Quasi-Hadamard Product of Certain Starlike and Convex Functions

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1. Introduction

Throughout this paper, let $S$ denote the functions of the form:

$$F(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \quad (a_1 > 0, a_k \geq 0),$$

(1.1)

$$F_r(z) = a_1 r z - \sum_{k=2}^{\infty} a_k r z^k \quad (r \in \mathbb{N}, a_1 r > 0, a_k r \geq 0),$$

(1.2)

$$g(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \quad (b_1 > 0, b_k \geq 0)$$

(1.3)

and

$$g_j(z) = b_1 j z - \sum_{k=2}^{\infty} b_k j z^k \quad (j \in \mathbb{N}, b_1 j > 0, b_k j \geq 0)$$

(1.4)

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

Let $S^*$ be the subclass of functions $S$ consisting of starlike functions in $U$. It is well known that $F \in S^*$ if and only if

$$\Re \left\{ \frac{z F'(z)}{F(z)} \right\} > 0, \quad (z \in U),$$

(1.5)
and \( C^* \) be the subclass of functions \( S \) consisting of convex functions in \( U \). It is well known that \( \mathcal{F} \in C^* \) if and only if

\[
\text{Re} \left\{ 1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} \right\} > 0, \quad (z \in U). \tag{1.6}
\]

Let \( S_x^* \) be the subclass of \( S \) consisting of functions of the form (1.1) satisfying

\[
\text{Re} \left\{ \frac{z\mathcal{F}'(z)}{\mathcal{F}(z) - \mathcal{F}(-z)} \right\} > 0, \quad (z \in U). \tag{1.7}
\]

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [10] (see also Robertson [9], Stankiewics [12] Wu [14] and Owa et al. [5]).

In [13], Sudharsan et al. introduced the class \( S_x^s(\alpha, \beta) \) consisting of functions of the form (1.1) and satisfying the following condition (see also [11])

\[
\left| \frac{z\mathcal{F}'(z)}{\mathcal{F}(z) - \mathcal{F}(-z)} - 1 \right| < \beta \left| \frac{z\mathcal{F}'(z)}{\mathcal{F}(z) - \mathcal{F}(-z)} + 1 \right| \tag{1.8}
\]

for some \( 0 \leq \alpha \leq 1, \quad 0 < \beta \leq 1 \) and \( z \in U \).

Let \( S_x^s(\alpha, \beta) \) denote the class of function \( \mathcal{F}(z) \) of the form (1.1) for which \( z\mathcal{F}' \in S_x^s(\alpha, \beta) \).

By using the same technique of Sudharsan et al. [13] and Aouf et al. [1], we get the following theorem.

**Theorem 1.** Let the function \( \mathcal{F}(z) \) defined by (1.1). Then

(i) \( \mathcal{F}(z) \in S_x^s(\alpha, \beta) \) if and only if

\[
\sum_{k=2}^\infty \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) a_{k,r} \right] \leq [\beta (2 + \alpha) - 1] a_{1,r} \tag{1.9}
\]

where \( 0 \leq \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 0 \leq \frac{2(1-\beta)}{1+\alpha \beta} < 1 \) and \( z \in U \).

(ii) \( \mathcal{F}(z) \in S_x^c(\alpha, \beta) \) if and only if

\[
\sum_{k=2}^\infty k \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) a_{k,r} \right] \leq [\beta (2 + \alpha) - 1] a_{1,r} \tag{1.10}
\]

where \( 0 \leq \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 0 \leq \frac{2(1-\beta)}{1+\alpha \beta} < 1 \) and \( z \in U \).

(iii) \( \mathcal{F}(z) \in S_x^{sh}(\alpha, \beta) \) if and only if

\[
\sum_{k=2}^\infty k^h \left\{ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \right\} a_{k,r} \leq [\beta (2 + \alpha) - 1] a_{1,r}, \tag{1.11}
\]

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where \(0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \frac{-1}{1 + \alpha \beta} < 1\) and \(z \in U\). Where \(h\) is an nonnegative real number.

We note that for every nonnegative real number \(h\), the class \(S_{k,h}^*(\alpha, \beta)\) is nonempty as the functions of the form

\[
F(z) = a_1 z - \sum_{k=2}^{\infty} \frac{\beta (2 + \alpha - 1)}{k^h \left[ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right]} a_1 \lambda_k z^k,
\]

where \(a_1 > 0, \lambda_k \geq 0\), and \(\sum_{k=2}^{\infty} \lambda_k \leq 0\), satisfy the inequality (1.12). It is evident that \(S_1^*(\alpha, \beta) \equiv S_c^*(\alpha, \beta)\) and, for \(c = 0\), \(S_c^*(\alpha, \beta)\) is identical to \(S_0^*(\alpha, \beta)\). Further, \(S_c^*(\alpha, \beta) \subset S_{k,\alpha, \beta}^*(\alpha, \beta)\) if \(c > k\), the containment being proper. Hence, for any positive integer \(c\), the inclusion relation

\[
S_c^*(\alpha, \beta) \subset S_{c-1}^*(\alpha, \beta) \subset \ldots \subset S_2^*(\alpha, \beta) \subset S_c^*(\alpha, \beta) \subset S_n^*(\alpha, \beta).
\]

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa [6, 7, 8], Kumar [2, 3, 4] and others. Accordingly, the quasi-Hadamard product of two functions \(F(z)\) and \(g(z)\) is given by

\[
F \ast g(z) = a_1 b_1 z - \sum_{k=2}^{\infty} a_k b_k z^k.
\]

II. THE MAIN THEOREMS

Theorem 2. A functions \(F_r(z)\) defined by (1.2) in the class \(S_{k,\alpha, \beta}^*(\alpha, \beta)\) for each \(r = 1, 2, ..., u\). Then we get the quasi-Hadamard product \(F_1 \ast F_2 \ast \ldots \ast F_u(z) \in S_{2(u-1)+1}^*(\alpha, \beta)\).

Proof. To prove the theorem, we need to show that

\[
\sum_{k=2}^{\infty} k^{2(u-1)+1} \left\{ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right\} \prod_{l=1}^{m} a_{k,l} \leq \left[ \beta (2 + \alpha - 1) \right] a_{1,r}.
\]

Since \(F_r(z) \in S_{\alpha, \beta}^*(\alpha, \beta)\), we have

\[
\sum_{k=2}^{\infty} k \left\{ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right\} a_{k,r} \leq \left[ \beta (2 + \alpha - 1) \right] a_{1,r},
\]

for each \(r = 1, 2, ..., u\). Therefore,

\[
k \left\{ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right\} a_{k,r} \leq \left[ \beta (2 + \alpha - 1) \right] a_{1,r}
\]

or

\[
a_{k,r} \leq \frac{\left[ \beta (2 + \alpha - 1) \right]}{k \left( (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right)} a_{1,r}
\]

where \(0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \frac{-1}{1 + \alpha \beta} < 1\) and \(z \in U\). Where \(h\) is an nonnegative real number.
for each \( r = 1, 2, \ldots, u \). The right-hand expression of this last inequality is not greater than \( k^{-2} a_{1,r} \). Hence

\[
a_{k,r} \leq k^{-2} a_{1,r}. \tag{2.3}
\]

for each \( r = 1, 2, \ldots, u \).

By (2.3) for each \( r = 1, 2, \ldots, u - 1 \), and (2.2) for \( r = u \), we get

\[
\sum_{k=2}^{\infty} k^{2(u-1)+1} \left\{ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right\} \Pi_{r=1}^{u} a_{k,r} \leq \left\{ k^{2(u-1)} \left[ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right] k^{-2(u-1)} \Pi_{r=1}^{u-1} a_{1,r} \right\} a_{k,u}
\]

\[
= \left[ \Pi_{r=1}^{u-1} a_{1,r} \right] \sum_{k=2}^{\infty} \left\{ k \left[ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] a_{k,u} \right] \right\}
\]

\[
\leq \left[ \beta (2 + \alpha) - 1 \right] \left[ \Pi_{r=1}^{u-1} a_{1,r} \right].
\]

Hence \( \mathcal{F}_1 \ast \mathcal{F}_2 \ast \ldots \ast \mathcal{F}_u(z) \in S_{2(u-1)+1}^* (\alpha, \beta) \). This completes the proof of Theorem 2.

\[ \square \]

**Theorem 3.** A functions \( \mathcal{F}_r(z) \) defined by (1.2) in the class \( S_{u}^* (\alpha, \beta) \) for each \( r = 1, 2, \ldots, u \). Then we get the quasi-Hadamard product \( \mathcal{F}_1 \ast \mathcal{F}_2 \ast \ldots \ast \mathcal{F}_u(z) \in S_{(u-1)}^* (\alpha, \beta) \).

**Proof.** Using \( \mathcal{F}_r(z) \in S_{u}^* (\alpha, \beta) \), we have

\[
\sum_{k=2}^{\infty} \left[ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] a_{k,r} \right] \leq \left[ \beta (2 + \alpha) - 1 \right] a_{1,r} \tag{2.4}
\]

for each \( r = 1, 2, \ldots, u \). Therefore,

\[
a_{k,r} \leq \left\{ \frac{\left[ \beta (2 + \alpha) - 1 \right]}{(1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right]} \right\} a_{1,r}
\]

and hence

\[
a_{k,r} \leq k^{-1} a_{1,r} \tag{2.5}
\]

for every \( r = 1, 2, \ldots, u \).

By (2.5) for \( r = 1, 2, \ldots, u - 1 \), and (2.4) for \( r = u \), we get

\[
\sum_{k=2}^{\infty} \left\{ k^{(u-1)} \left[ (1 + \alpha \beta) k + (\beta - 1) \left[ 1 - (-1)^k \right] \right] \Pi_{r=1}^{u} a_{k,r} \right\}
\]
Since $F_X$ we prove the theorem, we need to show that $g \leq \beta (2 + \alpha) - 1 \lfloor a_{1,r} \rfloor$

Hence $F_1 F_2 F_u(z) \in S_{(u-1)}^*(\alpha, \beta)$. This completes the proof of Theorem 3.

Theorem 4. A functions $g_r(z)$ defined by (1.2) in the class $S_c^*(\alpha, \beta)$ for each $r = 1, 2, ..., u$ and the functions $\tilde{g}_j(z) \in S_c^*(\alpha, \beta)$ for every $j = 1, 2, ..., q$. Then we get the Hadamard product $F_1 F_2 \cdot \cdot \cdot F_u  \bar{g}_1 \bar{g}_2 \cdot \cdot \cdot \bar{g}_q(z) \in S_{2u+q-1}^*(\alpha, \beta).

Proof. We denote the quasi-Hadamard product $F_1 F_2 \cdot \cdot \cdot F_u  \bar{g}_1 \bar{g}_2 \cdot \cdot \cdot \bar{g}_q(z)$ by the function $h(z)$, for the sake of the convenience. Clearly,

$$h(z) = \sum_{k=2}^{\infty} \left\{ (k)^{2u+q-1} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \left[ k^{-\beta - 1} a_{1,r} \right] \right) \right\}_k$$

To prove the theorem, we need to show that

$$\sum_{k=2}^{\infty} \left\{ (k)^{2u+q-1} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \left[ k^{-\beta - 1} a_{k,r} \right] \right) \right\}_k$$

$$\leq \beta (2 + \alpha) - 1 \lfloor a_{1,r} \rfloor .$$

(2.6)

Since $F_r(z) \in S_c^*(\alpha, \beta)$, the inequalities (2.2) and (2.3) hold for every $r = 1, 2, ..., u$. Further, since $\tilde{g}_j(z) \in S_c^*(\alpha, \beta), we have

$$\sum_{k=2}^{\infty} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \right] b_{k,r} \leq \beta (2 + \alpha) - 1 \lfloor b_{1,r} \rfloor .$$

(2.7)

for each $j = 1, 2, ..., q$. Whence we obtain

$$b_{k,j} \leq k^{-1} b_{1,j}$$

(2.8)

for each $j = 1, 2, ..., q$.

By (2.3) for $r = 1, 2, ..., u$, (2.8) for $j = 1, 2, ..., q-1$, and (2.7) for $j = q$, we get

$$\sum_{k=2}^{\infty} \left\{ (k)^{2u+q-1} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \right] \left[ k^{-\beta - 1} a_{k,r} \right] \right\}_k$$

for each $j = 1, 2, ..., q$.
\[
\leq \sum_{k=2}^{\infty} \left\{ k^{2u+q-1} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \right] \right\} \\
\times k^{-u(2)} \left( \prod_{r=1}^{u} a_{1,r} \Pi_{j=1}^{q-1} b_{k,j} \right)
\]
\[
\leq \sum_{k=2}^{\infty} \left\{ k^{-2u-(q-1)} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \right] \right\} \\
\times \left[ k^{-2u} k^{-(q-1)} \prod_{r=1}^{u} a_{1,r} \Pi_{j=1}^{q-1} b_{1,j} \right] b_{k,q}
\]
\[
= \left[ \prod_{r=1}^{u} a_{1,r} \Pi_{j=1}^{q-1} b_{1,j} \right] \sum_{k=2}^{\infty} \left[ (1 + \alpha \beta) k + (\beta - 1) \left( 1 - (-1)^k \right) \right] b_{k,q}
\]
\[
\leq \left[ \beta (2 + \alpha) - 1 \right] \left[ \prod_{r=1}^{u} a_{1,r} \Pi_{j=1}^{q-1} b_{1,j} \right].
\]

Hence \( h(z) \in S_{2u+q-1}^* (\alpha, \beta) \). This completes the proof of Theorem 4. \( \square \)

REFERENCES Références Referencias


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