On Dynamical Systems Induced by the Adele Ring

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Keywords: prime fields (or, p-adic number fields), the adele ring, p-adicvon neumann algebras, adele-ring von neumann algebras, p-adic dynamical systems, adelic dynamical systems.

I. INTRODUCTION

Continued from [10], in this paper, we consider how *primes* (or *prime numbers*) act on operator algebras. In particular, instead of acting each *p-adic number fields* $\mathbb{Q}_p$ to operator algebras, for every prime p, we act *the Adele ring* $\mathbb{A}_{\mathbb{Q}}$ on operator algebras. In [10], we act p-adic number fields $\mathbb{Q}_p$ on a given von Neumann algebra M, and construct a corresponding *dynamical system* generating its *crossed product algebra*. We have studied fundamental properties of such dynamical systems and crossed product W^*-algebras. Also, by applying *free probability*, we considered free-distributional data of certain operators. Here, based on results of [10], we act the Adele ring $\mathbb{A}_{\mathbb{Q}}$ on M.

The relations between primes and operator algebras have been studied in various different approaches. The main purposes of finding such relations are (i) to provide new tools for studying operator algebras, (ii) to apply operator-algebraic techniques (for example, free probability) to study *number theory*, and hence, (iii) to establish bridges between number theory and operator algebra theory. In [4], we studied how primes act "on" certain von Neumann algebras. Also, the primes as *operators* in certain von Neumann algebras have been studied, too, in [5] and [8]. In [6] and [7], we have studied primes as *linear functionals* acting on *arithmetic functions.* i.e., each prime induces a free-probabilistic structure on arithmetic functions. In such a case, one can understand arithmetic functions as *Krein-space operators* (for fixed primes), via certain *representations* (See [11] and [12]). These studies are all motivated by well-known number-theoretic results under free probability techniques.

Arveson studied *histories* as a group of actions induced by *real numbers* $\mathbb{R}$ on (type I subfactors of) $B(H)$, satisfying certain additional conditions, where H is an infinite dimensional separable *Hilbert space* (e.g., [1], [2] and cited papers therein). By understanding the field $\mathbb{R}$ as an additive group $(\mathbb{R}, +)$, he defined an E_0-*group* $\Gamma_{\mathbb{R}}$ of $*$-homomorphisms acting on $B(H)$ indexed by $\mathbb{R}$. By putting additional conditions on $\Gamma_{\mathbb{R}}$, he defined a *history* Γ acting on $B(H)$. We mimic Arveson's con-

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struction to establish our dynamical systems and corresponding crossed product algebras (e.g., [8], [9] and [10]).

In [9], by *framing* (e.g., also see [8]), a group Γ to groupoids generated by partial isometries, we studied possible *distortions* Γ_G of a history Γ. It shows that whenever a history Γ acts on H, a family of partial isometries distorts (or reduces, or restricts) the "original" historical property (in the sense of Arveson) of Γ. And such distortions are completely characterized by *groupoid actions*, sometimes called the E_0-*groupoid actions* induced by partial isometries on $B(H)$. The above framed (E_0-)groupoids Γ_G induce corresponding C^*-subalgebras $C^*(\Gamma_G)$ of $B(H)$, investigated by *dynamical system theory* and *free probability* (e.g., [15], [16] and [17]).

Independently, *p-adic analysis* provides a important tool for studying geometry at small distance (e.g., [18]). it is not only interested in various mathematical fields but also in physics (e.g., [3], [4], [5] and [18]). The *p-adic number fields* (or *p-prime fields*) $\mathbb{Q}_p$ and the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$ play key roles in modern *number theory*, *analytic number theory*, *L-function theory*, and *algebraic geometry* (e.g., [3], [13] and [14]). Also, analysis on such Adelic structures gives a way for understanding small-distance-measured geometry (e.g., [18]) and vector analysis under non-Archimedean metric (e.g., [5]). Thus, prime fields and the Adele ring are interesting topics both in mathematics and in other scientific fields.

We attempt to combine the above two topics; dynamical systems and *p*-adic analysis under Adelic settings; which seem independent from each other.

II. Definitions and Background

In this section, we introduce basic definitions and backgrounds of the paper.

a) *p-Adic Number Fields* $\mathbb{Q}_p$ *and The Adele Ring* $\mathbb{A}_{\mathbb{Q}}$. *Fundamental theorem of arithmetic* says that every positive integer in $\mathbb{N}$ except 1 can be expressed as a usual multiplication of *primes* (or prime numbers), equivalently, all positive integers which are not 1 are *prime-factorized* under multiplication. i.e., the primes are the building blocks of all positive integers except for 1. Thus, it is trivial that primes are playing key roles in both classical and advanced *number theory*.

The Adele ring $\mathbb{A}_{\mathbb{Q}}$ is one of the main topics in advanced number theory connected with other mathematical fields like *algebraic geometry* and *L-function theory*, etc.

Throughout this paper, we denote the set of all natural numbers (which are positive integers) by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of all rational numbers by $\mathbb{Q}$.

Let's fix a prime p. Define *the p-norm* $|.|_p$ on the rational numbers $\mathbb{Q}$ by

$$|q|_p = \left|p^r \tfrac{a}{b}\right|_p \overset{def}{=} \frac{1}{p^r},$$

whenever $q = p^r \tfrac{a}{b} \in \mathbb{Q}^{\times} = \mathbb{Q} \setminus \{0\}$, for some $r \in \mathbb{Z}$, with an additional identity:

$$|0|_p \overset{def}{=} 0 \ (\text{for all primes } p).$$

For example,

$$\left|-\tfrac{24}{5}\right|_2 = \left|2^3 \cdot (-\tfrac{3}{5})\right|_2 = \tfrac{1}{2^3} = \tfrac{1}{8}.$$

It is easy to check that

(i) $|q|_p \geq 0$, for all $q \in \mathbb{Q}$,

(ii) $|q_1 q_2|_p = |q_1|_p \cdot |q_2|_p$, for all $q_1, q_2 \in \mathbb{Q}$

(iii) $|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\}$,

for all $q_1, q_2 \in \mathbb{Q}$. In particular, by (iii), we verify that

(iii)$'$ $|q_1 + q_2|_p \leq |q_1|_p + |q_2|_p$,

for all $q_1, q_2 \in \mathbb{Q}$. Thus, by (i), (ii) and (iii)$'$, the *p*-norm $|.|_p$ is indeed a norm.

However, by (iii), this norm is "*non-Archimedean.*"

i. **Definition 2.1.** *We define a set $\mathbb{Q}_p$ by the norm-closure of the normed space $(\mathbb{Q}, |.|_p)$, for all primes p. We call $\mathbb{Q}_p$, the p-adic number field.*

For a fixed prime p, all elements of $\mathbb{Q}_p$ are formed by

$$p^r \left(\textstyle\sum_{k=0}^{\infty} a_k p^k \right), \text{ for } 0 \leq a_k < p, \tag{2.1.1}$$

for all $k \in \mathbb{N}$, and for all $r \in \mathbb{Z}$. For example,

$$-1 = (p-1)p^0 + (p-1)p + (p-1)p^2 + \cdots.$$

The subset of $\mathbb{Q}_p$, consisting of all elements formed by

$$\textstyle\sum_{k=0}^{\infty} a_k p^k, \text{ for } 0 \leq a_k < p \text{ in } \mathbb{N},$$

is denoted by $\mathbb{Z}_p$. i.e., for any $x \in \mathbb{Q}_p$, there exist $r \in \mathbb{Z}$, and $x_0 \in \mathbb{Z}_p$, such that

$$x = p^r x_0.$$

Notice that if $x \in \mathbb{Z}_p$, then $|x|_p \leq 1$, and vice versa. i.e.,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}. \tag{2.1.2}$$

So, the subset $\mathbb{Z}_p$ of (2.1.2) is said to be the *unit disk of* $\mathbb{Q}_p$. Remark that

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z}_p \supset \cdots.$$

It is not difficult to verify that

$$\mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset p^{-3}\mathbb{Z}_p \subset \cdots,$$

and hence

$$\mathbb{Q}_p = \overset{\infty}{\underset{k=-\infty}{\cup}} p^k \, \mathbb{Z}_p, \text{ set-theoretically.} \tag{2.1.3}$$

Consider the boundary U_p of $\mathbb{Z}_p$. By construction, the boundary U_p of $\mathbb{Z}_p$ is identical to $\mathbb{Z}_p \setminus p\mathbb{Z}_p$, i.e.,

$$U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : |x|_p = 1 = p^0\}. \tag{2.1.4}$$

Similarly, the subsets $p^k U_p$ are the boundaries of $p^k\mathbb{Z}_p$ satisfying

$$p^k U_p = p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}.$$

We call the subset U_p of $\mathbb{Q}_p$ in (2.1.4) the *unit circle of* $\mathbb{Q}_p$. And all elements of U_p are said to be *units of* $\mathbb{Q}_p$.

Therefore, by (2.1.3) and (2.1.4), we obtain that

$$\mathbb{Q}_p = \overset{\infty}{\underset{k=-\infty}{\sqcup}} p^k U_p, \text{ set-theoretically,} \tag{2.1.5}$$

where $\sqcup$ means the disjoint union. By [18], whenever $q \in \mathbb{Q}_p$ is given, there always exist $a \in \mathbb{Q}$, $k \in \mathbb{Z}$, such that

$$q \in a + p^k \, \mathbb{Z}_p, \text{ for } a, k \in \mathbb{Z}.$$

Fact (See [18]) *The p-adic number field $\mathbb{Q}_p$ is a Banach space. And it is locally compact. In particular, the unit disk $\mathbb{Z}_p$ is compact in $\mathbb{Q}_p$.* □

Define now the addition on $\mathbb{Q}_p$ by

$$\left(\textstyle\sum_{n=-N_1}^{\infty} a_n p^n \right) + \left(\textstyle\sum_{n=-N_2}^{\infty} b_n p^n \right) = \textstyle\sum_{n=-\max\{N_1,N_2\}}^{\infty} c_n \, p^n, \tag{2.1.6}$$

for $N_1, N_2 \in \mathbb{N}$, where the summands $c_n p^n$ satisfies that

$$c_n p^n \overset{def}{=} \begin{cases} (a_n + b_n)p^n & \text{if } a_n + b_n < p \\ p^{n+1} & \text{if } a_n + b_n = p \\ s_n p^{n+1} + r_n p^n & \text{if } a_n + b_n = s_n p + r_n, \end{cases}$$

18. V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, p-Adic Analysis and Mathematical Physics, Ser. Soviet & East European Math., vol 1, ISBN: 978-981-02-0880-6,(1994) World Scientific.

for all $n \in \{-\max\{N_1, N_2\}, ..., 0, 1, 2, ...\}$. Clearly, if $N_1 > N_2$ (resp., $N_1 < N_2$), then, for all $j = -N_1, ..., -(N_1 - N_2 + 1)$, (resp., $j = -N_2, ..., -(N_2 - N_1 + 1)$),

$$c_j = a_j \text{ (resp., } c_j = b_j).$$

And define the multiplication "on $\mathbb{Z}_p$" by

$$\left(\sum_{k_1=0}^{\infty} a_{k_1} p^{k_1}\right) \left(\sum_{k_2=0}^{\infty} b_{k_2} p^{k_2}\right) = \sum_{n=-N}^{\infty} c_n p^n, \tag{2.1.7}$$

where

$$c_n = \sum_{k_1+k_2=n} \left(r_{k_1,k_2} i_{k_1,k_2} + s_{k_1-1,k_2} \ i_{k_1-1,k_2}^c \right.$$
$$\left. + s_{k_1,k_2-1} i_{k_1,k_2-1}^c + s_{k_1-1,k_2-1} i_{k_1-1,\ k_2-1}^c \right),$$

where

$$a_{k_1} b_{k_2} = s_{k_1,k_2} \ p + r_{k_1,k_2},$$

by the division algorithm, and

$$i_{k_1,k_2} = \begin{cases} 1 & \text{if } a_{k_1} b_{k_2} < p \\ 0 & \text{otherwise,} \end{cases}$$

and

$$i_{k_1,k_2}^c = 1 - i_{k_1,k_2},$$

for all $k_1, k_2 \in \mathbb{N}$, and hence, "on $\mathbb{Q}_p$," the multiplication is extended to

$$\left(\sum_{k_1=-N_1}^{\infty} a_{k_1} p^{k_1}\right) \left(\sum_{k_2=-N_2}^{\infty} b_{k_2} p^{k_2}\right)$$
$$= \left(p^{-N_1}\right) \left(p^{-N_2}\right) \left(\sum_{k_1=0}^{\infty} a_{k_1-N_1} p^{k_1}\right) \left(\sum_{k_2=0}^{\infty} b_{k_1-N_2} p^{k_2}\right). \tag{2.1.7}$$

Then, under the addition (2.1.6) and the multiplication (2.1.7)′, the algebraic triple $(\mathbb{Q}_p, +, \cdot)$ becomes a field, for all primes p. Thus the p-prime fields $\mathbb{Q}_p$ are algebraically fields.

Fact *Every p-acid number field $\mathbb{Q}_p$, with the binary operations (2.1.6) and (2.1.7)′ is indeed a field.* □

Moreover, the Banach filed $\mathbb{Q}_p$ is also a (unbounded) Haar-measure space $(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p)$, for all primes p, where $\sigma(\mathbb{Q}_p)$ means the σ-algebra of $\mathbb{Q}_p$, consisting of all measurable subsets of $\mathbb{Q}_p$. Moreover, this measure ρ_p satisfies that

$$\begin{aligned} \rho_p\left(a + p^k \mathbb{Z}_p\right) &= \rho_p\left(p^k \mathbb{Z}_p\right) \\ &= \frac{1}{p^k} \\ &= \rho\left(p^k \mathbb{Z}_p^\times\right) = \rho\left(a + p^k \mathbb{Z}_p^\times\right), \end{aligned} \tag{2.1.8}$$

for all $a \in \mathbb{Q}$, and $k \in \mathbb{Z}$, where $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus \{0\}$. Also, one has

$$\begin{aligned} \rho_p(a + p^k U_p) &= \rho_p\left(p^k U_p\right) = \rho_p\left(p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p\right) \\ &= \rho_p\left(p^k \mathbb{Z}_p\right) - \rho_p\left(p^{k+1} \mathbb{Z}_p\right) \\ &= \frac{1}{p^k} - \frac{1}{p^{k+1}}, \end{aligned}$$

for all $a \in \mathbb{Q}$. Similarly, we obtain that

$$\rho_p\left(a + p^k U_p\right) = \rho\left(p^k U_p\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{2.1.9}$$

for all $a \in \mathbb{N}$, and $k \in \mathbb{Z}$ (See Chapter IV of [18]).

Fact *The Banach field $\mathbb{Q}_p$ is an unbounded Haar-measure space, where ρ_p satisfies (2.1.8) and (2.1.9), for all primes p.* □

The above three facts show that $\mathbb{Q}_p$ is a unbounded Haar-measured, locally compact Banach field, for all primes p.

ii. **Definition 2.2.** *Let $\mathcal{P} = \{all\ primes\} \cup \{\infty\}$. The Adele ring $\mathbb{A}_{\mathbb{Q}} = (\mathbb{A}_{\mathbb{Q}}, +, \cdot)$ is defined by the set*

$$\{(x_p)_{p \in \mathcal{P}} : x_p \in \mathbb{Q}_p,\ almost\ all\ x_p \in \mathbb{Z}_p,\ x_{\infty} \in \mathbb{R}\}, \qquad (2.1.10)$$

with identification $\mathbb{Q}_{\infty} = \mathbb{R}$, and $\mathbb{Z}_{\infty} = [0, 1]$, the closed interval in $\mathbb{R}$, equipped with

$$(x_p)_p + (y_p)_p = (x_p + y_p)_p, \qquad and \qquad (2.1.11)$$

$$(x_p)_p(y_p)_p = (x_p y_p)_p, \qquad (2.1.12)$$

for all $(x_p)_p$, $(y_p)_p \in \mathbb{A}_{\mathbb{Q}}$.

Indeed, this algebraic structure $\mathbb{A}_{\mathbb{Q}}$ forms a ring. Also, by the algebraic construction and the product topology, the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is also a locally compact Banach space equipped with the product measure. Set-theoretically,

$$\mathbb{A}_{\mathbb{Q}} \subseteq \underset{p \in \mathcal{P}}{\Pi} \mathbb{Q}_p = \mathbb{R} \times \left(\underset{p:prime}{\Pi} \mathbb{Q}_p \right).$$

In fact, the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is a *weak direct product* $\underset{p \in \mathcal{P}}{\Pi'} \mathbb{Q}_p$ of $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$, i.e.,

$$\mathbb{A}_{\mathbb{Q}} = \underset{p \in \mathcal{P}}{\Pi'} \mathbb{Q}_p.$$

i.e., whenever $(x_p)_p \in \mathbb{A}_{\mathbb{Q}}$, almost all x_q are in $\mathbb{Z}_q$, for primes q, except for finitely many x_p.

The product measure ρ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is given:

$$\rho = \underset{p \in \mathcal{P}}{\times} \rho_p,$$

with identification $\rho_{\infty} = \rho_{\mathbb{R}}$, the usual distance-measure (induced by $|.|_{\infty}$) on $\mathbb{R}$.

Fact *The Adele ring $\mathbb{A}_{\mathbb{Q}}$ is a unbounded-measured locally compact Banach ring.*
□

b) Dynamical Systems Induced by Algebraic Structures: In this section, we briefly discuss about *dynamical systems* induced by algebraic structures. Let X be an arbitrary algebraic structures, i.e., X is a semigroup, or a group, or a groupoid, or an algebra, etc (maybe equipped with topology).

Let M be an algebra over $\mathbb{C}$, and assume there exists a well-defined action α of X acting on M. i.e., $\alpha(x)$ is a well-defined function on X, satisfying that:

$$\alpha(x_1 \cdot x_2) = \alpha(x_1) \circ \alpha(x_2) \text{ on } M,$$

for all x_1, $x_2 \in X$, where $x_1 \cdot x_2$ means the operation on X, and $(\circ)$ means the usual functional composition. For convenience, we denote $\alpha(x)$ simply by α_x, for all $x \in X$.

Then the triple (X, M, α) is called the *dynamical system induced by X on M via α.* For such a dynamical system (X, M, α), one can define a *crossed product algebra*

$$\mathbb{M}_X = M \times_{\alpha} X,$$

by the algebra generated by M and $\alpha(X)$, satisfying that:

$$(m_1 \alpha_{x_1})(m_2 \alpha_{x_2}) = (m_1 \alpha_{x_1}(m_2)) \alpha_{x_1 x_2} \text{ in } \mathbb{M}_X,$$

where $\alpha_{x_j} = \alpha(x_j)$, for all $m_j \alpha_{x_j} \in \mathbb{M}_X$, for $j = 1, 2$.

If M is a $*$-algebra, then one may have an additional condition;

$$(m \alpha_x)^* = \alpha_x(m^*) \alpha_x^* \text{ in } \mathbb{M}_X,$$

for all $m \alpha_x \in \mathbb{M}_X$.

Of course, one can consider the cases where M is equipped with topology. More precisely, in $\mathbb{M}_X$, we may put a topology from the topology on M, making $\alpha(X)$ be continuous.

In this paper, we are interested in cases where given algebras M are *von Neumann algebras*. In such cases, we call the corresponding topological dynamical systems, W^*-*dynamical systems*, and the corresponding crossed product algebra, the *crossed product W^*-algebras*.

c) Free Probability: For more about free probability theory, see [16] and [17]. In this section, we briefly introduce *Speicher*'s combinatorial free probability (e.g., [16]), which is the combinatorial characterization of the original *Voiculescu*'s analytic free probability (e.g., [17]).

Let $B \subset A$ be von Neumann algebras with $1_B = 1_A$ and assume that there exists a *conditional expectation* $E_B : A \to B$ satisfying that:

(i) $E_B(b) = b$, for all $b \in B$,

(ii) $E_B(b \, a \, b') = b \, E_B(a) \, b'$, for all b, $b' \in B$ and $a \in A$,

(iii) E_B is bounded (or continuous), and

(iv) $E_B(a^*) = E_B(a)^*$, for all $a \in A$.

Then the pair (A, E_B) is called a B-*valued* (*amalgamated*) W^*-*probability space* (*with amalgamation over B*).

For any fixed B-valued random variables $a_1, ..., a_s$ in (A, E_B), we can have the B-*valued free distributional data* of them;

○ $(i_1, ..., i_n)$-*th B-valued joint $*$-moments*:

$$E_B \left(b_1 a_{i_1}^{r_1} \; b_2 a_{i_2}^{r_2} \; ... \; b_n \; a_{i_n}^{r_n} \right)$$

○ $(j_1, ..., j_m)$-*th B-valued joint $*$-cumulants*:

$$k_m^B \left(b_1' a_{j_1}^{t_1}, \; b_2' a_{j_2}^{t_2}, \; ..., \; b_m' a_{j_m}^{t_m} \right),$$

which provide the equivalent B-valued free distributional data of $a_1, ..., a_s$, for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $(j_1, ..., j_m) \in \{1, ..., s\}^m$, for all n, $m \in \mathbb{N}$, where $b_1, ..., b_n, b_1', ..., b_m' \in B$ are arbitrary and $r_1, ..., r_n, t_1, ..., t_m \in \{1, *\}$. By the *Möbius inversion*, indeed, they provide the same, or equivalent, B-valued free distributional data of $a_1, ..., a_s$. i.e., they satisfy

$$E_B \left(b_1 a_{i_1}^{r_1} \; ... \; b_n a_{i_n}^{r_n} \right) = \sum_{\pi \in NC(n)} k_\pi^B \left(b_1 a_{i_1}^{r_1}, \; ..., \; b_n a_{i_n}^{r_n} \right)$$

and

$$k_m^B \left(b_1' a_{j_1}^{r_1}, \; ..., \; b_m' a_{j_m}^{t_m} \right) = \sum_{\theta \in NC(m)} E_{B:\theta} \left(b_1' a_{j_1}^{t_1}, ..., b_m' a_{j_m}^{t_m} \right) \mu(\theta, 1_m),$$

where $NC(k)$ is the *lattice of all noncrossing partitions* over $\{1, ..., k\}$, for $k \in \mathbb{N}$, and $k_\pi^B(...)$ and $E_{B:\theta}(...)$ are the *partition-depending cumulant* and the *partition-depending moment*, and where μ is the *Möbius functional* in the *incidence algebra* I_2.

Recall that the *partial ordering on $NC(k)$* is defined by

$$\pi \leq \theta \overset{def}{\Longleftrightarrow} \forall \text{ blocks } V \text{ in } \pi, \exists \text{ blocks } B \text{ in } \theta \text{ s.t.} V \subseteq B.,$$

for all $k \in \mathbb{N}$. Under such a partial ordering $\leq$, the set $NC(k)$ is a lattice with its *maximal element* $1_k = \{(1, ..., k)\}$ and its *minimal element* $0_k = \{(1), (2), ..., (k)\}$. The notation $(...)$ inside partitions $\{...\}$ means the *blocks of the partitions*. For example, 1_k is the one-block partition and 0_k is the k-block partition, for $k \in \mathbb{N}$. Also, recall that the *incidence algebra I_2* is the collection of all functionals

$$\xi : \cup_{k=1}^{\infty} \left(NC(k) \times NC(k) \right) \to \mathbb{C},$$

satisfying $\xi(\pi, \theta) = 0$, whenever $\pi > \theta$, with its usual function addition $(+)$ and its *convolution* $(*)$ defined by

$$\xi_1 * \xi_2(\pi, \theta) \overset{def}{=} \sum_{\pi \leq \sigma \leq \theta} \xi_1(\pi, \sigma) \, \xi_2(\sigma, \theta),$$

Ref

17. D. Voiculescu, K. Dykemma, and A. Nica, Free Random Variables, CRM Monograph Series, vol 1., (1992)

for all ξ_1, $\xi_2 \in I_2$. Then this algebra I_2 has the *zeta functional* ζ, defined by

$$\zeta(\pi, \theta) \overset{def}{=} \begin{cases} 1 & \text{if } \pi \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

The *Möbius functional* μ is the convolution-inverse of ζ in I_2. So, it satisfies

$$\sum_{\pi \in NC(k)} \mu(\pi, 1_k) = 0, \text{ and } \mu(0_k, 1_k) = (-1)^{k-1} c_{k-1}, \quad (4.1.1)$$

for all $k \in \mathbb{N}$, where $c_m \overset{def}{=} \frac{1}{m+1} \begin{pmatrix} 2m \\ m \end{pmatrix}$ is the *m-th Catalan number*, for all $m \in \mathbb{N}$.

The *amalgamated freeness* is characterized by the amalgamated $*$-cumulants. Let (A, E_B) be given as above. Two W^*-subalgebras A_1 and A_2 of A, having their common W^*-subalgebra B in A, are *free over B in* (A, E_B), if and only if all their "mixed" $*$-cumulants vanish. Two subsets X_1 and X_2 of A are *free over B in* (A, E_B), if $vN(X_1, B)$ and $vN(X_2, B)$ are free over B in (A, E_B), where $vN(S_1, S_2)$ means the von Neumann algebra generated by S_1 and S_2. In particular, two B-valued random variable x_1 and x_2 are *free over B in* (A, E_B), if $\{x_1\}$ and $\{x_2\}$ are free over B in (A, E_B).

Suppose two W^*-subalgebras A_1 and A_2 of A, containing their common W^*-subalgebra B, are free over B in (A, E_B). Then we can construct a W^*-subalgebra $vN(A_1, A_2) = \overline{B[A_1 \cup A_2]}^w$ of A generated by A_1 and A_2. Such W^*-subalgebra of A is denoted by $A_1 *_B A_2$. If there exists a family $\{A_i : i \in I\}$ of W^*-subalgebras of A, containing their common W^*-subalgebra B, satisfying $A = \underset{i \in I}{*_B} A_i$, then we call A, the *B-valued free product algebra* of $\{A_i : i \in I\}$.

Assume now that the W^*-subalgebra B is $*$-isomorphic to $\mathbb{C} = \mathbb{C} \cdot 1_A$. Then the conditional expectation E_B becomes a linear functional on A. By φ, denote E_B. Then, for $a_1, ..., a_n \in (A, \varphi)$,

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, ..., a_n) \mu(\pi, 1_n)$$

by the Möbius inversion

$$= \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi_V(a_1, ..., a_n) \right) \mu(\pi, 1_n)$$

since the images of φ are in $\mathbb{C}$.
For example, if $\pi = \{(1, 3), (2), (4, 5)\}$ in $NC(5)$, then

$$\begin{aligned} \varphi_\pi(a_1, ..., a_5) &= \varphi(a_1\varphi(a_2)a_3)\varphi(a_4a_5) \\ &= \varphi(a_1a_3)\varphi(a_2)\varphi(a_4a_5). \end{aligned}$$

Remember here that, if φ is an arbitrary conditional expectation E_B, and if $B \neq \mathbb{C} \cdot 1_A$, then the above second equality does not hold in general.
So, we have

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi_V(a_1, ..., a_n)\mu(0_{|V|}, 1_{|V|}) \right) \quad (4.1.2)$$

by the multiplicativity of μ.

III. p-Adic W^*-Dynamical Systems

In this section, we introduce W^*-dynamical systems induced by p-adic number fields $\mathbb{Q}_p$, for $p \in \mathcal{P}$. They are defined by a certain semigroup(-or-monoidal) dynamical systems induced by semigroups (resp., monoids) $\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap)$. Throughout this section, we fix a von Neumann subalgebra M of $B(H)$, and a prime p.

a) p- Prime von Neumann Algebras $L^\infty(\mathbb{Q}_p)$: As a measure space, the p-adic number field $\mathbb{Q}_p$ has its corresponding L^2-*Hilbert space* H_p, defined by

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p, \ \rho_p\right), \text{ for all primes } p. \qquad (3.1.1)$$

We call H_p, the *p-prime Hilbert space.* Remark that all elements of H_p are the functions approximated by simple functions

$$\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \ \chi_S$$

(under limit), generated by *characteristic functions* χ_X,

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{Q}_p$, with $t_S \in \mathbb{C}$, for $S \in \sigma(\mathbb{Q}_p)$. So, one can understand each element f of H_p as an expression,

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \ \chi_S \text{ (possibly an infinite sum).}$$

By definition, the inner product $<,>_p$ on H_p is defined by

$$< f_1, \ f_2 >_p \overset{def}{=} \int_{\mathbb{Q}_p} f_1 \ \overline{f_2} \ d\rho_p,$$

for all $f_1, \ f_2 \in H_p$, having the corresponding norm $\|.\|_p$ on H_p,

$$\|f\|_p \overset{def}{=} \sqrt{< f, \ f >_p} = \sqrt{\int_{\mathbb{Q}_p} |f|^2 \ d\rho_p},$$

for all $f \in H_p$. Thus, if $f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$ in H_p, then

$$\int_{\mathbb{Q}_p} f \ d\rho_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \ \rho_p(S).$$

Let's fix a function $g \in L^\infty\left(\mathbb{Q}_p, \ \rho_p\right)$, which is an essential-norm bounded function. Similar to H_p-case, one can / may understand g as the approximation of simple functions. Then

$$gf \in H_p, \text{ too, for all } f \in H_p.$$

Definition 3.1. *The von Neumann subalgebras* $\mathfrak{M}_p = L^\infty\left(\mathbb{Q}_p, \ \rho_p\right)$ *of* $B(H_p)$ *are called the p-prime von Neumann algebras, for all* $p \in \mathcal{P}$.

By locally compactness, and Hausdorff property of $\mathbb{Q}_p$, for any $x \in \mathbb{Q}_p$, there exist $a \in \mathbb{Q}$, and $n \in \mathbb{Z}$, such that $x \in a + p^n U_p$ (See [18]). Therefore, we obtain the following lemma.

Lemma 3.1. *Let* $X \in \sigma(\mathbb{Q}_p)$ *be a measurable subset. Then there exists* $N \in \mathbb{N} \cup \{\infty\}$, *such that: (i) there are corresponding* $a_1, \ ..., \ a_N \in \mathbb{Q}$, *and* $n_1, \ ..., \ n_N \in \mathbb{Z}$, *and (ii)* X *is covered by the unions of* $a_k + p^{n_k} U_p$, *for* $k = 1, \ ..., \ N$, *i.e.,*

$$X \subseteq \bigcup_{k=1}^{N} \left(a_k + p^{n_k} U_p\right), \qquad (3.1.2)$$

where U_p *is the unit circle of* $\mathbb{Q}_p$, *which is the boundary of the unit disk* $\mathbb{Z}_p$. □

In (3.1.2), we show that every measurable subset X of $\mathbb{Q}_p$ is covered by a union of transformed boundaries $a + p^k U_p$ of $a + p^k \mathbb{Z}_p$ ($a \in \mathbb{Q}, \ k \in \mathbb{Z}$).

Lemma 3.2. *Let* X *be a measurable subset of the unit circle* U_p *in* $\mathbb{Q}_p$, *for primes* p. *Then there exists*

$$0 \le r_X \le 1 \text{ in } \mathbb{R}, \qquad (3.1.3)$$

such that

$$\rho_p(X) = r_X \left(1 - \frac{1}{p}\right).$$

□

By (3.1.3), we can obtain the following theorem.

Theorem 3.3. *(See [10]) Let χ_S be a characteristic function for $S \in \sigma(\mathbb{Q}_p)$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$, and $k_1, ..., k_N \in \mathbb{Z}$, $r_1, ..., r_N \in \mathbb{R}$, such that*

$$\int_{\mathbb{Q}_p} \chi_S \, d\rho_p = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right). \qquad (3.1.4)$$

□

The above formula (3.1.4) characterizes the identically-distributedness under the integral in $\mathfrak{M}_p$. By (3.1.4), one can obtain the following corollary.

Corollary 3.4. *Let $g = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \, \chi_S$ be an element of the p-prime von Neumann algebra $\mathfrak{M}_p$. Then there exist*

$$r_j \in [0, 1] \text{ in } \mathbb{R}, \; k_j \in \mathbb{Z}, \text{ and } t_j \in \mathbb{C},$$

and

$$h = \sum_{j=-\infty}^{\infty} (t_j r_j p^{k_j}) \, \chi_{U_p} \qquad (3.1.5)$$

such that g and h are identically distributed under the integral $\int_{\mathbb{Q}_p} \bullet \, d\rho_p$. □

b) p- Adic Semigroup W^*- Dynamical Systems: Now, let M be a fixed von Neumann algebra in the operator algebra $B(H)$ on the Hilbert space H, and $\mathbb{Q}_p$, a fixed p-adic number field, and let $\mathfrak{M}_p = L^{\infty}(\mathbb{Q}_p, \rho_p)$ be the p-prime von Neumann algebra in the sense of Section 3.1.

Let $\mathcal{H}_p$ be the *tensor product* Hilbert space $H \otimes H_p$ of the p-prime Hilbert space H_p and the Hilbert space H, where $\otimes$ means the topological tensor product of Hilbert spaces. i.e.,

$$\mathcal{H}_p = H \otimes H_p.$$

Understand the σ-algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$ as a monoid $(\sigma(\mathbb{Q}_p), \cap)$. It is not difficult to check indeed $\sigma(\mathbb{Q}_p)$ is a semigroup under the intersection $(\cap)$, with $(\cap)$-identity $\mathbb{Q}_p \in \sigma(\mathbb{Q}_p)$, i.e., it is a well-defined monoid.

Define an action α of the monoid $\sigma(\mathbb{Q}_p)$, acting on the von Neumann algebra M in $B(\mathcal{H}_p)$ by

$$\alpha(S)(m) \overset{def}{=} \chi_S \, m \, \chi_S^* = \chi_S \, m \, \chi_S, \qquad (3.2.1)$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$, in $B(\mathcal{H}_p)$, by understanding

$$\chi_S = \chi_S \otimes 1_M, \text{ and } m = 1_{\mathfrak{M}_p} \otimes m \text{ in } B(\mathcal{H}_p),$$

where $1_{\mathbb{Q}_p}$ is the identity map $\chi_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, and 1_M is the identity element of M.

Lemma 3.5. *(See [10]) The action α of $\sigma(\mathbb{Q}_p)$ in the sense of (3.2.1) acting on a von Neumann algebra M is a monoid action, and hence, the triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ forms a monoidal dynamical system.* □

Indeed, the morphism α of (3.2.1) satisfies that:

$$\alpha(S_1 \cap S_2) = \alpha(S_1) \circ \alpha(S_2) \text{ on } M,$$

for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$.

Remark that all elements f of the p-prime von Neumann algebra $\mathfrak{M}_p = L^{\infty}(\mathbb{Q}_p, \rho_p)$ is generated by the σ-algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$, in the sense that: every element $f \in \mathfrak{M}_p$ has its expression, $\sum_{S \in Supp(f)} t_S \chi_S$. So, the action α of (3.2.1) is extended to the linear morphism, also denoted by α, from $\mathfrak{M}_p$ into $B(\mathcal{H}_p)$, acting on M, with

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$$\alpha(f)(m) = \alpha\left(\sum_{S \in Supp(f)} t_S \chi_S\right)(m) \tag{3.2.2}$$

$$\overset{def}{=} \sum_{S \in Supp(f)} t_S \; \alpha(S)(m) = \sum_{S \in Supp(f)} t_S \; (\chi_S m \chi_S),$$

for all $f \in \mathfrak{M}_p$.

Definition 3.2. Let $\sigma(\mathbb{Q}_p)$ be the σ-algebra of the p-adic number field $\mathbb{Q}_p$, understood as a monoid $(\sigma(\mathbb{Q}_p), \cap)$, and let α be the action of $\sigma(\mathbb{Q}_p)$ on a von Neumann algebra M in the sense of (3.2.1). Then the mathematical triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ is called the p-adic (monoidal) W^*-dynamical system. For this p-adic W^*-dynamical system, define the crossed product W^*-algebra

$$\mathcal{M}_p \overset{def}{=} M \times_\alpha \sigma(\mathbb{Q}_p) \tag{3.2.5}$$

by the von Neumann subalgebra of $B(\mathcal{H}_p)$ generated by M and $\chi(\sigma(\mathbb{Q}_p))$ satisfying (3.2.2) (See Section 2.2 above).

The von Neumann algebra $\mathcal{M}_p$ is called the p-adic dynamical W^*-algebra induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$.

Note that, all elements of the p-adic dynamical W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ induced by the p-adic W^*-dynamical system $\mathcal{Q}(M, p)$ have their expressions,

$$\sum_{S \in \sigma(\mathbb{Q}_p)} m_S \; \chi_S, \text{ with } m_S \in M$$

(possibly infinite sums under topology). Define the *support* $Supp(T)$ of a fixed element $T = \sum_{S \in \sigma(\mathbb{Q}_p)} m_S \chi_S$ in $\mathcal{M}_p$ by

$$Supp(T) \overset{def}{=} \{S \in \alpha(\mathbb{Q}_p) : m_S \neq 0_M\}.$$

Now, let $m_1 \chi_{S_1}, m_2 \chi_{S_2} \in \mathcal{M}_p$, with $m_1, m_2 \in M$, $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. Then

$$(m_1 \chi_{S_1})(m_2 \chi_{S_2}) = m_1 \chi_{S_1} m_2 \chi_{S_1} \chi_{S_2}$$

$$= m_1 \chi_{S_1} m_2 \chi_{S_1}^2 \chi_{S_2} = m_1 \chi_{S_1} m_2 \chi_{S_1} \chi_{S_1} \chi_{S_2}$$

since $\chi_S = 1_M \otimes \chi_S$ (in $B(\mathcal{H}_p)$) are projections ($\chi_S^2 = \chi_S = \chi_S^*$), for all $S \in \sigma(\mathbb{Q}_p)$

$$= m_1 \; \alpha_{S_1}(m_2) \; \chi_{S_1} \chi_{S_2} = m_1 \; \alpha_{S_1}(m_2) \; \chi_{S_1 \cap S_2}.$$

Notation For convenience, if there is no confusion, we denote $\alpha_S(m)$ by m^S, for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$. $\square$

More generally, one has that:
(3.2.3)

$$\prod_{j=1}^{N} (m_j \chi_{S_j}) \quad = m_1 m_2^{S_1} m_3^{S_1 \cap S_2} ... m_N^{S_1 \cap ... \cap S_{N-1}} \chi_{S_1 \cap ... \cap S_N}$$

$$= \left(\prod_{j=1}^{N} m_j^{\overset{j-1}{\underset{i=0}{\cap}} S_i}\right) \left(\chi_{\overset{N}{\underset{j=1}{\cap}} S_j}\right)$$

for all $N \in \mathbb{N}$. Also, we obtain that

$$(m \chi_S)^* = \chi_S m^* \; \chi_S \chi_S = (m^*)^S \; \chi_S, \tag{3.2.4}$$

for all $m \chi_S \in \mathcal{M}_p$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.
So, let

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \; \chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, 2.$$

Notes

Then

$$T_1 T_2 = \sum_{(S_1, S_2) \in Supp(T_1) \times Supp(T_2)} m_{S_1} \chi_{S_1} m_{S_2} \chi_{S_2}$$

(3.2.5)

$$= \sum_{(S_1, S_2) \in Supp(T_1) \times Supp(T_2)} m_{S_1} m_{S_2}^{S_1} \chi_{S_1 \cap S_2},$$

by (3.2.3).

Also, if $T = \sum_{S \in Supp(T)} m_S \chi_S$ in $\mathcal{M}_p$, then

(3.2.6)

$$T^* = \sum_{S \in Supp(T)} (m_S^*)^S \chi_S,$$

by (3.2.4).

So, one can have that if

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, ..., n,$$

for $n \in \mathbb{N}$, then

$$T_1^{r_1} T_2^{r_2} \cdots T_n^{r_n} = \prod_{j=1}^{n} \left(\sum_{S_j \in Supp(T_j)} [m_{S_j}^{r_j}]^{S_j} \chi_{S_j} \right)$$

where

(3.2.7)

$$[m_{S_j}^{r_j}]^{S_j} \overset{def}{=} \begin{cases} m_{S_j} & \text{if } r_j = 1 \\ (m_{S_j}^*)^{S_j} & \text{if } r_j = *, \end{cases}$$

for $j = 1, ..., n$

$$= \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} Supp(T_j)} \left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \chi_{S_j} \right) \right)$$

(3.2.8)

$$= \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \right)^{\left(\bigcap_{i=1}^{j-1} S_i \right)} \right) \chi_{\bigcap_{j=1}^{n} S_j} \right),$$

for all $(r_1, ..., r_n) \in \{1, *\}^n$.

Lemma 3.6. *Let* $T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k}$ *be elements of the p-adic semigroup* W^*-*algebra* $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ *in* $B(\mathcal{H}_p)$, *for* $k = 1, ..., n$, *for* $n \in \mathbb{N}$. *Then*

(3.2.9)

$$\prod_{j=1}^{n} T_j^{r_j} = \sum_{(S_1, ..., S_n) \in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \right)^{\left(\bigcap_{i=1}^{j-1} S_i \right)} \right) \chi_{\bigcap_{j=1}^{n} S_j} \right),$$

for all $r_1, ..., r_n \in \{1, *\}$, *where* $[m_{S_j}^{r_j}]^{S_j}$ *are in the sense of (3.2.7).*

Proof. The proof of (3.2.9) is done by (3.2.8) with (3.2.7). ∎

c) Structure Theorems of $M \times_\alpha \sigma(\mathbb{Q}_p)$. Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic W^*-algebra induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$. In this section, we consider a structure theorem for this crossed product von Neumann algebra $\mathcal{M}_p$.

Define the usual tensor product W^*-subalgebra

$$\mathcal{M}_0 = M \otimes_{\mathbb{C}} \mathfrak{M}_p \text{ of } B(\mathcal{H}_p),$$

where $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ is the p-prime von Neumann algebra in the sense of Section 3.1, and where $\otimes_{\mathbb{C}}$ is the topological tensor product of topological operator algebras over $\mathbb{C}$. By definition, clearly, one can verify that $\mathcal{M}_p$ is a W^*-subalgebra of $\mathcal{M}_0$ in $B(\mathcal{H}_p)$, i.e.,

$$\mathcal{M}_p \overset{\text{Subalgebra}}{\subseteq} \mathcal{M}_0.$$

Now, define the "*conditional*" *tensor product* W^*-*algebra*

$$\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p,$$

induced by an action α of $\mathfrak{M}_p$ acting on M (in the sense of (3.2.4)), by a W^*-subalgebra of $\mathcal{M}_0$ dictated by the α-*relations*:

$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 m_2^{S_1}) \otimes \chi_{S_1}\chi_{S_2}, \qquad (3.3.1)$$

and

$$(m \otimes \chi_S)^* = (m^*)^S \otimes \chi_S^*, \qquad (3.3.2)$$

for all m_1, m_2, $m \in M$, and S_1, S_2, $S \in \sigma(\mathbb{Q}_p)$. i.e., the W^*-subalgebra $\mathcal{M}_0^p$ of $\mathcal{M}_0$ satisfying the α-relations (3.3.1) and (3.3.2) is the conditional tensor product W^*-algebra $M \otimes_\alpha \mathfrak{M}_p$.

Theorem 3.7. *(See [10]) Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic W^*-algebra induced by the p-adic W^*-dynamical system $\mathcal{Q}(M, p)$, and let $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$ be the conditional tensor product W^*-algebra of M and the p-prime von Neumann algebra $\mathfrak{M}_p$ satisfying the α-relations (3.3.1) and (3.3.2). Then these von Neumann algebras $\mathcal{M}_p$ and $\mathcal{M}_0^p$ are $*$-isomorphic in $B(\mathcal{H}_p)$, i.e.,*

(3.3.3)

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \overset{*\text{-}iso}{=} M \otimes_\alpha \mathfrak{M}_p = \mathcal{M}_0^p,$$

in $B(\mathcal{H}_p)$. □

IV. Free Probability on p-Adic Dynamical W^*-Algebras

In this section, we consider free probability on the p-adic dynamical W^*-algebra

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$$

induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$.

By Section 3.3, the von Neumann subalgebra $\mathcal{M}_p$ is $*$-isomorphic to the conditional tensor product W^*-algebra $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$ of a fixed von Neumann subalgebra M of $B(H)$ and the p-prime von Neumann algebra $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$, in $B(\mathcal{H}_p)$, for $p \in \mathcal{P}$. So, throughout this section, we understand $\mathcal{M}_p$ and $\mathcal{M}_0^p$, alternatively.

By understanding $\mathcal{M}_p$ as $\mathcal{M}_0^p$, we construct a well-defined conditional expectation

(4.1)

$$E_p : \mathcal{M}_0^p \overset{*\text{-iso}}{=} \mathcal{M}_p \to M_p,$$

where

$$M_p = M \otimes_\alpha \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p),\ S \subseteq U_p\}\right],$$

where U_p is the unit circle of $\mathbb{Q}_p$, which is the boundary $\mathbb{Z}_p - p\mathbb{Z}_p$ of the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$, satisfying that:

$$E_p(m\chi_S) = E_p(m \otimes \chi_S) \overset{def}{=} m\,\chi_{S \cap U_p},$$

for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

Define now a morphism

$$F_p : M_p \to M_p$$

by a linear transformation satisfying that:

$$F_p(m\chi_S) = m\left(r_S \chi_{U_p}\right), \qquad (4.1)'$$

for all $S \in \sigma(\mathbb{Q}_p)$, where $r_S \in [0, 1]$ in $\mathbb{R}$, making

R_{ef}

$$\int_{\mathbb{Q}_p} \left(\chi_{S \cap U_p} \right) d\rho_p = r_S \int_{\mathbb{Q}_p} \left(\chi_{U_p} \right) d\rho_p$$

$$\Longleftrightarrow$$

$$\rho_p \left(r_S \chi_{U_p} \right) = r_S \left(1 - \tfrac{1}{p} \right) = \rho_p \left(\chi_{S \cap U_p} \right),$$

by the identically distributedness (3.1.4) (and (3.1.5)), i.e.,

$$\rho_p(S \cap U_p) \overset{def}{=} r_S \left(1 - \tfrac{1}{p} \right) = r_S \rho_p(U_p). \qquad (4.1)''$$

Define now a linear functional

$$\gamma : M_p \to \mathbb{C}$$

by

$$\gamma \overset{def}{=} \left(\ \otimes \int_{\mathbb{Q}_p} \bullet \, d\rho_p \right) \circ F_p, \qquad (4.2)$$

where F_p is in the sense of $(4.1)'$. More precisely, it satisfies that:

$$\begin{aligned} \gamma \left(m \ \otimes \ \chi_S \right) &\overset{def}{=} (m) \int_{\mathbb{Q}_p} \left(r_S \chi_{U_p} \right) d\rho_p \\ &= r_S \ (m) \left(1 - \tfrac{1}{p} \right). \end{aligned}$$

And then define a linear functional

$$\gamma_p : \mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0 \to \mathbb{C}$$

by

$$\gamma_p = \gamma \circ E_p, \qquad (4.3)$$

where γ and E_p are in the sense of (4.2) and (4.1), respectively. i.e., for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$,

$$\gamma_p \left(m \ \chi_S \right) = \gamma \left(E_p(m \ \chi_S) \right)$$

$$= \gamma \left(m^{Y \cap U_p} \left(r \chi_{U_p} \right) \right) = (m) \int_{\mathbb{Q}_p} \left(r \ \chi_{U_p} \right) d\rho_p$$

$$= r \ \psi(m) \left(1 - \tfrac{1}{p} \right),$$

for some $r \in [0, 1]$, satisfying $(4.1)''$.

Then the pair $(\mathcal{M}_p, \gamma_p)$ is a W^*-probability space in the sense of Section 2.3. We consider the free distributional data of certain elements of $(\mathcal{M}_p, \gamma_p)$.

Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic dynamical W^*-algebra in $B(\mathcal{H}_p)$, understood also as its $*$-isomorphic von Neumann algebra, $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$. Let $\gamma_p = \gamma \circ E_p$ be the linear functional in the sense of (4.3) on $\mathcal{M}_0^p = \mathcal{M}_p$, where γ is in the sense of (4.2) and E_p is in the sense of (4.1), with $(4.1)''$. i.e., γ_p is a linear functional on $\mathcal{M}_p$, satisfying that:

$$\gamma_p(m\chi_S) = \gamma \left(E_p(m\chi_S) \right) = \gamma \left(m(r\chi_{U_p}) \right) = r\psi(m) \left(1 - \tfrac{1}{p} \right),$$

for some $r \in [0, 1]$, satisfying $(4.1)''$, for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

By [10], the morphism $\gamma_p = \gamma \circ E_p : \mathcal{M}_p \to \mathbb{C}$ of (4.3) is indeed a well-defined bounded linear functional on $\mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0$.

Definition 4.1. *The pair $(\mathcal{M}_p, \gamma_p)$ is called the p-adic dynamical W^*-probability space.*

The following lemmas are obtained by the straightforward computations.

Lemma 4.1. *(See [10]) Let $m\chi_S$ be a free random variable in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Then*

$$\gamma_p \left((m\chi_S)^n \right) = r_S \, \psi(m(m^S)^{n-1}) \left(1 - \tfrac{1}{p} \right), \qquad (4.4)$$

for all $n \in \mathbb{N}$, where $r_S \in [0, 1]$ satisfies (4.1)''. □

Lemma 4.2. *(See [10]) Let* $m_1 \chi_{S_1}, ..., m_n \chi_{S_n}$ *be free random variables in the p-adic dynamical W^*-probability space* $(\mathcal{M}_p, \gamma_p)$, *with* $m_k \in M$, $S_k \in \sigma(\mathbb{Q}_p)$, *for* $k = 1, ..., n$, *for* $n \in \mathbb{N}$. *Then there exists* $r_0 \in [0, 1]$, *such that:*

$$\gamma_p \left(\prod_{j=1}^{n} m_j \chi_{S_j} \right) = r_0 \left(\psi \left(\prod_{j=1}^{N} m_j^{\overset{j-1}{\underset{i=0}{\cap}} S_i} \right) \right) \left(1 - \frac{1}{p} \right). \qquad (4.5)$$

□

By (4.4) and (4.5), we obtain the following free distributional data of free random variables of the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$.

Theorem 4.3. *(See [10]) Let* $(\mathcal{M}_p, \gamma_p)$ *be the p-adic dynamical W^*-probability space, and let*

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k}, \text{ for } k = 1, ..., n,$$

be free random variables in $(\mathcal{M}_p, \gamma_p)$, *for* $n \in \mathbb{N}$. *Then*

$$\gamma_p \left(\prod_{j=1}^{n} T_j \right) = \sum_{(S_1, ..., S_n) \in \overset{n}{\underset{j=1}{\prod}} Supp(T_j)} r_{(S_1, ..., S_n)} \left(\psi \left(\prod_{j=1}^{n} \left(m_{S_j} \right)^{\left(\overset{j-1}{\underset{i=1}{\cap}} S_i \right)} \right) \right) \left(1 - \frac{1}{p} \right).$$

$$(4.6)$$

□

So, by the Möbius inversion of Section 2.3, one can obtain that:

$$k_n \left(\left(m_1 \chi_{S_1} \right)^{r_1}, ..., \left(m_n \chi_{S_n} \right)^{r_n} \right)$$
$$= \sum_{\pi \in NC(n)} (\gamma_p)_\pi \left([m_1^{r_1}]^{S_1} \chi_{S_1}, ..., [m_n^{r_n}]^{S_n} \chi_{S_n} \right) \mu(\pi, 1_n)$$
$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} (\gamma_p)_V \left([m_1^{r_1}]^{S_1} \chi_{S_1}, ..., [m_n^{r_n}]^{S_n} \chi_{S_n} \right) \mu \left(0_{|V|}, 1_{|V|} \right) \right)$$

by the Möbius inversion (See Section 4.1)
$$= \sum_{\pi \in NC(n)} \left(\prod_{V = (i_1, ..., i_k) \in \pi} \gamma_p \left([m_{i_1}^{r_{i_1}}]^{S_{i_1}} \chi_{S_{i_1}} \cdots [m_{i_k}^{r_{i_k}}]^{S_{i_k}} \chi_{S_{i_k}} \right) \mu(0_k, 1_k) \right) \quad (4.7)$$
$$= \sum_{\pi \in NC(n)} \left(\prod_{V = (i_1, ..., i_k) \in \pi} \left(r_V \left(\psi \left(\prod_{t=1}^{k} \left([m_{i_t}^{r_{i_t}}]^{S_{i_t}} \right)^{\overset{k-1}{\underset{t=1}{\cap}} S_{i_t}} \right) \right) \left(1 - \frac{1}{p} \right) \right) \mu(0_k, 1_k) \right),$$

by (4.6), where $r_V \in [0, 1]$ satisfy (4.1)''.

By (4.7), we obtain the following inner free structure of the p-adic dynamical W^*-algebra $\mathcal{M}_p$, with respect to γ_p.

Theorem 4.4. *(See [10]) Let* $m_1 \chi_S$, *and* $m_2 \chi_S$ *be free random variables in the p-adic dynamical W^*-probability space* $(\mathcal{M}_p, \gamma_p)$, *with* $m_1, m_2 \in M$, *and* $S \in \sigma(\mathbb{Q}_p) \setminus \{\varnothing\}$. *Also, assume that S is not a measure-zero element in* $\sigma(\mathbb{Q}_p)$. *Then* $\{m_1, m_1^S\}$ *and* $\{m_2, m_2^S\}$ *are free in the W^*-probability space* (M, ψ), *if and only if* $m_1 \chi_S$ *and* $m_2 \chi_S$ *are free in* $(\mathcal{M}_p, \gamma_p)$. □

Now, let $m_1 \chi_S$ and $m_2 \chi_{U_p} \in \mathcal{M}_p$, with $m_1, m_2 \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Assume that $S \cap U_p$ is empty. Since $S \cap U_p = \varnothing$, all mixed cumulants of $m_1 \chi_S$ and $m_2 \chi_{U_p}$ have $r_V = 0$, for some $V \in \pi$ in (4.7), for all $\pi \in NC(n)$. Therefore, one obtains the following inner freeness condition of $(\mathcal{M}_p, \gamma_p)$.

Theorem 4.5. *(See [10]) Let* $S_1 \neq S_2 \in \sigma(\mathbb{Q}_p)$ *such that* $S_1 \cap S_2 = \varnothing$. *Then the subsets*

$\{m \chi_{S_1} : m \in M\}$ *and* $\{a \chi_{S_2} : a \in M\}$
are free in $(\mathcal{M}_p, \gamma_p)$. □

One may do the same process by fixing $p^k U_p$ instead of fixing U_p, for $k \in \mathbb{Z}$. Recall that $p^k U_p$ are the boundaries $p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p$ of $p^k \mathbb{Z}_p$, for all $k \in \mathbb{Z}$ (See Section 2.1). i.e., for a fixed $k \in \mathbb{Z}$, define

(4.8)
$$M_{p:k} \overset{def}{=} M \otimes_\alpha \mathbb{C}\left[\{\chi_{p^k U_p}\}\right] \overset{\text{*-iso}}{=} M,$$

Then $M_p = M \otimes_\alpha \mathbb{C}\left[\{\chi_{U_p}\}\right]$ of (4.1) is identical to $M_{p:0}$ in the sense of (4.8).

Similar to (4.1), construct a conditional expectation
$$E_{p:k} : \mathcal{M}_p = \mathcal{M}_p^0 \to M_{p:k}$$

by a linear morphism satisfying that:

(4.9)
$$E_{p:k}(m\chi_S) = m \, \chi_{S \cap p^k U_p}^0,$$

with
$$\chi_{S \cap p^k U_p}^0 = r \, \chi_{p^k U_p},$$

where $r \in [0, 1]$ satisfying

(4.9)'
$$\int_{\mathbb{Q}_p} \chi_{S \cap p^k U_p}^0 \, d\rho_p = r \int_{\mathbb{Q}_p} \chi_{p^k U_p} \, d\rho_p = r\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right).$$

Then, just like (4.1), $E_{p:k}$ is a well-defined conditional expectation from $\mathcal{M}_p$ onto $M_{p:k} = M$.

And then, for $k \in \mathbb{Z}$, define a linear functional
$$\gamma_k : M_{p:k} \to \mathbb{C}$$

by

$$\gamma_k\left(m \, \chi_{p^k U_p}\right) \overset{def}{=} \psi(m) \int_{\mathbb{Q}_p}\left(\chi_{p^k U_p}\right) d\rho_p \qquad (4.10)$$

$$= \psi(m)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right).$$

Then one has a well-defined linear functional
$$\gamma_{p:k} : \mathcal{M}_p \to \mathbb{C}$$

defined by

$$\gamma_{p:k} \overset{def}{=} \gamma_k \circ E_{p:k}, \text{ for all } k \in \mathbb{Z}. \qquad (4.11)$$

Note that our linear functional γ_p in the sense of (4.3) is identified with $\gamma_{p:0}$ of (4.11).

Observation 4.1 Let's replace $M_p = M_{p:0}$ of (4.1) to $M_{p:k}$, for $k \in \mathbb{Z}$. Then the formulae (4.4), (4.5), (4.6) and (4.7) can be re-obtained by replacing factors $\left(1 - \frac{1}{p}\right)$ to $\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)$. So, the freeness of the above two theorems are same under $(\mathcal{M}_p, \gamma_{p:k})$-settings.

For instance, if $m_j \chi_{S_j} \in (\mathcal{M}_p, \gamma_{p:k})$, for $j = 1, ..., n$, for $n \in \mathbb{N}$, then

$$\gamma_{p:k}\left(\prod_{j=1}^n m_j \chi_{S_j}\right) = r_0\left(\psi\left(\prod_{j=1}^N m_j^{\cap_{i=0}^{j-1} S_i}\right)\right)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right),$$

for some $r_0 \in [0, 1]$, satisfying (4.9)'. □

The above **Observation 4.1** shows that we have systems of W^*-probability spaces

$$\left\{(\mathcal{M}_p, \gamma_{p:k})\right\}_{k \in \mathbb{Z}},$$

sharing similar free probability with $(\mathcal{M}_p, \gamma_p = \gamma_{p:0})$.

V   Adelic W^*-Dynamical Systems

In this section, we consider a W^*-dynamical system induced by the σ-algebra $\sigma(\mathbb{A}_{\mathbb{Q}})$ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$. Similar to the p-adic cases of Sections 3 and 4, one may understand $\sigma(\mathbb{A}_{\mathbb{Q}})$ as a monoid

$$\sigma(\mathbb{A}_{\mathbb{Q}}) = (\sigma(\mathbb{A}_{\mathbb{Q}}), \cap),$$

equipped with its binary operation $\cap$, the set intersection. Like in Section 3, we fix a von Neumann algebra M embedded in an operator algebra $B(H)$ on a Hilbert space H.

We will define a suitable action, also denoted by α, of the monoid $\sigma(\mathbb{A}_{\mathbb{Q}})$ acting on M in $B(\mathcal{H}_{\mathbb{Q}})$, where

$$\mathcal{H}_{\mathbb{Q}} = H \otimes H_{\mathbb{Q}}.$$

Before proceeding, we introduce weak tensor product structures in Section 5.1.

a)   Weak Tensor Product Structures: Let X_i be arbitrary sets, for $i \in \Lambda$, where Λ means any countable index set. Let

$$g_i : X_i \to X_i \tag{5.1.1}$$

be well-defined functions, for all $i \in \Lambda$.

Now, let X be the *Cartesian product* $\underset{i \in \Lambda}{\Pi} X_i$ *of* $\{X_i\}_{i \in \Lambda}$. Define the subset $\mathcal{X}$ of X by

$$\mathcal{X} = \left\{ (x_i)_{i \in \Lambda} \in X \;\middle|\; \begin{array}{c} \text{finitely many } x_i \in X_i, \text{ and} \\ \text{almost of all other } x_j \in g_j(X_j), \\ \text{for } i, \ j \in \Lambda \end{array} \right\}, \tag{5.1.2}$$

determined by a system $g = \{g_i\}_{i \in \Lambda}$ of (5.1.1). We denote this subset $\mathcal{X}$ of (5.1.2) in X by

$$\mathcal{X} = \underset{i \in \Lambda}{\Pi_g} X_i.$$

It is clear that $\mathcal{X}$ is a subset of X. If g_i are bijections, for all $i \in \Lambda$, then $\mathcal{X}$ is *equipotent* (or *bijective*) to X. However, in general, $\mathcal{X}$ is a subset of X.

Definition 5.1. *The subset* $\mathcal{X} = \underset{i \in \Lambda}{\Pi_g} X_i$ *of* $X = \underset{i \in \Lambda}{\Pi} X_i$, *in the sense of (5.1.2), is called the weak tensor product set of* $\{X_i\}_{i \in \Lambda}$ *induced by a system* $g = \{g_i\}_{i \in \Lambda}$ *of functions* g_i.

Let $\mathbb{Q}_p$ be our p-adic number fields, for all $p \in \mathcal{P}$. Define a function

$$g_p : \mathbb{Q}_p \to \mathbb{Q}_p$$

by

$$g_p \left(p^{-N} \left(\sum_{j=0}^{\infty} a_j p^j \right) \right) \overset{def}{=} \sum_{j=0}^{\infty} a_j p^j, \tag{5.1.3}$$

for all $p^{-N} \sum_{j=0}^{\infty} a_j p^j \in \mathbb{Q}_p$ (with $N \in \mathbb{N} \cup \{0\}$), for all $p \in \mathcal{P}$. Then the image $g_p(\mathbb{Q}_p)$ is identical to the compact subset $\mathbb{Z}_p$, the unit disk of $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. Therefore, the Adele ring

$$\mathbb{A}_{\mathbb{Q}} = \underset{p \in \mathcal{P}}{\Pi'} \mathbb{Q}_p$$

is identified with

$$\mathbb{A}_{\mathbb{Q}} = \underset{p \in \mathcal{P}}{\Pi_g} \mathbb{Q}_p,$$

by (2.1.10) and (5.1.2), where $g = \{g_p\}_{p \in \mathcal{P}}$ is the system of functions g_p of (5.1.3).

Remark here that, for example, if we have real number r in $\mathbb{R} = \mathbb{Q}_{\infty}$, with its decimal notation

$$|r| = \sum_{k\in\mathbb{Z}} t_k \cdot 10^{-k} = \cdots t_{-2} t_{-1} t_0 . \, t_1 t_2 t_3 \cdots$$

with $0 \le t_k < 10$ in $\mathbb{N}$, then

$$g_\infty(r) = 0. \, t_1 t_2 t_3 \cdots, \tag{5.1.4}$$

with identification $g_\infty(\pm 1) = 1$.

Traditionally, we simply write $\mathbb{A}_{\mathbb{Q}} = \Pi'_{p\in\mathcal{P}} \mathbb{Q}_p$ as before, if there is no confusion.

Remark also that, X_i's of (5.1.1) and (5.1.2) may / can be algebraic structures (e.g., semigroups, or groups, or monoids, or groupoids, or vector spaces, etc), or topological spaces (e.g., Hilbert spaces, or Banach spaces, etc). One may put product topology on the weak tensor product, with continuity on $\{g_i\}_{i\in\Lambda}$. Similarly, if X_i's are topological algebras (e.g., Banach algebras, or C^*-algebras, or von Neumann algebras, etc), then we may have suitable product topology, with bounded (or continuous) linearity on the system $\{g_i\}_{i\in\Lambda}$.

Notation In topological-$*$-algebraic case, to distinguish with other situations, we use the notation $\underset{i\in\Lambda}{\otimes_\Phi}$, instead of using $\underset{i\in\Lambda}{\Pi_\Phi}$, for a system Φ of functions. $\square$

b) *The Adele-Ring von Neumann Algebra* $\mathfrak{M}$*.* In this section, we establish a von Neumann algebra $\mathfrak{M}$ generated by the Adele ring $\mathbb{A}_{\mathbb{Q}}$. Recall that the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is a unbounded-measured product topological ring induced by $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$. In particular, it is a weak tensor product of $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, i.e.,

$$\mathbb{A}_{\mathbb{Q}} = \Pi'_{p\in\mathcal{P}} \mathbb{Q}_p = \Pi_{g}_p \mathbb{Q}_p,$$

where $g = \{g_p\}_{p\in\mathcal{P}}$ is the system of functions (5.1.3) satisfying (5.1.4).

By understanding $\mathbb{A}_{\mathbb{Q}}$ as a measure space $(\mathbb{A}_{\mathbb{Q}}, \sigma(\mathbb{A}_{\mathbb{Q}}), \rho)$ (e.g., see Section 2.1), we have the L^2-Hilbert space $H_{\mathbb{Q}}$, defined by

(5.2.1)

$$H_{\mathbb{Q}} \overset{def}{=} L^2(\mathbb{A}_{\mathbb{Q}}, \rho).$$

It has its inner product $<,>$, defined by

(5.2.1)$'$

$$< F_1, F_2 > \overset{def}{=} \int_{\mathbb{A}_{\mathbb{Q}}} F_1 \, \overline{F_2} \, d\rho,$$

for all $F_1, F_2 \in H_{\mathbb{Q}}$. And similar to Section 3.1, the von Neumann algebra $\mathfrak{M}$ is defined by

$$\mathfrak{M} \overset{def}{=} L^\infty(\mathbb{A}_{\mathbb{Q}}, \rho). \tag{5.2.2}$$

Definition 5.2. *We call the Hilbert space $H_{\mathbb{Q}}$ of (5.2.1) the Adele-ring Hilbert space. The von Neumann algebra $\mathfrak{M}$ of (5.2.2) is said to be the Adele-ring von Neumann algebra.*

Let $F = \sum_{Y\in\sigma(\mathbb{A}_{\mathbb{Q}})} t_Y \, \chi_Y$ be an element of the Adele-ring von Neumann algebra $\mathfrak{M}$. Then

$$\begin{aligned}
\int_{\mathbb{A}_{\mathbb{Q}}} F \, d\rho &= \int_{\mathbb{A}_{\mathbb{Q}}} \left(\sum_{Y\in\sigma(\mathbb{A}_{\mathbb{Q}})} t_Y \, \chi_Y \right) d\rho \\
&= \sum_{Y\in\sigma(\mathbb{A}_{\mathbb{Q}})} t_Y \, \rho(Y).
\end{aligned} \tag{5.2.3}$$

By construction, if Y is a subset of the Adele ring $\mathbb{A}_{\mathbb{Q}}$, then

$$Y = \Pi_{p\in\mathcal{P}} Y_p, \tag{5.2.4}$$

where Y_p's are subsets of $\mathbb{Q}_p$, for $p \in \mathcal{P}$. So, by (2.1.10), (5.2.3) and (5.2.4), one has

$$\rho(Y) = \rho \left(\underset{p \in \mathcal{P}}{\Pi} Y_p \right) = \underset{p \in \mathcal{P}}{\Pi} \rho_p(Y_p), \qquad (5.2.5)$$

by identifying ρ_∞ with the usual distance measure on $\mathbb{Q}_\infty = \mathbb{R}$.

As we discussed in (3.1.3) and (3.1.4), if $S \in \sigma(\mathbb{Q}_p)$, for a prime p, then the element χ_S is identically distributed with

$$\sum_{j=1}^{N} r_j \, \chi_{p^{k_j} U_p},$$

for some $N \in \mathbb{N} \cup \{\infty\}$, $r_j \in [0, 1]$ in $\mathbb{R}$, and $k_j \in \mathbb{Z}$, for $j = 1, ..., N$. Therefore, we obtain the following theorem.

Theorem 5.1. *Let $Y \in \sigma(\mathbb{A}_\mathbb{Q})$ and let χ_Y be a generating element of the Adele-ring von Neumann algebra $\mathfrak{M}$. Then there exist $N_p \in \mathbb{N} \cup \{\infty\}$, $r_{p:j} \in [0, 1]$ in $\mathbb{R}$, and $k_{p:j} \in \mathbb{Z}$, for $j = 1, ..., N_p$, for $p \in \mathcal{P}$, such that*

$$\int_{\mathbb{A}_\mathbb{Q}} \chi_Y \, d\rho = \underset{p \in \mathcal{P}}{\Pi} \left(\sum_{j=1}^{N_p} r_{p:j} \left(\frac{1}{p^{k_{p:j}}} - \frac{1}{p^{k_{p:j}+1}} \right) \right). \qquad (5.2.6)$$

Proof. Let $Y \in \sigma(\mathbb{A}_\mathbb{Q})$. Then, by (5.2.4), there exist $Y_p \in \sigma(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$, such that $Y = \underset{p \in \mathcal{P}}{\Pi} Y_p$. By Section 3.1, for each $p \in \mathcal{P}$, the ρ_p-measurable subsets Y_p has $N_p \in \mathbb{N} \cup \{\infty\}$, and $k_{p:1}, ..., k_{p:N_p} \in \mathbb{Z}$, and $r_{p:1}, ..., r_{p:N_p} \in [0, 1]$, such that:

$$\rho_p(Y_p) = \sum_{j=1}^{N_p} r_{p:j} \left(\frac{1}{p^{k_{p:j}}} - \frac{1}{p^{k_{p:j}+1}} \right) = \int_{\mathbb{Q}_p} \chi_{Y_p} \, d\rho_p.$$

Therefore, by the product measure $\rho = \underset{p \in \mathcal{P}}{\times} \rho_p$ on the Adele ring $\mathbb{A}_\mathbb{Q}$, we have that:

$$\int_{\mathbb{A}_\mathbb{Q}} \chi_Y \, d\rho = \rho(Y) = \left(\underset{p \in \mathcal{P}}{\times} \rho_p \right) \left(\underset{p \in \mathcal{P}}{\Pi} Y_p \right)$$

$$= \underset{p \in \mathcal{P}}{\Pi} \rho_p(Y_p)$$

$$= \underset{p \in \mathcal{P}}{\Pi} \left(\sum_{j=1}^{N_p} r_{p:j} \left(\frac{1}{p^{k_{p:j}}} - \frac{1}{p^{k_{p:j}+1}} \right) \right).$$

Therefore, the formula (5.2.6) holds. ∎

The formula (5.2.6) characterizes the identically-distributedness on elements of the Adele-ring von Neumann algebra $\mathfrak{M}$.

The following theorem provides a structure theorem of the Adele-ring von Neumann algebra $\mathfrak{M}$ in terms of the p-prime von Neumann algebras $\{\mathfrak{M}_p\}_{p \in \mathcal{P}}$.

Theorem 5.2. *Let $\mathfrak{M} = L^\infty(\mathbb{A}_\mathbb{Q}, \rho)$ be the Adele-ring von Neumann algebra, and let $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ be the p-prime von Neumann algebras, for $p \in \mathcal{P}$. Then $\mathfrak{M}$ is $*$-isomorphic to the weak tensor product von Neumann algebra $\otimes_\varphi \underset{p \in \mathcal{P}}{\mathfrak{M}_p}$ of $\{\mathfrak{M}_p\}_{p \in \mathcal{P}}$, induced by the system of functions $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$, i.e.,*

$$\mathfrak{M} \overset{*-iso}{=} \underset{p \in \mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p, \text{ with } \mathfrak{M}_\infty = L^\infty(\mathbb{R}), \qquad (5.2.8)$$

where the weak tensor product $\otimes_\varphi$ is not only algebraic, but also topological, satisfying

$$\varphi_p \left(\sum_{X \in \sigma(\mathbb{Q}_p)} t_X \, \chi_X \right) \overset{def}{=} \sum_{X \in \sigma(\mathbb{Q}_p)} t_X \, \chi_{X \cap \mathbb{Z}_p}, \qquad (5.2.9)$$

for all $p \in \mathcal{P}$.

Proof. By the construction of the Adele ring $\mathbb{A}_{\mathbb{Q}}$, it is the weak tensor product $\Pi'_{p \in \mathcal{P}} \mathbb{Q}_p$ of $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$ as we discussed in Sections 2.1 and 5.1. Therefore,

$$\mathfrak{M} \stackrel{def}{=} L^\infty(\mathbb{A}_{\mathbb{Q}}, \rho) \stackrel{*\text{-iso}}{=} L^\infty \left(\Pi'_{p \in \mathcal{P}} \mathbb{Q}_p, \underset{p \in \mathcal{P}}{\times} \rho_p \right)$$

$$\stackrel{*\text{-iso}}{=} \underset{p \in \mathcal{P}}{\otimes_\varphi} L^\infty(\mathbb{Q}_p, \rho_p) = \underset{p \in \mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p,$$

where $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$ is the family of $*$-homomorphisms $\varphi_p : \mathbb{Q}_p \to \mathbb{Z}_p$ of (5.2.9).

Now, it suffices to show that φ_p are $*$-homomorphisms, for all $p \in \mathcal{P}$. Trivially φ_p are linear and bounded, by the very definition. Also, it satisfies that

$$\varphi_p \left(\chi_{S_1} \chi_{S_2} \right) = \varphi_p \left(\chi_{S_1 \cap S_2} \right) = \chi_{S_1 \cap S_2 \cap \mathbb{Z}_p}$$

$$= \chi_{(S_1 \cap \mathbb{Z}_p) \cap (S_2 \cap \mathbb{Z}_p)} = \chi_{S_1 \cap \mathbb{Z}_p} \chi_{S_2 \cap \mathbb{Z}_p}$$

$$= \varphi_p(S_1) \, \varphi_p(S_2),$$

for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. So, for any $g_1, g_2 \in \mathfrak{M}_p$,

$$\varphi_p(g_1 g_2) = \varphi_p(g_1) \, \varphi_p(g_2).$$

Now, observe that

$$\varphi_p \left((t \, \chi_S)^* \right) = \varphi_p \left(\bar{t} \, \chi_S^* \right) = \bar{t} \, \varphi_p(\chi_S)$$

$$= \bar{t} \, \chi_{S \cap \mathbb{Z}_p} = \bar{t} \, \overline{\chi_{S \cap \mathbb{Z}_p}} = \bar{t} \left(\chi_{S \cap \mathbb{Z}_p} \right)^*$$

$$= \left(t \chi_{S \cap \mathbb{Z}_p} \right)^* = \left(\varphi_p(t \chi_S) \right)^*,$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $t \in \mathbb{C}$. Therefore, for $g \in \mathfrak{M}_p$,

$$\varphi_p(g^*) = \left(\varphi_p(g) \right)^*.$$

Therefore, φ_p are well-defined $*$-homomorphisms, for all $p \in \mathcal{P}$. So, the family $\{\varphi_p\}_{p \in \mathcal{P}}$ is a system of $*$-homomorphisms.

Therefore, indeed, $\mathfrak{M}$ is $*$-isomorphic to the weak tensor product $\otimes_\varphi \mathfrak{M}_p$, as a $p \in \mathcal{P}$
well-defined W^*-subalgebra of the usual tensor product W^*-algebra $\underset{p \in \mathcal{P}}{\otimes_{\mathbb{C}}} \mathfrak{M}_p$. ∎

The above theorem shows that, to study the Adele-ring von Neumann algebra $\mathfrak{M}$, we can investigate the system of conditional summands $\mathfrak{M}_p$, the p-prime von Neumann algebras, for $p \in \mathcal{P}$.

c) Adele W^*-Dynamical Systems

Let's fix an arbitrary von Neumann algebra M in an operator algebra $B(H)$, and let $\mathfrak{M} = L^\infty(\mathbb{A}_{\mathbb{Q}}, \rho)$ be the Adele-ring von Neumann algebra, which is $*$-isomorphic to the weak tensor product W^*-algebra $\otimes_\varphi \mathfrak{M}_p$ of p-prime von Neumann algebras $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$, where $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$
$p \in \mathcal{P}$
is in the sense of (5.2.9). Thus, in this section, we understand $\mathfrak{M}$ and $\underset{p \in \mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p$,
alternatively.

Consider the σ-algebra $\sigma(\mathbb{A}_{\mathbb{Q}})$ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$ as a monoid $(\sigma(\mathbb{A}_{\mathbb{Q}}), \cap)$, with its identity $\mathbb{A}_{\mathbb{Q}}$. Define a monoidal action α of $\sigma(\mathbb{A}_{\mathbb{Q}})$ acting on M in $B(\mathcal{H}_{\mathbb{Q}})$ by

$$\alpha_S(m) \stackrel{def}{=} \chi_S \, m \, \chi_S^* = \chi_S \, m \, \chi_S, \qquad (5.3.1)$$

for all $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, and $m \in M$, where

$$\mathcal{H}_{\mathbb{Q}} = H \otimes H_{\mathbb{Q}}.$$

Then the action α of $\sigma(\mathbb{A}_\mathbb{Q})$ is indeed a well-defined monoidal action, since

$$
\begin{aligned}
\alpha_{S_1 \cap S_2}(m) &= \chi_{S_1 \cap S_2}\, m\, \chi_{S_1 \cap S_2} = \chi_{S_1 \cap S_2}\, m\, \chi_{S_2 \cap S_1} \\
&= \chi_{S_1}\, \chi_{S_2}\, m\, \chi_{S_2}\, \chi_{S_1} = \chi_{S_1}\, (\alpha_{S_2}(m))\, \chi_{S_1} \\
&= \alpha_{S_1}\, (\alpha_{S_2}(m)) = (\alpha_{S_1} \circ \alpha_{S_2})\,(m),
\end{aligned}
$$

and

$$
(\alpha_{S_1}(m))^* = \left(\chi_{S_1} m \chi_{S_1}\right)^* = \chi_{S_1}\, m^*\, \chi_{S_1} = \alpha_{S_1}(m^*),
$$

for all $S_1,\ S_2 \in \sigma(\mathbb{A}_\mathbb{Q})$, and $m \in M$. Thus,

$$
(\alpha_{S_1 \cap S_2})\,(m) = (\alpha_{S_1} \circ \alpha_{S_2})\,(m), \text{ and } (\alpha_{S_1}(m))^* = \alpha_{S_1}(m^*), \tag{5.3.2}
$$

for all $S_1,\ S_2 \in \sigma(\mathbb{A}_\mathbb{Q})$, for all $m \in M$.

The action α compresses operators of M in $B(\mathcal{H}_\mathbb{Q})$, and hence it is bounded. So, α is a well-defined monoidal action of $\sigma(\mathbb{A}_\mathbb{Q})$ acting on M in $B(\mathcal{H}_\mathbb{Q})$, by (5.3.2).

Notation Similar to Section 3.2, we denote $\alpha_S(m)$ simply by m^S, for all $S \in \sigma(\mathbb{A}_\mathbb{Q})$ and $m \in M$. $\square$

Notice that there exists an action χ of the monoid $\sigma(\mathbb{A}_\mathbb{Q})$ acting on $H_\mathbb{Q} = L^2(\mathbb{A}_\mathbb{Q}, \rho)$, such that

$$
\chi(S) = \chi_S, \text{ the characteristic function of } S, \tag{5.3.3}
$$

for all $S \in \sigma(\mathbb{A}_\mathbb{Q})$. By construction,

$$
H_\mathbb{Q} = \overline{\text{linear span of } \chi(\sigma(\mathbb{A}_\mathbb{Q}))}^{<\cdot,\cdot>},
$$

under the Hilbert space topology induced by $<,>$ of $(5.2.1)'$.

Definition 5.3. *The triple $A_M = (M, \sigma(\mathbb{A}_\mathbb{Q}), \alpha)$ of a fixed von Neumann algebra M in $B(H)$, the σ-algebra $\sigma(\mathbb{A}_\mathbb{Q})$ of the Adele ring $\mathbb{A}_\mathbb{Q}$, understood as a monoid equipped with $(\cap)$, and the monoidal action α of $\sigma(\mathbb{A}_\mathbb{Q})$ in the sense of (5.3.1), is called an Adele W^*-dynamical system in $B(\mathcal{H}_\mathbb{Q})$. For an Adele W^*-dynamical system A_M, define the corresponding crossed product W^*-algebra*

$$
\mathbb{M}_\mathbb{Q} = M \times_\alpha \sigma(\mathbb{A}_\mathbb{Q})
$$

by the W^*-subalgebra of $B(\mathcal{H}_\mathbb{Q})$ generated by M and $\alpha\,(\chi\,(\sigma(\mathbb{A}_\mathbb{Q})))$, where χ is in the sense of (5.3.3), consisting of all elements

$$
\sum_{S \in \sigma(\mathbb{A}_\mathbb{Q})} m_S\, \chi_S \text{ with } m_S \in M.
$$

This W^*-subalgebra $\mathbb{M}_\mathbb{Q}$ of $B(\mathcal{H}_\mathbb{Q})$ is said to be the Adele dynamical W^*-algebra induced by A_M.

Let $A_M = (M, \sigma(\mathbb{A}_\mathbb{Q}), \alpha)$ be an Adele W^*-dynamical system, and let $\mathbb{M}_\mathbb{Q} = M \times_\alpha \sigma(\mathbb{A}_\mathbb{Q})$ be the Adele dynamical W^*-algebra induced by A_M. Let $m_j \chi_{S_j}$ be elements of $\mathbb{M}_\mathbb{Q}$, with $m_j \in M$, and $S_j \in \sigma(\mathbb{A}_\mathbb{Q})$, for $j = 1, ..., n$, for $n \in \mathbb{N}$. Then one can obtain that

$$
\prod_{j=1}^{n} (m_j\, \chi_{S_j}) = \left(m\left(\prod_{j=2}^{n} m^{\overset{j-1}{\underset{i=1}{\cap}} S_i} \right) \right) \left(\chi_{\underset{j=1}{\overset{n}{\cap}} S_j} \right), \tag{5.3.4}
$$

since

$$
\begin{aligned}
(m_1\chi_{S_1})(m_2\chi_{S_2}) &= m_1\chi_{S_1} m_2\chi_{S_1}^2 \chi_{S_2} \\
&= m_1\chi_{S_1} m_2\chi_{S_1}\chi_{S_1 \cap S_2} \\
&= m_1 m_2^{S_1}\chi_{S_1 \cap S_2}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
(m\chi_S)^* &= \chi_S^* m^* = \chi_S m^* \\
&= \chi_S m^*\chi_S^2 = \chi_S m^*\chi_S\chi_S \\
&= (m^*)^S\chi_S = (m^*)^S\chi_S^*,
\end{aligned} \tag{5.3.5}
$$

for all $m \, \chi_S \in \mathbb{M}_{\mathbb{Q}}$, with $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$.

So, the Adele dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$ is a W^*-subalgebra of $B(\mathcal{H}_{\mathbb{Q}})$ generated by M and $\chi(\sigma(\mathbb{A}_{\mathbb{Q}}))$, satisfying the conditions (5.3.4) and (5.3.5).

Let $\mathfrak{M} = L^\infty(\mathbb{A}_{\mathbb{Q}}, \rho)$ be the Adele-ring von Neumann algebra. For a fixed von Neumann algebra M, construct the tensor product W^*-algebra

$$\mathcal{M}_0 = M \otimes_{\mathbb{C}} \mathfrak{M},$$

which is a W^*-subalgebra of $B(\mathcal{H}_{\mathbb{Q}})$. Define now a W^*-subalgebra $\mathcal{M}_{\mathbb{Q}}$ of $\mathcal{M}_0$ by the "conditional" tensor product W^*-algebra

(5.3.6)
$$\mathcal{M}_{\mathbb{Q}} \overset{def}{=} M \otimes_\alpha \mathfrak{M},$$

satisfying the following α-relations (5.3.7) and (5.3.8);

(5.3.7)
$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 m_2^{S_1}) \otimes \chi_{S_1} \chi_{S_2},$$

and

(5.3.8)
$$(m \otimes \chi_S)^* = (m^*)^S \otimes \chi_S^*,$$

for all $m_1, m_2, m \in M$, and $S_1, S_2, S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Of course, the α-relations; (5.3.7) and (5.3.8); are determined under linearity.

Similar to Section 3.3, we obtain the following structure theorem for $\mathbb{M}_{\mathbb{Q}}$.

Theorem 5.3. *Let* $\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ *be the Adele dynamical W^*-algebra in* $B(\mathcal{H}_{\mathbb{Q}})$ *induced by an Adele W^*-dynamical system A_M, and let $\mathfrak{M}$ be the Adele-ring von Neumann algebra. Then $\mathbb{M}_{\mathbb{Q}}$ and the conditional tensor product W^*-algebra $M \otimes_\alpha \mathfrak{M}$ of (5.3.6) are $*$-isomorphic, i.e.,*

(5.3.9)
$$\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}}) \overset{*\text{-}iso}{=} M \otimes_\alpha \mathfrak{M} = \mathcal{M}_{\mathbb{Q}}.$$

Proof. Let $\mathbb{M}_{\mathbb{Q}}$ be the Adele dynamical W^*-algebra $M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ induced by an Adele W^*-dynamical system A_M, and let $\mathcal{M}_{\mathbb{Q}} = M \otimes_\alpha \mathfrak{M}$ be the conditional tensor product W^*-algebra (5.3.6) of a fixed von Neumann algebra M, and the Adele-ring von Neumann algebra $\mathfrak{M}$ in $B(\mathcal{H}_{\mathbb{Q}})$, satisfying the α-relations (5.3.7) and (5.3.8).

Define now a morphism

$$\Phi : \mathcal{M}_{\mathbb{Q}} \to \mathbb{M}_{\mathbb{Q}}$$

by a linear transformation satisfying

$$\Phi(m \otimes \chi_S) = m \chi_S, \qquad (5.3.10)$$

for all $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Then it is generator-preserving, and hence, it is bijective and bounded. Also, it satisfies that

$$\Phi\left((m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2})\right) = \Phi\left((m_1 m_2^{S_1}) \otimes \chi_{S_1} \chi_{S_2}\right)$$
$$= (m_1 m_2^{S_1}) \, \chi_{S_1 \cap S_2} = (m_1 \chi_{S_1})(m_2 \chi_{S_2})$$
$$= \Phi(m_1 \otimes \chi_{S_1}) \, \Phi(m_2 \otimes \chi_{S_2}),$$

for all $m_1, m_2 \in M$, and $S_1, S_2 \in \sigma(\mathbb{A}_{\mathbb{Q}})$.

Thus, for any $T_1, T_2 \in \mathcal{M}_{\mathbb{Q}}$, we have

$$\Phi(T_1 T_2) = \Phi(T_1) \, \Phi(T_2) \text{ in } \mathbb{M}_{\mathbb{Q}}, \qquad (5.3.11)$$

by the linearity of Φ. Furthermore,

$$\Phi\left((m \otimes \chi_S)^*\right) = \Phi\left((m^*)^S \otimes \chi_S^*\right)$$
$$= (m^*)^S \, \chi_S = (m \, \chi_S)^* = (\Phi(m \otimes \chi_S))^*,$$

for all $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$.

So, for any $T \in \mathcal{M}_{\mathbb{Q}}$,

$$\Phi(T^*) = \Phi(T)^* \text{ in } \mathbb{M}_{\mathbb{Q}}. \tag{5.3.12}$$

Therefore, by (5.3.11) and (5.3.12), the bijective linear transformation Φ of (5.3.10) is a $*$-isomorphism from $\mathcal{M}_{\mathbb{Q}}$ onto $\mathbb{M}_{\mathbb{Q}}$. ▮

The above structure theorem (5.3.9) shows that, just like the p-adic cases, Adelic dynamical W^*-algebras $M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ are understood as conditional tensor product W^*-algebras $M \otimes_\alpha \mathfrak{M}$. As in Section 3, we handle von Neumann algebras $\mathbb{M}_{\mathbb{Q}}$ and $\mathcal{M}_{\mathbb{Q}}$, alternatively.

One of the most interesting results of the above structure theorem (5.3.9) is the following structure theorem.

Theorem 5.4. Let $\mathbb{M}_{\mathbb{Q}}$ be the Adele dynamical W^*-algebra induced by an Adele W^*-dynamical system A_M. Then $\mathbb{M}_{\mathbb{Q}}$ is $*$-isomorphic to the weak tensor product W^*-algebra $\otimes_{\varphi_M} \mathcal{M}_p$ of the p-adic dynamical W^*-algebras $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ in $p \in P$ the sense of (3.2.5), for $p \in \mathcal{P}$, i.e.,

$$\mathbb{M}_{\mathbb{Q}} \overset{*\text{-}iso}{=} \underset{p \in \mathcal{P}}{\otimes_{\varphi_M}} \mathcal{M}_p, \tag{5.3.13}$$

with the system φ_M,

$$\varphi_M \overset{def}{=} 1_M \otimes \varphi = \{1_M \otimes \varphi_p\}_{p \in \mathcal{P}}, \tag{5.3.14}$$

where $\varphi = \{\varphi_p\}_{p \in \mathcal{P}}$ is in the sense of (5.2.9).

Proof. By (5.3.9), the given Adele dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$ is $*$-isomorphic to $\mathcal{M}_{\mathbb{Q}} = M \otimes_\alpha \mathfrak{M}$;

$$\mathbb{M}_{\mathbb{Q}} \overset{*\text{-}iso}{=} \mathcal{M}_{\mathbb{Q}}.$$

Also, by (5.2.8), the Adele-ring von Neumann algebra $\mathfrak{M}$ is $*$-isomorphic to $\otimes_\varphi \mathfrak{M}_p$, where φ is in the sense of (5.2.9); $p \in \mathcal{P}$

$$\mathfrak{M} \overset{*\text{-}iso}{=} \underset{p \in \mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p,$$

where $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ are p-prime von Neumann algebras, for $p \in \mathcal{P}$. Thus, one can have that

$$\mathbb{M}_{\mathbb{Q}} \overset{*\text{-}iso}{=} M \otimes_\alpha \mathfrak{M} \overset{*\text{-}iso}{=} M \otimes_\alpha \left(\underset{p \in \mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p \right)$$

$$\overset{*\text{-}iso}{=} \underset{p \in \mathcal{P}}{\otimes_{\varphi_M}} (M \otimes_\alpha \mathfrak{M}_p)$$

where $\varphi_M = \{1_M \otimes \varphi_p\}_{p \in \mathcal{P}}$

$$\overset{*\text{-}iso}{=} \underset{p \in \mathcal{P}}{\otimes_{\varphi_M}} \mathcal{M}_p,$$

by (3.3.3). ▮

The structure theorem (5.3.13) provides a useful tool for studying our Adelic dynamical W^*-algebras $\mathbb{M}_{\mathbb{Q}}$ in terms of p-adic dynamical W^*-algebras $\mathcal{M}_p$'s.

VI. Adelic Dynamical W^*-Algebras

Let M be a fixed von Neumann algebra in $B(H)$, and let

$$\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$$

be the Adele dynamical W^*-algebra induced by an Adele W^*-dynamical system

$$A_M = (M, \sigma(\mathbb{A}_\mathbb{Q}), \alpha) \text{ in } B(\mathcal{H}_\mathbb{Q}).$$

In Section 5, we showed that $\mathbb{M}_\mathbb{Q}$ is $*$-isomorphic to the conditional tensor product W^*-algebra

$$\mathcal{M}_\mathbb{Q} = M \otimes_\alpha \mathfrak{M}$$

of M and the Adele von Neumann algebra $\mathfrak{M} = L^\infty(\mathbb{A}_\mathbb{Q}, \rho)$ by (5.3.9). And hence, it is $*$-isomorphic to the weak tensor product von Neumann algebra

$$\mathbf{M}_\mathbb{Q} = \underset{p \in \mathcal{P}}{\otimes_{\varphi_M}} \mathcal{M}_p$$

of p-adic dynamical W^*-algebras $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$, for $p \in \mathcal{P}$, by (5.3.13).

We understand these three von Neumann algebras $\mathbb{M}_\mathbb{Q}$, $\mathcal{M}_\mathbb{Q}$ and $\mathbf{M}_\mathbb{Q}$, as the same von Neumann algebras $\mathbb{M}_\mathbb{Q}$. Especially, case-by-case, we use a suitable one among $\{\mathbb{M}_\mathbb{Q}, \mathcal{M}_\mathbb{Q}, \mathbf{M}_\mathbb{Q}\}$ as $\mathbb{M}_\mathbb{Q}$.

First, recall that, if $Y \in \sigma(\mathbb{A}_\mathbb{Q})$, then there exist $Y_p \in \sigma(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$, such that

$$Y = \underset{p \in \mathcal{P}}{\Pi} Y_p,$$

where most of Y_q's are identical to $Y_q \cap \mathbb{Z}_q$ (i.e., $Y_q \subseteq \mathbb{Z}_q$), for $q \in \mathcal{P}$.

For instance, the subset U of $\mathbb{A}_\mathbb{Q}$,

$$U = \underset{p \in \mathcal{P}}{\Pi} U_p \tag{6.0.1}$$

is a well-determined element of $\sigma(\mathbb{A}_\mathbb{Q})$, where U_p are the unit circles of $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. We call U, the *unit circle of the Adele ring* $\mathbb{A}_\mathbb{Q}$. Indeed, for any element $(u_p)_{p \in \mathcal{P}} \in U$, we have

$$|(u_p)_{p \in \mathcal{P}}|_\mathbb{Q} = \underset{p \in \mathcal{P}}{\Pi} |u_p|_p = 1,$$

where $|.|_\mathbb{Q}$ is the non-Archimedean norm on $\mathbb{A}_\mathbb{Q}$ induced by the p-norms $\{|.|_p\}_{p \in \mathcal{P}}$ (e.g., see [18]).

Define now a conditional expectation

$$E : \mathbb{M}_\mathbb{Q} = \mathcal{M}_\mathbb{Q} \to M \otimes_\alpha \mathbb{C}[\{\chi_U\}] \overset{\text{*-iso}}{=} M$$

by a linear morphism satisfying that:

$$E(m\chi_Y) = m(r\chi_U), \tag{6.0.2}$$

where $r \in [0, 1]$ satisfies that:

$$\int_{\mathbb{A}_\mathbb{Q}} \chi_{Y \cap U} \, d\rho = r \int_{\mathbb{A}_\mathbb{Q}} \chi_U \, d\rho = r \left(\underset{p \in \mathcal{P}}{\Pi} \left(1 - \frac{1}{p} \right) \right). \tag{6.0.3}$$

Remark here that the quantity $\underset{p \in \mathcal{P}}{\Pi} \left(1 - \frac{1}{p} \right)$ on the right-hand side of $(6.0.2)'$ satisfies that:

$$\underset{p \in \mathcal{P}}{\Pi} \left(1 - \frac{1}{p} \right) = \left(1 - \frac{1}{\infty} \right) \left(\underset{p:primes}{\Pi} \left(1 - \frac{1}{p} \right) \right)$$

$$= \underset{p:primes}{\Pi} \left(1 - \frac{1}{p} \right) = \frac{1}{\zeta(1)},$$

where

$$\zeta(s) \overset{def}{=} \sum_{n=1}^\infty \frac{1}{n^s} = \underset{p:primes}{\Pi} \frac{1}{1 - p^{-s}} = \frac{1}{\underset{p:primes}{\Pi} \left(1 - \frac{1}{p^s} \right)}$$

is the *Riemann zeta function*, satisfying that:

$$\frac{1}{\zeta(s)} = \underset{p:prime}{\Pi} \left(1 - \frac{1}{p^s} \right), \text{ for } s \in \mathbb{C}.$$

By definition, it is clear that

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and hence,

$$\frac{1}{\zeta(1)} = 0.$$

Thus, one can verify that the formula (6.0.3) becomes 0, for all $m\chi_Y \in \mathbb{M}_{\mathbb{Q}}$, with $m \in (M, \psi)$ and $Y \in \sigma(\mathbb{A}_{\mathbb{Q}})$. In other words, we cannot directly mimic the p-adic dynamical free-probabilistic approaches as in Section 4.

Therefore, we consider a new, but similar approach to establish a suitable free probability model on our Adelic dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$.

a) Adelic Dynamical W^*-Probability Spaces $\{(\mathbb{M}_{\mathbb{Q}}, \varphi_P)\}_{P \subset \mathcal{P}}$. As we have discussed at the beginning of this section, we cannot directly mimic the free-probabilistic settings from the p-adic dynamical W^*-probability spaces to our Adelic W^*-probability settings. So, we construct suitable linear functionals differently from those of Section 4 (and those of [10]).

Take first a "finite" subset P of $\mathcal{P}$, say

$$P = \{p_1, ..., p_n\}, \qquad (6.1.1)$$

for some $n \in \mathbb{N}$, in particular, suppose all $p_1, ..., p_n$ of P are primes (not ∞) in $\mathcal{P}$. We call such subsets P of $\mathcal{P}$, *finite prime (sub)sets* of $\mathcal{P}$.

Let P be a finite prime set (6.1.1) of $\mathcal{P}$. Define an element U_P of $\sigma(\mathbb{A}_{\mathbb{Q}})$ by

$$U_P \stackrel{def}{=} \left(\prod_{p \in P} U_p \right) \times \left(\prod_{q \in \mathcal{P} \setminus P} \mathbb{Z}_q \right), \qquad (6.1.2)$$

under possible re-arrangement. i.e., for all p in P, take the unit circle U_p of $\mathbb{Q}_p$, and for almost all other q in $\mathcal{P}$, take $\mathbb{Z}_q$ of $\mathbb{Q}_q$, and then product them to construct a ρ-measurable subset U_P in the Adele ring $\mathbb{A}_{\mathbb{Q}}$.

Then define a subalgebra M_P of $\mathcal{M}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{Q}} = \mathbf{M}_{\mathbb{Q}}$ by

$$M_P \stackrel{def}{=} M \otimes_{\alpha} \mathbb{C}\left[\{ \chi_S : S \in \sigma(\mathbb{A}_{\mathbb{Q}}), \ S \subseteq U_P \} \right]. \qquad (6.1.3)$$

Define now a conditional expectation

$$E_P : \mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} \to M_P$$

by a linear morphism satisfying that:

$$E_P(m\chi_Y) = m \ \chi_{Y \cap U_P}. \qquad (6.1.4)$$

Now, let's check the morphism E_P of (6.1.4) is indeed a conditional expectation:
(6.1.5) For any $m \ \chi_S \in M_P$, one has

$$E_P(m\chi_S) = m\chi_{S \cap U_P} = m\chi_S,$$

since $S \subseteq U_P$, and hence, for any $x \in M_P$, we have

$$E_P(x) = x, \text{ under linearity.}$$

(6.1.6) For $m_j \chi_{S_j} \in M_P$, for $j = 1, 2$, and $m \ \chi_Y \in \mathbb{M}_{\mathbb{Q}}$, we have
$$E_P\left((m_1\chi_{S_1})(m\chi_Y)(m_2\chi_{S_2}) \right)$$
$$= E_P\left(m_1 m^{S_1} m_2^{S_1 \cap Y} \ \chi_{S_1 \cap Y \cap S_2} \right)$$
$$= E_P\left(m_1 m^{S_1} m_2^{S_1 \cap Y} \ \chi_{S_1 \cap Y \cap S_2} \right)$$
$$= \left(m_1 m^{S_1} m_2^{S_2 \cap Y} \right) \chi_{S_1 \cap Y \cap S_2},$$

and

$$(m_1\chi_{S_1}) \left(E_P(m\chi_Y) \right) (m_2\chi_{S_2})$$
$$= (m_1\chi_{S_1}) \left(m\chi_{Y \cap U_P} \right) (m_2\chi_{S_2})$$
$$= \left(m_1 m^{S_1} m_2^{S_1 \cap Y \cap U_P} \right) \chi_{S_1 \cap Y \cap S_2},$$
$$= \left(m_1 m^{S_1} m_2^{S_1 \cap Y} \right) \chi_{S_1 \cap Y \cap S_2},$$

because $S_1 \cap U_P = S_1$, so, $S_1 \cap Y \cap U_P = S_1 \cap Y$, and hence,

$$E_P\left((m_1\chi_{U_P})(m\chi_Y)(m_2\chi_{U_P})\right)$$
$$= \left(m_1\chi_{U_P}\right)\left(E_P(m\chi_Y)\right)\left(m_2\chi_{U_P}\right).$$

Thus, under linearity, we have that:

$$E_P\left(x_1 y x_2\right) = x_1\ E_P(y)\ x_2,$$

for all x_1, $x_2 \in M_P$ and $y \in \mathbb{M}_\mathbb{Q}$.

(6.1.7) Also, one has that:

$$E_P\left((m\chi_Y)^*\right) \ = E_P\left((m^*)^Y \chi_Y\right)$$
$$= (m^*)^Y (\chi_{Y \cap U_P}) = (E_P(m\chi_Y))^*,$$

and hence, for all $y \in \mathbb{M}_\mathbb{Q}$, we have

$$E_P(y^*) = E_P(y)^*.$$

Proposition 6.1. *The morphism E_P of (6.1.4) is a well-defined conditional expectation from $\mathbb{M}_\mathbb{Q}$ onto M_P, for any finite prime set P of $\mathcal{P}$.*

Proof. By definition, the morphism E_P of (6.1.4) is bounded and linear. So, it is a conditional expectation because of (6.1.5), (6.1.6) and (6.1.7). ∎

Define now a morphism $F_P : M_P \to M_P$ by a linear morphism satisfying that:

$$F_P\left(m\chi_Y\right) = m\left(r_Y \chi_{U_P}\right), \text{ for some } r_Y \in [0, 1]. \quad (6.1.8)$$

In particular, the quantity r_Y in (6.1.8) is determined as follows in $[0, 1]$ of $\mathbb{R}$:

$$\int_{\mathbb{A}_\mathbb{Q}} \chi_{Y \cap U_P}\ d\rho = \rho\left(Y \cap U_P\right)$$

$$= \rho\left(\left(\prod_{p \in \mathcal{P}} Y_p\right) \cap \left(\prod_{p \in \mathcal{P}} V_p\right)\right)$$

where $U_p = \prod_{p \in \mathcal{P}} V_p$ satisfies (6.1.2) (under possible re-arrangement)

$$= \left(\underset{p \in \mathcal{P}}{\times} \rho_p\right)\left(\prod_{p \in \mathcal{P}} (Y_p \cap V_p)\right)$$

$$= \prod_{p \in \mathcal{P}} \rho_p\left(Y_p \cap V_p\right)$$

$$= \left(\prod_{p \in P} \rho_p\left(Y_p \cap U_p\right)\right)\left(\prod_{q \in \mathcal{P} \setminus P} \rho_p\left(Y_q \cap \mathbb{Z}_q\right)\right)$$

by (6.1.2)

$$= \left(\prod_{p \in P} r_p\left(1 - \tfrac{1}{p}\right)\right)\left(\prod_{q \in \mathcal{P} \setminus P} r_q \cdot 1\right)$$

for $r_w \in [0, 1]$, since

$$\rho_w(U_w) = 1 - \tfrac{1}{w}, \text{ and } \rho_w(\mathbb{Z}_w) = 1$$

for all $w \in \mathcal{P}$, and hence, we have

$$\int_{\mathbb{A}_\mathbb{Q}} \chi_{Y \cap U_P} d\rho = \left(\prod_{q \in \mathcal{P}} r_q\right)\left(\prod_{p \in P}\left(1 - \tfrac{1}{p}\right)\right). \quad (6.1.9)$$

Define r_Y in $[0, 1]$ by

$$r_Y = \prod_{q \in \mathcal{P}} r_q, \quad (6.1.10)$$

where the quantity of the right-hand side of (6.1.10) is from (6.1.9).

i.e., the morphism F_P on M_P satisfies

$$F_P\left(m\chi_Y\right) = m\left(r_Y \chi_{U_P}\right),$$

where $r_Y \in [0, 1]$ satisfy (6.1.10), for all $m\chi_Y \in M_P$.

By Section 5.2, without loss of generality, one can verify that: if

$$Y = \prod_{p \in \mathcal{P}} Y_p \in \sigma(\mathbb{A}_{\mathbb{Q}}), \text{ with } Y_p \in \sigma(\mathbb{Q}_p),$$

then almost all Y_q's are identical to $\mathbb{Z}_q$.

Assumption If we take $Y = \prod_{p \in \mathcal{P}} Y_p$ in $\sigma(\mathbb{A}_{\mathbb{Q}})$, with $Y_p \in \sigma(\mathbb{Q}_p)$, then we assume almost all Y_q's are identical to $\mathbb{Z}_q$. $\square$

Define now a linear functional

$$\gamma_0 : M_P \to \mathbb{C}$$

by

$$\gamma_0 \overset{def}{=} \left(\otimes \int_{\mathbb{A}_{\mathbb{Q}}} \bullet \, d\rho \right) \circ F_P \tag{6.1.11}$$

i.e., γ_0 is a linear morphism satisfying that:

$$\gamma_0 (m \, \chi_Y) \overset{def}{=} (m) \left(\int_{\mathbb{A}_{\mathbb{Q}}} r_Y \chi_{U_P} d\rho \right) \tag{6.1.12}$$

$$= r_Y \ (m) \left(\prod_{p \in P} \left(1 - \frac{1}{p} \right) \right),$$

for all $m \in (M, \psi)$ and $Y \in \sigma(\mathbb{A}_{\mathbb{Q}})$, where $r_Y \in [0, 1]$ is in the sense of (6.1.10). The linear morphism γ_0 of (6.1.12) is indeed a well-defined linear functional on M_P.

Define now a linear functional γ_P on $\mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} = \mathbf{M}_{\mathbb{Q}}$ by

$$\gamma_P \overset{def}{=} \gamma_0 \circ E_P, \tag{6.1.13}$$

for any fixed finite prime sets P of $\mathcal{P}$.

Since γ_0 is a bounded linear functional, and E_P is a bounded conditional expectation, γ_P of (6.1.13) is indeed a well-defined linear functional on the Adelic dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$.

Definition 6.1. *Let $\mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} = \mathbf{M}_{\mathbb{Q}}$ be an Adelic dynamical W^*-algebra over a W^*-probability space (M, ψ). Let P be a finite prime set of $\mathcal{P}$, and γ_P, the corresponding linear functional in the sense of (6.1.13). Then the pair $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$ is called the Adelic dynamical W^*-probability space induced by a finite prime set P of $\mathcal{P}$.*

By definition, for any $m \chi_S \in \mathbb{M}_{\mathbb{Q}}$, one has that:

$$\gamma_P (m\chi_S) = \gamma_0 (E_P(m\chi_S))$$
$$= \gamma_0 (m\chi_{S \cap U_P})$$
$$= r_{S \cap U_P} \ (m) \left(\prod_{p \in P} \left(1 - \frac{1}{p} \right) \right), \tag{6.1.14}$$

where $r_{S \cap U_P} \in [0, 1]$ satisfies (6.1.10) and (6.1.12).

Notice now that

$$U_p = \left(\prod_{p \in P} U_p \right) \times \left(\prod_{q \in \mathcal{P} \setminus P} \mathbb{Z}_q \right),$$

under possible re-arrangement. Like in Section 4, if we replace U_p's to $p^k U_p$, for some $k \in \mathbb{Z}$, i.e., if we define

$$U_{P:k} \overset{def}{=} \left(\prod_{p \in P} U_{p:k} \right) \times \left(\prod_{q \in \mathcal{P} \setminus P} \mathbb{Z}_q \right),$$

where $U_{p:k} = p^k U_p$, as in (4.8), then we have similar structures, for all $k \in \mathbb{Z}$, with identity:

$$U_P = U_{P:0}.$$

However, in such cases, the formula (6.1.9) will be replaced by

$$\int_{\mathbb{A}_{\mathbb{Q}}} U_{P:k} \, d\rho = \prod_{p \in P} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}} \right).$$

In this paper, we only consider the case where we have $U_P = U_{P:0}$.

b) Free Structure of $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$. Let $\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ be the Adelic dynamical W^*-algebra in $B(\mathcal{H}_{\mathbb{Q}})$ induced by an Adele W^*-dynamical system $A_M = (M, \sigma(\mathbb{A}_{\mathbb{Q}}), \alpha)$. As before, we understand $\mathbb{M}_{\mathbb{Q}}$ as its $*$-isomorphic von Neumann algebras $\mathcal{M}_{\mathbb{Q}} = M \otimes_\alpha \mathfrak{M}$, and $\mathbf{M}_{\mathbb{Q}} = \otimes_{\varphi_M} \mathcal{M}_p$, case-by-case, and let γ_P be the $p \in \mathcal{P}$ linear functional in the sense of (6.1.13), satisfying that:

$$\gamma(m\chi_S) = r_S \quad (m) \left(\prod_{p \in P} \left(1 - \frac{1}{p} \right) \right),$$

where $r_S \in [0, 1]$ in $\mathbb{R}$ satisfying (6.1.10) and (6.1.12), for all $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, for all finite primes sets P of $\mathcal{P}$.

Throughout this section, we fix a finite prime set P of $\mathcal{P}$, and concentrate on free probabilistic structure on $\mathbb{M}_{\mathbb{Q}}$ in terms of γ_P of (6.1.13). The following lemma is obtained by the straightforward computations.

Lemma 6.2. *Let $m\chi_S$ be a free random variable in the Adelic dynamical W^*-probability space $(\mathbb{M}_{\mathbb{Q}}, \gamma)$, with $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Then*

$$\gamma_P \left((m\chi_S)^n \right) = r_{S \cap U_P} \left(\ (m(m^S)^{n-1}) \right) \left(\prod_{p \in P} \left(1 - \frac{1}{p} \right) \right), \qquad (6.2.1)$$

for all $n \in \mathbb{N}$, where $r_{S \cap U_P} \in [0, 1]$ satisfies (6.1.10) and (6.1.12).

Proof. If $m\chi_S \in \mathbb{M}_{\mathbb{Q}}$, with $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, then

$$(m\chi_S)^n = m m^S m^{S \cap S} \dots m^{S \cap S \cap \dots \cap S} \chi_{S \cap \dots \cap S}$$
$$= m \, m^S \, m^S \dots m^S \, \chi_S = m \, (m^S)^{n-1} \, \chi_S,$$

for all $n \in \mathbb{N}$. Therefore, one can have that

$$\gamma_P \left((m\chi_S)^n \right) = \gamma_P \left(m(m^S)^{n-1} \chi_S \right)$$
$$= r_{S \cap U_P} \quad \left(m(m^S)^{n-1} \right) \left(\prod_{p \in P} \left(1 - \frac{1}{p} \right) \right)$$

for all $n \in \mathbb{N}$, by (6.1.14), where $r_{S \cap U_P} \in [0, 1]$ satisfies (6.1.10) and (6.1.12). $\blacksquare$

More general to (6.2.1), we obtain the following lemma.

Lemma 6.3. *Let $m_1 \chi_{S_1}, \dots, m_n \chi_{S_n}$ be free random variables in an Adelic dynamical W^*-probability space $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$, with $m_k \in M$, $S_k \in \sigma(\mathbb{Q}_p)$, for $k = 1, \dots, n$, for $n \in \mathbb{N}$. Then*

$$\gamma \left(\prod_{j=1}^{n} m_j \chi_{S_j} \right) = r_{\left(\bigcap_{j=1}^{n} S_j \right) \cap U_P} \left(\psi \left(\prod_{j=1}^{N} m_j^{\bigcap_{i=0}^{j-1} S_i} \right) \right) \left(\rho \left(\bigcap_{j=1}^{n} S_j \right) \right), \qquad (6.2.2)$$

where $r_{\left(\bigcap_{j=1}^{n} S_j \right) \cap U_P} \in [0, 1]$, satisfying (6.1.10) and (6.1.12).

Proof. If $m_k \chi_{S_k} \in (\mathcal{M}_p, \gamma_p)$ are given as above, for $k = 1, \dots, n$, then

$$\prod_{j=1}^{n} (m_j \chi_{S_j}) = m_1 m_2^{S_1} m_3^{S_1 \cap S_2} \dots m_N^{S_1 \cap \dots \cap S_{n-1}} \chi_{S_1 \cap \dots \cap S_n}$$

$$= \left(\prod_{j=1}^{n} m_j^{\bigcap_{i=0}^{j-1} S_i} \right) \left(\chi_{\bigcap_{j=1}^{n} S_j} \right),$$

in $\mathbb{M}_{\mathbb{Q}}$, for all $n \in \mathbb{N}$.

Thus, one has that:

$$\gamma_P\left(\prod_{j=1}^{n}(m_j\chi_{S_j})\right) = \gamma_P\left(\left(\prod_{j=1}^{n}m_j^{\overset{j-1}{\underset{i=0}{\cap}}S_i}\right)\left(\chi_{\overset{n}{\underset{j=1}{\cap}}S_j}\right)\right)$$

$$= r_{\left(\overset{n}{\underset{j=1}{\cap}}S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}m_j^{\overset{j-1}{\underset{i=0}{\cap}}S_i}\right)\right)\left(\prod_{p\in P}\left(1-\frac{1}{p}\right)\right),$$

by (6.1.14), where $r_{\left(\overset{n}{\underset{j=1}{\cap}}S_j\right)\cap U_P} \in [0,1]$ satisfies (6.1.10) and (6.1.12). ∎

Notation 6.2 *In the following, we denote* $\prod_{p\in P}\left(1-\frac{1}{p}\right)$ *by* ζ_P^-, *for convenience.*
□

By (6.2.1) and (6.2.2), we obtain the following free-distributional data of free random variables of $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$.

Theorem 6.4. *Let* $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$ *be an Adelic dynamical W^*-probability space determined by a finite prime set P of $\mathcal{P}$, and let*

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k}\chi_{S_k}, \text{ for } k = 1, ..., n,$$

be free random variables, for $n \in \mathbb{N}$. Then
(6.2.3)

$$\gamma_P\left(\prod_{j=1}^{n}T_j\right) =$$

$$\zeta_P^-\left(\sum_{(S_1,...,S_n)\in\overset{n}{\underset{j=1}{\prod}}Supp(T_j)} r_{\left(\overset{n}{\underset{i=1}{\cap}}S_i\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\overset{j-1}{\underset{i=1}{\cap}}S_i\right)}\right)\right)\right),$$

*where ζ_P^- is in the sense of **Notation 6.2**, and where* $r_{r_{\left(\overset{j-1}{\underset{i=1}{\cap}}S_i\right)\cap U_P}} \in [0,1]$ *satisfy*

(6.1.10) *and* (6.1.12).

Proof. Inductively, one can get that

$$\prod_{j=1}^{n}T_j = \sum_{(S_1,...,S_n)\in\overset{n}{\underset{j=1}{\prod}}Supp(T_j)}\left(\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\overset{j-1}{\underset{i=1}{\cap}}S_i\right)}\right)\left(\chi_{\overset{n}{\underset{j=1}{\cap}}S_j}\right)\right),$$

for all $j = 1, ..., n$. So,

$$\gamma_P\left(\prod_{j=1}^{n}T_j\right) = \gamma_P\left(T_1 T_2 \ ... \ T_n\right)$$

$$= \gamma_P\left(\sum_{(S_1,...,S_n)\in\overset{n}{\underset{j=1}{\prod}}Supp(T_j)}\left(\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\overset{j-1}{\underset{i=1}{\cap}}S_i\right)}\right)\left(\chi_{\overset{n}{\underset{j=1}{\cap}}S_j}\right)\right)\right)$$

$$= \sum_{(S_1,...,S_n)\in\overset{n}{\underset{j=1}{\prod}}Supp(T_j)}\left(\gamma_P\left(\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\overset{j-1}{\underset{i=1}{\cap}}S_i\right)}\right)\left(\chi_{\overset{n}{\underset{j=1}{\cap}}S_j}\right)\right)\right)$$

$$= \sum_{(S_1,...,S_n)\in\overset{n}{\underset{j=1}{\prod}}Supp(T_j)}\left(r_{\left(\overset{n}{\underset{i=1}{\cap}}S_i\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\overset{j-1}{\underset{i=1}{\cap}}S_i\right)}\right)\right)(\zeta_P^-)\right)$$

by (6.2.2), where $r_{\left(\bigcap\limits_{i=1}^{n} S_i\right) \cap U_P} \in [0, 1]$ satisfy (6.1.10) and (6.1.12), and where ζ_P^- is in the sense of **Notation 6.2**. ∎

Thanks to (6.2.3), we obtain the following corollary.

Corollary 6.5. *Let* $T = \sum\limits_{S \in Supp(T)} m_S \, \chi_S$ *be a free random variable in* $(\mathcal{M}_p, \gamma_p)$. *Then*

$$\gamma_P\left(T^n\right) = \zeta_P^-\left(\sum_{(S_1,\ldots,S_n)\in Supp(T)^n}\left(r_{\left(\bigcap\limits_{j=1}^{n}S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}\right)^{\binom{j-1}{\bigcap\limits_{i=1}S_i}}\right)\right)\right)\right), \quad (6.2.4)$$

$$\gamma_P\left((T^*)^n\right) =$$
$$\zeta_P^-\left(\sum_{(S_1,\ldots,S_n)\in Supp(T)^n}\left(r_{\left(\bigcap\limits_{j=1}^{n}S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}\left((m_{S_j}^*)^{S_j}\right)^{\binom{j-1}{\bigcap\limits_{i=1}S_i}}\right)\right)\right)\right), \quad (6.2.5)$$

for all $n \in \mathbb{N}$, *where* $r_{\left(\bigcap\limits_{j=1}^{n}S_j\right)\cap U_P} \in [0, 1]$ *satisfy (6.1.10) and (6.1.12).* □

Let $m_1\chi_{S_1}, \ldots, m_n\chi_{S_n}$ be free random variables in $(\mathbb{M}_\mathbb{Q}, \gamma)$, for $n \in \mathbb{N}$, where $m_1, \ldots, m_n \in M$, and $S_1, \ldots, S_n \in \sigma(\mathbb{A}_\mathbb{Q})$. Then, by (6.2.3), one can obtain that:
$$k_n^P\left(m_1\chi_{S_1}, \ldots, m_n\chi_{S_n}\right)$$

$$= \sum_{\pi\in NC(n)} (\gamma_P)_\pi\left(m_1\chi_{S_1}, \ldots, m_n\chi_{S_n}\right)\mu(\pi, 1_n)$$

$$= \sum_{\pi\in NC(n)}\left(\prod_{V\in\pi}(\gamma_P)_V\left(m_1\chi_{S_1}, \ldots, m_n\chi_{S_n}\right)\mu\left(0_{|V|}, 1_{|V|}\right)\right)$$

by the Möbius inversion (See Section 2.3)

$$= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,\ldots,i_k)\in\pi}\gamma_P\left(m_{i_1}\chi_{S_{i_1}} \cdots m_{i_k}\chi_{S_{i_k}}\right)\mu\left(0_k, 1_k\right)\right)$$

$$(6.2.7)$$

$$= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,\ldots,i_k)\in\pi}\left(\zeta_P^-\left(r_V\psi\left(\prod_{t=1}^{k}\left(m_{i_t}\right)^{\binom{k-1}{\bigcap\limits_{t=1}S_{i_t}}}\right)\right)\right)\mu\left(0_k, 1_k\right)\right),$$

where $k_n^P(\ldots)$ mean free cumulants induced by γ_V in the sense of Section 2.3.

By (6.2.7), we obtain the following inner free structure of the given Adelic dynamical W^*-algebra $\mathbb{M}_\mathbb{Q}$, with respect to γ_P.

Theorem 6.6. *Let* $m_1\chi_S$, *and* $m_2\chi_S$ *be free random variables in an Adelic dynamical* W^*-*probability space* $(\mathbb{M}_\mathbb{Q}, \gamma_P)$, *with* $m_1, m_2 \in M$, *and* $S \in \sigma(\mathbb{A}_\mathbb{Q})$, *with* $\rho(S) \neq 0$. *Moreover, assume that* S *contains* U_P. *i.e., suppose*

$$S = \prod_{q\in\mathcal{P}}S_q \text{ in } \mathbb{A}_\mathbb{Q}, \text{ and } S_p \supseteq U_p, \text{ for all } p \in P. \quad (6.2.8)$$

Then $\{m_1, m_1^S\}$ *and* $\{m_2, m_2^S\}$ *are free in the* W^*-*probability space* (M, ψ), *if and only if* $m_1\chi_S$ *and* $m_2\chi_S$ *are free in* $(\mathbb{M}_\mathbb{Q}, \gamma_P)$. *i.e.,*

$$\{m_1, m_1^S\} \text{ and } \{m_2, m_2^S\} \text{ are free in } (M, \psi) \quad (6.2.9)$$

$$\Longleftrightarrow$$

$$m_1\chi_S \text{ and } m_2\chi_S \text{ are free in } (\mathbb{M}_\mathbb{Q}, \gamma_P),$$

under the condition (6.2.8).

Proof. ($\Rightarrow$) Assume that $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in (M, ψ). Then, by definition, all mixed free $*$-cumulants of them (with respect to the linear functional ψ) vanish (See Section 2.3, or [16]). i.e.,

$$k_n^\psi \left(u_{i_1}^{r_1}, \ ..., \ u_{i_n}^{r_n} \right) = 0 \text{ in } \mathbb{C},$$

for all $n \in \mathbb{N} \setminus \{1\}$, where $(u_{i_1}, \ ..., \ u_{i_n}) \in \{m_1, m_2, m_1^S, m_2^S\}$ are "mixed," and $(i_1, \ ..., \ i_n) \in \{1, 2\}^n$, and $(r_1, \ ..., \ r_n) \in \{1, *\}^n$, where $k_n (...)$ mean free cumulants induced by ψ.

Consider mixed free $*$-cumulants of $m_1\chi_S$ and $m_2\chi_S$ in $(\mathbb{M}_\mathbb{Q}, \gamma)$, for a fixed nonzero ρ-measurable set $S \in \sigma(\mathbb{A}_\mathbb{Q})$. By (6.2.7), one has that

$$k_n^P \left((m_{i_1}\chi_S)^{r_1}, \ ..., \ (m_{i_n}\chi_S)^{r_n} \right)$$

$$= \left(\zeta_P^- \right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} r_V \ \psi \left(\overset{k}{\underset{t=1}{\Pi}} \left([m_{j_t}^{r_{j_t}}]^{S_{i_t}} \right)^{\overset{k-1}{\underset{t=1}{\cap}} S_{j_t}} \right) \mu \left(0_k, \ 1_k \right) \right)$$

where all S_{j_t} are identical to S, and r_V satisfy (6.1.10) and (6.1.12), and where

$$\left([m_j^r]^S \right)^Y \overset{def}{=} \begin{cases} m_j^Y & \text{if } r = 1 \\ (m_j^*)^{S \cap Y} & \text{if } r = *, \end{cases}$$

for all $j, r \in \{1, *\}$ and $S, Y \in \sigma(\mathbb{A}_\mathbb{Q})$, and hence,

$$= \left(\zeta_P^- \right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} r_V \ \left(\overset{k}{\underset{t=1}{\Pi}} \left([m_{j_t}^{r_{j_t}}]^S \right)^S \right) \mu \left(0_k, \ 1_k \right) \right)$$

$$= \left(\zeta_P^- \right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} \ \left(\overset{k}{\underset{t=1}{\Pi}} \left([m_{j_t}^{r_{j_t}}]^S \right)^S \right) \mu \left(0_k, \ 1_k \right) \right)$$

by the condition (6.2.8) (Since the assumption (6.2.8) holds, $r_V = 1$, for all $V \in \pi$, for all $\pi \in NC(n)$, for all $n \in \mathbb{N}$)

$$= \left(\zeta_P^- \right) \left(k_n (u_{i_1}^{r_1}, \ ..., \ u_{i_n}^{r_n}) \right) = \left(\zeta_P^- \right) \cdot 0$$
$$= 0,$$

for all $n \in \mathbb{N} \setminus \{1\}$. It shows that, if $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in $(M,$), then $\{m_1\chi_S, m_2\chi_S\}$ are free in $(\mathbb{M}_\mathbb{Q}, \gamma_P)$, under the condition (6.2.8).

($\Leftarrow$) Assume now that two free random variables $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathbb{M}_\mathbb{Q}, \gamma_P)$, where S satisfies $\rho(S) \neq 0$ and the condition (6.2.8), i.e.,

$$k_n^P \left((m_{i_1}\chi_S)^{r_1}, \ ..., \ (m_{i_n}\chi_S)^{r_n} \right)$$
(6.2.10)

$$= \left(\zeta_P^- \right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} \left(r_V \ \left(\overset{k}{\underset{t=1}{\Pi}} \left([m_{j_t}^{r_{j_t}}]^S \right)^S \right) \right) \mu \left(0_k, \ 1_k \right) \right)$$

$$= 0,$$

whenever $(i_1, \ ..., \ i_n)$ are "mixed" in $\{1, 2\}^n$, for $(r_1, \ ..., \ r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N} \setminus \{1\}$.

The formula (6.2.10) is identical to

$$\left(\zeta_P^- \right) \left(k_n(u_{i_1}^{r_1}, \ ..., \ u_{i_n}^{r_n}) \right),$$

since $r_V = 1$, by (6.2.8), for the mixed n-tuple $(u_{i_1}, \ ..., \ u_{i_n})$ of $\{m_1, m_1^S\} \cup \{m_2, m_2^S\}$.

Since $\rho(S) \neq 0$, and since the condition (6.2.8) is assumed, $\rho(S \cap U_P) \neq 0$, and hence,

Notes

$$\left(\zeta_P^-\right)\left(k_n^\psi(u_{i_1}^{r_1}, \, ..., \, u_{i_n}^{r_n})\right) = 0,$$

as in (6.2.10), equivalently,

$$k_n^\psi(u_{i_1}^{r_1}, \, ..., \, u_{i_n}^{r_n}) = 0,$$

for all mixed n-tuple $(u_{i_1}, \, ..., \, u_{i_n}) \in \{m_1, m_1^S, m_2, m_2^S\}$. Equivalently, $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in (M, ψ). ∎

The above theorem shows that, the freeness of (M, ψ) acts like a certain kind of free-filterizations for the inner freeness of $(\mathbb{M}_\mathbb{Q}, \gamma_P)$, under the assumption (6.2.8).

The following corollary is a direct consequence of the above theorem.

Corollary 6.7. Let M_1 and M_2 be W^*-subalgebras of M in $B(H)$, and assume that the subsets $\{M_1, \alpha_S(M_1)\}$ and $\{M_2, \alpha_S(M_2)\}$ are free in (M, ψ), for $S \in \sigma(\mathbb{A}_\mathbb{Q})$, with $\rho(S \cap U_P) \neq 0$, satisfying the condition (6.2.8). Then two subsets

$$M_1 \otimes_\alpha \{\chi_S\} \text{ and } M_2 \otimes_\alpha \{\chi_S\} \text{ of } \mathcal{M}_\mathbb{Q} = \mathbb{M}_\mathbb{Q},$$

are free in $(\mathbb{M}_\mathbb{Q}, \gamma_P)$, for a fixed finite prime set P of $\mathcal{P}$.

Conversely, if $M_1 \otimes \{\chi_S\}$ and $M_2 \otimes \{\chi_S\}$ are free in $(\mathcal{M}_p, \gamma_p)$, where S satisfies (6.2.8), then $\{M_1, \alpha_S(M_1)\}$ and $\{M_2, \alpha_S(M_2)\}$ are free in (M, ψ), too. □

Let U_P be in the sense of (6.1.2) for a fixed finite prime set P of $\mathcal{P}$. Assume now that $S_1, S_2 \in \sigma(\mathbb{A}_\mathbb{Q})$ satisfies

$$S_1 \cap U_P \neq \varnothing \text{ and } S_2 \cap U_P = \varnothing. \tag{6.2.11}$$

For example, "$S_2 \cap U_P = \varnothing$" means that, if $S_2 = \prod_{q \in \mathcal{P}} S_2^q$, with $S_2^q \in \sigma(\mathbb{Q}_q)$, then

$$S_2^p \cap U_p = \varnothing, \text{ for all } p \in P,$$

and

$$S_2^q \cap \mathbb{Z}_q = \varnothing, \text{ for all } q \in \mathcal{P} \setminus P.$$

By (6.2.11), it is clear that

$$(S_1 \cap U_P) \cap (S_2 \cap U_P) = \varnothing, \tag{6.2.12}$$

even though $S_1 \cap S_2 \neq \varnothing$.

Theorem 6.8. Let $m_1 \chi_{S_1}, m_2 \chi_{S_2}$ be free random variables in an Adelic dynamical W^*-probability space $(\mathbb{M}_\mathbb{Q}, \gamma_P)$. If S_1 and S_2 satisfy the condition (6.2.11) in $\sigma(\mathbb{A}_\mathbb{Q})$, then they are free in $(\mathbb{M}_\mathbb{Q}, \gamma_P)$. i.e.,

$$S_1 \text{ and } S_2 \text{ satisfy (6.2.11)} \tag{6.2.13}$$

$$\Longrightarrow$$

$$M \otimes_\alpha \mathbb{C}\left[\{\chi_{S_1}\}\right] \text{ and } M \otimes_\alpha \mathbb{C}[\{\chi_{S_2}\}] \text{ are free}$$

in $(\mathbb{M}_\mathbb{Q}, \gamma_P)$.

Proof. Suppose $S_1, S_2 \in \sigma(\mathbb{A}_\mathbb{Q})$ satisfy the condition (6.2.11). Then, with respect to U_P of (6.1.2), they also satisfy the condition (6.2.12). Therefore, one has that:

$$k_n^P \left((m_{i_1}\chi_S)^{r_1}, \, ..., \, (m_{i_n}\chi_S)^{r_n}\right) \tag{6.1.14}$$

$$= \left(\zeta_P^-\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(i_1,...,i_k) \in \pi} r\left(\bigcap_{t=1}^k S_{i_t}\right) \cap U_P \; \psi\left(\prod_{t=1}^k (m_{i_t})^{\binom{k-1}{\cap} S_{i_t}}\right) \mu(0_k, \, 1_k) \right)$$

by (6.2.7), where $\left(\bigcap_{t=1}^{k-1} S_{i_t}\right) \cap U_P \in [0, 1]$ satisfy (6.1.10) and (6.1.12), and the elements $\left([m_j^r]^S\right)^Y$ are in the sense of the proof of the above Theorem .

Assume that a block $V = (i_1, \, ..., \, i_k)$ of π in (6.1.13) is mixed in $\{1, 2\}^k$. Then the corresponding quantity

$$r_{\left(\underset{t=1}{\overset{k}{\cap}} S_{i_t}\right) \cap U_P} = 0 \text{ in } [0, 1],$$

by (6.2.12). Therefore, whenever a noncrossing partition π of $NC(n)$ contains at least one mixed block, then the corresponding summand vanishes. Even though a noncrossing partition θ of $NC(n)$ does not contain a mixed block, since it contains a block corresponding S_2, one obtains the quantity

$$r_{S_2 \cap S_2 \cap ... \cap S_2 \cap U_P} = r_{S_2 \cap U_P} = 0 \text{ in } [0, 1],$$

for at least one block of θ. Thus, even though θ does not contain a mixed block, the corresponding partition-depending free moment vanishes.

i.e., whenever $(i_1, ..., i_n) \in \{1, 2\}^n$ are mixed for $n \in \mathbb{N} \setminus \{1\}$, then the free cumulants (6.1.13) vanish. Equivalently, $m_1 \chi_{S_1}$ and $m_2 \chi_{S_2}$ are free in $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$. ∎

With a freeness characterization (6.2.9) (under (6.2.8)), the above freeness necessary condition (6.2.13) provide inner free structures of the Adelic W^*-algebra $\mathbb{M}_{\mathbb{Q}}$ in terns of linear functionals γ_P, for finite prime sets P of $\mathcal{P}$.

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Notes

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