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Neighborhood Properties of Generalized Bessel Function

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Abstract- Let A denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \, (a_n \geq 0, n \in \mathbb{N}),$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$ In this paper, the new subclasses $Q_{n,c}(\gamma,k,\beta)$, $H_{n,c}(\gamma,k,\beta;\mu)$, $Q^{\alpha}_{n,c}(\gamma,k,\beta)$ and $H^{\alpha}_{n,c}(\gamma,k,\beta;\mu)$ of *A* which are de.ned by using generalized Bessel Function are introduced. Certain properties of neighborhood for functions belonging to these classes are studied.

Keywords: univalent functions, neighborhoods, starlike functions, convex functions and bessel operator.

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NEIGHBORHOODPROPERTIESOFGENERALIZEDBESSELFUNCTION

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Neighborhood Properties of Generalized Bessel Function

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Keywords: *univalent functions, neighborhoods, starlike functions, convex functions and bessel operator.*

I. Introduction

Let A denote the class of functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \quad (a_n\geq 0, n\in\mathbb{N}). \tag{1.1}$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$.

For any function $f(z)\in A$, $z\in U$ and $\eta\geq 0$,we define

$$N_{n,\eta}f(z)=\left\{g\in A:g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } \sum_{n=2}^{\infty}n\left|a_n-b_n\right|\leq\eta\right\}, \tag{1.2}$$

which is the $(n,\eta)-$neighborhood of $f(z)$.

For $e(z)=z$, we see that

$$N_{n,\eta}e(z)=\left\{g\in A:g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } \sum_{n=2}^{\infty}n\left|b_n\right|\leq\eta\right\}. \tag{1.3}$$

The concept of neighborhoods was first introduced by Goodman [3] .

In this paper, we discuss certain properties of $(n,\eta)-$neighborhood results for functions in the classes $Q_{n,c}(\gamma,k,\beta)$, $H_{n,c}(\gamma,k,\beta;\mu)$, $Q^{\alpha}_{n,c}(\gamma,k,\beta)$ and $H^{\alpha}_{n,c}(\gamma,k,\beta;\mu)$ of A .

The subclass $S^{*}_{n}(\gamma)$[4] of A, is the class of functions of complex order γ satisfying

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0 \quad (z\in U,\gamma\in\mathbb{C}\backslash\{0\}).$$

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The subclass $K_n(\gamma)$[4] of A, is the class of functions of complex order γ satisfying

$$\mathrm{Re}\left\{1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\}>0 \quad (z\in U,\gamma\in\mathbb{C}\setminus\{0\}).$$

The Hadamard product of two power series

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \text{ and } g(z)=z+\sum_{n=2}^{\infty}b_nz^n.$$

is defined as $(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$.

we recall here a generalized Bessel function $w(z)$ of the first kind of order γ, defined in [2] and given by

$$w_{\gamma,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-c)^n}{n!\Gamma(\gamma+n+\frac{b+1}{2})}(\frac{z}{2})^{2n+\gamma} \quad (z\in U)$$

where stands for $\Gamma-$ Euler function. Which is the particular solution of the second - order homogeneous differential equation (see [5])

$$z^2w''(z)+bzw'(z)+\left[cz^2-\gamma^2+(1-b)\gamma\right]w(z)=0,$$

where $z\in U$. Now we consider the function $\varphi(z)$ defined by

$$\varphi_{\gamma,b,c}(z)=2^{\gamma}\Gamma(\gamma+\frac{b+1}{2})z^{1-\frac{\gamma}{2}}w\left(\sqrt{z}\right).$$

By using the well-know Pochhammer symbol $(x)_\mu$ defined for $x,\mu\in U$ and in the terms of the Euler gamma function, by

$$(x)_\mu=\frac{\Gamma(x+n)}{\Gamma(x)}\begin{cases}1 & (\mu=0)\\ x(x+1)\ldots(x+n-1) & (\mu\in N=\{1,2,3,\ldots\})\end{cases}$$

we can express $\varphi_{\gamma,b,c}(z)=\varphi_{k,c}(z)$ as

$$\varphi_{k,c}(z)=z+\sum_{n=1}^{\infty}\frac{\left(\frac{-c}{4}\right)^n}{(k)_n(n+1)}z^{n+1} \quad (k:=\gamma+\frac{b+1}{2}\notin z)$$

where $z_0=\{0,-1,-2,\ldots\}$.

Now, by using idea of Dziok and Srivastava [1], and we introduced the B_k^c operator as follows:

$$B_k^cf(z)=\varphi(z)*f(z)=z-\sum_{n=2}^{\infty}\frac{(-c)^{n-1}a_nz^n}{4^{n-1}(k)_{n-1}(n-1)!}. \tag{1.4}$$

Definition 1. The subclass $Q_{n,c}(\gamma,k,\beta)$ of A is defined as the class of functions f such that

$$\left|\frac{1}{\gamma}\left(\frac{z\left[B_k^cf(z)\right]'}{B_k^cf(z)}-1\right)\right|<\beta \tag{1.5}$$

where , $\gamma\in\mathbb{C}\setminus\{0\}$, $0<\beta\leq1$, $c\in N_0$ *and* $z\in U$.

Definition 2. Let the subclass $H_{n,c}(\gamma,k,\beta;\mu)$ of A is defined as the class of functions f such that

$$\left|\frac{1}{\gamma}\left[(1-\mu)\frac{B_k^cf(z)}{z}+\mu(B_k^cf(z))'-1\right]\right|<\beta \tag{1.6}$$

where , $\gamma\in\mathbb{C}\setminus\{0\}$, $0<\beta\leq1$, $0\leq\mu\leq1$, $c\in N_0$ *and* $z\in U$.

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2. E. Deniz, H. Orhan and H. M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, Taiwanse J. Math. 15(2) (2011), 883-917.

II. Neighborhood for Classes $Q_{n,c}(\gamma, k, \beta)$ and $H_{n,c}(\gamma, k, \beta; \mu)$

Notes

In this section, we obtain inclusion relations involving $N_{n,\eta}$ for functions in the classes $Q_{n,c}\left(\gamma, k, \beta\right)$ and $H_{n,c}(\gamma, k, \beta; \mu)$.

Lemma 1. A function $f(z) \in Q_{n,c}\left(\gamma, k, \beta\right)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!} \left[n - 1 + \beta \left|\gamma\right|\right] a_n \leq \beta \left|\gamma\right|. \tag{2.1}$$

Proof. Let $f(z) \in Q_{n,c}\left(\gamma, k, \beta\right)$. Then , by (1.5) we can write,

$$\operatorname{Re}\left\{\frac{z\left[B_k^c f(z)\right]'}{B_k^c f(z)} - 1\right\} > -\beta\left|\gamma\right| \qquad (z \in U). \tag{2.2}$$

Using (1.1) and (1.4), we have,

$$\operatorname{Re}\left\{\frac{-\sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!}\left[n-1\right] a_n z^n}{z - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!} a_n z^n}\right\} > -\beta\left|\gamma\right|, \quad (z \in U). \tag{2.3}$$

Letting $z \to 1$,through the real values, the inequality (2.3) yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$,we obtain,

$$\begin{aligned} \left|\frac{z\left[B_k^c f(z)\right]'}{B_k^c f(z)} - 1\right| &= \left|\frac{\sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!}\left[n-1\right] a_n z^n}{z - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!} a_n z^n}\right| \\ &\leq \frac{\sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!}\left[n-1\right] a_n}{1 - \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!} a_n} \\ &\leq \beta\left|\gamma\right|. \end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in Q_{n,c}(\gamma, k, \beta)$, which establishes the required result. □

On similar lines , we have the following Lemma.

Lemma 2. A function $f(z) \in H_{n.c}(\gamma, k, \beta; \mu)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!} \left[1 + \mu(n-1)\right] a_n \leq \beta\left|\gamma\right|. \tag{2.4}$$

Theorem 1. Let $c < 0$. if

$$\eta = \frac{2\beta\left|\gamma\right|}{\frac{(-c)}{4(k)}\left[1 + \beta\left|\gamma\right|\right]}, \qquad \left(\ \left|\gamma\right| < 1\right), \tag{2.5}$$

then $Q_{n,c}(\gamma, k, \beta) \subset N_{n,\eta}\left(e\right)$.

Proof. Let $f(z) \in Q_{n,k}(\gamma, k, \beta)$. By Lemma 1, we have,

$$\frac{(-c)}{4(k)}\left[1 + \beta\left|\gamma\right|\right] \sum_{n=2}^{\infty} a_n \leq \beta\left|\gamma\right|,$$

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which implies,

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta|\gamma|}{\frac{(-c)}{4(k)}[1+\beta|\gamma|]}. \tag{2.6}$$

Using (2.1) and (2.6), we have,

$$\begin{aligned}\frac{(-c)}{4(k)}\sum_{n=2}^{\infty} na_n &\leq \beta|\gamma| + \frac{(-c)}{4(k)}[1-\beta|\gamma|]\sum_{n=2}^{\infty} a_n \\ &\leq \frac{2\beta|\gamma|}{[1+\beta|\gamma|]} = \eta.\end{aligned}$$

That is,

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\beta|\gamma|}{\frac{(-c)}{4(k)}[1+\beta|\gamma|]} = \eta.$$

Thus, by the definition given by (1.3), $f(z) \in N_{n,\eta}(e)$, which completes the proof. □

Theorem 2. Let $c<0$. *If*

$$\eta = \frac{2\beta|\gamma|}{(1+\mu)\frac{(-c)}{4(k)}}, \quad (|\gamma|<1), \tag{2.7}$$

then $H_{n,c}(\gamma,k,\beta;\mu) \subset N_{n,\delta}(e)$.

Proof. Let $f(z) \in H_{n,c}(\gamma,k,\beta;\mu)$. Then, by Lemma 2, we have,

$$\frac{(-c)}{4(k)}(1+\mu)\sum_{n=2}^{\infty} a_n \leq \beta|\gamma|,$$

which gives the following coefficient inequality,

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta|\gamma|}{\frac{(-c)}{4(k)}(1+\mu)}. \tag{2.8}$$

Using (2.4) and (2.8), we also have,

$$\begin{aligned}\mu\frac{(-c)}{4(k)}\sum_{n=2}^{\infty} na_n &\leq \beta|\gamma| + (\mu-1)\frac{(-c)}{4(k)}\sum_{n=2}^{\infty} a_n \\ &\leq \beta|\gamma| + (\mu-1)\frac{\beta|\gamma|}{(1+\mu)}.\end{aligned}$$

That is,

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\beta|\gamma|}{(1+\mu)\frac{(-c)}{4(k)}} = \eta.$$

Thus, by the definition given by (1.3), $f(z) \in N_{n,\eta}(e)$, which completes the proof. □

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III. Neighborhood for Classes $Q^{\alpha}_{n,c}(\gamma, k, \beta)$ and $H^{\alpha}_{n,c}(\gamma, k, \beta; \mu)$

In this section, we define the subclasses $Q^{\alpha}_{n,c}(\gamma,k,\beta)$ and $H^{\alpha}_{n,c}(\gamma,k,\beta;\mu)$ of A and neighborhoods of these classes are obtained.

For $0 \le \alpha < 1$ and $z \in U$, a function $f(z) \in Q^{\alpha}_{n,c}(\gamma,k,\beta)$ if there exists a function $g(z) \in Q_{n,c}(\gamma,k,\beta)$ such that

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$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha. \tag{3.1}$$

For $0 \le \alpha < 1$ and $z \in U$, a function $f(z) \in H^{\alpha}_{n,c}(\gamma,k,\beta;\mu)$ if there exists a function $g(z) \in H_{n,c}(\gamma,k,\beta;\mu)$ such that the inequality (3.1) holds true.

Theorem 3. If $g(z) \in Q_{n,c}(\gamma,k,\beta)$ and

$$\alpha = 1 - \frac{\eta \frac{(-c)}{4(k)} \left[1 + \beta\left|\gamma\right|\right]}{2\left[\frac{(-c)}{4(k)}\left[1+\beta\left|\gamma\right|\right] - \beta\left|\gamma\right|\right]}, \tag{3.2}$$

then $N_{n,\eta}(g) \subset Q^{\alpha}_{n,c}(\gamma,k,\beta)$.

Proof. Let $f(z) \in N_{n,\eta}(g)$. Then,

$$\sum_{n=2}^{\infty} n\left|a_n - b_n\right| \le \eta, \tag{3.3}$$

which yields the coefficient inequality,

$$\sum_{n=2}^{\infty} \left|a_n - b_n\right| \le \frac{\eta}{2}, \quad (n \in \mathbb{N}). \tag{3.4}$$

Since $g(z) \in Q_{n,c}(\gamma,k,\beta)$ by (2.6), we have ,

$$\sum_{n=2}^{\infty} b_n \le \frac{\beta\left|\gamma\right|}{\frac{(-c)}{4(k)}\left[1+\beta\left|\gamma\right|\right]}, \tag{3.5}$$

so that,

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{n=2}^{\infty}\left|a_n - b_n\right|}{1 - \sum_{n=2}^{\infty} b_n} \\ &\le \frac{\eta}{2}\frac{\frac{(-c)}{4(k)}\left[1+\beta\left|\gamma\right|\right]}{\frac{(-c)}{4(k)}\left[1+\beta\left|\gamma\right|\right] - \beta\left|\gamma\right|} \\ &= 1 - \alpha.\end{aligned}$$

Thus, by definition, $f(z) \in Q^{\alpha}_{n,c}(\gamma,k,\beta)$ for α given by (3.2), which establishes the desired result. □

On similar lines, we can prove the following theorem .

Theorem 4. If $g(z) \in H_{n,c}(\gamma,k,\beta;\mu)$ and

$$\alpha = 1 - \frac{\eta \frac{(-c)}{4(k)}(1+\mu)}{2\left[\frac{(-c)}{4(k)}(1+\mu) - \beta\left|\gamma\right|\right]} \tag{3.6}$$

then $N_{n,\delta}(g) \subset H^{\alpha}_{n,c}(\gamma,k,\beta;\mu)$.

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