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On Certain Type Fractional Integration of Special Functions Via Pathway Operator

By Hemlata Saxena & Rajendra Kumar Saxena
Career Point University, India

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Ref

5. Mathai, A. M., A pathway to matrix variate gamma and normal densities, Linear Algebra and its Applicaations, 396(2005), 317-328.

On Certain Type Fractional Integration of Special Functions Via Pathway Operator

Hemlata Saxena [α] & Rajendra Kumar Saxena [σ]

Abstract- In this present paper we study product of some special functions via pathway fractional integral operator. Our results are quite general in nature .Some known and new results are also obtain here.

Keywords: Pathway fractional integral operator, M-series, New generalized Mittag-Leffler function.

I. Introduction

Definition: Let $f(x) \in L(a,b), \eta \in C, R(\eta) > 0, a > 0$ and let us take a "Pathway parameter" $\alpha < 1$. Then the pathway fractional integration operator is defined by Nair [8]

$$\left(P_{0^+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left(\frac{x}{a(1-\alpha)}\right)} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt \qquad \text{......(1.1)}$$

when $\alpha = 0, \ a = 1$ *and* η is replaced by $\eta - 1$ in (1.1) it yields

$$\left(I_{0+}^{\eta} f\right)(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt \qquad \text{...(1.2)}$$

which is the left – sided Riemann-Liouville fractional integral defined by Samko et. al.[9].

The pathway model is introduced by Mathai [5] and studied further by Mathai and Houbold [6] [7].

For $R(\alpha) > 0$, the pathway model for scalar random variables is represented by the following probability density function.

$$f(x) = c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \qquad \text{.......(1.3)}$$

$$\gamma > 0, \delta > 0, \beta \geq 0, \left\{1 - a(1-\alpha)|x|^{\delta}\right\} > 0, -\infty < x < \infty,$$

where c is the normalizing constant and α is pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2} \frac{\delta\left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{1-\alpha} + 1\right)}, \qquad \alpha < 1 \qquad \text{.......(1.4)}$$

Author α σ: Career Point University, Kota (Rajasthan) India. e-mails: hemlata_19_75@yahoo.co.in, rajendrasaxenacpu@gmail.com

$$= \frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma(\frac{\beta}{1-\alpha})}{\Gamma(\frac{\gamma}{\delta})\,\Gamma(\frac{\beta}{1-\alpha}-\frac{\gamma}{\delta})}\ , \text{ for } \frac{1}{1-\alpha}-\frac{\gamma}{\delta}>0, \alpha>1 \qquad ......(1.5)$$

$$= \frac{1}{2}\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})}\ , \text{ for } \alpha\to 1 \qquad ......(1.6)$$

For $\alpha<1$, it is a finite range density with $[1-a(1-\alpha)|x|^{\delta}]>0$ and (1.3) remains in the extended generalized type-1 beta family. For $\alpha<1$, the pathway density in (1.3) includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

when $\alpha>1$, we write $1-\alpha=-(\alpha-1)$, then

$$\left(P_{0^+}^{(\eta,\alpha)}f\right)(x) = x^{\eta}\int_0^{\left(-\frac{x}{a(\alpha-1)}\right)}\left[1+\frac{a(\alpha-1)t}{x}\right]^{-\frac{\eta}{(\alpha-1)}} f(t)dt$$

$$f(x) = c\,|x|^{\gamma-1}[1+a(\alpha-1)|x|^{\delta}]^{-\frac{\beta}{\alpha-1}} \qquad ......\ (1.7)$$

Where $\alpha>1, \delta>0, \beta\geq 0, -\infty<x<\infty$,
which is extended generalized type-2 beta model for real x, It includes the type-2 beta density, the F density, the student-t density,the Cauchy density and many more.

Here, we consider only the case of pathway parameter $\alpha<1$. For $\alpha\to 1$ both (1.3) and (1.7) take the exponential form, since.

$$\lim_{\alpha\to 1} c\ |x|^{\gamma-1}[1-a(1-\alpha)|x|^{\delta}]^{\frac{\eta}{1-\alpha}}$$

$$= \lim_{\alpha\to 1} c\ |x|^{\gamma-1}[1+a(\alpha-1)|x|^{\delta}]^{-\frac{\eta}{\alpha-1}}$$

$$= c|x|^{\gamma-1}e^{-a\eta|x|^{\delta}} \qquad ....(1.8)$$

For $\alpha\to 1_-$, $\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} \to e^{-\frac{a\eta}{x}t}$, the operator (1.1) reduces to the following form

$$\left(P_{0^+}^{(\eta,1)}f\right)(x) = x^{\eta}\int_0^{\infty} e^{-\frac{a\eta}{x}t}\ f(t)dt$$

$$= x^{\eta}L_f(\frac{a\eta}{x}) \qquad ......(1.9)$$

It reduces to the Laplace integral transform of f with parameter $\frac{a\eta}{x}$.

In this paper we will integrate product of M-series, Fox's H-function and generalized Mittag-Leffler function by means of pathway model.

The generalized M-series is defined and studied by Sharma and Jain [10] as follows

$$ {}_{\rho}^{\alpha',\beta'}M_{\sigma}(z) = \sum_{k=0}^{\infty}\frac{(a_1')_k \ldots\ldots(a_\rho')_k}{(b_1')_k\ldots\ldots(b_\sigma')_k}\frac{z^k}{\Gamma(\alpha' k+\beta')}$$

Ref

10. Sharma, Manoj and Jain, Renu, A note on a generalized M-series as a special function of fractional calculus, Fract. Cal. Appl. Anal. 12(4) (2009), 449-452.

$$= \psi_1(k) \qquad \ldots(1.10)$$

Where $z, \alpha', \beta' \in C, Re(\alpha') > 0$

Here $(a_j')_k, (b_j')_k$ are known as Pochammer symbols. The series (1.10) is defined when none of the parameters $b_{js}', j = 1,2, \ldots \sigma$ is negative integer or zero. The series in (1.10) is convergent for all z if $\rho \le \sigma$, it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $\rho = \sigma + 1$ and divergent, if $\rho > \sigma + 1$. When $\rho > \sigma + 1$ and $|z| < \delta$, the series can converge on conditions depending on the parameters.

The series representation of Fox H- function studied by Fox C [2] as follows:

$$H_{P,Q}^{M,N}\left[z\middle|\begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right] = \sum_{h=1}^{N}\sum_{v=0}^{\infty}\frac{(-1)^v X(\xi)}{v!\,E_h},\left(\frac{1}{z}\right)^{\xi} \qquad \ldots(1.11)$$

where $\xi = \frac{e_h - v - 1}{E_h}$ and $(h = 1,2, \ldots \ , N)$

and

$$X(\xi) = \frac{\prod_{j=1}^{M}\Gamma(f_j + F_j\xi)\prod_{\substack{j=1\\j\neq h}}^{N}\Gamma(1 - e_j + E_j\xi)}{\prod_{j=m+1}^{Q}\Gamma(1 - f_j - F_j\xi)\prod_{j=N+1}^{P}\Gamma(e_j + E_j\xi)} \qquad \ldots..(1.12)$$

Following are the convergence conditions :

$$T_1 = \sum_{i=1}^{N}E_i - \sum_{i=N+1}^{P}E_i + \sum_{i=1}^{M}F_i - \sum_{i=M+1}^{Q}F_i \qquad \ldots.(1.13)$$

$$T_2 = \sum_{i=1}^{n}\alpha_i - \sum_{i=n+1}^{q}\alpha_i + \sum_{i=1}^{m}\beta_i - \sum_{i=m+1}^{q}\beta_i \qquad \ldots(1.14)$$

Recently, a new generalization of Mittag-Leffler function was defined by Faraj and Salim [3] as follows:

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\, z^n}{\Gamma(\alpha n + \beta)\,(\delta)_{pn}} \qquad \ldots...(1.15)$$

Where $z, \alpha, \beta, \gamma, \delta \in C; Min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0, p, q > 0$.

Further, generalization of Mittag- Leffler function was defined by Khan and Ahmed [4] as follows:

$$E_{\alpha,\beta,v,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z) = \sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}\,(\gamma)_{qn}\, z^n}{\Gamma(\alpha n + \beta)(v)_{\sigma n}\ (\delta)_{pn}} \qquad \ldots(1.16)$$

Where $\alpha, \beta, \gamma, \delta, \mu, v, \rho, \sigma \in C; p, q > 0$ and $q \le Re(\alpha) + \rho p$, and

$$Min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta), Re(\mu), Re(v), Re(\rho), Re(\sigma)\} > 0$$

If we take $\mu = v, \rho = \sigma$ in (1.16) it reduces to eq. (1.15).

Write generalized hypergeometric function was defined by Srivastava and Manocha [11] as follows:

Ref

2. Fox, C., The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math soc., 98 (1961), 395-429.

$$ {}_p\psi_q[(a_1,A_1),\dots(a_p,A_p);(b_1,B_1),\dots,(b_q,B_q);z] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_in)z^n}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)n!} \quad \dots.(1.17)$$

II. Main Results

Theorem-1 Let $\eta,\gamma,\delta,q,p,\omega,\rho \in C,\ c,b \in R, Re(\beta)>0, Re(\delta)>0, Re(\eta)>0, Re(\gamma)>0, Re(\omega)>0, Re\left(1+\frac{\eta}{1-\alpha}\right)>0, Re(\rho)>0, \alpha<1, b\in R, c\in R, Re\left(\omega+\delta\frac{f_j}{F_j}\right)>0, |\arg c|<\frac{1}{2}T_1\pi, T_1T_2>0, \rho\le\sigma, |d|<\alpha'^{\alpha'}, \beta^*>0, j=1,\dots,Q;$
Then

$$P_{0^+}^{(\eta,\alpha)}\left\{t^{\omega-1}\,{}_{\rho}^{\alpha',\beta'}M_{\sigma}\left[dt^{-\beta^*}\right].\ H_{P,Q}^{M,N}\left[ct^{\delta'}\Big|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right].E_{\alpha,\beta,p}^{\gamma,\delta,q}(bt^{\rho})\right\}$$

$$=\psi_1(k)\frac{d^kx^{\eta+\omega-\beta^*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\Gamma(\delta)}{\Gamma(\gamma)[a(1-\alpha)]^{\omega-\beta^*k}}\,{}_2\psi_3\left[\frac{bx^{\rho}}{[a(1-\alpha)]^{\rho}}\middle|\begin{matrix}(\omega-\delta\xi-\beta^*k,\rho)(\gamma,q)\\(\omega,\rho)(\delta,p)\left(1+\omega+\frac{\eta}{1-\alpha}-\delta\xi-\beta^*k,\rho\right)\end{matrix}\right].$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\delta'}}{[a(1-\alpha)]^{\delta'}}\middle|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right] \quad \dots\dots\dots..(2.1)$$

Proof: The theorem -1 can be evaluated by using the definitions (1.1),(1.10),(1.11) and (1.15) then by interchange the order of integrations and summations, evaluate the inner integral by making use of beta function formula, we arrive at the desired result (2.1).

Theorem-2 Let $\eta,\gamma,\delta,q,p,\beta,T_1,T_2>0, \mu,\rho,\gamma,\vartheta,\beta,v,\sigma,\delta\in C, Re(\eta)>0, Re(\gamma)>0, Re(\beta)>0, Re\left(1+\frac{\eta}{1-\alpha}\right)>0, b,c\in R, \alpha<1, Re\left(\omega+\delta\frac{f_j}{F_j}\right)>0, |\arg c|<\frac{1}{2}T_1\pi, \rho\le\sigma$ *and* $|d|<\alpha'^{\alpha'}, \beta^*>0, j=1,\dots,Q$ *and* $\min(Re(\vartheta), Re(\beta), Re(\gamma), Re(\delta), Re(\mu), Re(v), Re(\rho), Re(\sigma))>0$

Then

$$P_{0^+}^{(\eta,\alpha)}\left\{t^{\beta-1}\,{}_{\rho}^{\alpha',\beta'}M_{\sigma}\left[dt^{-\beta^*}\right].\ H_{P,Q}^{M,N}\left[ct^{\delta'}\Big|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right].E_{\vartheta,\beta,v,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(bt^{\vartheta})\right\}$$

$$=\psi_1(k)\frac{d^kx^{\eta+\beta-\beta^*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\Gamma(v)}{[a(1-\alpha)]^{\beta-\beta^*k}\Gamma(\mu)\Gamma(\gamma)}.$$

$${}_3\psi_3\left[\frac{bx^{\vartheta}}{[a(1-\alpha)]^{\vartheta}}\middle|\begin{matrix}(\mu,\rho)\quad(\gamma,q)\ (\beta-\delta'\xi-\beta^*k,\vartheta)\\(\beta,\vartheta)(v,\sigma)\left(1+\beta+\frac{\eta}{1-\alpha}-\delta'\xi-\beta^*k,\vartheta\right)\end{matrix}\right].$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\delta'}}{[a(1-\alpha)]^{\delta'}}\middle|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right] \quad \dots\dots\dots(2.2)$$

Proof: The theorem -2 can be evaluated by using the definitions (1.1),(1.10) (1.11) and (1.16) then by interchange the order of integrations and summations, evaluate the inner integral by making use of beta function formula, we arrive at the desired result (2.2).

Notes

Notes

III. Special Cases

1. If we take $\delta = p = q = 1, \rho = \beta, \alpha = \beta,\ \beta = \omega$ and in H- function $\delta' = \delta$ in theorem -1 then we at once arrive at the known result of [1,Theorem-2].
2. If we take $\delta = p = 1$ in theorem -1 then we get the following particular case of the solution (2.1)

 Corollary-1

The following formula holds

$$P_{0^+}^{(\eta,\alpha)}\left\{t^{\omega-1}\ {}_{\rho}\square_{M}^{\alpha',\beta'}{}_{\sigma}\,[dt^{-\beta^*}].\ H_{P,Q}^{M,N}\left[ct^{\delta'}\,\middle|\,\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right].E_{\rho,\omega}^{\gamma,q}(bt^{\rho})\right\}$$

$$= \psi_1(k)\frac{d^k x^{\eta+\omega-\beta^*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma(\gamma)[a(1-\alpha)]^{\omega-\beta^*k}}\ {}_2\square\psi_2\left[\frac{bx^{\rho}}{[a(1-\alpha)]^{\rho}}\,\middle|\,\begin{matrix}(\omega-\delta\xi-\beta^*k,\rho)(\gamma,q)\\(\omega,\rho)\left(1+\omega+\frac{\eta}{1-\alpha}-\delta\xi-\beta^*k,\rho\right)\end{matrix}\right]$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\delta'}}{[a(1-\alpha)]^{\delta'}}\,\middle|\,\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right]$$

Where $\eta,\gamma,q,\omega,\rho \in C,\ c,b \in R, Re(\beta) > 0, Re(\delta) > 0, Re(\eta) > 0, Re(\gamma) > 0, Re(\omega) > 0, Re\left(1+\frac{\eta}{1-\alpha}\right) > 0, Re(\rho) > 0, \alpha < 1, b \in R, c \in R, Re\left(\omega+\delta\frac{f_j}{F_j}\right) > 0, |\arg c| < \frac{1}{2}T_1\pi, T_1T_2 > 0, \rho \leq \sigma, |d| < \alpha'^{\alpha'}, \beta^* > 0, j = 1,\ldots,Q;$

3. If we take $\mu = v, \rho = \sigma, \delta = p = q = 1$ and $\vartheta \to \beta, \beta \to \omega$ in H function $\delta' = \delta$ in theorem-2 then we at once arrive at the known result of [1, Theorem-1] .
4. If we take $\mu = v, \rho = \sigma$ then we at once arrive at the theorem-1.
5. Making $\beta^*, \delta' \to 0$ and $\delta = p = q = 1, \rho = \beta$ in the result (2.1) and $\beta^*, \delta' \to 0, \mu = v, \rho = \sigma, \delta = p = q = 1$ in result (2.2) then we at once arrive at the known result of Nair in[8].

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