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Structure of Regular Semigroups

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Abstract- This paper concerned with basic concepts and some results on (idempotent) semigroup satisfying the identities of three variables. The motivation of taking three for the number of variables has come from the fact that many important identities on idempotent semigroups are written by three or fewer independent variables. We consider the semigroup satisfying the property abc = ac and prove that it is left semi-normal and right quasi-normal. Again an idempotent semigroup with an identity aba = ab and aba = ba (ab = a, ab = b) is always a semilattices and normal. An idempotent semigroup is normal if and only if it is both left quasi-normal and right quasi-normal. If a semigroup is rectangular then it is left and right semi-regular.

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STRUCTUREOFREGULARSEMIGROUPS

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Structure of Regular Semigroups

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Abstract- This paper concerned with basic concepts and some results on (idempotent) semigroup satisfying the identities of three variables. The motivation of taking three for the number of variables has come from the fact that many important identities on idempotent semigroups are written by three or fewer independent variables. We consider the semigroup satisfying the property abc = ac and prove that it is left semi-normal and right quasi-normal. Again an idempotent semigroup with an identity aba = ab and aba = ba (ab = a, ab = b) is always a semilattices and normal. An idempotent semigroup is normal if and only if it is both left quasi-normal and right quasi-normal. If a semigroup is rectangular then it is left and right semi-regular.

I. Preliminaries and Basic Properties of Regular Semigroups

In this section we present some basic concepts of semigroups and other definitions needed for the study of this chapter and the subsequent chapters.

- *Definition:* A semigroup (S, .) is said to be left(right) singular if it satisfies the identity ab = a (ab = b) for all a,b in S
- *Definition:* A semigroup (S, .) is rectangular if it satisfies the identity aba = a for all a, b in S.
- *Definition:* A semigroup (S, .) is called left(right) regular if it satisfies the identity aba = ab (aba = ba) for all a,b in S.
- *Definition:* A semigroup (S, .) is called regular if it satisfies the identity abca = abaca for all a,b,c in S
- *Definition:* A semigroup (S, .) is said to be total if every element of S can be written as the product of two elements of S. i.e, $S^2 = S$.
- *Definition:* A semigroup (S, .) is said to be left(right) normal if abc = acb (abc = bac) for all a,b,c in S.
- *Definition:* A semigroup (S, .) is said to be normal if satisfies the identity abca = acba for all a,b,c in S.
- *Definition:* A semigroup (S, .) is said to be left(right) quasi-normal if it satisfies the identity abc = acbc (abc = abac) for all a,b,c in S.
- *Definition:* A semigroup (S, .) is said to be left (right) semi-normal if it satisfies the identity abca = acbca (abca =abcba) for all a,b,c in S.
- *Definition:* A semigroup (S, .) is said to be left(right) semi-regular if it satisfies the identity abca =abacabca (abca = abcabaca) if for all a,b,c in S.
- *Result:* [11,12] A left (right) singular semigroup is rectangular.
- *Theorem:* Every left(right) singular semigroup is total.

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Proof: Let (S, .) be a left(right) singular semigroup.

Then $ab = a$ for any a,b in S

To prove that S is total we have to prove $S^2 = S$

We know that $S \subseteq S^2$

To prove $S^2 \subseteq S$

Let $x \in S^2 \Rightarrow x = a.b$ for any a,b in S

$x = a$

$x \in S$

$S^2 \subseteq S$

$\therefore S = S^2$

Hence (S, .) is total.

- *Note:* From the result 2.1.11 every left(right) singular semigroup is rectangular and again from theorem 2.1.12 it is total. Hence every rectangular semigroup is total.
- *Theorem:* A semigroup satisfying the singular properties is always a semilattice.
- *Theorem:* Let (S, .) be a semigroup. If S is left and right regular then S is a semilattice.

Proof: Let (S, .) be a semigroup with left regular and right regular then

$aba = ab$ (left regular)

$aba = ba$ (right regular) for all a,b in S.

From the above we have $ab = ba$

$\therefore$ (S, .) is a commutative.

To prove that (S, .) is band, let $aba = ab \Rightarrow a(ba) = ab \Rightarrow abab = ab$ ($ba = bab$)

$\Rightarrow (ab)(ab) = ab \Rightarrow (ab)^2 = ab$ put $a = b \Rightarrow (a.a)^2 = a.a \Rightarrow (a.a)^2 = a^2$ $a.a = a$

$\therefore$ (S, .) is a band

So, if S is left and right regular then (S, .) is commutative band

Hence (S, .) is a semilattice.

- *Lemma:* A left (right) regular semigroup is regular.
- *Note:* Every regular semigroup need not be a left (right) regular.
- *Lemma:* A right (left) singular semigroup is regular.
- *Lemma:* A left (right) normal semigroup is normal.
- *Theorem:* Let (S, .) be a semigroup. If S is both left and right regular then S is normal.
- *Lemma:* A left (right) regular semigroup is right (left) quasi-normal.
- *Lemma:* A left (right) regular semigroup is right (left) semi-normal.
- *Theorem:* A left (right) regular semigroup is left (right) semi-regular
- *Theorem:* If an idempotent semigroup satisfies the right (left) quasi-normal then it is right(left) semi-regular.
- *Lemma:* A semigroup (S, .) with left (right) quasi-normal is right (left) semi-normal.
- *Theorem:* If a semigroup (S, .) is rectangular then (S, .)is right (left) semi-regular.
- *Note:* Similarlly, we prove that,

Notes

1. A semigroup(S, .) is regular then (S, .) is left (right) semi-regular.
2. A semigroup (S, .) is satisfies the left (right) semi-normal property is right (left) semi-regular.

- *Theorem:* An idempotent semigroup (S, .) is normal if and only if it is both left and right quasi-normal.

Proof: Let (S, .) be an idempotent semigroup.

Notes

Assume that (S, .) is both left and right quasi-normal

then abc = acbc (leftquasi-normal)

abc = abac (right quasi-normal)

To prove that (S, .) is normal. Since S is right quasi-normal, we have

abc = abac ⇒abc= a(bac) ⇒ abc=abcac (left quasi-normal)

⇒ abc=(abc)ac ⇒ abc=acbcac (left quasi normal)

⇒ abc=ac(bcac) ⇒ abc = acbac (left quasi normal)

⇒ abca = acbaca⇒abc= ac(baca) ⇒abc = acbca ⇒abc = a(cbca) ⇒ abc = acba

⇒ abca = acba.

Hence (S, .) is normal.

Conversely; let (S, .) be normal then abca = acba

we show that (S, .) is both left quasi-normal and right quasi-normal.

Consider abc = abc.c ⇒abc = a(bc)c⇒abc= acbc ⇒ abc = acbc.

(S, .) is left quasi-normal.

Symilarlly, abc = a.abc ⇒abc = a(ab)c⇒abc = abac ⇒ abc = abac.

Therefore (S, .) is right quasi-normal.

II. Semigroup Satisfying the Identity abc = ac.

In this section we discuss if a semigroup S satisfying the identity abc = ac for any three variables a,b,c∈S, then the following conditions are equivalent to one another.

a) left semi-normal
b) left semi-regular
c) right semi-normal
d) right semi-regular
e) regular
f) normal
g) left quasi-normal
h) right quasi-normal

- *Theorem:* A semigroup S with an identity abc = ac, for any a,b,c∈S is left (right) semi-normal if and only if it is left (right) semi-regular.

Proof: Let S be a semigroup with an identity abc = ac for any a,b,c∈S.

Assume that S be a left semi-normal.

Now we show that S is left semi-regular

Since S is left semi-normal we have abca = acbca ⇒abca = (ac)bca

⇒abca= abcbca (ac = abc) ⇒abca = a(bc)bca ⇒abca = abacbca (bc = bac)

⇒abca = aba(cb)ca ⇒abca = abacabca (cb = cab) ⇒ abca = abacabca.

∴S is left semi-regular.

Conversely, let S be left semi regular then abca = abacabca

⇒abca = a(bac)abca ⇒abca = abcabca (bac = bc) ⇒abca = (abc)abca

Global Journal of Science Frontier Research (F) Volume XV Issue III Version I Year 2015

⇒abca = acabca (abc = ac) ⇒abca = a(cab)ca ⇒abca = acbca (cab = cb) abca = acbca.
Hence S is left semi-normal.

- *Theorem:* An idempotent semigroup S with an identity abc = ac for any a,b,c in S is left (right) semi-regular if and only if it is regular.

Proof: Let S be an idempotent semigroup with an identity abc = ac for any a,b,c in S. Assume that S be a regular semigroup then abca = abaca

⇒ abca = ab(ac)a ⇒abca = ababca (ac = abc) ⇒abca = a(ba)bca ⇒abca = abcabca (ba = bca)

⇒abca = a(bc)abca ⇒abca = abacabca (bc = bac) ⇒ abca = abacabca
Hence S is left semi-regular.

Conversely, let S be left semi-regular then abca = abacabca⇒abca = abac(abc)a

⇒abca = abacaca (abc = ac) ⇒abca = abac(aca) ⇒abca=abaca.a (aca = a.a)

⇒abca = abaca (a.a = a) ⇒ abca = abaca
Hence S is regular

- *Theorem:* A semigroup S with an identity abc = ac for any a,b,c in S is left (right) semi-regular if and only if it is normal.

Proof: Let S be a semigroup with an identity abc = ac for any a,b,c∈S and assume that S be normal then abca = acba ⇒abca = (ac)ba ⇒abca = abcba (ac=abc) ⇒abca = abc(ba)

⇒abca = abcbca (ba=bca) ⇒abca = a(bc)bca ⇒abca = abacbca (bc = bac)

⇒abca = aba(cb)ca ⇒ abca = abacabca (cb = cab) ⇒abca = abacabca

S is left semi-regular

Conversely, assume that S is left semi-regular then abca = a(bac)abca

⇒abca = abcabca (bac=bc) ⇒abca =(abc)abca ⇒abca = acabca (abc = ac)

⇒abca = a(cab)ca ⇒ abca = acbca (cab=cb) ⇒abca = ac(bca) ⇒abca = acba (bca = ba)

⇒ abca = acba
Hence S is normal.

- *Theorem:* A semigroup S with an identity abc = ac where a,b,c in S is regular if and only if it is left (right) semi-normal

Proof: Let S be a semigroup with an identity abc = ac for all a,b,c in S.

Assume that S is regular then abca= (ab)aca ⇒abca = acbaca (ab=acb) ⇒abca = ac(bac)a

⇒abca =acbc a (bac=bc) ⇒ abca = acbca
S is left semi-normal

Conversely, let S be left semi-normal. Then abca = (ac)bca ⇒abca = abcbca (ac = abc) ⇒abca = a(bc)bca ⇒abca = abacbca (bc= bac) ⇒abca = abac(bca) ⇒abca = abacba (bca=ba)

⇒abca = aba(cba) ⇒ abca = abaca (cba = ca)
Hence S is regular.

- *Lemma:* A semigroup S satisfying the property abc = ac for any a,b,c ∈ S is left (right) semi-regular if and only if it is right(left) semi-regular.

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- *Lemma:* A semigroup S with an identity abc = ac is left(right) semi-normal if and only if it is right (left) semi-normal
- *Note:* A semigroup S with an identity abc = ac where a,b,c∈S is left(right) quasi-normal if and only if it is right (left) quasi-normal.
- *Lemma:* A semigroup S with the property abc = ac for all a,b,c∈S is normal if and only if it is left(right) semi-normal
- *Theorem:* A semigroup with an identity abc = ac, for any a,b,c∈ S is normal if and only if it is left(right) quasi-normal.

Proof: Let S be a semigroup with an identity abc = ac for any a,b,c ∈S

Let S be a left quasi-normal. Then, abc = acbc ⇒ abca = acbca ⇒abca = ac(bca)

⇒abca = acba (bca = ba) ⇒ abca = acba

S is normal.

Conversely, let S be normal. Then, abca = acba ⇒ abcac = acbac

⇒ a(bca)c = acbac ⇒ abac = acbac (bca = ba) ⇒ a(bac) = acbac

⇒ abc = acbac (bac = bc) ⇒ abc = ac(bac) ⇒ abc = acbc.

∴S is a left quasi-normal.

Similarlly, we can prove that a semigroup S with an identity abc = ac for any a,b,c∈ S is regular if and only if it is left (right) quasi -normal.

- *Theorem:* A semigroup S with an identity abc = ac for all a,b,c ∈S is regular if and only if it is normal.

Proof: Let s be a semigroup with an identity abc = ac for any a,b,c∈S.

Let S be a normal semigroup then abca = acba ⇒abca = (ac)ba

⇒abca = abcba (ac = abc) ⇒abca = abc(ba) ⇒abca = abcbca (ba = bca)

⇒abca = a(bc)bca ⇒abca = abacbca ⇒abca = abac(bca) ⇒abca = abacba (bca = ba)

⇒abca = aba(cba) ⇒abca = abaca (cba = ca) ⇒ abca = abaca.

Hence S is a regular.

Conversely, let S be a regular semigroup then abca = abaca ⇒abca = (ab)aca

⇒abca = acbaca (ab = acb) ⇒abca = ac(bac)a ⇒abca = acbca (bac = bc)

⇒abca = ac(bca) ⇒abca = acba (bca = ba) ⇒ abca = acba.

∴S is normal.

III. Semigroup Satisfies the Identity ab = a (ab = b)

we present some results on semigroup with an identity ab = a (ab = b) for all a,b in a semigroup S. We prove that the necessary and sufficient conditions for a semigroup S to be regular, normal, left (right) normal, left (right) semi-normal, right(left) semi-regular, left (right) regular, left (right) quasi-normal.

- *Theorem:* A semigroup S with an identity ab = a for any a,b∈S is normal if and only if it is regular.

Proof: Let S be a semigroup satisfying the identity ab = a for all a,b∈S.

Assume that S is normal. Then abca = (a)cba ⇒abca = abcba (a = ab)

⇒abca = a(b)cba ⇒abca = abacba (b = ba) ⇒abca = aba(cb)a ⇒abca = abaca (cb = c)

Global Journal of Science Frontier Research (F) Volume XV Issue III Version I Year 2015

$\Rightarrow$ abca = abaca.

Therefore S is regular.

Conversely, let S be a regular semigroup then abca = abaca $\Rightarrow$abca = (ab)aca

$\Rightarrow$abca = aaca (ab = a) $\Rightarrow$abca = (aa)ca $\Rightarrow$abca = aca (aa = a) $\Rightarrow$abca = a(c)a

$\Rightarrow$abca = acba (c = cb) $\Rightarrow$ abca = acba.

$\therefore$S is normal.

- *Theorem:* A semigroup S satisfying the identity ab = a for any a,b in S is left (right) regular if and only if it is regular.
- *Theorem:* A semigroup S with an identity ab = a for all a,b in S is left(right) semi-normal if and only if it is right(left) semi-normal.
- *Theorem:* A semigroup S with an identity ab = a for any a,b in S is left (right) semi-normal if and only if it is regular.

Proof: Let S be a semigroup with an identity ab = a for any a,b$\in$S.

Let S be left semi-normal then abca = acbca $\Rightarrow$abca = ac(bc)a $\Rightarrow$abca = acba (bc = b)

$\Rightarrow$abca = (a)cba $\Rightarrow$abca = abcba (a = ab) $\Rightarrow$abca = a(b)cba $\Rightarrow$abca = abacba (b = ba)

$\Rightarrow$abca = aba(cb) a $\Rightarrow$abca = abaca (cb = c) $\Rightarrow$ abca = abaca

Hence S is regular.

Conversely, let S be regular then abca = abaca $\Rightarrow$abca = (ab)aca $\Rightarrow$abca = aaca (ab = a)

$\Rightarrow$abca = (aa)ca (aa = a) $\Rightarrow$abca = aca $\Rightarrow$abca = a(c)a $\Rightarrow$abca = acba (c = cb)

$\Rightarrow$abca = ac(b)a

$\Rightarrow$abca = acbca (b = bc) $\Rightarrow$ abca = acbca

$\therefore$S is left semi-normal.

- *Lemma:* A semigroup S with an identity ab = a for any a,b in S is left(right) semi-normal if and only if it is normal.
- *Theorem:* Let S be a semigroup and assume that S satisfies the identity ab = a then S is left(right) normal if and only if it is normal
- *Theorem:* A semigroup S satisfying the identity ab = a for any a,b in S is left(right) semi-regular if and only if it is right(left) semi regular
- *Theorem:* A semigroup S with an identity ab = a, for any a,b$\in$S is left(right) semi-regular if and only if it is normal.

Proof: Let S be a semigroup with an identity ab = a for any a,b in S

Assme that S be left semi-regular then abca = abacabca $\Rightarrow$abca = (ab)acabca

$\Rightarrow$abca = aacabca (ab = a) $\Rightarrow$abca = (aa)cabca $\Rightarrow$abca = acabca (aa = a) $\Rightarrow$abca = a(ca)bca

$\Rightarrow$abca = acbca (ca = c) $\Rightarrow$abca = ac(bc)a $\Rightarrow$abca = acba (bc = b) $\Rightarrow$ abca = acba.

$\therefore$S is normal.

Conversely, let S be normal then abca = acba $\Rightarrow$abca = (a)cba $\Rightarrow$abca = abcba (a = ab)

$\Rightarrow$abca = a(b)cba $\Rightarrow$abca = abacba (b = ba) $\Rightarrow$abca = aba(c)ba $\Rightarrow$abca = abacaba (c = ca)

Notes

Global Journal of Science Frontier Research (F) Volume XV Issue III Version I Year 2015

Notes

$\Rightarrow$abca = abaca(b)a $\Rightarrow$abca = abacabca (b = bc) $\Rightarrow$ abca = abacabca.
Therefore S is left semi-regular.

- *Theorem:* A semigroup S with an identity ab = a for all a,b in S is left(right) semi-regular if and only if it is left(right) semi-normal

Proof: Let S be a semigroup with the identity ab = a for any a,b$\in$S.
Assume that S be a left semi-regular semigroup then, we have abca = abacabca

$\Rightarrow$abca = (ab)acabca $\Rightarrow$abca = aacabca (a = ab) $\Rightarrow$abca = (aa)cabca $\Rightarrow$abca = acabca (aa = a)

$\Rightarrow$abca = a(ca)bca $\Rightarrow$abca = acbca (ca = c) $\Rightarrow$ abca = acbca.
Hence S is left semi-normal.

Conversely, let S be left semi-normal then abca = acbca $\Rightarrow$abca = (a)cbca

$\Rightarrow$abca = abcbca (a = ab) $\Rightarrow$abca = a(b)cbca $\Rightarrow$abca = abacbca (b = ba) $\Rightarrow$abca = aba(c)bca

$\Rightarrow$abca = abacabca (c = ca) $\Rightarrow$ abca = abacabca.

Hence S is left semi-regular.

- *Theorem:* Let S be a semigroup and assume that S satisfy the left singular property then S is left(right) quasi-normal if and only if it is normal
- *Theorem:* A semigroup S with an identity ab = a for any a,b in S is left (right) quasi-normal if and only if it is left(right) semi-regular.

Proof: Let S be a semigroup and it satisfies the identity ab = a for all a,b $\in$S

Assume that S be left quasi-normal then abc = acbc $\Rightarrow$ ab(c) = acb(c)

$\Rightarrow$ abca = acbca (c = ca) $\Rightarrow$ abca = (a)cbca $\Rightarrow$abca = abcbca (a = ab) $\Rightarrow$abca = a(b)cbca

$\Rightarrow$abca = abacbca (b = ba) $\Rightarrow$abca = aba(c)bca $\Rightarrow$abca = abacabca (c = ca)

$\Rightarrow$abca = abacabca

$\therefore$S is left semi-regular.

Conversely, let S be left semi-regular then abca = abacabca $\Rightarrow$ ab(ca) = abacab(ca)

$\Rightarrow$ abc = abacabc (ca = a) $\Rightarrow$abc = (ab)acabc$\Rightarrow$abc = aacabc (ab = a) $\Rightarrow$abc = (aa)cabc

$\Rightarrow$abc = acabc (aa = a) $\Rightarrow$abc = a(ca)bc $\Rightarrow$abc = acbc $\Rightarrow$ abc = acbc.
Hence S is left quasi-normal.

- *Note:* Similarly, we can prove that,

a) a semigroup S with an identity ab = a for any a,b, in S is left(right) semi-regular if and only if it is right(left) semi-normal.
b) a semigroup S satisfies the identity ab = a , for any a,b$\in$S is left(right) quasi-normal if and only if it is any one of the following:

(1) Regular.

(2) Left(right semi-normal.

(3) Left(right) semi-regular.

(4) Left(right) regular.

(5) Left(right) normal.

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