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The Variational Iteration Method for Analytic Treatment of Homogeneous and Inhomogeneous Partial Differential Equations

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Abstract- In this paper, the author used the variational iteration method (VIM) to find the analytical solution to homogeneous and inhomogeneous partial differential equations. Few numerical examples were presented to show the effectiveness and efficiency of the method. It was observed that by carefully chosen a very good choice of initial guess leads to a solution in a closed form. The method is elegant and reliable.

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I. Introduction

Partial differential equations; linear or nonlinear, homogenous or inhomogeneous has many applications to real life problems that arise in science, engineering and technology.

There are many numerical methods for the solution of different types of differential equations such as Adomian decomposition method [1-3], homotopy perturbation method [4], variational iteration method [5-9], modified variational iteration method 14-15]. Results by various researchers [1-15] have shown reliability, efficiency and applicability of these methods.

In this paper, a variational iteration method for the solution of homogeneous and inhomogeneous partial differential equations is presented. It is to be noted that the Lagrange multiplier reduces the iteration on integral operator and also minimizes the computational time. The method requires no transformation and or linearisation of any forms. Some numerical problems and results are presented to show the reliability of the method.

II. The Variational Iteration Method

The basic idea of the He's Variational Iteration Method (VIM) [5-9], can be explained by considering the following nonlinear partial differential equations

$$Lu + Nu = g(x) \qquad (1)$$

Ref

4. Hossein Jafari etal (2008): Application of Homotopy Perturbation Method for Solving Gas Dynamic Equation. Applied Mathematical Sciences, Vol.2,2008. No 48,2393-2396.

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Where L is the linear operator, N is the nonlinear operator and g(x) is the inhomogeneous term. According to the method, we can construct a correction functional as follows

The corresponding variational iteration method for solving (1) is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left[Lu_n(s) + N\bar{u}_n(s) - g(s) \right] ds, \tag{2}$$

where λ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript n denote the nth approximation, $\bar{u}_n$ is considered as a restricted variation i.e $\delta \bar{u}_n = 0$. The successive approximation $u_{n+1}, n \geq 0$ of the solution u can be easily obtained by determine the Lagrange multiplier and the initial guess u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

III. Application and Numerical Results

In this section, six problems will be presented to illustrate the efficiency of the method.

Example 3.1

Consider the following inhomogeneous equation

$$u_x - u_y = 1 + 2x + 2y,\ \ u(0,y) = y + y^2, u(x,0) = 2x + 3x^2 \tag{3}$$

The correction functional is given by

$$u_{n+1}(x,y) = y + y^2 + \int_0^x \lambda(s) \left[\frac{\partial u_n(s,y)}{\partial s} - \frac{\partial u_n(s,y)}{\partial y} - 1 - 2s - 2y) \right] ds \tag{4}$$

Making the correction functional stationary to obtain $\lambda(s) = -1$, hence, the iterative formula becomes

$$u_{n+1}(x,y) = y + y^2 - \int_0^x \left[\frac{\partial u_n}{\partial x} - \frac{\partial u_n}{\partial y} - 1 - 2x - 2y) \right] ds \tag{5}$$

Consequently, following approximants are obtained

$$u_1(x,y) = y + y^2 + 4xy + x^2 + 2x \tag{6}$$

$$u_2(x,y) = y + y^2 + 4xy + 3x^2 + 2x \tag{7}$$

$$u_3(x,y) = y + y^2 + 4xy + 3x^2 + 2x \tag{8}$$

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the closed form and exact solution is given as

$$u(x,y) = y + y^2 + 4xy + 3x^2 + 2x \tag{9}$$

Notes

Example 3.2

Consider the following inhomogeneous equation

$$u_x + u_y = 2xy^2 + 2x^2 y, \quad u(0,y) = 0, u(x,0) = 0 \tag{10}$$

The correction functional and iterative formula becomes:

Notes

$$u_{n+1}(x,y) = 0 - \int_0^x \lambda(s)\left[\frac{\partial u_n(s,y)}{\partial s} + \frac{\partial u_n(s,y)}{\partial y} - 2sy^2 - 2s^2 y)\right] ds \tag{11}$$

then, following approximants are obtained

$$u_1(x,y) = x^2y^2 + \frac{2}{3}x^3 y \tag{12}$$

$$u_2(x,y) = x^2y^2 + \frac{1}{6}x^4 \tag{13}$$

$$u_3(x,y) = x^2y^2 \tag{14}$$

$$u_4(x,y) = x^2y^2 \tag{15}$$

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The exact solution is given by

$$\lim_{n\to\infty} u(x,y) = x^2y^2 \tag{16}$$

Example 3.3

Consider the following equation

$$u_x + u_y = 2u, \quad u(0,y) = e^y, u(x,0) = e^x \tag{17}$$

The correction functional is given by

$$u_{n+1}(x,y) = e^y - \int_0^x \left[\frac{\partial u_n(s,y)}{\partial s} + \frac{\partial u_n(s,y)}{\partial y} - 2u(s,y)\right] ds \tag{18}$$

from (18), following iterations are obtained

$$u_1(x,y) = e^y + xe^y \tag{19}$$

$$u_2(x,y) = e^y + xe^y + \frac{x^2}{2}e^y \tag{20}$$

$$u_3(x,y) = e^y + xe^y + \frac{x^2}{2}e^y + \frac{x^3}{6}e^y \tag{21}$$

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The series solution is given by

$$u(x,y) = e^{y} + xe^{y} + \frac{x^2}{2!}e^{y} + \frac{x^3}{3!}e^{y} + \frac{x^4}{4!}e^{y} + \frac{x^5}{5!}e^{y} + \frac{x^6}{6!}e^{y} + \dots$$
$$= e^{y}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots \tag{22}$$

and the closed form solution is

$$u(x,y) = e^{x+y} \tag{23}$$

which is the exact solution.

Example 3.4

Consider the following equation

$$u_x + u_y + u_z = 3,\ \ u(0,x,y) = y + z, u(x,0,z) = x + z,\ u(x,y,0) = x + y \tag{24}$$

The correction functional and formula is given by

$$u_{n+1}(x,y,z) = y + z - \int_0^x \left[\frac{\partial u_n(s,y,z)}{\partial s} + \frac{\partial u_n(s,y,z)}{\partial y} + \frac{\partial u_n(s,y,z)}{\partial z} - 3) \right] ds \tag{25}$$

Consequently, following approximants are obtained

$$u_1(x,y) = x + y + z \tag{26}$$

$$u_2(x,y) = x + y + z \tag{27}$$

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$$u(x,y) = x + y + z \tag{28}$$

This is the exact solution to (24).

Example 3.5

Consider the following equation

$$u_x + u_y + u_z = 3u,\ \ u(0,x,y) = e^{\,y+z}, u(x,0,z) = e^{\,x+z},\ u(x,y,0) = e^{\,x+y} \tag{29}$$

The correction functional is given by

$$u_{n+1}(x,y,z) = e^{y+z} - \int_0^x \left[\frac{\partial u_n(s,y,z)}{\partial s} + \frac{\partial u_n(s,y,z)}{\partial y} + \frac{\partial u_n(s,y,z)}{\partial z} - 3u(s,y,z)) \right] ds \tag{30}$$

therefore,

$$u_1(x,y) = e^{y+z} + xe^{y+z} \tag{31}$$

$$u_2(x,y) = e^{y+z} + xe^{y+z} + \frac{x^2}{2!}e^{y+z} \tag{32}$$

Notes

$$u_3(x,y) = e^{y+z} + xe^{y+z} + \frac{x^2}{2!}e^{y+z} + \frac{x^3}{3!}e^{y+z} + \ldots \tag{33}$$

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Notes

The series solution is given by

$$u(x,y) = e^{y+z}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots \tag{34}$$

and the closed form solution is

$$u(x,y) = e^{x+y+z} \tag{35}$$

Example 3.6

Consider the following equation

$$u_x - u_y = 0, \;\; u(0,y) = \sin y . u(x,0) = \sin x \tag{36}$$

The correction functional is given by

$$u_{n+1}(x,y) = \sin y - \int_0^x \left[\frac{\partial u_n(s,y)}{\partial s} + \frac{\partial u_n(s,y)}{\partial y} \right] ds \tag{37}$$

Consequently, following approximant are obtained

$$u_1(x,y) = \sin y + x\cos y \tag{38}$$

$$u_2(x,y) = \sin y + x\cos y - \frac{x^2}{2!}\sin y \tag{39}$$

$$u_3(x,y) = \sin y + x\cos y - \frac{x^2}{2!}\sin y - \frac{x^3}{3!}\cos y \tag{40}$$

$$u_4(x,y) = \sin y + x\cos y - \frac{x^2}{2!}\sin y - \frac{x^3}{3!}\cos y - \frac{x^4}{4!}\sin y \tag{41}$$

$$u_5(x,y) = \sin y + x\cos y - \frac{x^2}{2!}\sin y - \frac{x^3}{3!}\cos y - \frac{x^4}{4!}\sin y + \frac{x^5}{5!}\cos y \tag{42}$$

The series solution is given by

$$u(x,y) = \sin y(1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots) + \cos y(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots) \tag{43}$$

and the closed form solution is

$$u(x,y) = \sin x coxy + \sin y \cos x = \sin(x+y) \tag{44}$$

which is the exact solution.

IV. Conclusion

The paper has successfully described and applied the variational iteration method to some partial differential equations of physical significance. The method provides the solutions in terms of rapidly convergent series. It is also clear and remarkable that approximate solutions are in good agreement with analytical solution. The VIM was used in a direct way without using linearization, perturbation or restrictive assumptions. The method is elegant and reliable.

Notes

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