Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 7 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Extended Exp($-\varphi(\xi)$)-Expansion Method for Solving the Generalized Hirota-Satsuma Coupled KdV System

By Mostafa M. A. Khater
Mansoura University, Egypt

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GJSFR-F Classification : *FOR Code: MSC 2010: 35Q20 - 35K99 - 35P05.*

Strictly as per the compliance and regulations of :

Ref

12. Malfliet, W. Solitary wave solutions of nonlinear wave equation. *Am. J. Phys. 60* (1992), 650–654.

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I. Introduction

No one can deny the important role which played by the nonlinear partial differential equations in the description of many and a wide variety of phenomena not only in physical phenomena, but also in plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. For examples tanh - sech method [12],[16] and [18], extended tanh - method [13], [6] and [20], sine - cosine method [19], [17] and [22], homogeneous balance method [4], the exp($-\varphi(\xi)$)-expansion Method [11], Jacobi elliptic function method [3], [5], [14] and [24], F-expansion method [2], [21] and [9], exp-function method [8] and [7], trigonometric function series method [32], $(\frac{G'}{G})-$ expansion method [10], [15], [29] and [26], the modified simple equation method [1], [27], [30], [28], [31] and [25] and so on.

The objective of this article is to apply the extended exp($-\varphi(\xi)$)-expansion method for finding the exact traveling wave solution of the generalized Hirota-Satsuma coupled KdV system [23], which play an important role in mathematical physics.

The rest of this paper is organized as follows: In section 2, we give the description of the e exp($-\varphi(\xi)$)-expansion method. In section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In section 4, conclusions are given.

Author: *Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt.*
e-mail: mostafa.khater2024@yahoo.com

II. Description of Method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \tag{2.1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following,we give the main steps of this method
Step 1. We use the wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - ct, \tag{2.2}$$

where c is a positive constant, to reduce Eq.(2.1)to the following ODE:

$$P(u, u', u'', u''', ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives,while $u' = \frac{du}{d\xi}$.
Step 2. Suppose that the solution of ODE (2.3) can be expressed by a polynomial in $exp(-\varphi(\xi))$ as follows

$$u(\xi) = \sum_{i=-m}^{m} a_i \left(exp\left(-\varphi\left(\xi\right)\right)\right)^i, \tag{2.4}$$

Since a_m $(0 \leq m \leq n)$ are constants to be determined, such that $(a_m\, or\, a_{-m}) \neq 0$.
the positive integer m can be determined by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in Eq.(2.3). Moreover precisely, we define the degree of $u\left(\xi\right)$ as $D\left(u\left(\xi\right)\right) = m$, which gives rise to degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q,\ D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s\left(n+q\right).$$

Therefore, we can find the value of m in Eq.(2.3), where $\varphi = \varphi(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi) = exp\left(-\varphi\left(\xi\right)\right) + \mu exp\left(\varphi\left(\xi\right)\right) + \lambda, \tag{2.5}$$

the solutions of ODE (2.3) are
when $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$\varphi(\xi) = ln\left(\frac{-\sqrt{\lambda^2-4\mu}\ tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\xi+C_1\right)\right)-\lambda}{2\mu}\right), \tag{2.6}$$

and

$$\varphi(\xi) = ln\left(\frac{-\sqrt{\lambda^2-4\mu}\ coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\xi+C_1\right)\right)-\lambda}{2\mu}\right), \tag{2.7}$$

when $\lambda^2 - 4\mu > 0, \mu = 0,$

$$\varphi(\xi) = -ln\left(\frac{\lambda}{exp\left(\lambda\left(\xi+C_1\right)\right)-1}\right), \tag{2.8}$$

when $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0,$

$$\varphi(\xi) = ln\left(-\frac{2\left(\lambda\left(\xi+C_1\right)+2\right)}{\lambda^2\left(\xi+C_1\right)}\right), \tag{2.9}$$

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when $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0,$

$$\varphi(\xi) = ln\left(\xi + C_1\right), \tag{2.10}$$

when $\lambda^2 - 4\mu < 0,$

$$\varphi(\xi) = ln\left(\frac{\sqrt{4\mu - \lambda^2}\, tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\ (\xi + C_1)\right) - \lambda}{2\mu}\right), \tag{2.11}$$

and

$$\varphi(\xi) = ln\left(\frac{\sqrt{4\mu - \lambda^2}\, cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\ (\xi + C_1)\right) - \lambda}{2\mu}\right), \tag{2.12}$$

where $a_m, \ldots\ldots, \lambda, \mu$ are constants to be determined later,

Step 3. After we determine the index parameter m, we substitute Eq.(2.4) along Eq.(2.5) into Eq.(2.3) and collecting all the terms of the same power $exp\left(-m\varphi(\xi)\right)$, $m = 0, 1, 2, 3, \ldots$. and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i.

Step 4. substituting these values and the solutions of Eq.(2.5) into Eq.(2.3) we obtain the exact solutions of Eq.(2.3).

III. Application

Here, we will apply the extended exp($-\varphi(\xi)$)-expansion method described in Sec.2 to find the exact traveling wave solutions and the solitary wave solutions of the generalized Hirota-Satsuma coupled KdV system[23]. We consider the generalized Hirota-Satsuma couple KdV system

$$\begin{cases} u_t = \frac{1}{4}u_{xxx} + 3uu_x + 3\left(-v^2 + w\right)_x, \\ v_t = -\frac{1}{2}v_{xxx} - 3uv_x, \\ w_t = -\frac{1}{2}w_{xxx} - 3uw_x. \end{cases} \tag{3.1}$$

When $w = 0$, Eq.(3.1) reduce to be the well known Hirota-Satsuma couple KdV equation. Using the wave transformation $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, $w(x,t) = w(\xi)$, $\xi = k(x - \lambda_1 t)$ carries the partial differential equation (3.1) into the ordinary differential equation

$$\begin{cases} -\lambda_1\, k\, u' = \frac{1}{4}k^3 u''' + 3\, k\, u\, u' + 3\, k\, \left(-v^2 + w\right)', \\ -\lambda_1\, k\, v' = -\frac{1}{2}k^3 v''' - 3\, k\, u\, v', \\ -\lambda_1\, k\, w' = -\frac{1}{2}k^3 w''' - 3\, k\, u\, w'. \end{cases} \tag{3.2}$$

Suppose we have the relations between $(u\, and\, v)$ and $(w\, and\, v) \Rightarrow \left(u = \alpha v^2 + \beta v + \gamma\right)$ and $(w = Av + B)$ where α, β, γ, A and B are arbitrary constants. Substituting this relations into second and third equations of Eq.(3.2) and integrating them , we get the same equation and integrate it once again we obtain

$$k^2 v'^2 = -2\alpha v^4 - 2\beta v^3 + 2\left(\lambda_1 - 3\gamma\right) v^2 + 2c_1 v + c_2, \tag{3.3}$$

where c_1 and c_2 is the arbitrary constants of integration, and hence, we obtain

$$\begin{aligned} k^2 u'' &= 2\alpha k^2 v'^2 + k^2\left(2\alpha v + \beta\right) v'' \\ &= 2\alpha\left[-\alpha v^4 - 2\beta v^3 + 2\left(\lambda_1 - 3\gamma\right) v^2 + 2c_1 v + c_2\right] \\ &+ \left(2\alpha v + \beta\right)\left[-2\alpha v^3 - 3\beta v^2 + 2\left(\lambda_1 - 3\gamma\right) v + c_1\right]. \end{aligned} \tag{3.4}$$

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23. Yan, Z. The extended jacobian elliptic function expansion method and its application in the generalized hirota Ŭsatsuma coupled kdv system. *Chaos, Solitons & Fractals* 3, 15(2003), 575–583.

So that, we have

$$P'' + lP - mP^3 = 0. \tag{3.5}$$

Where

$$c_1 = \frac{1}{2\alpha^2\left(\beta^2 + 2\lambda_1\alpha\beta - 6\alpha\beta\gamma\right)},\quad v(\xi) = aP(\xi) - \frac{\beta}{2\alpha},\quad \alpha = \frac{\beta^2-4}{4\left(\gamma-\lambda_1\right)},\quad A = \frac{4\beta\left(\lambda_1-\gamma\right)}{\beta^2-4},$$

$$B = \frac{1}{6\left(-\gamma+\lambda_1\right)\left(\beta^2-4\right)^2}(16c_3\lambda_1\beta^2 - 2c_3\lambda_1\beta^4 - 16c_3\gamma\beta^2 + 3c_3\gamma\beta^4 + 56\lambda_1^2\gamma\beta^2$$

$$-48\gamma^2\lambda_1\beta^2 - 16c_2 + c_2\beta^6 - 12c_2\beta^4 + 12c_2\beta^2 - 16\gamma^2\lambda_1 - 32\lambda_1^2\gamma - 8\lambda_1^3\beta^2 + \beta^4\gamma^3$$

$$-2\beta^4\lambda_1^3 + 32c_3\gamma - 32c_3\lambda_1 + 48\gamma^3 + \beta^4\gamma^2\lambda_1),$$

$$l = \frac{-a}{k^2}\left(\frac{3\beta^2}{2\alpha} + 2\lambda_1 - 6\gamma\right),\quad m = \frac{-2\alpha a^3}{k^2}.$$

Balancing between the highest order derivatives and nonlinear terms appearing in P'' and P^3 $\Rightarrow (N+2=3N) \Rightarrow (N=1)$. So that, by using Eq.(2.4) we get the formal solution of Eq.(3.5)

$$P(\xi) = a_{-1}exp(\varphi(\xi)) + a_0 + a_1exp(-\varphi(\xi)). \tag{3.6}$$

Substituting Eq.(3.6) and its derivative into Eq.(3.5) and collecting all term with the same power of $[exp(-3\varphi(\xi)), exp(-2\varphi(\xi)), ..., exp(+3\varphi(\xi))]$ we obtained:

$$2\,a_1 + m{a_1}^3 = 0, \tag{3.7}$$

$$3\,\lambda\,a_1 + 3\,ma_0{a_1}^2 = 0, \tag{3.8}$$

$$2\,\mu\,a_1 + \lambda^2a_1 + la_1 + 3\,ma_{-1}{a_1}^2 + 3\,m{a_0}^2a_1 = 0, \tag{3.9}$$

$$\lambda\,a_{-1} + \mu\,\lambda\,a_1 + la_0 + 6\,ma_{-1}a_0a_1 + m{a_0}^3 = 0, \tag{3.10}$$

$$2\,\mu\,a_{-1} + \lambda^2a_{-1} + la_{-1} + 3\,m{a_{-1}}^2a_1 + 3\,ma_{-1}{a_0}^2 = 0, \tag{3.11}$$

$$3\,\mu\,\lambda\,a_{-1} + 3\,m{a_{-1}}^2a_0 = 0, \tag{3.12}$$

$$2\,\mu^2a_{-1} + m{a_{-1}}^3 = 0. \tag{3.13}$$

Solving above system by using maple 16, we get:

Case 1.

$$l = 4\,\mu,\ m = \frac{-2}{{a_1}^{-2}},\ \lambda = 0,\ a_{-1} = \mu\,a_1,\ a_0 = 0,\ a_1 = a_1.$$

Case 2.

$$l = -8\,\mu,\ m = \frac{-2}{{a_1}^{-2}},\ \lambda = 0,\ a_{-1} = -\mu\,a_1,\ a_0 = 0,\ a_1 = a_1.$$

Thus the solution is

For Case 1.

$$p(\xi) = \mu\,a_1exp(\varphi(\xi)) + a_1exp(-\varphi(\xi)). \tag{3.14}$$

For Case 2.

$$p(\xi) = -\mu\,a_1exp(\varphi(\xi)) + a_1exp(-\varphi(\xi)). \tag{3.15}$$

Notes

Let us now discuss the following cases:

For Case 1. When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$P(\xi) = \mu\, a_1 \left(\frac{-\sqrt{\lambda^2 - 4\mu}\, tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C_1)\right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{-\sqrt{\lambda^2 - 4\mu}\, tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C_1)\right) - \lambda} \right). \tag{3.16}$$

and

$$P(\xi) = \mu\, a_1 \left(\frac{-\sqrt{\lambda^2 - 4\mu}\, coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C_1)\right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{-\sqrt{\lambda^2 - 4\mu}\, coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C_1)\right) - \lambda} \right). \tag{3.17}$$

When $\lambda^2 - 4\mu > 0, \mu = 0$,

$$P(\xi) = \mu\, a_1 \left(\frac{exp\left(\lambda\left(\xi + C_1\right)\right) - 1}{\lambda} \right) + a_1 \left(\frac{\lambda}{exp\left(\lambda\left(\xi + C_1\right)\right) - 1} \right). \tag{3.18}$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$P(\xi) = \mu\, a_1 \left(-\frac{2\left(\lambda\left(\xi + C_1\right) + 2\right)}{\lambda^2\left(\xi + C_1\right)} \right) + a_1 \left(-\frac{\lambda^2\left(\xi + C_1\right)}{2\left(\lambda\left(\xi + C_1\right) + 2\right)} \right). \tag{3.19}$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$P(\xi) = \mu\, a_1 \left(\xi + C_1\right) + a_1 \frac{1}{\left(\xi + C_1\right)}. \tag{3.20}$$

When $\lambda^2 - 4\mu < 0$,

$$P(\xi) = \mu\, a_1 \left(\frac{\sqrt{4\mu - \lambda^2}\, tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\,(\xi + C_1)\right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\, tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\,(\xi + C_1)\right) - \lambda} \right). \tag{3.21}$$

and

$$P(\xi) = \mu\, a_1 \left(\frac{\sqrt{4\mu - \lambda^2}\, cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\,(\xi + C_1)\right) - \lambda}{2\mu} \right) \tag{3.22}$$

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$$+ a_1 \left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\, cot \left(\frac{\sqrt{4\mu-\lambda^2}}{2} \;(\xi + C_1) \right) - \lambda} \right).$$

For Case 2. When $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$P(\xi) = -\mu\, a_1 \left(\frac{-\sqrt{\lambda^2 - 4\mu}\, tanh \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{-\sqrt{\lambda^2 - 4\mu}\, tanh \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (\xi + C_1) \right) - \lambda} \right). \tag{3.23}$$

and

$$P(\xi) = -\mu\, a_1 \left(\frac{-\sqrt{\lambda^2 - 4\mu}\, coth \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{-\sqrt{\lambda^2 - 4\mu}\, coth \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (\xi + C_1) \right) - \lambda} \right). \tag{3.24}$$

When $\lambda^2 - 4\mu > 0, \mu = 0,$

$$P(\xi) = -\mu\, a_1 \left(\frac{exp\,(\lambda\,(\xi + C_1)) - 1}{\lambda} \right) + a_1 \left(\frac{\lambda}{exp\,(\lambda\,(\xi + C_1)) - 1} \right). \tag{3.25}$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0,$

$$P(\xi) = -\mu\, a_1 \left(-\frac{2\,(\lambda\,(\xi + C_1) + 2)}{\lambda^2\,(\xi + C_1)} \right) + a_1 \left(-\frac{\lambda^2\,(\xi + C_1)}{2\,(\lambda\,(\xi + C_1) + 2)} \right). \tag{3.26}$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0,$

$$P(\xi) = -\mu\, a_1\,(\xi + C_1) + a_1 \frac{1}{(\xi + C_1)}. \tag{3.27}$$

When $\lambda^2 - 4\mu < 0,$

$$P(\xi) = -\mu\, a_1 \left(\frac{\sqrt{4\mu - \lambda^2}\, tan \left(\frac{\sqrt{4\mu-\lambda^2}}{2} \;(\xi + C_1) \right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\, tan \left(\frac{\sqrt{4\mu-\lambda^2}}{2} \;(\xi + C_1) \right) - \lambda} \right). \tag{3.28}$$

Notes

and

$$P(\xi) = -\mu\, a_1 \left(\frac{\sqrt{4\mu - \lambda^2}\, cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\ (\xi + C_1) \right) - \lambda}{2\mu} \right) + a_1 \left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\, cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\ (\xi + C_1) \right) - \lambda} \right). \tag{3.29}$$

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23. Yan, Z. The extended jacobian elliptic function expansion method and its application in the generalized hirota Űsatsuma coupled kdv system. *Chaos, Solitons & Fractals 3*, 15(2003), 575–583.

IV. Conclusion

The extended exp(−φ(ξ))-expansion method has been successfully used to find the the wide range of exact and solitary traveling wave solutions for the generalized Hirota-Satsuma couple KdV system. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the generalized Hirota-Satsuma couple KdV system are new and different from those obtained in [23]. It can be concluded that this method is reliable and propose a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations. The solutions represent the solitary traveling wave solution for the generalized Hirota-Satsuma couple KdV system.

Competing interests

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

Author's contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied

V. Acknowledgment

The author thanks the referees for their suggestions and comments.

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