Structure of Regular Semigroups

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Abstract- This paper concerned with basic concepts and some results on (idempotent) semigroup satisfying the identities of three variables. The motivation of taking three for the number of variables has come from the fact that many important identities on idempotent semigroups are written by three or fewer independent variables. We consider the semigroup satisfying the property abc = ac and prove that it is left semi-normal and right quasi-normal. Again an idempotent semigroup with an identity aba = ab and aba = ba (ab = a, ab = b) is always a semilattices and normal. An idempotent semigroup is normal if and only if it is both left quasi-normal and right quasi-normal. If a semigroup is rectangular then it is left and right semi-regular.

GJSFR-F Classification : FOR Code : MSC 2010: 18B40

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Structure of Regular Semigroups

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Abstract: This paper concerns with basic concepts and some results on (idempotent) semigroup satisfying the identities of three variables. The motivation of taking three for the number of variables has come from the fact that many important identities on idempotent semigroups are written by three or fewer independent variables. We consider the semigroup satisfying the property abc = ac and prove that it is left semi-normal and right quasi-normal. Again an idempotent semigroup with an identity aba = ab and aba = ba (ab = a, ab = b) is always a semilattices and normal. An idempotent semigroup is normal if and only if it is both left quasi-normal and right quasi-normal. If a semigroup is rectangular then it is left and right semi-regular.

I. Preliminaries and Basic Properties of Regular Semigroups

In this section we present some basic concepts of semigroups and other definitions needed for the study of this chapter and the subsequent chapters.

• Definition: A semigroup \((S, .)\) is said to be left(right) singular if it satisfies the identity \(ab = a (ab = b)\) for all \(a, b\) in \(S\).

• Definition: A semigroup \((S, .)\) is rectangular if it satisfies the identity \(aba = a\) for all \(a, b\) in \(S\).

• Definition: A semigroup \((S, .)\) is called left(right) regular if it satisfies the identity \(aba = ab (aba = ba)\) for all \(a, b\) in \(S\).

• Definition: A semigroup \((S, .)\) is called left(right) quasi-normal if it satisfies the identity \(abc = acbc (abc = abac)\) for all \(a, b, c\) in \(S\).

• Definition: A semigroup \((S, .)\) is said to be total if every element of \(S\) can be written as the product of two elements of \(S\). i.e, \(S^2 = S\).

• Definition: A semigroup \((S, .)\) is said to be left(right) normal if \(abc = acb (abc = bac)\) for all \(a, b, c\) in \(S\).

• Result: [11,12] A left(right) singular semigroup is rectangular.

• Theorem: Every left(right) singular semigroup is total.
Proof: Let \((S, .)\) be a left(right) singular semigroup. Then \(ab = a\) for any \(a, b\) in \(S\)
To prove that \(S\) is total we have to prove \(S^2 = S\)
We know that \(S \subseteq S^2\)
To prove \(S^2 \subseteq S\)
Let \(x \in S^2 \Rightarrow x = a \cdot b\) for any \(a, b\) in \(S\)
\[x = a\]
\[x \in S\]
\[S^2 \subseteq S\]
\[\therefore S = S^2\]

Hence \((S, .)\) is total.

- **Note:** From the result 2.1.11 every left(right) singular semigroup is rectangular and again from theorem 2.1.12 it is total. Hence every rectangular semigroup is total.
- **Theorem:** A semigroup satisfying the singular properties is always a semilattice.
- **Theorem:** Let \((S, .)\) be a semigroup. If \(S\) is left and right regular then \(S\) is a semilattice.

**Proof:** Let \((S, .)\) be a semigroup with left regular and right regular then
\[aba = ab\] (left regular)
\[aba = ba\] (right regular) for all \(a, b\) in \(S\).
From the above we have \(ab = ba\)
\[\therefore (S, .)\] is a commutative.
To prove that \((S, .)\) is band, let \(aba = ab \Rightarrow a(ba) = ab \Rightarrow abab = ab\) (\(ba = bab\))
\[\Rightarrow (ab)(ab) = ab \Rightarrow (ab)^2 = ab\] put \(a = b \Rightarrow (a.a)^2 = a.a \Rightarrow (a.a)^2 = a^2\)
\[a.a = a\]
\[\therefore (S, .)\] is a band

So, if \(S\) is left and right regular then \((S, .)\) is commutative band
Hence \((S, .)\) is a semilattice.

- **Lemma:** A left (right) regular semigroup is regular.
- **Note:** Every regular semigroup need not be a left (right) regular.
- **Lemma:** A right (left) singular semigroup is regular.
- **Lemma:** A left (right) normal semigroup is normal.
- **Theorem:** Let \((S, .)\) be a semigroup. If \(S\) is both left and right regular then \(S\) is normal.
- **Lemma:** A left (right) regular semigroup is right (left) quasi-normal.
- **Lemma:** A left (right) regular semigroup is right (left) semi-normal.
- **Theorem:** A left (right) regular semigroup is left (right) semi-regular
- **Theorem:** If an idempotent semigroup satisfies the right (left) quasi-normal then it is right(left) semi-regular.
- **Lemma:** A semigroup \((S, .)\) with left (right) quasi-normal is right (left) semi-normal.
- **Theorem:** If a semigroup \((S, .)\) is rectangular then \((S, .)\) is right (left) semi-regular.
- **Note:** Similarly, we prove that,
1. A semigroup $(S, .)$ is regular then $(S, .)$ is left (right) semi-regular.
2. A semigroup $(S, .)$ is satisfies the left (right) semi-normal property is right (left) semi-regular.

- **Theorem:** An idempotent semigroup $(S, .)$ is normal if and only if it is both left and right quasi-normal.

**Proof:** Let $(S, .)$ be an idempotent semigroup. Assume that $(S, .)$ is both left and right quasi-normal
then
\[ abc = acbc \quad (\text{left quasi-normal}) \]
\[ abc = abac \quad (\text{right quasi-normal}) \]
To prove that $(S, .)$ is normal. Since $S$ is right quasi-normal, we have
\[ abc = abac \quad \Rightarrow abc = a(bac) \Rightarrow abc = abcac \quad (\text{left quasi-normal}) \]
\[ \Rightarrow abc = (abc)ac \Rightarrow abc = acbcac \quad (\text{left quasi normal}) \]
\[ \Rightarrow abc = ac(bcac) \Rightarrow abc = acbc \quad (\text{left quasi normal}) \]
\[ \Rightarrow abca = acbaca \Rightarrow abc = ac(baca) \Rightarrow abc = a(cbca) \Rightarrow abc = acbcac \quad (\text{left quasi normal}) \]
\[ \Rightarrow abc = ac(bac) \Rightarrow abc = abcac \quad (\text{left quasi-normal}) \]
\[ \Rightarrow abca = acba. \]
Hence $(S, .)$ is normal.
Conversely; let $(S, .)$ be normal then $abca = acba$
we show that $(S, .)$ is both left quasi-normal and right quasi-normal.
Consider
\[ abc = ab.c \quad \Rightarrow abc = a(bc)c \Rightarrow abc = acbc \quad \Rightarrow abc = acbc. \]
$(S, .)$ is left quasi-normal.
Symilarlly, $abc = a.abc \Rightarrow abc = a(ab)c \Rightarrow abc = abac \quad \Rightarrow abc = abac.$
Therefore $(S, .)$ is right quasi-normal.

II. **Semigroup Satisfying the Identity $abc = ac.$**

In this section we discuss if a semigroup $S$ satisfying the identity $abc = ac$ for any three variables $a, b, c \in S$, then the following conditions are equivalent to one another.

a) left semi-normal
b) left semi-regular
c) right semi-normal
d) right semi-regular
e) regular
f) normal
g) left quasi-normal
h) right quasi-normal

- **Theorem:** A semigroup $S$ with an identity $abc = ac$, for any $a, b, c \in S$ is left (right) semi-normal if and only if it is left (right) semi-regular.

**Proof:** Let $S$ be a semigroup with an identity $abc = ac$ for any $a, b, c \in S$.
Assume that $S$ be a left semi-normal.
Now we show that $S$ is left semi-regular
Since $S$ is left semi-normal we have $abca = acbca \Rightarrow abca = (ac)bca$
\[ \Rightarrow abca = abcaca \quad (ac = abc) \Rightarrow abca = a(bc)bca \Rightarrow abca = abacbca \quad (bc = bac) \]
\[ \Rightarrow abca = aba(cb)ca \Rightarrow abca = abacabc \quad (cb = cab) \Rightarrow abca = abacabca. \]
\[ \therefore S \text{ is left semi-regular.} \]
Conversely, let $S$ be left semi regular then $abca = abacabca$
\[ \Rightarrow abca = a(bac)abca \Rightarrow abca = abcabca \quad (bac = bc) \Rightarrow abca = (abc)abca. \]
⇒\(abca = acabca\) \((abc = ac)\) ⇒\(abca = a\(cab\)ca\) ⇒\(abca = acbca\) \((cab = cb)\) \(abca = acbca\).

Hence S is left semi-normal.

- **Theorem:** An idempotent semigroup S with an identity \(abc = ac\) for any \(a,b,c\) in S is left (right) semi-regular if and only if it is regular.

  **Proof:** Let S be an idempotent semigroup with an identity \(abc = ac\) for any \(a,b,c\) in S. Assume that S be a regular semigroup then \(abca = abaca\)
  \(⇒\) \(abca = ab(ac)a\) ⇒\(abca = ababca\) \((ac = abc)\) ⇒\(abca = a(ba)bca\) ⇒\(abca = abcabca\) \((ba = bca)\)
  \(⇒\) \(abca = a(bc)abca\) ⇒\(abca = abacabca\) \((bc = bac)\) ⇒ \(abca = abacabca\)
  Hence S is left semi-regular.

  Conversely, let S be left semi-regular then \(abca = abacabca\) ⇒\(abca = abac(abc)a\)
  \(⇒\) \(abca = abaca\) \((a.a = a)\) ⇒ \(abca = abaca\)
  Hence S is regular.

- **Theorem:** A semigroup S with an identity \(abc = ac\) for any \(a,b,c\) in S is left (right) semi-regular if and only if it is normal.

  **Proof:** Let S be a semigroup with an identity \(abc = ac\) for any \(a,b,c\) in S and assume that S be normal then \(abca = acba\) ⇒\(abca = (ac)ba\) ⇒\(abca = acbca\) \((ac = abc)\) ⇒\(abca = abc(ba)\) ⇒\(abca = ac(ba)\) ⇒\(abca = acba\)
  \((bca = ba)\)
  \(⇒\) \(abca = acba\)
  Hence S is normal.

- **Theorem:** A semigroup S with an identity \(abc = ac\) where \(a,b,c\) in S is regular if and only if it is left (right) semi-normal.

  **Proof:** Let S be a semigroup with an identity \(abc = ac\) for all \(a,b,c\) in S.

  Assume that S is regular then \(abca = (ab)aca\) ⇒\(abca = acbaca\) \((ab = acb)\) ⇒\(abca = acbca\) \((a = abc)\) ⇒\(abca = acbca\)
  \((bac = bc)\) ⇒ \(abca = acbca\)
  Hence S is normal.

- **Lemma:** A semigroup S satisfying the property \(abc = ac\) for any \(a,b,c\) in S is left (right) semi-regular if and only if it is right(left) semi-regular.
Lemma: A semigroup $S$ with an identity $abc = ac$ is left(right) semi-normal if and only if it is right (left) semi-normal.

Note: A semigroup $S$ with an identity $abc = ac$ where $a, b, c \in S$ is left(right) quasi-normal if and only if it is right (left) quasi-normal.

Lemma: A semigroup $S$ with the property $abc = ac$ for all $a, b, c \in S$ is normal if and only if it is left(right) semi-normal.

Theorem: A semigroup $S$ with an identity $abc = ac$ for any $a, b, c \in S$ is normal if and only if it is left(right) quasi-normal.

Proof: Let $S$ be a semigroup with an identity $abc = ac$ for any $a, b, c \in S$. Let $S$ be a left quasi-normal. Then, $abc = acbc \Rightarrow abca = acbca \Rightarrow abca = ac(bca)$

$\Rightarrow abca = acba \quad (bca = ba) \Rightarrow abca = acba$

$S$ is normal.

Conversely, let $S$ be normal. Then, $abca = acba \Rightarrow abcac = acbac$

$\Rightarrow a(bc)a = acbac \Rightarrow abac = acbac \quad (bca = ba) \Rightarrow (a)bac = acbac$

$\Rightarrow abca = acbac \quad (bac = bc) \Rightarrow abc = ac(bac) \Rightarrow abc = acbc.$

$\therefore S$ is a left quasi-normal.

Similarly, we can prove that a semigroup $S$ with an identity $abc = ac$ for any $a, b, c \in S$ is regular if and only if it is left (right) quasi-normal.

Theorem: A semigroup $S$ with an identity $abc = ac$ for all $a, b, c \in S$ is regular if and only if it is normal.

Proof: Let $S$ be a semigroup with an identity $abc = ac$ for any $a, b, c \in S$. Let $S$ be a normal semigroup then $abca = acba \Rightarrow abca = (ac)ba$

$\Rightarrow abca = abcba \quad (ac = abc) \Rightarrow abca = abc(ba) \Rightarrow abca = abc(ba) \Rightarrow (a)bc = acbac$

$\Rightarrow abca = a(bc)bca \Rightarrow abca = abacba \Rightarrow abca = abc(bca) \Rightarrow abca = abc(bca)$

$\Rightarrow abca = abaca \quad (cb = c)$

Hence $S$ is a regular.

Conversely, let $S$ be a regular semigroup then $abca = abaca \Rightarrow abca = abc(bca) \Rightarrow abca = abaca \Rightarrow abca = abaca \Rightarrow abca = (bc = ba) \Rightarrow abca = abaca$.

$\therefore S$ is normal.

III. Semigroup Satisfies the Identity $ab = a \ (ab = b)$

We present some results on semigroup with an identity $ab = a \ (ab = b)$ for all $a, b$ in a semigroup $S$. We prove that the necessary and sufficient conditions for a semigroup $S$ to be regular, normal, left (right) normal, left (right) semi-normal, right(left) semi-regular, left (right) regular, left (right) quasi-normal.

Theorem: A semigroup $S$ with an identity $ab = a$ for any $a, b \in S$ is normal if and only if it is regular.

Proof: Let $S$ be a semigroup satisfying the identity $ab = a$ for all $a, b \in S$.

Assume that $S$ is normal. Then $abca = (a)cba \Rightarrow abca = abcba \quad (a = ab)$

$\Rightarrow abca = a(b)cba \Rightarrow abca = abacba \quad (b = ba) \Rightarrow abca = aba(cb)a \Rightarrow abca = abaca$.

(c = cb)
⇒ \text{abca = abaca.}
Therefore S is regular.
Conversely, let S be a regular semigroup then \( \text{abca} = \text{abaca} \Rightarrow \text{abca} = (ab)aca \)
⇒ \text{abca} = aaca \ (ab = a) \Rightarrow \text{abca} = (aa)ca \Rightarrow \text{abca} = aca \ (aa = a) \Rightarrow \text{abca} = a(c) a \)
⇒ \text{abca} = acba \ (c = cb) \Rightarrow \text{abca} = acba.
\therefore \text{S is normal.}

• \textbf{Theorem:} A semigroup S satisfying the identity \( ab = a \) for any \( a, b \) in S is left (right) regular if and only if it is regular.
• \textbf{Theorem:} A semigroup S with an identity \( ab = a \) for all \( a, b \) in S is left(right) semi-normal if and only if it is right(left) semi-normal.
• \textbf{Theorem:} A semigroup S with an identity \( ab = a \) for any \( a, b \) in S is left (right) semi-normal if and only if it is regular.

\textbf{Proof:} Let S be a semigroup with an identity \( ab = a \) for any \( a, b \in S \).
Let S be left semi-normal then \( \text{abca} = \text{acbca} \Rightarrow \text{abca} = \text{ac(b)cba} \Rightarrow \text{abca} = \text{acba} \ (bc = b) \)
⇒ \text{abca} = (a)cba \Rightarrow \text{abca} = abcba \ (a = ab) \Rightarrow \text{abca} = a(b)cba \Rightarrow \text{abca} = abacba \ (b = ba) \)
⇒ \text{abca} = abacba \ (c = cb) \Rightarrow \text{abca} = acba.
Hence S is regular.
Conversely, let S be regular then \( \text{abca} = \text{acbca} \Rightarrow \text{abca} = (ab)aca \Rightarrow \text{abca} = aaca \ (ab = a) \)
⇒ \text{abca} = (aa)ca \ (aa = a) \Rightarrow \text{abca} = acba \ (aa = a) \Rightarrow \text{abca} = acba \ (c = cb) \)
⇒ \text{abca} = ac(b)a \Rightarrow \text{abca} = abcba \ (a = ab) \Rightarrow \text{abca} = a(b)cba \Rightarrow \text{abca} = abacba \ (b = ba) \)
⇒ \text{abca} = abacba \ (c = ca) \Rightarrow \text{abca} = acba.
\therefore \text{S is normal.}

Notes
\[ abca = abaca(b)a \Rightarrow abca = abacabca \quad (b = bc) \Rightarrow abca = abacabca. \]

Therefore \( S \) is left semi-regular.

- **Theorem:** A semigroup \( S \) with an identity \( ab = a \) for all \( a,b \) in \( S \) is left(right) semi-regular if and only if it is left(right) semi-normal

**Proof:** Let \( S \) be a semigroup with the identity \( ab = a \) for any \( a,b \in S \).

Assume that \( S \) be a left semi-regular semigroup then, we have \( abca = abacabca \)
\[ \Rightarrow abca = (ab)acabca \Rightarrow abca = aacabca \quad (a = ab) \Rightarrow abca = (aa)cabca \Rightarrow abca = acabca \quad (aa = a) \]
\[ \Rightarrow abca = a(ca)bca \Rightarrow abca = acbca \quad (ca = c) \Rightarrow abca = acbca. \]

Hence \( S \) is left semi-normal.

Conversely, let \( S \) be left semi-normal then \( abca = acbca \Rightarrow abca = (a)cdba \)
\[ \Rightarrow abca = acbca \quad (a = ab) \Rightarrow abca = a(b)cdba \Rightarrow abca = abacbca \quad (b = ba) \Rightarrow abca = aba(c)bca \]
\[ \Rightarrow abca = abacabca \quad (c = ca) \Rightarrow abca = abacabca. \]

Hence \( S \) is left semi-regular.

- **Theorem:** Let \( S \) be a semigroup and assume that \( S \) satisfy the left singular property then \( S \) is left(right) quasi-normal if and only if it is normal

- **Theorem:** A semigroup \( S \) with an identity \( ab = a \) for any \( a,b \) in \( S \) is left(right) quasi-normal if and only if it is any one of the following:

1. Regular.
2. Left(right) semi-normal.
3. Left(right) semi-regular.
4. Left(right) regular.
5. Left(right) normal.

\[ \therefore S \text{ is left semi-regular.} \]

Conversely, let \( S \) be left semi-regular then \( abca = abacabca \Rightarrow ab(ca) = abacab(ca) \)
\[ \Rightarrow abc = abacabc \quad (ca = a) \Rightarrow abc = (ab)acabc \Rightarrow abc = aacabc \quad (ab = a) \Rightarrow abc = (aa)cabc \]
\[ \Rightarrow abc = acabc \quad (aa = a) \Rightarrow abc = a(ca)bc \Rightarrow abc = acbc \Rightarrow abc = acbc. \]

Hence \( S \) is left quasi-normal.

- **Note:** Similarly, we can prove that,

a) a semigroup \( S \) with an identity \( ab = a \) for any \( a,b \), in \( S \) is left(right) semi-regular if and only if it is right(left) semi-normal.

b) a semigroup \( S \) satisfies the identity \( ab = a \), for any \( a,b \in S \) is left(right) quasi-normal if and only if it is any one of the following:

(1) Regular.
(2) Left(right) semi-normal.
(3) Left(right) semi-regular.
(4) Left(right) regular.
(5) Left(right) normal.
REFERENCES Références Referencias