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Linear Elliptic Systems with Nonlinear Boundary Conditions without Landesman-Lazer Conditions

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Abstract- The boundary value problem is examined for the system of elliptic equations of from $-\Delta u+A(x)u=$ in Ω, where $A(x)$ is positive semidefinite matrix on $\mathbb{R}^{k\times k}$, and $\frac{\partial u}{\partial \nu}+g(u)=h(x)$ on $\partial\Omega$. It is assumed that $g\in C(\mathbb{R}^k, \mathbb{R}^k)$ is a bounded function which may vanish at infinity. The proofs are based on Leray-Schauder degree methods.

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Linear Elliptic Systems with Nonlinear Boundary Conditions without Landesman-Lazer Conditions

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Abstract- The boundary value problem is examined for the system of elliptic equations of from $-\Delta u + A(x)u =$ in Ω, where $A(x)$ is positive semidef inite matrix on $\mathbb{R}^{k\times k}$ and $\frac{\partial u}{\partial \nu}+g(u)=h(x)$ on $\partial\Omega$. It is assumed that $g \in C(\mathbb{R}^k, \mathbb{R}^k)$ is a bounded function which may vanish at infinity. The proofs are based on Leray-Schauder degree methods.

I. Introduction

Let $\mathbb{R}^k$ be real $k-$dimensional space, if $w \in \mathbb{R}^k$, then $|w|_E$ denotes the Euclidean norm of w. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with boundary $\partial\Omega$ of class C^∞. Let $g \in C^1(\mathbb{R}^k, \mathbb{R}^k)$, $h \in C(\partial\Omega, \mathbb{R}^k)$, and the matrix

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1k}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}(x) & a_{k2}(x) & \cdots & a_{kk}(x) \end{bmatrix}.$$

Verifies the following conditions:

(A1) The functions $a_{ij} : \Omega \to \mathbb{R}$, $\forall\, i,\ j \in \{1, \cdots, k\}$.

(A2) $A(x)$ is positive semidefinite matrix on $\mathbb{R}^{k\times k}$, almost everywhere $x \in \Omega$, and $A(x)$ is positive definite on a set of positive measure with $a_{ij} \in L^p(\Omega)\ \forall\, i,\ j \in \{1, \cdots, k\}$ for $p > \frac{N}{2}$ when $N \geq 3$, and $p > 1$ when $N = 2$.

We will study the solvability of

$$\begin{aligned} -\Delta u + A(x)u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + g(u) &= h(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The interest in this problem is the resonance case at the boundary with a bounded nonlinearity, we will assume that g a bounded function, and there is a constant $R > 0$ such that

$$|g(w(x))|_E \leq R \quad \forall\, w \in \mathbb{R}^k \ \&\ x \in \partial\Omega. \tag{1.2}$$

Our assumptions allow that g is not only bounded, but also may be vanish at infinity i.e.;

$$\lim_{|w|_E \to \infty} g(w) = 0 \in \mathbb{R}^k. \tag{1.3}$$

Condition (1.3) is not required by our assumptions, but allowing for it is the main result of this paper.

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In case of the scalar equation $i.e.$; $k = 1$ and g doesn't satisfy condition (1.3) but satisfying the Landesman-Lazer condition

$$g_- < \bar{h} < g^+$$

where $\lim_{w\to-\infty} g(w) = g_-$, $\bar{h} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} h\,dx$, $\lim_{w\to\infty} g(w) = g^+$,
and $A(x) = 0 \in \mathbb{R}^{k\times k}$. Then it is well know that there is a solution for (1.1). The first results when the nonlinearity in the equation in scalar case was done by Landesman and Lazaer [1] in 1970. Their work led to great interest and activity on boundary value problems at resonance which continuous to this day. A particularly interesting extension of Landesman and Lazer's work to systems was done by Nirenberg [2], [3] in case of system and the nonlinearity in the equation was done by Ortega and Ward [4], in the scalar case without Landesman-Lazer condition was done by Iannacci and Nkashama [5], Ortega and Sánchez [6], more completely the case for periodic solutions of the system of ordinary differential equations with bounded nonlinear g satisfying Nirenberg's condition. They studied periodic so solutions

$$u'' + cu' + g(u) = p(t),$$

for $u \in \mathbb{R}^k$.
In case $c = 0$ was done by Mawhin [7]. In case the nonlinear terms vanish at infinity, as in (1.3), the Landesman-Lazer conditions fail. We would like to know what we can do in this case, and what conditions on a bounded nonlinearity that vanishes at infinity might replace that ones of the Landesman-Lazer type. Several authors have considered the case when the nonlinearity $g\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ is a scalar function satisfies Carathéodory conditions i,e.;

i: $g(.,u)$ is measurable on $\partial\Omega$, for each $u \in \mathbb{R}$,
ii: $g(x,.)$ is continuous on $\mathbb{R}$, for $a.e.x \in \partial\Omega$,
iii: for any constant $r > 0$, there exists a function $\gamma_r \in L^2(\partial\Omega)$, such that

$$|g(x,u)| \le \gamma_r(x), \tag{1.4}$$

for $a.e.x \in \Omega$, and all $u \in \mathbb{R}$ with $|u| \le r$,

was done by Fadlallah [8] and the others have considered the case when the nonlinearity does not decay to zero very rapidly. For example in case the nonlinearity in the equation if $g = g(t)$ is a scalar function, the condition

$$\lim_{|t|\to\infty} tg(t) > 0. \tag{1.5}$$

and related ones were assumed in [9], [10], [11], [12], [13], [14], [15], [16], [17]. These papers all considered scalar problem, but also considered the Dirichlet (Neumann) problem at resonance (non-resonance) at higher eigenvalues (Steklov-eigenproblems). The work in some of these papers makes use of Leray-Schauder degree arguments, and the others using critical point theory both the growth restrictions like (1.5) and Lipschitz conditions have been removed (see [15], [17]). In this paper we study systems of elliptic boundary value problems with nonlinear boundary conditions Neumann type and the nonlinearities at boundary vanishing at the infinity. We do not require the problem to be in variational from.
Let S^{k-1} be the unit sphere in $\mathbb{R}^k$. We will assume that $S^{k-1}\cap\partial\Omega \neq \emptyset$ and Let $\mathbb{S} = S^{k-1}\cap\partial\Omega$.

1.1. *Assumptions*

*G*1**:** $g \in C^1(\mathbb{R}^k,\mathbb{R}^k)$ and g is bounded with $g(w) \neq 0$ for $|w|_E$ large.

*G*2**:** For each $z \in \mathbb{S}$ the $\lim_{r\to\infty} \dfrac{g(rz)}{|g(rz)|_E} = \varphi(z)$ exists, and the limits is uniform for $z \in \mathbb{S}$. It follows that $\varphi \in C(\mathbb{S},\mathbb{S})$ and the topological degree of φ is defined.

*G*3**:** $deg(\varphi) \neq 0$

Ref

1. E. Landesman and A. Lazer: *Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609-623.*

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1.2. *Notations*

- Let $\langle .,. \rangle_{L^2}$ denote the inner product in $L^2 := L^2(\Omega, \mathbb{R}^k)$ where L^2 is Lebesgue space
- Let $\langle .,. \rangle_E$ denote the standard inner product in $\mathbb{R}^k$
- Assume that $((A1)$-$(A2))$ holds, then define

$$E(u,v) := \sum_{i=1}^{k} \langle \bigtriangledown u_i, \bigtriangledown v_i \rangle_{L^2} + \langle a_{ij}(x) u_i, v_i \rangle_{L^2}, \; j = 1, \ldots, k,$$

for $u, v \in H^1 = H^1(\Omega, \mathbb{R}^k)$ where H^1 the Sobolev space.

We note that it follows from the assumptions $G1:$ - $G3:$ that on large balls

$$B(R) := \{y : |y|_E \leq R\},$$

the $deg(g, B(R), 0) \neq 0$ see [18],[19].

We modify the Lemma 1 and Theorem 1 in [4] to fit our problem.

Lemma 1.1. Assume that $G1:$ and $G2:$ hold and $C > 0$ is a given constant. Then there exists $R > 0$ such that

$$\int_{\partial\Omega} g(u(x))\, dx \neq 0,$$

for each function $u \in C(\partial\Omega, \mathbb{R}^k)$ (we can write $u = \bar{u} + \tilde{u}$ where $\bar{u} = \int_{\partial\Omega} u(x)dx = 0$, and $\bar{u} \perp \tilde{u}$) with $|\bar{u}|_E \geq R$ and $||u - \bar{u}||_{L^\infty(\partial\Omega)} \leq C$

Proof. By the way of contradiction. Assume that for some $C > 0$ there is exist a sequence of functions $\{u_n\}_{n=1}^{\infty} \in C(\bar{\Omega}, \mathbb{R}^k)$, with

$$|\bar{u}_n|_E \to \infty, \; ||u_n - \bar{u}_n||_{L^\infty(\partial\Omega)} \leq C$$

and

$$\int_{\partial\Omega} g(u_n(x))\, dx = 0. \tag{1.6}$$

We constructed a subsequence of u_n one can assume that $\bar{z}_n = \frac{\bar{u}_n}{|\bar{u}_n|_E}$ converges to some point $z \in \mathbb{S}$. The uniform bound on $u_n - \bar{u}_n$ implies that also $\frac{u_n}{|u_n|_E}$ converges to z and this convergence is uniform with respect to $x \in \bar{\Omega}$. It follows from the assumption $G2:$ that

$$\lim_{n\to\infty} \frac{g(u_n(x)}{|g(u_n(x))|_E} = \varphi(z)$$

uniformly in $\bar{\Omega}$. Since $\varphi(z)$ is in the unit sphere one can find an integer n_0 such that if $n \geq n_0$ and $x \in \bar{\Omega}$, then

$$\langle \frac{g(u_n(x)}{|g(u_n(x))|_E}, \varphi(z) \rangle_E \geq \frac{1}{4}$$

Define

$$\gamma_n(x) = |g(u_n(x))|_E.$$

By $G1:$ clearly $\gamma_n > 0$ everywhere. For $n \geq n_0$

$$\langle \int_{\partial\Omega} g(u_n(x))\, dx, \varphi(z) \rangle_E = \int_{\partial\Omega} \langle g(u_n(x)), \varphi(z) \rangle_E\, dx$$

$$= \int_{\partial\Omega} \gamma_n(x) \langle \frac{g(u_n(x))}{\gamma_n(x)}, \varphi(z) \rangle_E\, dx \geq \frac{1}{4} \int_{\partial\Omega} \gamma_n(x)\, dx > 0$$

Therefore, $\int_{\partial\Omega} g(u_n(x))\, dx > 0$. Now we have contradiction with (1.6)

The proof completely of the lemma. □

Ref

4. R. Ortega, and J.R. Ward, *A Semilinear elliptic system with vanishing nonlinearities, proceedings of the fourth international conference on dynamical systems and differential equations May 24 – 27, 2002, Wilmington, NC.*

II. Main Result

Let

$$Qu = Nu. \tag{2.1}$$

Be linear elliptic equation with nonlinear boundary condition. Suppose N is continuous and bounded (i.e.; $|Nu|_E \leq C$ for all u). If Q has a compact inverse Q^{-1} then by Leray-Schauder theory (2.1) has a solution. On the other hand if Q is not invertible the existence of a solution depends on the behavior of N and its interaction with the null space of Q see [19].

Theorem 2.1. Suppose $g \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ satisfies $G1$:, $G2$:, and $G3$:. If $h \in C(\partial\Omega, \mathbb{R}^k)$, satisfies $\bar{h} = 0$. Then, (1.1) has at least one solution.

Proof. Define

$$J : H^1(\Omega) \to R$$

be continuous map in $H^1(\Omega)$ with the $L^2(\Omega)$ norm

$$J(v) = E(u, v)$$

Define

$$Dom(L) := \{u \in H^1(\Omega) : -\Delta u + A(x)u = 0\}$$

Define an operator L on $L^2 = L^2(\Omega, \mathbb{R}^k)$ for $u \in Dom(L)$ and each $v \in H^1(\Omega)$ by

$$E(u, v) =< Lu, v >_{L^2(\Omega)},$$

we use the embedding theorem see [20] since you know that $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and the trace theorem $(H^1 \to L^2(\partial\Omega))$. Thus, $L : Dom(L) \subset L^2(\partial\Omega) \to L^2(\partial\Omega)$ then the equation

$$E(u, v) =< h, v >_{L^2(\partial\Omega)} \ \ \forall\ v \in H^1(\partial\Omega),$$

if and only if

$$Lu = h.$$

The latter equation is solvable if and only if

$$Ph := \frac{1}{|\partial\Omega|}\int_{\partial\Omega} h = 0.$$

Now if $h \in L^\infty(\partial\Omega, \mathbb{R}^k)$ and $Ph = 0$. Then, each solution $u \in H^1(\Omega)$ is Hölder continuous, so $u \in C^\gamma(\bar{\Omega}, \mathbb{R}^k)$ for some $\gamma \in (0, 1)$. Since we know that there is constant $r_1 > 0$ such that

$$||u||_\gamma \leq r_1 \left(||u||_{L^2(\partial\Omega)} + ||h||_{L^\infty(\partial\Omega)}\right).$$

When $Ph = 0$ there is a unique solution $Kh = \tilde{u} \in H^1(\Omega)$ with $P\tilde{u} = 0$ to

$$Lu = h,$$

and if $h \in C(\partial\Omega) = C(\partial\Omega, \mathbb{R}^k)$ then

$$||Kh||_\gamma \leq r_1 \left(||Kh||_{L^2(\partial\Omega)} + ||h||_{L^\infty(\partial\Omega)}\right) \leq r_2||h||_{C(\partial\Omega)}$$

and K maps $C(\partial\Omega)$ into itself take compact set to compact set i.e.; compactly.
Let Q be the restriction of L to $L^{-1}(C(\partial\Omega)) = KC(\partial\Omega) + \mathbb{R}^k$. We define $N : C(\partial\Omega) \to C(\partial\Omega)$ by

$$N(w)(x) := h(x) - g(w(x)) \ \forall w \in C(\partial\Omega)$$

is continuous. Now (1.1) can be written as

$$Qu = Nu$$

Ref

19. J. Mawhin; *Topological degree methods in nonlinear boundary-value problems, in NSFCBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1979.*

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and $\ker Q = ImP$, $ImQ = \ker P$. The linear map Q is a Fredholm map (see [16]) and N is $Q-$compact (see [19]). Now we define the Homotopy equation as follows Let $\lambda \in [0,1]$ such that

$$Qu = \lambda Nu. \tag{2.2}$$

The a priori estimates (i.e.; the possible solutions of (2.2) are uniformly bounded in $C(\partial\Omega)$). Now we show that the possible solutions of (2.2) are uniformly bounded in $C(\partial\Omega)$ independent of $\lambda \in [0,1]$ Since we know that $u = \bar{u} + \tilde{u}$ where $\bar{u} = Pu$. Then

$$||\tilde{u}||_\gamma = ||\lambda KNu||_\gamma \leq r_2 ||Nu||_{C(\partial\Omega)} \leq R_1,$$

where R_1 is a constant (g is abounded function). It remains to show that $\bar{u} \in \mathbb{R}^k$ is bounded, independent of $\lambda \in [0,1]$. By the way of contradiction assume is not the case (i.e.; $\bar{u}$ unbounded). Then there are sequence $\{\lambda_n\} \subset [0,1]$, and $\{u_n\} \subset Dom(Q)$ with $||\tilde{u}_n||_\gamma \leq R_1$,

$$Qu_n = \lambda_n Nu_n \text{ and } |\bar{u}_n|_E \to \infty,$$

we get that

$$PNu_n = PN(\tilde{u}_n + \bar{u}_n) = -\int_{\partial\Omega} g(\tilde{u}_n(x) + \bar{u}_n(x))\, dx = 0.$$

Now $u_n = \tilde{u}_n + \bar{u}_n$ so $||u_n - \bar{u}_n||_{L^\infty(\partial\Omega)} = ||\tilde{u}_n||_{L^\infty(\partial\Omega)} \leq R_1$ and $||\bar{u}_n||_{L^\infty(\partial\Omega)} \to \infty$. It follows from Lemma1.1 that for all sufficiently large n

$$\int_{\partial\Omega} g(u_n(x))\, dx \neq 0.$$

We have reached a contradiction, and hence all possible solutions of (2.2) are uniformly bounded in $C(\partial\Omega)$ independent of $\lambda \in [0,1]$

Let $\bar{B}(0,r) = \{x : |x|_E \leq r\}$ denote the ball in $C(\partial\Omega, \mathbb{R}^k)$. Now you can apply Leray-Schauder degree theorem see ([18],[19]), the only thing left to show is that

$$deg(PN, \bar{B}(0,r) \cap \ker Q, 0) \neq 0,$$

for large $r > 0$. So $deg(PN, \bar{B}(0,r) \cap \ker Q, 0) = deg(g, \bar{B}_r, 0)$, where $\bar{B}_r$ is the ball in $\mathbb{R}^k$ of radius r. Since for $|x|_E$ large, and $deg(\varphi) \neq 0$ we have that $deg(g, \bar{B}_r, 0) \neq 0$ for large r. Therefore $deg(PN, \bar{B}(0,r) \cap \ker Q, 0) \neq 0$ By Leray-Schauder degree theorem equation (2.2) has a solution when $\lambda = 1$. Therefore, equation (1.1) has at least one solution. This proves the theorem. □

We will give one example.

Example 2.1. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with boundary $\partial\Omega$ of class C^∞. Let

$$\begin{aligned} -\Delta u + A(x)u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \frac{u}{1+|u|_E^2} &= h(x) \quad \text{on } \partial\Omega \end{aligned} \tag{2.3}$$

where $A(x)$ is positive semidefinite matrix on $\mathbb{R}^{2\times 2}$, and where $u = (u_1, u_2) \in \mathbb{R}^2$ and h real valued function and continuous on $\partial\Omega$, and $\int_{\partial\Omega} h(x)\, dx = 0$ and $g(u) = \frac{u}{1+|u|_E^2}$

$$\lim_{|u|_E \to \infty} g(u) = \lim_{|u|_E \to \infty} \frac{u}{1+|u|_E^2} = 0$$

$g(u)$ vanishes at infinity, clearly $g \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and bounded with $g(u) \neq 0$, for $|u|_E$ large. Therefore g satisfies $G1$:.

$$\frac{g(ru_1, ru_2)}{|g(ru_1, ru_2)|} = \frac{g(ru)}{|g(ru)|} = \frac{\frac{ru}{1+|ru|_E^2}}{\left|\frac{ru}{1+|ru|_E^2}\right|} = \frac{u}{|u|_E} = u$$

Ref

16. N. Mavinga, M. N. Nkashama, *Nonresonance on the boundary and strong solutions of elliptic equations with nonlinear boundary conditions. Journal of Applied functional Analysis, Vol.7, NO.3,248-257, copyright 2012 Eudoxus Press, LLC.*

For all u in $\mathbb{S}$ and $r > 0$. Therefore $G2$: holds.
And $\varphi(u) = u$ so that $deg(\varphi) \neq 0$. Therefore $G3$: holds. By Theorem 2.1. Then, equation (2.3) has at least one solution.

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