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Quasi-Hadamard Product of Certain Starlike and Convex Functions

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QUASIHADAMARDPRODUCTOFCERTAINSTARLIKEANDCONVEXFUNCTIONS

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Quasi-Hadamard Product of Certain Starlike and Convex Functions

H. E. Darwish [α], A. Y. Lashin [σ] & A. N. Alnayyef [ρ]

Abstract- In this paper, we establish certain results concerning the quasi-Hadmard product for two classes related to starlike and convex univalent functions with respect to symmetric points.

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I. Introduction

Throughout this paper, let S denote of the functions of the form :

$$\mathcal{F}(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_1 > 0, a_k \geq 0), \tag{1.1}$$

$$\mathcal{F}_r(z) = a_{1,r} z - \sum_{k=2}^{\infty} a_{k,r} z^k \qquad (r \in \mathbb{N}, a_{1,r} > 0, a_{k,r} \geq 0), \tag{1.2}$$

$$\breve{g}(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \qquad (b_1 > 0, b_k \geq 0) \tag{1.3}$$

and

$$\breve{g}_j(z) = b_{1,j} z - \sum_{k=2}^{\infty} b_{k,j} z^k \qquad (j \in \mathbb{N}, b_{1,j} > 0, b_{k,j} \geq 0) \tag{1.4}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

Let $S^\star$ be the subclass of functions S consisting of starlike functions in U. It is well known that $\mathcal{F} \in S^*$ *if and only if*

$$\operatorname{Re}\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\} > 0, \quad (z \in U), \tag{1.5}$$

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and $C^{\star}$ be the subclass of functions S consisting of convex functions in U. It is well known that $\mathcal{F} \in C^{\star}$ *if and only if*

$$\mathrm{Re}\left\{1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right\}>0, \quad (z\in U). \tag{1.6}$$

Let $S_s^{\star}$ be the subclass of S consisting of functions of the form (1.1) satisfying

$$\mathrm{Re}\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)-\mathcal{F}(-z)}\right\}>0, \quad (z\in U). \tag{1.7}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [10] (see also Robertson [9] , Stankiewics [12] Wu [14] and Owa et al. [5]).

In [13], Sudharsan et al. introduced the class $S_s^{\star}(\alpha,\beta)$ consisting of functions of the form (1.1) and satisfying the following condition (see also [11])

$$\left|\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)-\mathcal{F}(-z)}-1\right|<\beta\left|\alpha\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)-\mathcal{F}(-z)}+1\right| \tag{1.8}$$

for some $0\leq\alpha\leq 1$, $0<\beta\leq 1$ and $z\in U$.

Let $S_c^{\star}(\alpha,\beta)$ denote the class of function $\mathcal{F}(z)$ of the from (1.1) for which $z\mathcal{F}'\in S_s^{\star}(\alpha,\beta)$.

By using the same technique of Sudharsan et al. [13] and Aouf et al. [1], we get the following theorem.

Theorem 1. Let the function $\mathcal{F}(z)$ *defined by* (1.1). *Then*

(i) $\mathcal{F}(z)\in S_s^{\star}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\left[(1+\alpha\beta)k+(\beta-1)\left[1-(-1)^{k}\right]a_{k,r}\right]\leq[\beta(2+\alpha)-1]\,a_{1,r} \tag{1.9}$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\frac{2(1-\beta)}{1+\alpha\beta}<1$ and $z\in U$.

(ii) $\mathcal{F}(z)\in S_c^{\star}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}k\left[(1+\alpha\beta)k+(\beta-1)\left[1-(-1)^{k}\right]a_{k,r}\right]\leq[\beta(2+\alpha)-1]\,a_{1,r} \tag{1.10}$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\frac{2(1-\beta)}{1+\alpha\beta}<1$ and $z\in U$.

(iii) $\mathcal{F}(z)\in S_{s,h}^{\star}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}k^{h}\left\{(1+\alpha\beta)k+(\beta-1)\left[1-(-1)^{k}\right]\right\}a_{k,r}\leq[\beta(2+\alpha)-1]\,a_{1,r}, \tag{1.11}$$

Ref

5. S. Owa, Z. Wu and F. Ren, A note on certain subclass of Sakaguchin functions, Bull. Soc. Roy. Liege 57 (1988), 143-150.

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \frac{-}{1+\alpha\beta} < 1$ and $z \in U$. Where h is an nonnegative real number.

We note that for every nonnegative real number h, the class $S^{\star}_{s,h}(\alpha, \beta)$ is nonempty as the functions of the from

$$\mathcal{F}(z) = a_1 z - \sum_{k=2}^{\infty} \frac{\beta(2+\alpha)-1}{k^h\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right]} a_1 \lambda_k z^k, \tag{1.12}$$

where $a_1 > 0$, $\lambda_k \geq 0$,and $\sum_{k=2}^{\infty} \lambda_k \leq 0$, satisfy the inequality (1.12). It is evident that $S^{\star}_1(\alpha, \beta) \equiv S^{\star}_c(\alpha, \beta)$ and, for $c = 0$, $S^{\star}_c(\alpha, \beta)$ is identical to $S^{\star}_0(\alpha, \beta)$. Further, $S^{\star}_c(\alpha, \beta) \subset S^{\star}_k(\alpha, \beta)$ if $c > k$, the containment being proper. Hence, for any positive integer$c$, the inclusion relation

$$S^{\star}_c(\alpha, \beta) \subset S^{\star}_{c-1}(\alpha, \beta) \subset ... \subset S^{\star}_2(\alpha, \beta) \subset S^{\star}_c(\alpha, \beta) \subset S^{\star}_s(\alpha, \beta).$$

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa [6, 7, 8], Kumar [2, 3, 4] and others. Accordingly, the quasi-Hadamard product of two functions $\mathcal{F}(z)$ and $\breve{g}(z)$ is given by

$$\mathcal{F} * \breve{g}(z) = a_1 b_1 z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

II. The Main Theorems

Theorem 2. A functions $\mathcal{F}_i(z)$ defined by (1.2) in the class $S^{\star}_c(\alpha, \beta)$ for each $r = 1, 2, ..., u$. Then we get the quasi-Hadamard product $\mathcal{F}_1 *$

$$\mathcal{F}_2 * ... * \mathcal{F}u(z) \in S^{\star}_{2(u-1)+1}(\alpha, \beta).$$

Proof. To prove the theorem, we need to show that

$$\sum_{k=2}^{\infty} k^{2(u-1)+1}\left\{(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right\} \Pi_{r=1}^{m} a_{k,r} \leq \left[\beta(2+\alpha)-1\right] a_{1,r}. \tag{2.1}$$

Since $\mathcal{F}_r(z) \in S^{\star}_c(\alpha, \beta)$,we have

$$\sum_{k=2}^{\infty} k\left\{(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right\} a_{k,r} \leq \left[\beta(2+\alpha)-1\right] a_{1,r}, \tag{2.2}$$

for each $r = 1, 2, ..., u$.Therefore,

$$k\left\{(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right] a_{k,r}\right\} \leq \left[\beta(2+\alpha)-1\right] a_{1,r}$$

or

$$a_{k,r} \leq \left\{\frac{\left[\beta(2+\alpha)-1\right]}{k\left((1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right)}\right\} a_{1,r}$$

Ref

2. V. Kumar, Hadamard product of certain starlike functions, J. Math. Anal. Appl. 110 (1985), 425-428.

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for each $r = 1, 2, ..., u$. The right-hand expression of this last inequality is not greater then $k^{-2} a_{1,r}$. Hence

$$a_{k,r} \leq k^{-2} a_{1,r}. \tag{2.3}$$

for each $r = 1, 2, ..., u$.

By (2.3) for each $r = 1, 2, ..., u-1$, and (2.2) for $r = u$, we get

$$\sum_{k=2}^{\infty} k^{2(u-1)+1} \left\{ (1+\alpha\beta) k + (\beta - 1) \left[1 - (-1)^k \right] \right\} \Pi_{r=1}^{u} a_{k,r}$$

$$\leq \left\{ k^{2(u-1)+1} \left[(1+\alpha\beta) k + (\beta - 1) \left[1 - (-1)^k \right] k^{-2(u-1)} \Pi_{r=1}^{u-1} a_{1,r} \right] a_{k,u} \right\}$$

$$= \quad \left[\Pi_{r=1}^{u-1} a_{1,r} \right] \sum_{k=2}^{\infty} \left\{ k \left[(1+\alpha\beta) k + (\beta - 1) \left[1 - (-1)^k \right] a_{k,u} \right] \right\}$$

$$\leq \quad \left[\beta (2+\alpha) - 1 \right] \left[\Pi_{r=1}^{u} a_{1,r} \right] \; .$$

Hence $\mathcal{F}_1 * \mathcal{F}_2 * ... * \mathcal{F}_u(z) \in S^{\star}_{2(u-1)+1}(\alpha, \beta)$.This completes the proof of Theorem 2. □

Theorem 3. A functions $\mathcal{F}_r(z)$ *defined by* (1.2) *in the class* $S^{\star}_s(\alpha, \beta)$ *for each* $r = 1, 2, ..., u$. *Then we get the quasi-Hadamard product* $\mathcal{F}_1 *$

$$\mathcal{F}_2 * ... * \mathcal{F}_u(z) \in S^{\star}_{(u-1)}(\alpha, \beta).$$

Proof. Using $\mathcal{F}_r(z) \in S^{\star}_s(\alpha, \beta)$,we have

$$\sum_{k=2}^{\infty} \left[(1+\alpha\beta) k + (\beta - 1) \left[1 - (-1)^k \right] a_{k,r} \right] \leq \left[\beta (2+\alpha) - 1 \right] a_{1,r} \tag{2.4}$$

for each $r = 1, 2, ..., u$. Therefore,

$$a_{k,r} \leq \left\{ \frac{\left[\beta (2+\alpha) - 1 \right]}{(1+\alpha\beta) k + (\beta - 1) \left[1 - (-1)^k \right]} \right\} a_{1,r}$$

and hence

$$a_{k,r} \leq k^{-1} a_{1,r} \tag{2.5}$$

for every $r = 1, 2, ..., u$.

By (2.5) for $r = 1, 2, ..., u-1$,and (2.4) for $r = u$, we get

$$\sum_{k=2}^{\infty} \left\{ k^{(u-1)} \left[(1+\alpha\beta) k + (\beta - 1) \left[1 - (-1)^k \right] \Pi_{r=1}^{u} a_{k,r} \right] \right\}$$

Notes

$$\leq \sum_{k=2}^{\infty}\left\{k^{(u-1)}\left[(1+\alpha\beta)\,k+(\beta-1)\left[1-(-1)^{k}\right]\left[k^{-(u-1)}\Pi_{r=1}^{u-1}a_{1,i}\right]\right]a_{k,u}\right\}$$

$$= \left[\Pi_{r=1}^{u-1}a_{1,r}\right]\sum_{k=2}^{\infty}\left\{\left[(1+\alpha\beta)\,k+(\beta-1)\left[1-(-1)^{k}\right]\right]a_{k,u}\right\}$$

$$\leq \left[\beta\,(2+\alpha)-1\right]\left[\Pi_{r=1}^{u}a_{1,r}\right]$$

Notes

Hence $\mathcal{F}_1 * \mathcal{F}_2 * ... * \mathcal{F}_u(z) \in S^{\star}_{(u-1)}(\alpha,\beta)$. This completes the proof of Theorem 3. □

Theorem 4. A functions $\mathcal{F}r\,(z)$ defined by (1.2) *in the class $S^{\star}_{c}(\alpha,\beta)$ for each $r = 1, 2, ..., u$; and the functions $\breve{g}_j(z)$in the class $S^{\star}_{s}(\alpha,\beta)$ for every $j = 1, 2, ..., q$. Then we get the Hadamard product $\mathcal{F}_1 * \mathcal{F}_2 * ... * \mathcal{F}_u * \breve{g}_1 * \breve{g}_2 * ... * \breve{g}_q\,(z) \in S^{\star}_{2u+q-1}(\alpha,\beta)$.*

Proof. We denote the quasi-Hadamard product $\mathcal{F}_1 * \mathcal{F}_2 * ... * \mathcal{F}_u * \breve{g}_1 * \breve{g}_2 * ... * \breve{g}_q\,(z)$ by the function $h\,(z)$, for the sake of the convenience.

Clearly,

$$h(z) = \left[\Pi_{r=1}^{u}a_{1,i}.\Pi_{j=1}^{q}b_{1,j}\right]z - \sum_{k=2}^{\infty}\left[\Pi_{r=1}^{u}a_{k,r}.\Pi_{j=1}^{q}b_{k,j}\right]z^{k}.$$

To prove the theorem, we need to show that

$$\sum_{k=2}^{\infty}\left\{(k)^{2u+q-1}\left[(1+\alpha\beta)\,k+(\beta-1)\left(1-(-1)^{k}\right)\right]\left[\Pi_{r=1}^{u}a_{k,r}.\Pi_{j=1}^{q}b_{k,j}\right]\right\}$$
$$\leq \left[\beta\,(2+\alpha)-1\right]\left(\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q}b_{1,j}\right). \tag{2.6}$$

Since $\mathcal{F}_r\,(z) \in S^{\star}_{c}(\alpha,\beta)$,the inequalities (2.2) and (2.3) hold for every $r = 1, 2, ..., u$. Further, since $\breve{g}_j\,(z) \in S^{\star}_{s}(\alpha,\beta)$,we have

$$\sum_{k=2}^{\infty}\left[(1+\alpha\beta)\,k+(\beta-1)\left[1-(-1)^{k}\right]b_{k,r}\right] \leq \left[\beta\,(2+\alpha)-1\right]b_{1,r} \tag{2.7}$$

for each $j = 1, 2, ..., q$. Whence we obtain

$$b_{k,j} \leq k^{-1}b_{1,j} \tag{2.8}$$

for each $j = 1, 2, ..., q$.

By (2.3) for $r = 1, 2, ..., u$, (2.8) for $j = 1, 2, ..., q-1$, and (2.7) for $j = q$, we get

$$\sum_{k=2}^{\infty}\left\{k^{2u+q-1}\left[(1+\alpha\beta)\,k+(\beta-1)\right]\left[1-(-1)^{k}\right]\left[\Pi_{r=1}^{u}a_{k,r}.\Pi_{j=1}^{q}b_{k,j}\right]\right\}$$

$$\leq \sum_{k=2}^{\infty}\left\{k^{2u+q-1}\left[(1+\alpha\beta)k+(\beta-1)\left(1-(-1)^{k}\right)\right]\right\}$$

$$\times k^{-u(2)}\left(\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q}b_{k,j}\right)$$

$$\leq \sum_{k=2}^{\infty}\left\{k^{-2u-(q-1)}\left[(1+\alpha\beta)k+(\beta-1)\left(1-(-1)^{k}\right)\right]\right\}$$

$$\times\left[k^{-2u}.k^{-(q-1)}\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q-1}b_{1,j}\right]b_{k,q}$$

$$=\left[\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q-1}b_{1,j}\right]\sum_{k=2}^{\infty}\left[(1+\alpha\beta)k+(\beta-1)\left(1-(-1)^{k}\right)\right]b_{k,q}$$

$$\leq [\beta(2+\alpha)-1]\left[\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q-1}b_{1,j}\right].$$

Hence $h(z)\in S^{\star}_{2u+q-1}(\alpha,\beta)$. This completes the proof of Theorem 4. □

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