Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 2 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Chebychev Polynomials of the first Kind and Whittaker's Constant

By A. Anjorin & S. Akinbode
Lagos State University Ojo Lagos State Nigeria, Nigeria

Abstract- In this paper, we provide the conditions required for the Chebychev polynomials of the first kind $\{T_n(z)\}_{n\geq 0}$ to be a basic set. Then, we prove that the domain of effectiveness is a unit disc $D(R) = (0,1)$ related to the radius R of convergence of the associated basic series of $\{T_n(z)\}_{n\geq 0}$. We then give the Cannon condition satisfied by $\{T_n(z)\}_{n\geq 0}$ and the corresponding Whittaker's constant which is better than that obtained in the previous works using the Goncharov's polynomials. The order and type of the polynomials are also given.

GJSFR-F Classification : FOR Code : MSC 2010: 11S05 , 33C15

CHEBYCHEVPOLYNOMIALSOFTHEFIRSTKINDANDWHITTAKERSCONSTANT

Strictly as per the compliance and regulations of :

Chebychev Polynomials of the first Kind and Whittaker's Constant

A. Anjorin [α] & S. Akinbode [σ]

Abstract- In this paper, we provide the conditions required for the Chebychev polynomials of the first kind $\{T_n(z)\}_{n\geq 0}$ to be a basic set. Then, we prove that the domain of effectiveness is a unit disc $D(R) = (0,1)$ related to the radius R of convergence of the associated basic series of $\{T_n(z)\}_{n\geq 0}$. We then give the Cannon condition satisfied by $\{T_n(z)\}_{n\geq 0}$ and the corresponding Whittaker's constant which is better than that obtained in the previous works using the Goncharov's polynomials. The order and type of the polynomials are also given.

I. Introduction

The basic sets of polynomials continue to be at the core of many investigations [1]-[19] since the work of Whittaker [12]. The properties of series of the form $C_0P_0(z) + C_1P_1(z) + \cdots +$ where $P_0(z), P_1(z)\cdots$ are prescribed polynomials differ widely according to the particular polynomials choosen. For example, the region of convergence may be a circle (Taylor Series), an ellipse (series of Legendre polynomials), a half-plane (Newton's interpolation series) etc. Whittaker [12], in his attempt to find common properties exhibited by all these polynomials, introduced the subject of basic sets of polynomials. In his work, he gave the definition of basic set, basic series and effectiveness of basic sets. Cannon [13] obtained the necessary and sufficient conditions for the effectiveness of basic sets for classes of functions of finite radii of regularity and of entire functions. Nassif and Adepoju [18] investigated the zeros of polynomials belonging to simple sets. Wakid and Maker [2] also contributed to the investigations of the zeros of polynomials belonging to simple sets. Initially, the subject has been approached through the classical treatment. Then Newns [19] laid down the treatment of the subject based on functional analysis consideration. Over the years, this approach has received further advancement through the works of Falgas [16], Adepoju [3] and Kishka and El-Sayed Ahmed [1].

Definition 1.1 *[1] A sequence $\{P_n(z)\}_{n\geq 0}$ of polynomials is said to form a basic set if and only if polynomial $P_i(z), i = 0, 1, \cdots$ admits a unique finite linear combination of the polynomials of the set:*

$$P(z) = \sum_{k=0}^{n} C_k P_k(z), \quad where \;\; n < \infty. \tag{1.1}$$

Indeed, the polynomials $\{P_n(z)\}_{n\geq 0}$ are linearly independent, i.e.,

Author α σ: Lagos State University Ojo, Lagos NIgeria. e-mails: anjomathss@yahoo.com, akinbode_abayomi@yahoo.com

Ref

1. Z.M.G. Kishka and A. El-Sayed Ahmed, On the order and type of basic and composite sets of polynomials in complete Reinhardt domains, period. Math. Hung., 46(1), 67-76 (2003).

$$\sum_{k=0}^{n} C_k P_k(z) = 0$$

implies

$$C_0 = C_1 = C_2 = \cdots = C_n = 0.$$

The set of Polynomials $\{z^n\}_{n\geq 0}$ has a unique representation of the form

$$z^n = \sum_{k=0}^{n} \pi_{n,k} P_k(z), \quad \text{where} \quad \pi_{n,k} = C_k^n$$

so that

$$1 = \sum_{k=0}^{0} \pi_{0,k} P_k(z), \quad z = \sum_{k=0}^{1} \pi_{1,k} P_k(z), \quad z^2 = \sum_{k=0}^{2} \pi_{2,k} P_k(z), \quad \cdots, z^n = \sum_{k=0}^{n} \pi_{n,k} P_k(z). \tag{1.2}$$

In general, any polynomial of the form

$$P(z) = \sum_{i=0}^{k} P_i z^i$$

can be written

$$P(z) = \sum_{i=0}^{k} P_i z^i = \sum_{i=0}^{k} \left(\sum_{j=0}^{i} \pi_{i,j} P_j(z) \right) \tag{1.3}$$

$$= \sum_{i=0}^{k} P_i \left(\pi_{i,0} P_0(z) + \pi_{i,1} P_1(z) + \cdots \right) \tag{1.4}$$

$$= \sum_{i=0}^{k} P_i \pi_{i,0} P_0(z) + \sum_{i=0}^{k} P_i \pi_{i,1} P_1(z) + \cdots \tag{1.5}$$

$$= \sum_{n} C_n P_n(z), \tag{1.6}$$

so that the representation is unique.
To every basic set there corresponds an associated basic series. So, if $\{P_n(z)\}_{n\geq 0}$ forms a basic set, then the corresponding basic series can be written as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \left\{ \left(\sum_{k=0}^{n} \pi_{n,k} P_k(z) \right) \right.$$

$$= \sum_{n=0}^{\infty} a_n \left(\pi_{n,0} P_0(z) + \pi_{n,1} P_1(z) + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \pi_n f(0) P_n(z), \tag{1.7}$$

where

$$\pi_n f(0) = a_0 \pi_{0,n} + a_1 \pi_{1,n} + a_2 \pi_{2,n} + \cdots = \sum_{k=0}^{\infty} a_k \pi_{k,n}.$$

Notes

Substituting $a_k \equiv f^k(0)/k!$, we have

$$\pi_n f(0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \pi_{k,n} = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_{k,n} \frac{d^k f(0)}{dz^k} \tag{1.8}$$

so that the corresponding basic series takes the form

$$f(z) = \sum_{n=0}^{\infty} \pi_n f(0) P_n(z), \tag{1.9}$$

where $\pi_j, j = 1, 2, \cdots, n$ can be regarded as the elements of the set of operators of the basic set, $\{\pi_n\}_{n\geq 0}$, given by

$$\pi_n = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_{n,k} \frac{d^k}{dz^k}. \tag{1.10}$$

This set is called a basic set of operators if these operators are associated with a basic set of polynomials $\{P_n(z)\}_{n\geq 0}$. Prior to the definition of the effectiveness of basic sets, we recall that in the complex plane, a domain is an open connected set. A regular closed curve is usually denoted by $D(C)$, the domain interior to C; its closure is denoted by $\bar{D}(C)$ and the class of functions regular in $D(C)$ is written as $H(C)$. When C is a circle of radius $|z| = r$, the above entities are referred to as $D(r), \bar{D}(r)$ and $H(r)$, respectively.

Definition 1.2 *Let $f(z)$ be a function regular in the domain $D(R)$. The basic series $\sum_{n=0}^{\infty} \pi_n f(0) P_n(z)$ of (1.7) is said to represent $f(z)$ in the domain if it converges uniformly to $f(z)$ in $D(R)$. If the domain is a circle, we simply state that the basic set $\{P_n(z)\}_{n\geq 0}$ represents $f(z)$ in $|z| \leq R$. We write this as*

$$f(z) \sim \sum_{n=0}^{\infty} \pi_n f(0) P_n(z) \tag{1.11}$$

Definition 1.3 *[3] A basic series (or a basic set) $\{P_n(z)\}_{n\geq 0}$ is effective in a domain $D(R)$ if every function $f(z)$ regular in $D(R)$ is represented by the basic series.*

The following is relevant for the effectiveness in closed circle. Suppose the set $\{P_n(z)\}_{n\geq 0}$ is basic. We denote by

$$M_k(R) = \max_{|z|=r} |P_k(z)| \tag{1.12}$$

the maximum value of $P_k(z) \in \{P_n(z)\}_{n\geq 0}$ over $|z| = R$. The Cannon sum [13] $W_n(R)$ of the basic set $\{P_n(z)\}_{n\geq 0}$ is defined by

$$W_n(R) = \sum_{k=0}^{\infty} |\pi_{n,k}| M_k(R), \tag{1.13}$$

and the Cannon function is denoted by

$$\lambda(R) = \overline{\lim}_{n\to\infty} \{W_n(R)\}^{\frac{1}{n}} = \lim_{n\to\infty} \sup \{W_n(R)\}^{\frac{1}{n}}. \tag{1.14}$$

Ref

3. J. A. Adepoju, Basic set of Goncarov polynomials and Faber regions, Ph.D. Thesis (1979). Lagos.

This definition is satisfied in both open and closed circles with the notation $<$ and $\leq$ respectively. Let us now define the Cannon condition.

Definition 1.4 *[3] A set $\{P_n(z)\}_{n\geq 0}$ of polynomials in which the polynomial $P_n(z)$ is of degree n, is necessarly basic and is called a simple set. Let $\{P_n(z)\}_{n\geq 0}$ be a basic set of polynomials. Then the number $V_{(n)}$ is defined as the number of polynomials of the set whose degree is less than n. The polynomials of the set $\{P_n(z)\}_{n\geq 0}$ are linearly independent if $V_{(n)} \leq n$ for $n \geq 1$. The number D_n is also defined as the degree of polynomials of the highest degree in the representation (1.1).*

For

$$z^n = \sum_{k=0}^{n} \pi_{n,k} P_k(z), \quad \text{we have} \quad D_n \geq n.$$

Definition 1.5 *[1] [3] Let N_n be the number of non-zero terms in the representation. Then the basic set $\{P_n(z)\}_{n\geq 0}$ is called the Cannon set if $N_n^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$. This condition is called the Cannon condition.*

In this paper, we are interested in the investigation of properties of the Chebychev basic set. In Section 2, we prove that the set of Chebychev polynomials $\{T_n(z)\}_{n\geq 0}$ is a basic set. In Section 3, we provide the corresponding associated basic series. In section 4, we show the effectiveness of the basic set. In section 5, we prove that $\{T_n(z)\}_{n\geq 0}$ forms a Cannon set. Finally, in section 6, we infer that the Chebychev basic set gives better improvement for the Whittaker constant than the other sets of classical polynomials.

II. BASIC SET OF CHEBYCHEV POLYNOMIALS

Let us prove the following

Theorem 2.1 *The set of Chebychev polynomials $\{T_n(z)\}_{n\geq 0}$ is a basic set.*

(i) We first show that the representation

$$T_n(z) = \sum_{k=0}^{[n/2]} \binom{n}{2k} z^{n-2k} \left(z^2-1\right)^k \quad \text{is unique.} \tag{2.1}$$

Since

$$\begin{aligned} T_0(z) &= 1 = \sum_{k=0}^{0} \binom{0}{2k} z^{0-2(0)} \left(z^2-1\right)^0 = 1 \\ T_1(z) &= \sum_{k=0}^{0} \binom{1}{2k} z^{1-2(0)} \left(z^2-1\right)^0 = z \\ T_2(z) &= \sum_{k=0}^{1} \binom{2}{2k} z^{2-2(k)} (z^2-1)^k = 2z^2-1 \\ &\vdots \\ T_n(z) &= \sum_{k=0}^{[n/2]} \binom{n}{2k} z^{n-2k} (z^2-1)^k. \end{aligned}$$

In general, given any polynomial

$$T_n(z) = \sum_{i=0}^{[n/2]} t_i T_i(z) \tag{2.2}$$

and using $T_n(z)$ representation, we can write

$$\begin{aligned} T_n(z) &= \sum_{i=0}^{[n/2]} t_i \left(\sum_{j=0}^{[i/2]} \binom{i}{2j} z^{i-2j}(z^2-1)^j \right) \\ &= \sum_{i=0}^{[n/2]} t_i \left\{ \binom{i}{0} z^i + \binom{i}{2} z^{i-2(1)}(z^2-1) + \binom{i}{4} z^{i-2(2)}(z^2-1)^2 + \cdots \right\} \\ &= \sum_{i=0}^{[n/2]} t_i \binom{i}{0} z^i + \sum_{i=0}^{[n/2]} t_i \binom{i}{2} z^{i-2}(z^2-1) + \sum_{i=0}^{[n/2]} t_i \binom{i}{4} z^{i-2(2)}(z^2-1)^2 + \cdots \\ &= \sum_{k=0}^{[n/2]} t_k \binom{n}{2k} z^{n-2k}(z^2-1). \end{aligned} \tag{2.3}$$

Hence, the set$\{T_n(z)\}_{n\geq 0}$ is well represented. To complete the proof, we state the following

Lemma 2.2 The sequence $\{T_n(z)\}_{n\geq 0}$ of Chebychev polynomials in which $T_n(z)$ is of degree n is basic.

We have $T_0(z) = 1T_0$, since $T_0(z)$ is a polynomial of degree zero. Dividing through by T_0, we have $1 = \frac{1}{T_0}T_0(z) = \pi_{0,0}T_0(z)$ with $\pi_{0,0} = \binom{0}{0} = 1$. Let also $T_1(z) = T_{1,0} + T_{1,1}z$ where $T_{1,1} \neq 0$. Dividing through by $T_{1,1}$, we have $\frac{1}{T_{1,1}}T_1(z) = \frac{T_{1,0}}{T_{1,1}} \cdot 1 + z$ so that $z = -\frac{T_{1,0}}{T_{1,1}} \cdot 1 + \frac{1}{T_{1,1}}T_1(z) = \frac{-T_{1,0}}{T_{1,1}} \cdot \pi_{0,0}T_0(z) + \frac{1}{T_{1,1}}T_1(z) = -\pi_{1,0}T_0(z) + \pi_{1,1}T_1(z)$.
Hence the representation is true for $n = 0, 1$. Suppose it is true for $2, 3, \cdots n-1$ and let

$$\begin{aligned} T_n^*(z) &= T_{n,0} + T_{n,1}z + T_{n,2}(2z^2-1) + T_{n,3}(4z^2-3z) + \cdots \\ &\quad + T_{n,n-1}\left[\sum_{k=n-1}^{[n/2]} \binom{n-1}{2k} z^{n-2k}(z^2-1)^k \right] \\ &\quad + T_{n,n} \sum_{k=n}^{[n/2]} \binom{n}{2k} z^{n-2k}(z^2-1)^k \end{aligned}$$

where $T_{n,n} \neq 0$. Then

$$\begin{aligned} T_n^*(z) &= T_{n,0}\binom{0}{0} z^0 + T_{n,1}\left\{ \binom{1}{0} z^{1-0}(z^2-1)^0 \right\} \\ &\quad + T_{n,2}\left\{ \binom{2}{0} z^{2-0}(z^2-1)^0 + \binom{2}{2} (z^2-1) \right\} + \cdots \end{aligned}$$

$$+T_{n,n-1}\left\{\sum_{k=n-1}^{[n/2]}\begin{pmatrix} n-1 \\ 2k \end{pmatrix} z^2(z^2-1)^k\right\}$$

$$+T_{n,n}\left\{\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k\right\}, \qquad (2.4)$$

what we can write in a compact form as

$$\begin{aligned} T_n^*(z) &= T_{n,0}+T_{n,1}z+T_{n,2}(2z^2-1)+T_{n,3}(4z^3-3z)+\cdots \\ &+T_{n,n}\left\{\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k\right\}. \end{aligned} \qquad (2.5)$$

Let $T_{n,j}=\alpha_j$. Then we get

$$T_{n,0}=\alpha_0, T_{n,1}=\alpha_1, \cdots, T_{n,n}=\alpha_n,$$

so that

$$T_n^*(z)=\alpha_0T_0(z)+\alpha_1T_1(z)+\alpha_2T_2(z)+\cdots+\alpha_nT_n(z)$$

where

$$T_0(z)=1, T_1(z)=z, T_2(z)=(2z^2-1), \cdots.$$

Consequently,

$$\begin{aligned} T_{n,n}\left\{\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k\right\} &= T_n^*(z)-\{\alpha_0T_0(z) \\ &+\alpha_1T_1(z)+\cdots+\alpha_{n-1}T_{n-1}(z)\} \end{aligned}$$

from which we get on dividing by $T_{n,n}$:

$$\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k=\frac{T_n^*(z)}{T_{n,n}}-\frac{1}{T_{n,n}}\left\{\alpha_0T_0(z)+\alpha_1T_1(z)+\cdots+\alpha_nT_n(z)\right\}. \qquad (2.6)$$

The right hand side can be denoted by $\tilde{T}_n^\star(z)$. Hence, we obtain

$$\tilde{T}_n^\star(z)=\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k. \qquad (2.7)$$

For $0<k<n$, we have

$$\tilde{T}_n^\star(z)=\sum_{k=0}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k. \qquad (2.8)$$

By unique representation $T_n^\star(z)=T_n(z)$. Furthermore, the polynomials are linearly independent. Indeed, one can immediately prove that

Notes

$$\begin{aligned} a_0T_0(z)+a_1T_1(z)+\cdots+a_nT_n(z) &= a_01+a_1z+a_2(2z^2-1)+a_3(4z^3-3z) \\ &\quad +a_4(8z^4-8z^2+1) \\ &\quad +\cdots+a_n\sum_{k=0}^{[n/2]}\binom{n}{2k}z^{n-2k}(z^2-1)^k \\ &= 0 \end{aligned} \tag{2.9}$$

Notes

leads to $a_i=0$ for $i=0,1,2,\cdots,n$. Thus, the set $\{T_n(z)\}_{n\geq 0}$ of Chebychev polynomials is basic.

III. Associated Basic Series of $\{T_n(z)\}_{n\geq 0}$

In this section, we are interested in the investigation of existence of basic series associated with the basic set $\{T_n(z)\}_{n\geq 0}$.

Theorem 3.1 *The set $\{T_n(z)\}_{n\geq 0}$ has an associated basic series.*

Proof: Consider the function

$$f(z)=\sum_{k=0}^{\infty}a_nT_n(z). \tag{3.1}$$

Expanding it, we get

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}a_n\left(\sum_{k=0}^{[n/2]}\binom{n}{2k}z^{n-2k}\left(z^2-1\right)^k\right) \\ &= \sum_{n=0}^{\infty}a_n\left\{\binom{n}{0}z^n+\binom{n}{2}z^{n-2}(z^2-1)+\cdots+\binom{n}{2j}z^{n-2j}(z^2-1)^j+\cdots\right\} \\ &= \sum_{n=0}^{\infty}a_n\left\{\binom{n}{0}Q_0(z)+\binom{n}{2}Q_1(z)+\cdots+\binom{n}{2j}Q_j(z)+\cdots\right\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} Q_0(z) &= z^n, \quad Q_1(z)=z^{n-2}(z^2-1), \quad Q_2(z)=z^{n-4}(z^2-1)^2 \\ &\vdots \\ Q_j(z) &= z^{n-2j}(z^2-1)^j. \end{aligned} \tag{3.3}$$

Then, we can write

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\left\{a_n\binom{n}{0}Q_0(z)+a_n\binom{n}{2}Q_1(z)+\cdots+a_n\binom{n}{2j}Q_j(z)+\cdots\right\} \\ &= a_0\binom{0}{0}Q_0(z)+a_1\binom{1}{0}Q_0(z)+\cdots+a_n\binom{n}{0}Q_0(z) \\ &+ a_0\binom{0}{2}Q_1(z)+a_1\binom{1}{2}Q_1(z)+\cdots+a_n\binom{n}{2}Q_1(z) \\ &\quad +a_0\binom{0}{4}Q_2(z)+a_1\binom{1}{4}Q_2(z)+\cdots+a_n\binom{n}{4}Q_2(z)+\cdots \end{aligned}$$

$$= \sum_{k=0}^{\infty} a_k \begin{pmatrix} k \\ 2n \end{pmatrix} Q_n(z) \equiv \sum_{n=0}^{\infty} \pi_n f(0) Q_n(z),$$

where $\pi_n f(0) = a_k \begin{pmatrix} k \\ 2n \end{pmatrix}$. Hence, the basic series associated with the basic set $\{T_n(z)\}_{n\geq 0}$ is $f(z) = \sum_{k=0}^{\infty} a_k \begin{pmatrix} k \\ 2n \end{pmatrix} Q_n(z)$, where $\{Q_n(z)\}_{n\geq 0}$ forms a basis for $\{T_n(z)\}_{n\geq 0}$

Notes

IV. Effectiveness of Chebychev Polynomials $D(R)$ or $D_+(r)$

In this section, we investigate the effectiveness of $\{T_n(z)\}_{n\geq 0}$ in the domain $D(R)$ or $D_+(r)$.

Theorem 4.1 Let $\{T_n(z)\}_{n\geq 0}$ be a basic set of polynomials and suppose that, for any value $R > 0, \lambda(R) = \sigma \geq R$, then the basic series is effective in $\tilde{D}(R)$ for the class $\bar{H}(\sigma)$.

Proof:

Let $f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$ be any function regular in $|z| < \sigma$. Then $\sum_{n=0}^{\infty} a_n T_n(z)$ converges and $\lim_{n\to\infty} a_n z^{n-2k}(z^2-1)^k = 0$. So, we can choose n large enough so that $|a_n z^{n-2k}(z^2 - 1)^k| \to 0$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} |a_n z^{n-2k}(z^2-1)^k| \leq \lim_{n\to\infty} |a_n|\sigma^n < 1$$

This implies that

$$\lim_{n\to\infty} |a_n|^{1/n} < \frac{1}{\sigma}. \tag{4.1}$$

Now, consider the series

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z).$$

We have, for $|z| \leq R$,

$$\left| a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z) \right| \leq |a_n| W_n(R).$$

By (4.1)

$$\overline{\lim}_{n\to\infty} \{|a_n| W_n(R)\}^{\frac{1}{n}} \leq \overline{\lim}_{n\to\infty} |a_n|^{\frac{1}{n}} \cdot \lambda(R) < \frac{\sigma}{\sigma} = 1.$$

Thus, the series

$$\sum_{n=0}^{\infty} |a_n| W_n(R)$$

Notes

is convergent and since

$$|\sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z)| \leq \sum_{n=0}^{\infty} |a_n| W_n(R),$$

then by Weierstrass-M-test, the series

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z)$$

is uniformly and absolutely convergent in $|z| \leq R$. Re-arranging the terms of the series, we conclude that the resulting basic series

$$\sum_{n=0}^{\infty} \pi_n f(0) Q_n(z)$$

converges uniformly to $f(z)$ in $|z| \geq R$. Thus, the basic series represents $f(z)$ in $|z| \leq R$.

Corollary 4.2 If for any value of $R > 0, \lambda(R) = R$, then the basic set $\{T_n(z)\}_{n \geq 0}$ will be effective in $|z| \leq R$.

Proof: If $R > 0$ and $\lambda(R) = R$, from Theorem 4.1, the basic series represents in $|z| \leq R$ every function regular in $|z| \leq R$. That is to say, the basic series (or the basic set) will be effective in $|z| \leq R$. Thus the condition that $\lambda(R) = R$ is a sufficient condition for effectiveness in $|z| \leq R$.

Theorem 4.3 The necessary and sufficient condition for the set $\{T_n(z)\}_{n \geq 0}$ to be effective in $|z| \geq R$ is that $\lambda(R) = R$.

Proof:

Necessity: If the set $\{T_n(z)\}_{n \geq 0}$ is effective in $|z| \leq R$, then $\lambda(R) = R$. If on the contrary, $\lambda(R) > R$, then for any number ρ for which $R < \rho < \lambda(R)$, there exists a function $f(z)$ of radius of regularity ρ, that is $f(z)$ is regular in $|z| \leq R$ and that the basic series cannot represent in $|z| \leq R$. Thus, the set will not be effective in $|z| \leq R$.
Sufficienty: This follows directly from Corollary 4.2 since for any value of $R > 0, \lambda(R) = R$. Then the basic set $\{T_n(z)\}_{n \geq 0}$ will be effective in $|z| \leq R$. Since, $\{T_n(z)\}_{n \geq 0}$ is effective in $D(R)$ (or $D_+(r)$), then it is represented in $D(R)$ (or $D_+(r)$) by a basic series of the form

$$f(z) = \sum_{n=0}^{\infty} \pi_n f(0) Q_n(z),$$

where $f(z)$ belongs to $H(R)$ (or $H(r)$), the class of all holomorphic functions. Hence, there exists $f(z) \in H(R)$ (or $H(r)$) denoted by $f(z) = \sum_{n=0}^{\infty} \pi_n f(0) Q_n(z)$, representing the $\{T_n(z)\}_{n \geq 0}$ in $D(R)$ (or $D_+(r)$). Hence, $\{T_n(z)\}_{n \geq 0}$ is effective in $D(R)$ (or $D_+(r)$).

Suppose by definition

$$M_n(R) = \max_{|z|=R} |T_k(z)| = \max_{|z|=R} \left| z^{n-2k}(z^2-1)^k \right| \leq R^{n-2k}(R^2-1)^k \leq R^n(R^2-1)^k.$$

Then

$$W_n(R) = \sum_{k=0}^{[n/2]} |\pi_{n,2k}|\, M_n(R) < \sum_{k=0}^{[n/2]} |\pi_{n,2k}|\, R^n(R^2-1)^k$$

and the Cannon function $\lambda(R)$ is

$$\lambda(R) = \overline{\lim}_{n\to\infty} \{W_n(R)\}^{1/n}.$$

This implies that

$$\lambda(R) \leq \overline{\lim}_{n\to\infty} \left\{ \sum_{k=0}^{[n/2]} |\pi_{n,2k}| \right\}^{1/n} \{R^n\}^{1/n} (R^2-1)^{k/n} = R.$$

Since $\lambda(R)$ is non-negative, i.e. $\lambda(R) > 0$, we have that $\lambda(R) = R$. Then the domain of effectiveness $D(R) = (0,1)$ is a disc.

V. Cannon Condition

In this section, we investigate the condition for the set $\{T_n(z)\}_{n\geq 0}$ to be a Cannon set. This can be stated in the following result.

Theorem 5.1 *The set $\{T_n(z)\}_{n\geq 0}$ forms a Cannon set.*

Proof: It suffices to show that if $N_n = C_{2k}^n$ is the number of non-zero terms in the unique representation of $\{T_n(z)\}_{n\geq 0}$, $N^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$. Using Stirling formula, i.e. $n! = \sqrt{2\pi n^n}\, e^{-n}$, we have

$$N_n = \frac{n!}{(2k)!(n-2k)!} = \frac{\sqrt{2\pi n^n} e^{-n}}{\sqrt{2\pi (2k)^{2k}}\, e^{-2k}} \times \left(\sqrt{2\pi (n-2k)^{(n-2k)}}\, e^{-(n-2k)} \right)^{-1}.$$

As n approaches infinity, we have that , $N_n^{1/n} \longrightarrow 1$. Hence $\{T_n(z)\}_{n\geq 0}$ is a Cannon set.

VI. Improved Whittaker's Constant using Chebychev Polynomials of the First Kind

a) Generalities

Over the years there has been intensive investigation [3]-[12] [18], on the best approximation of the so called Whittaker's constant noted W. This constant was introduced by Whittaker in his work [12] on interpolation of function. The problem is the following: given a function $f(z)$ in a complex plane $\mathbb{C}$, what is the upper bound for which this function is entire and is of exponential type c such that in a domain $D(R)$ which could be a disc, one has $f(z) = f'(z) = f''(z) = \cdots = f^{(n)}(z) = 0$. In other words, what range does the constant W lie? Ever since there remains the question of the range which best approximates this constant. To this question tackle many others since the

Ref

18. Nassif, M, and Adepoju, J.A., (1978), "Zeros of polynomials belonging to simple set" National Academy of Science letters ; Vol. 1, pp. 223-224.

historical work by Whittaker. Up till now, to our best knowledge of the literature, the best range is obtained using Goncharov polynomials and this has been computed to be $0.7259 \leq W \leq 0.7380$. In this section, we state this range can be improved using Chebychev polynomials of the first kind instead of the other classical polynomials. The Chebychev polynomials are of considerable interest in interpolation theory [12]. The degree of accuracy of the interpolation result obtained depends on the bound which can be found for these polynomials. With $|z_k| \leq 1 \quad k = 0, 1, \cdots$ the maximum value of these polynomials will be denoted by M_n as follows:

$$M_n = |T_n(z, z_0, z_1, \cdots, z_n)|.$$

We claim that for $n \geq 1$, there exists r such that

$$M_n < r^{n+1}, \tag{6.1}$$

where r is some positive number, $r > 0$. This leads to an improved value of the Whittaker's constant W defined as the least upper bound of a number c such that the function $f(z)$ is an entire function of exponential type c and if $f(z)$ and each of its derivatives have at least one zero in the unit circle, then $f(z) = f'(z) = \cdots = f^{(n)}(z) = 0$ or equivalently $f(z) \equiv 0$. Let now the set $\{T_n(z)\}_{n\geq 0}$ be a basic set of Chebychev polynomial, effective in the circle $|z| \leq 1$. Then $\{T_n(z)\}_{n\geq 0}$ is said to be of order 1 if its zeros lie in $|z| \leq 1$. In this case, the polynomial $\{T_n(z)\}_{n\geq 0}$ can be expanded in the power series

$$T_n(z) = \sum_{k=0}^{\infty} e^{\pi ik/2} \frac{z^k}{k!}.$$

Let W_j be the Whittaker's constant, $0.7259 \leq W \leq 0.7380$ with $j = 0, 1, 2, \cdots, n$ and $W_1 \neq W_2 \neq \cdots \neq W_n$. Then the polynomial corresponding to W_j is of the type $\sigma = \frac{1}{W_j}$, where W_j is the modulos of the zeros of $\{T_n(z)\}_{n\geq 0}$. Assume $f(z)$ be an entire function exponential type c. Then if

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!},$$

it follows that $a_n = O(c+\epsilon)^n, \epsilon > 0$. That is, for any $b > c$, it follows that for sufficiently large n

$$|a_n| < b^n. \tag{6.2}$$

Denote by $\{z_k, k = 0, 1, \cdots, n\}$ the points, inside the unit circle where $f(z)$ and its derivative vanish. Then $\{T_n(z)\}_{n\geq 0}$ will be represented by the power series

$$f(z) = \sum_{k=0}^{\infty} \frac{a_{n+k}}{k!} T_k(z).$$

For large n and $|z| \leq 1$, we get from (6.1) and (6.2)

$$|f(z)| \quad \leq \quad b^{n+k} r^{n+k+1} = \frac{(br)^n r}{(1 - br)}.$$

Ref

12. J. M. Whittaker, "Interpolatory Function Theory", The University Press, Cambridge, 1935.

Thus as $n \longrightarrow \infty$ we obtain

$$|f(z)| \leq \frac{r}{1-br}.$$

In this case the Whittaker's constant $W > \frac{1}{r}$.

b) Main result

Here, we aim at proving that the Levinson method [9] applied to Chebychev polynomials instead of Goncharov one's, provides a better approximation of the range of the Whittaker's constant. Furthermore, the Chebychev polynomials reveal to be the best set for the computation of the closest boundary value of this range than the other classical polynomials (Lagauerre, Legendry, Jacobi, Bessel etc.) commonly used in Mathematical Physics. Following Levinson method, we get the following statement.

Theorem 6.1 *For an entire function $T_n(z)$ of Chebychev polynomials, there exists a positive number r such that $|T_n(z)| = M_n < r^{n+1}$. Then, for Chebychev polynomials of the first kind, the Whittaker's constant has an upper bound not exceeding 0.7380 and a lower bound not exceeding 0.73778.*

Proof: Using Chebychev polynomial representation, we obtain

$$\frac{\partial T_n}{\partial z_k} = -T_k(z, z_1, z_2, \cdots, z_n)T_{n-k-1}(z_k, z_{k+1}, \cdots, z_{k-1})$$

Using Euler's formula for homogeneous function of degree n allows us to write

$$\begin{aligned} nT_n(z) &= \frac{z\partial T_n}{\partial z} + \frac{z_0\partial T_n}{\partial z_0} + \cdots + z_{n-1}\frac{\partial T_n}{\partial z_{n-1}} \\ nT_n(z) &= zT_{n-1}(z, z_1, \cdots, z_{n-1}) \\ &\quad - \sum_{k=0}^{n-1} z_k T_k(z, \cdots, z_k)T_{n-k-1}(z_k, \cdots, z_{n-1}) \end{aligned} \quad . \tag{6.3}$$

Taking absolute value of both sides of the relation (6.3), we obtain

$$M_n \leq M_{n-1} + \sum_{k=0}^{n-1} M_k M_{n-k-1}. \tag{6.4}$$

To get M_n we will make the use of the Taylor series expansion of the function $T_n(z, z_0, z_1, \cdots, z_n)$. First, let us define the new function H_n such that

$$\begin{aligned} H_n(z_0, z_1, \cdots, z_{n-1}) &= T_n(0, z_0, \cdots, z_{n-1}) \\ \frac{\partial H_n}{\partial z_0} &= -T_{n-1}(z_0, z_1. \cdots, z_{n-1}) \\ \frac{\partial^2 H_n}{\partial z_0^2} &= -T_{n-2}(z_0, z_2. \cdots, z_{n-1}) \end{aligned} \tag{6.5}$$

Ref

9. N. Levinson, A Theorem of Boars, Duke Math. J. 8(1941) 181-182.

Notes

Thus, by Taylor's theorem, we get

$$H_n(z_0, z_1, \cdots, z_{n-1}) = -z_0 H_{n-1}(z_1, \cdots, z_{n-1}) - \frac{z_0^2}{2!} H_{n-2}(z_2, \cdots, z_{n-1}) - \cdots - \frac{z_0^{n-1}}{(n-1)!} H_1(z_{n-1}). \quad (6.6)$$

so that

$$T_n(z, z_0, z_1, \cdots, z_{n-1}) = (z-z_0) H_{n-1}(z_1, z_2, \cdots, z_{n-1}) + \frac{(z-z_0)^2}{2!} H_{n-2}(z_2, \cdots z_{n-1}) + \frac{(z-z_0)^3}{3!} H_{n-3}(z_3, z_4, \cdots, z_{n-1}) + \cdots + \frac{(z-z_0)^n}{n!} H_0, \quad (6.7)$$

with

$$H_1(z) = -z_0, \quad H_2(z_0, z_1) = z_0 z_1 - \frac{z_0^2}{2!}, \quad H_3(z_0, z_1, z_2) = -z_0 z_1 + \frac{z_0 z_1^2}{2!} + \frac{z_0^2 z_1}{2!} - \frac{z_0^3}{3!}.$$

Now, as $|z| \leq 1$, assuming $z = -1, z_0 = 1$ and using (6.4), we obtain

$$M_1 \leq |1 - (-1)| = 2.$$

Similarly

$$M_2 \leq \left(\frac{2|z - z_0|}{2} \right), H_1(z_0) = 2 \sin \theta.1 \equiv f(\theta)$$

which attains its maximum at $\theta = \pi/2$. Hence $M_2 \leq 2$. By the same way, we find $M_3 \leq 3, M_4 \leq 5.06759$, etc. so that by (6.4), we get

$$M_n \leq 3M_{n-1} + 4M_{n-2} + 4M_{n-3} + 6M_{n-4} + 10.13518 M_{n-5}.$$

Then, assuming the existence of a number r_0 such that $M_k < r_0^{k+1}$, we obtain

$$M_n \leq 3r^n + 4r^{n-1} + 4r^{n-2} + 6r^{n-3} + 10.13518 r^{n-4} \cdots \quad (6.8)$$

As the right side of (6.8) is not greater than nr^{n+1}, for $r_0 < r$, we have

$$nr^{n+1} > M_n$$

so that for $n = 10$

$$10r^{11} > 3r^{10} + 4r^9 + 4r^8 + 6r^7 + 10.13518 r^6 \cdots \quad (6.9)$$

Dividing by r^6, (6.9) gives four complex roots with one real root stated below

$$\begin{array}{ll} (i)\ -0.75884 - 0.49888895i; & (ii)\ -0.75884 + 0.49888895i; \\ (iii)\ 0.231357 - 0.923815i; & (iv)\ 0.231357 + 0.923815i; \\ (v)\ 1.35497 \cong 1.3550. & \end{array} \quad (6.10)$$

The inverse of the real root gives the Whittaker's constant W such that

$$W > \frac{1}{r} = \frac{1}{1.3550} = 0.7380. \quad (6.11)$$

By Newton Ralpson iteration there exists r^* such that

$$1.354967 \leq r^* \leq 1.35542 \quad \text{and} \quad \frac{1}{1.354967} \geq \frac{1}{r^*} \geq \frac{1}{1.35542}$$

which gives

$$0.7380 \geq W \geq 0.73778.$$

So taking into account (6.11), we obtain the Whittaker constant $W = 0.7380$ which improves the earlier known result by Levinson [9] and Macintyre [6]. So, the new range of the Whittaker's constant becomes

$$0.73778 \leq W \leq 0.7380. \tag{6.12}$$

Theorem 6.2 Let $\{T_n(z)\}$ be the set of Chebychev polynomials associated with the points $\alpha\beta^n$ where α and β are complex numbers. If $\beta > 1$, the set will be of infinite order and if $\beta = 1$, the set will be of order 1 and type $\sigma = \frac{|\alpha|}{\tau}$ where τ is the modulo of a zero of the function

$$f(\theta) = \sum_{t=0}^{\infty} \beta^{t\pi i/2} \theta^t / t!$$

of the least modulo.

Proof: The Chebychev polynomial of the first kind can be written as

$$f(z) = \cos(n \cos^{-1}(z)).$$

Let $\cos^{-1}(z) = \theta$ so that

$$f(\theta) = \cos(n\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}$$

or equivalently, putting $t = 2n$ and replacing the exponential function e by β,

$$f(\theta) = \sum_{t=0}^{\infty} \beta^{\pi i t/2} \frac{\theta^t}{t!}.$$

Hence, the Chebychev polynomial is of order 1. Suppose $f(\theta)$ has no zero on the finite plane $-1 \leq \theta \leq 1$. Then $f(\theta)$ is constant. It follows that τ is the modulo of a zero of $f(\theta)$ nearest to the origin. Thus there should exist a function $h(\theta) = \frac{1}{f(\theta)}$ regular on the $|\theta| < \tau$, where

$$h(\theta) = \sum_{t=0}^{\infty} \beta^{-\pi i t/2} t! \theta^- t,$$

which can be written as

$$h(\theta) = \sum_{t=0}^{\infty} c_t \theta^{-r}, \tag{6.13}$$

Ref

6. S.A. Macintyre, An upper bound for the Whittaker's constant, London Maths. Soc. J. 22 (1947), 305-311.

Notes

with $c_t = t!\beta^{-\pi i t/2}$. We assume that

$$\lim_{t\to\infty} \{c_t\}^{\frac{1}{t}} = \frac{1}{\tau} > 0.$$

Suppose now there exist a number $q < \dfrac{1}{\tau}$ and a positive integer m such that

$$c_t > q^m; 0 \leq m \leq t. \tag{6.14}$$

The Cannon function $\lambda(h(\theta):[r])$ is defined as

$$\lambda(h(\theta):[r]) = \overline{\lim_{t\to\infty}} \{W_t(h(\theta):[r])\}^{\frac{1}{t}},$$

where $W_t(h(\theta):[r])$ is the Cannon sum expressed as

$$W_t(h(\theta):[r]) \;=\; \sum_{t=0}^{\infty} \left| t!\beta^{-t\pi i/2} \right| M_t(\theta) = \sum_{t=0}^{\infty} |c_t|\, M_t(\theta),$$

where

$$M_t(\theta) = \max_{|\theta|=r} |\theta^t|.$$

Hence

$$W_t(h(\theta):[r]) \geq |q^m|\, r^{-t},$$

that ensures that the order of the set $\{T_n(z)\}_{n\geq 0}$ is infinite in the case $\beta > 1$.
Now, let us examine the case $\beta = 1$. Then, by similar arguments, an integral function defined by

$$f(\theta) = \sum_{t=0}^{\infty} \frac{\theta^t}{t!}$$

is of order 1.
By analogy, following step by step the previous development, the case $|\alpha| = 1$ and $\beta \to 1$ as $n \to \infty$ leads to the same conclusion as above. Therefore, the order and type of the Chebychev set of polynomials are well defined.
To complete the full analysis regarding the better approximation of the lower and upper bounds of the Whittaker's constant, we achieve the computation of range on which lies this constant for all classical commonly used polynomials in Mathematical Physics i.e. Laguerre, Jacobi and Bessel polynomials. These sets of polynomials can be categorised into two groups: A-basic set and B-basic set. A-basic set contains those polynomials of the form $\sum_{k=0}^{n} a_k z^k$, (e.g. monomials (z^n), Laguerre polynomials $\left.\sum_{k=0}^{n} \frac{(-n)_k z^k}{(k!)^2}\right)$, etc), while B-basic set is the set of the polynomials of the form $\sum_{k=0}^{[n/2]} a_k z^k$ (such as Chebychev polynomials of the first kind, Hermite, Legendre Jacobi, Gaugenbauer polynomials, etc). Hence, the polynomials belonging to B-basic set have a better Whittaker's constant than those polynomials of A-basic set. See data provided in the table.

Polynomials	Range of Whittaker's constant	Order	Type
A-basic set	$0.7220 \leq W \leq 0.7378$	1 if of the form $\sum_{t=0}^{\infty} \frac{\beta^{\pi it/2}\theta^t}{t!}$ or otherwise ∞.	$1.35538 \leq \sigma \leq 1.3589$
B-basic set	$0.7230 \leq W \leq 0.7380$	1 if ultraspherical or the form $\sum_{t=0}^{\infty} \frac{\beta^{\pi it/2}\theta^t}{t!}$ or otherwise ∞	$1.35542 \leq \sigma \leq 1.38408$

Notes

VII. Concluding Remarks

In this paper, we have shown that the set of Chebychev polynomials of the first kind $\{T_n(z)\}_{n\geq 0}$ forms a basic sets and provides the best improved Whittaker's constant than other classical basic sets. The classical basic sets can be splited into two classes denoted, respectively, A-basic sets and B-basic sets of polynomials. In general, the Whittaker's constant is best improved upon by the B-basic sets.

References Références Referencias

1. Z.M.G. Kishka and A. El-Sayed Ahmed, On the order and type of basic and composite sets of polynomials in complete Reinhardt domains, period. Math. Hung., 46(1), 67-76 (2003).
2. R.H. Maker and O.M. Wakid, On the order and type of basic sets of polynomials associated with functions of algebraic semi-block matrices, period. Math. Hung., 8, 3-7, (1977).
3. J. A. Adepoju, Basic set of Goncarov polynomials and Faber regions, Ph.D. Thesis (1979). Lagos.
4. M. A. Evgrafor, "Interpolyacionnaya Sadaca Abelya - Goncarova" (The Abel-Goncharov Interpolation Problem), Goncharov interpolation problem, Gosudavstr. Izadat. Tohn.-Teor. Lit., Moscow, 1954.
5. S.S. Macintyre, On the zeros of successive derivatives of integral functions, Trans. Amer. Math. Soc. 67 (1949). 241-251.
6. S.A. Macintyre, An upper bound for the Whittaker's constant, London Maths. Soc. J. 22 (1947), 305-311.
7. N Levinson, The Goncharov polynomials, Duke Math. J. 11(1944), 729-733; idem J. 12 (1945), 335.
8. R.P. Boas, Functions of exponential type, 11, IV, Duke Math., J. 11 (1944), 17-22 and 799.
9. N. Levinson, A Theorem of Boars, Duke Math. J. 8(1941) 181-182.
10. R.P. Boas, Univalent derivatives of entire functions, Duke Math. J. 6 (1940), 719-721.
11. I.J. Shoenberge, On the zeros of successive derivatives of integral functions, Trans. Amer. Math. Soc. 40 (1936), 12-23.
12. J. M. Whittaker, "Interpolatory Function Theory", The University Press, Cambridge, 1935.
13. Cannon, B. (1937) "On the convergence of series of polynomials" Proceedings of the London Math. Soc. Ser. 2, Vol. 43; pp. 348-364.

14. Copson, E. T. (1955) "Theory of functions of a complex variable" (Oxford) (From corrected sheets of ˙rst edition), pp. 164-165.
15. Dienes, P. (1931) "The Taylor Series" (Oxford).
16. Falgas, M. (1964) "Sur les s˙eries de base de polynˆomes" Annales Scienti˙ques de L'Ecole NormalSuperieure 3e serie, vol. 8, pp. 1-76.
17. Gupta, S. L. and Rani, N. (1975) "Fundamental real analysis" (India), pp.164-266.
18. Nassif, M, and Adepoju, J.A., (1978), "Zeros of polynomials belonging to simple set" National Academy of Science letters ; Vol. 1, pp. 223-224.
19. Newns, W. F., (1953), On the representation of analytic functions by in˙nite series, Phil. Trans. of the Roy. Soc. of London, Ser. A 245, 439 - 468.

Notes

This page is intentionally left blank