau\left(t\right)\geq t$.

Theorem 2.3. Let the condition (S_2) *holds, and there exists a function* $\zeta(t)$ *such that*

$$
\zeta'\left(t\right)\geq 0, \zeta\left(t\right)>t \text{ and } \delta\left(t\right)=g\left(\zeta\left(\zeta\left(t\right)\right),b\right)<t
$$

If the first order delay differential equation

$$
z'\left(t\right)+Q_1(t)f^{1/\beta}\left(z\left(\delta\left(t\right)\right)\right)=0, \tag{2.13}
$$

where

$$
Q_1=\left(\frac{1}{r_1(t)}\int_t^{\zeta(t)}\left(\frac{1}{r_2(u)}\int_u^{\zeta(t)}\Theta_2(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1},
$$

Ref

15. G. S. Ladde, V. lakshmikantham and B. G. Zhang, Oscillation Theory of Differential Equations With Deviating Arguments, Macel Dekker, New York, 1987.

is oscillatory. Then every solution $x(t)$ *of Eq. (1.1) is either oscillatory or* $\limsup_{t\to\infty} |x(t)| = \infty$.

Proof. Let x be a non-oscillatory solution of Eq. (1.1) on the interval I. Then, without loss of generality, there exists a $t_1 \geq t_0$ such that $x(t) > 0$ and $x(g(t,\xi)) > 0$ for $t \geq t_1$. By Lemma **??**, we have $z(t)$ has only one of the two Cases (P_1) or (P_2). For the Case (P_1). Since $z(t) > 0$, $z'(t) > 0$ and $\frac{d}{dt}E_1(z(t)) > 0$, $\lim_{t\to\infty} z(t) = \infty$, and from definition of $z(t)$, we get

$$\limsup_{t\to\infty} |x(t)| = \infty.$$

In the Case (P_2). Since $z'(t) < 0$ and $\tau(t) \geq t$, we obtain

$$x(t) \geq z(t)(1 - p(t)),$$

for $t \geq t_2$. Thus, from (A_5), there exists a $t_3 \geq t_2$ with $g(t,\xi) \geq t_2$ for $t \geq t_3$ such that

$$x(g(t,\xi)) \geq z(g(t,\xi))(1 - p(g(t,\xi))).$$

Hence, Eq. (**??**) and (S_2) yield

$$-\frac{d}{dt}E_2(z(t)) \geq \int_a^b q(t,\xi)\, f(z(g(t,\xi)))\, f(1 - p(g(t,\xi)))\, d\sigma(\xi)$$

Then, from (A_5), we get

$$-\frac{d}{dt}E_2(z(t)) \geq \Theta_2(t)\, f(z(g(t,b)))$$

By integrating this inequality from t to $\zeta(t)$, we have

$$\frac{d}{dt}E_1(z(t)) \geq r_2^{-1/\alpha_2}(t)\, f^{1/\beta}(z(g(\zeta(t),b))) \left(\int_t^{\zeta(t)} \Theta_2(s)\, ds\right)^{1/\alpha_2}$$

Again, integrate the above inequality from t to $\zeta(t)$, we obtain

$$-z'(t) \geq Q_1(t)\, f^{1/\beta}(z(\delta(t))).$$

Hence, By Theorem 1.1, there exists a positive solution of Eq. (2.13) with $\lim_{t\to\infty} z(t) = 0$ which contradicts that Eq. (2.13) is oscillatory. This completes the proof of Theorem 2.3. □

Remark 2.2. If $\alpha_1 = 1, r_1(t) = 1, a = b = 1, q(t,\xi) = q(t), g(t,\xi) = g(t)$ *and* $\sigma(\xi) = \xi$ *then Theorems 2.2, 2.3 extend and improve Theorem 2.4, 2.10 in* [3].

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Notes

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