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Global Exponential Stability of Impulsive Functional Differential Equations with Effect of Delay at the Time of Impulses

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GLOBALEXPONENTIALSTABILITYOFIMPULSIVEFUNCTIONALDIFFERENTIALEQUATIONSWITHEFFECTOFDELAYATTHETIMEOFIMPULSES

Strictly as per the compliance and regulations of :

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1. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific,Singapore, 1989.

Global Exponential Stability of Impulsive Functional Differential Equations with Effect of Delay at the Time of Impulses

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Abstract- This paper studies the global exponential stability of impulsive functional differential system with the effect of delay at the time of impulses by using Lyapunov functions and Razumikhin technique. This result extends some results existing in the literature. The obtained result also shows that the derivative of Lyapunov function may not be negative even then impulses can make the system globally exponentially stabilized.

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I. Introduction

The impulsive differential equations represent a framework for mathematical modeling of many real life situations in the field of engineering, biology, chemistry, physics, control systems, population dynamics and many more [1]. In last two decades the stability analysis of these have been extensively explored [4,5,7,15-19]. In [17] the criteria for global exponential stability for impulsive functional differential equations is obtained by using Lyapunov function and Razumikhin technique. Moreover, it has been shown that impulses may make the system exponentially stable even if derivative of Lyapunov function is not negative. It is supposed that the state variables on impulses are related to present state variables but it is also possible that state variables on impulses are related to time delay. The aim of this paper is to get global exponential stability criteria for impulsive functional differential equation when state variables are dependent on both present and past state variables.

This paper organized as follows. In section II, some notations and definitions are given. We proved some criteria of global exponential stability for impulsive functional differential equations in section III, At last some concluding remarks are given in section IV.

II. Preliminaries

Let R^n denotes the n-dimensional real space and N denotes the set of positive integers. For given constant $\tau > 0$, the linear space $PC([-\tau,0],R^n)$ with norm $\|.\|$ defined by

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$$\|\psi\| = \sup_{r\in[-\tau,0]} \|\psi(r)\|.$$

Consider the differential system :

$$x'(t) = f(t, x_t), t \ge t_0, t \ne t_k$$
$$\Delta x(t_k) = I_k(x(t_k^-)) + J_k(x(t_k^- - \tau)), k = 1,2,3,.....$$
$$x_{t_0} = \psi \qquad (1)$$

Where

$$f : R_+ \times PC([-\tau,0], R^n) \to R^n;$$

$$I_k, J_k \in C[R^n, R^n]; \psi \in PC([-\tau,0], R^n);$$

$$0 \le t_0 < t_1 < t_2 < .... < t_k < ...., \quad \text{with} \quad t_k \to \infty \text{ as } k \to \infty \text{ ;}$$

$$\Delta x(t) = x(t) - x(t^-); x(t), x(t^-) \in R^n$$

Throughout in this paper, we assume that f, I_k, J_k, $k \in N$ satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t \ge t_0$ [6]. For any $\psi \in PC([-\tau,0], R^n)$, there exists a unique solution of (1) denoted by $x(t) = x(t, t_0, \psi)$. We further assume that all solutions x(t) of (1) are continuous except at $t = t_k, k \in N$,at which x(t) is right continuous i.e. $x(t_k^+) = x(t_k), k \in N$ and left limit i.e. $x(t_k^-)$ exists.

Definition 1: The function $V : R_+ \times R^n \to R_+$ is said to belong to the class ν_0 if the following conditions hold:

1) V is continuous in each of the sets $[t_{k-1}, t_k) \times R^n$,and for each $x \in R^n, t \in [t_{k-1}, t_k)$ $k \in N$, $lim_{(t,w)\to(t_k^-,x)} V(t, w) = V(t_k^-, x)$ exists.

2) $V(t, x)$ is locally Lipschitzian in all $x \in R^n$, and for all $t \ge t_0$, $V(t, 0) \equiv 0$.

Definition 2: Given a function $V: R_+ \times R^n \to R_+$, the upper right-hand derivative of V with respect to system (1) is defined by

$$D^+V(t,\varphi(0)) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta}[V(t+\delta, \varphi(0) + \delta f(t,\psi)) - V(t,\varphi(0))]$$

for $(t,\psi) \in R_+ \times PC([-\tau,0,], R^n)$.

Definition 3: The trivial solution of the system (1) is said to be globally exponentially stable if there exist some constants a >0 and M ≥1such that for any initial data $x_{t_0} = \psi$

$$\|x(t,t_0,\psi)\| \le M\|\psi\| e^{-a(t-t_0)}, t \ge t_0 \text{ , where } (t_0,\psi) \in R_+ \times PC([-\tau,0,], R^n).$$

III. Main Results

Now in this section, we shall establish criteria for global exponential stability of impulsive functional differential equation in which state variables on impulses are related to time delay. We have the followings results.

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6. G. Ballinger, X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dynam. Contin. Discrete Impuls. Systems 5 (1999) 579–591.

Notes

Theorem 1: Assume that there exist a function $V \in \nu_0$ and some constants $p, b, b_1, b_2 > 0$ and $l > \tau, \lambda > b$ such that

(i) $b_1\|x\|^p \leq V(t,x) \leq b_2\|x\|^p$, for any $t \in R_+$ and $x \in R^n$

(ii) $D^+V(t, \varphi(0)) \leq bV(t, \varphi(0))$, for all $t \in [t_{k-1}, t_k)$ $k \in N$

Whenever $hV(t,\varphi(0)) \geq V(t+r,\varphi(r))$ for $r \in [-\tau, 0]$, where $h \geq e^{2l\lambda}$ is a constant

(iii) for all $\varphi \in PC([-\tau,0];R^n)$

$V(t_k,\varphi(0)+I_k(\varphi(0))+J_k(\varphi(r)) \leq z_k\left[V(t_k^-,\varphi(0))+\sup_{r\in[-\tau,0]} V(t_k^-+r,\varphi(r))\right]$, where $z_k > 0, k \in N$ are constants.

(iv) $\tau \leq t_k - t_{k-1} \leq l$ and $z_k < \dfrac{e^{-\lambda l}.e^{-\lambda(t_{k+1}-t_k)}}{1+e^{\lambda\tau}}$

Then the trivial solution of (1) is globally exponentially stable.

Proof :- Choose $M \geq 1$ such that

$$b_2\|\psi\|^p < M\|\psi\|^p e^{-\lambda(t_1-t_0)}e^{-lb} < M\|\psi\|^p e^{-\lambda(t_1-t_0)} \leq hb_2\|\psi\|^p \tag{2}$$

Let $x(t) = x(t,t_0,\psi)$ be any solution of (1) with $x_{t_0} = \psi$ and v(t) =V(t, x). We shall now show that

$$v(t) \leq M\|\psi\|^p e^{-\lambda(t_k-t_0)}, t \in [t_{k-1},t_k), k \in N \tag{3}$$

We shall prove this by induction, so firstly we shall show that result is true for k = 1 i.e.

$$v(t) \leq M\|\psi\|^p e^{-\lambda(t_1-t_0)}, t \in [t_0,t_1) \tag{4}$$

From condition (i) and (2) for $t \in [t_0-\tau,t_0]$

$$v(t) \leq b_2\|x\|^p \leq b_2\|\psi\|^p < M\|\psi\|^p e^{-\lambda(t_1-t_0)}e^{-lb}$$

If (4) is not true, then there exist some $\hat{t} \in (t_0,t_1)$ such that

$$v(\hat{t}) > M\|\psi\|^p e^{-\lambda(t_1-t_0)} > M\|\psi\|^p e^{-\lambda(t_1-t_0)}e^{-lb} > b_2\|\psi\|^p \geq v(t_0+r) \tag{5}$$

where $r \in [-\tau,0]$

Which implies that there exist $\overset{\#}{t} \in \left(t_0,\hat{t}\right)$ such that

$$v(\overset{\#}{t}) = M\|\psi\|^p e^{-\lambda(t_1-t_0)} \text{ and } v(t) \leq M\|\psi\|^p e^{-\lambda(t_1-t_0)}, t_0-\tau \leq t \leq \overset{\#}{t} \tag{6}$$

Then there exist $\overset{\#\#}{t} \in \left(t_0, \overset{\#}{t}\right)$ such that

$$v(\overset{\#\#}{t}) = b_2 \|\psi\|^p \text{ and } v(t) \geq b_2 \|\psi\|^p, \overset{\#\#}{t} \leq t \leq \overset{\#}{t} \tag{7}$$

Then for any $t \in \left[\overset{\#\#}{t}, \overset{\#}{t}\right]$, we got

$$v(t+r) \leq M \|\psi\|^p e^{-\lambda(t_1 - t_0)} \leq h b_2 \|\psi\|^p \leq h v(t) \tag{8}$$

And therefore from condition (ii), we get

$D^+ v(t) \leq b v(t)$, for $t \in \left[\overset{\#\#}{t}, \overset{\#}{t}\right]$ and then we have $v(\overset{\#\#}{t}) \geq v(\overset{\#}{t}) e^{-lb}$ i.e.

$b_2 \|\psi\|^p \geq M \|\psi\|^p e^{-\lambda(t_1 - t_0)} e^{-lb}$ which contradicts (2) .Hence (4) holds that means result (3) is true for k=1

Now assume that result (3) holds for k =1,2,3,4….m

$$\text{i.e. } v(t) \leq M \|\psi\|^p e^{-\lambda(t_k - t_0)}, t \in [t_{k-1}, t_k), k = 1,2,3,\ldots\ldots m \tag{9}$$

from condition (iii) and (9) ,we get

$$\begin{aligned}
v(t_m) &\leq z_m \left[V(t_m^-, \varphi(0)) + \sup_{r \in [-\tau, 0]} V(t_m^- + r, \varphi(r)) \right] \\
&\leq z_m \left[M \|\psi\|^p e^{-\lambda(t_m - t_0)} + M \|\psi\|^p e^{-\lambda(t_m + r - t_0)} \right] \\
&\leq z_m M \|\psi\|^p e^{-\lambda(t_m - t_0)} (1 + e^{-\lambda r}) \\
&\leq z_m M \|\psi\|^p e^{-\lambda(t_m - t_0)} (1 + e^{\lambda \tau}) \\
&< e^{-\lambda l} e^{-\lambda(t_{m+1} - t_m)} M \|\psi\|^p e^{-\lambda(t_m - t_0)} \\
&< e^{-\lambda l} M \|\psi\|^p e^{-\lambda(t_{m+1} - t_0)} \\
&< M \|\psi\|^p e^{-\lambda(t_{m+1} - t_0)}
\end{aligned} \tag{10}$$

next we shall show that (3) holds for k = m+1

$$\text{i.e. } v(t) \leq M \|\psi\|^p e^{-\lambda(t_{m+1} - t_0)}, t \in [t_m, t_{m+1}) \tag{11}$$

suppose that (11) is not true then we can define $\bar{t} = \inf\{t \in [t_m, t_{m+1}]; v(t) > M \|\psi\|^p e^{-\lambda(t_{m+1}, t_0)}\}$ from (11) we know that $\bar{t} \neq t_m$. By the continuity of v(t) in $[t_m, t_{m+1})$,we have

$$v(\bar{t}) = M \|\psi\|^p e^{-\lambda(t_{m+1} - t_0)} \text{ and } v(t) \leq M \|\psi\|^p e^{-\lambda(t_{m+1} - t_0)}, t \in [t_m, \bar{t}] \tag{12}$$

From (10) we have since $v(t_m) < e^{-\lambda l} M \|\psi\|^p e^{-\lambda(t_{m+1} - t_0)} < v(\bar{t})$ which implies that there exist some

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$\overset{*}{t} \in \left(t_m, \bar{t}\right)$ such that

$$v(\overset{*}{t}) = e^{-\lambda l} M \|\psi\|^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(\overset{*}{t}) \le v(t) \le v(\bar{t}), t \in [\overset{*}{t}, \bar{t}] \tag{13}$$

Since $\tau \le t_k - t_{k-1} \le l$ therefore $t + r \in [t_{m-1}, \bar{t}]$ for $t \in [\overset{*}{t}, \bar{t}], r \in [-\tau, 0]$ .By (9), (12) and (13), we get for $t \in [\overset{*}{t}, \bar{t}]$

$$\begin{aligned} v(t+r) &\le M\|\psi\|^p e^{-\lambda(t_m - t_0)} \\ &= M\|\psi\|^p e^{-\lambda(t_{m+1}-t_0)} e^{\lambda(t_{m+1}-t_m)} \\ &\le e^{\lambda l} M\|\psi\|^p e^{-\lambda(t_{m+1}-t_0)} \\ &= e^{2\lambda l} v(\overset{*}{t}) \\ &\le hv(t) \end{aligned}$$

Then from condition (ii), we get $D^+ v(t) \le bv(t)$, since $\lambda > b$ from (13) we have

$$v(\bar{t}) \le v(\overset{*}{t}) e^{lb} = e^{-\lambda l} M\|\psi\|^p e^{-\lambda(t_{m+1}-t_0)} e^{lb} < v(\bar{t})$$

Which is contradiction

Thus (3) also hold for k = m+1

Hence by principle of mathematical induction (3) holds and we have

$$v(t) \le M\|\psi\|^p e^{-\lambda(t - t_0)}, t \in \left[t_{k-1}, t_k\right)$$

Then by condition (i)

$$\begin{aligned} & b_1\|x\|^p \le v(t) \le M\|\psi\|^p e^{-\lambda(t-t_0)} \\ & \Rightarrow b_1\|x\|^p \le M\|\psi\|^p e^{-\lambda(t-t_0)} \\ & \Rightarrow \|x\| \le \left(\frac{M}{b_1}\right)^{\frac{1}{p}} \|\psi\| \; e^{-\frac{\lambda}{p}(t-t_0)} \\ & \Rightarrow \|x\| \le M^* \|\psi\| \; e^{-\frac{\lambda}{p}(t-t_0)} \end{aligned}$$

Where $M^* \ge \max\{1, \left(\frac{M}{b_1}\right)^{\frac{1}{p}}\}$

Therefore the trivial solution of system (1) is globally exponentially stable with rate of convergence $\frac{\lambda}{p}$

Remark 1: If we want to remove the restriction $\lambda > b$ in above theorem then we need to modify conditions (ii) and (iv) as follows:

Theorem 2: Assume that there exist a function $V \in \nu_0$ and some constants $p, b, b_1, b_2 > 0$ and $l > \tau$ such that

(i) $b_1 \|x\|^p \leq V(t, x) \leq b_2 \|x\|^p$, for any $t \in R_+$ and $x \in R^n$

(ii) $D^+V(t, \varphi(0)) \leq bV(t, \varphi(0))$, for all $t \in [t_{k-1}, t_k)$ $k \in N$

Whenever $hV(t,\varphi(0)) \geq V(t+r, \varphi(r))$ for $r \in [-\tau, 0]$, where $h \geq \max\{e^{2l\lambda}, e^{lb}\}$ is a constant

(iii) for all $\varphi \in PC([-\tau,0]; R^n)$

$$V(t_k, \varphi(0) + I_k(\varphi(0)) + J_k(\varphi(r)) \leq z_k \left[V(t_k^-, \varphi(0)) + \sup_{r \in [-\tau,0]} V(t_k^- + r, \varphi(r)) \right], \text{ where}$$

$z_k > 0, k \in N$ are constants .

(iv) $\tau \leq t_k - t_{k-1} \leq l$ and $z_k < \dfrac{e^{-(\lambda+b)l} . e^{-\lambda(t_{k+1}-t_k)}}{1+e^{\lambda\tau}}$

Then the trivial solution of (1) is globally exponentially stable.

Proof:- The proof of this theorem is omitted as it is almost same as that of Theorem 1

Remark 2:- As we know that the derivative of the Lyapunov function should be negative for a delay differential system to be stable but in these theorems derivative may be positive which does not ensure the stability of the differential system . So it is clear that the impulses can contribute to make a system exponentially stable.

IV. Conclusion

In this paper, global exponential stability criteria for impulsive functional differential system have been extended to a system in which state variables on impulses are related to time delay. These results widen the scope of stability theory and are more general as compared to some existing results.

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