Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 6 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Harmonic Univalent Function with Varying Arguments Defined by using Salagean Integral Operator with Fixed Point

By Dr. Poonam Dixit, Nikhil Kumar & Puneet Shukla
C. S. J. M. University, India

Abstract- In this paper authors study a new class of harmonic functions defined by using Salagean Integral operator with varying arguments. We obtain coefficient inequalities, extreme points and distortion bounds.

Keywords: harmonic, analytic, varying arguments, sense preserving, salagean integral operataor, fixed point.

GJSFR-F Classification : FOR Code : MSC 2010: 30F15

HARMONICUNIVALENTFUNCTIONWITHVARYINGARGUMENTSDEFINEDBYUSINGSALAGEANINTEGRALOPERATORWITHFIXEDPOINT

Strictly as per the compliance and regulations of :

Ref

7. Libera, R. J., (1965), Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16, pp.755-758.

Harmonic Univalent Function with Varying Arguments Defined by using Salagean Integral Operator with Fixed Point

Dr. Poonam Dixit [α], Nikhil Kumar [σ] & Puneet Shukla [ρ]

Abstract- In this paper authors study a new class of harmonic functions defined by using Salagean Integral operator with varying arguments. We obtain coefficient inequalities, extreme points and distortion bounds .

Keywords: harmonic, analytic, varying arguments, sense preserving, salagean integral operataor, fixed point.

I. Introduction

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \bar{g} \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|, z \in D$ (see[7]).

Denote by S_H The class of functions f of the form (1) that the harmonic univalent and sense preserving in the unit disc U=$\{z \in C : |z| < 1\}$ for which

$$f(0) = h(0) = f_z'(0) - 1 = 0$$

then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = a_1 z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1 \tag{2}$$

In 1984 Clunie and Shell-Small [7] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds . Since then , there have been several related papers on S_H and its subclasses.

Author α: Department of Mathematics, UIET, C. S. J. M. University, Kanpur. e-mail: dixit_poonam14@rediffmail.com
Author σ: e-mail: nikhils820@gmail.com
Author ρ: e-mail: puneetshukla05@gmail.com

Salagean integral operator I^n is defined as follows (see[9])

(a) $I^0 f(z) = f(z)$

(b) $I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt$

....

....

(c)$I^n f(z) = I(I^{n-1} f(z))$ $(n \in N = \{1, 2, 3, ...\})$.

In [4], Cotirla defined Salagean integral operator for harmonic univalent functions $f(z)$ such that h(z) and g(z)are given by (2) as follows

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \qquad (3)$$

where

$$I^n h(z) = a_1 z + \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k$$

With the help of the modified Salagean integral operator we let$E_H(m, n, \gamma, \rho,)$ be the family of harmonic functions $f = h + \bar{g}$, which satisfy the following condition [11]

$$R_e \left[(1 + \rho e^{i\alpha}) \frac{I^n f(z)}{I^m f(z)} - \rho e^{i\alpha} \right] \geq \gamma \qquad (4)$$

$(\alpha \in R, 0 \leq \gamma < 1, \rho \geq 0, m \in N, n \in N_0 = N \cup \{0\}, m > n$, and $z \in U)$,

where $I^n f$ is defined by (3) , we note that

(a) taking $\alpha = 0$, $E_H(n+1, n; 2\beta - 1, 1)$=$H(n, \beta)(0 \leq \beta < 1)$(see Cotirla[4])

(b) taking $m = n + q$, $E_H(n + q, n; \gamma, \rho)$ =$H_{\rho,q}$ (n, γ) $(q \in N)$ (see Guney and Sakar [5]).

also we note that , by the special choices of α, γ, ρ, m and n , we obtain the following special cases

(a)Taking $\alpha = 0, then E_H(m, n, 2\beta - 1, 1) = H(m, n; \beta) = \{f \in S_H :$

$$R_e \left[\frac{I^n f(z)}{I^m f(z)} \right] > \beta(0 \leq \beta < 1; m \in N; n \in N_0; m > n; z \in U)\}$$

(b)$E_H(n + 1, n; \gamma, \rho) = E_H(n; \gamma, \rho) = \{f \in S_H$:

$$R_e \left[(1 + \rho e^{i\alpha}) \frac{I^n f(z)}{I^{n+1} f(z)} - \rho e^{i\alpha} \right] \geq \gamma(\alpha \in R; 0 \leq \gamma < 1; \rho \geq 0; n \in N_0; z \in U)\}$$

Ref

5. Guney, H. O., Sakar, F.M.,(2011), On uniformaly starlike functions defined by an integral operator, Acta Universities Apulensis, 28, pp.293-301.

Notes

(c) $E_H(1,0;\gamma,\rho) = E_H(\gamma,\rho) = \{f \in S_H :$

$$R_e\left[(1+\rho e^{i\alpha})\frac{f(z)}{If(z)} - \rho e^{i\alpha}\right] \geq \gamma(\alpha \in R; 0 \leq \gamma < 1; \rho \geq 0; z \in U)\}$$

also we define the subclass $V_{\bar{H}}(m,n;\gamma,\rho)$ consists of harmonic functions $f_n = h + \bar{g_n}$ in $E_H(m,n;\gamma,\rho)$ such that h and g_n are the form

$$h(z) = a_1 z + \sum_{k=2}^{\infty} a_k z^k, g_n(z) = \sum_{k=1}^{\infty} b_k z^k \tag{5}$$

and there exists a real number ϕ such that , mod 2π,

$$arg(a_k) + (k-1)\phi \equiv \pi, k \geq 2 \; and \; arg(b_k) + (k+1)\phi \equiv (n-1)\pi, k \geq 1. \tag{6}$$

also we note that, by the special choices of α, γ, m and n, we obtain:

(a) taking $\alpha = 0$, $V_{\bar{H}}(n+1,n;2\beta-1,1)$=$V_{\bar{H}}(n,\beta)$;

(b) taking $\alpha = 0$, $V_{\bar{H}}(m,n;2\beta-1,1)$=$V_{\bar{H}}(m,n,\beta)$;

(c) $V_{\bar{H}}(n+1,n;\gamma,\rho)$=$V_{\bar{H}}(n;\gamma,\rho)$;

(d) $V_{\bar{H}}(1,0;\gamma,\rho)$=$V_{\bar{H}}(\gamma,\rho)$;

II. Main Result

Unless otherwise mentioned , we assume in the reminder of this paper that , $\alpha \in R, 0 \leq \gamma < 1, \rho \geq 0, m \in N, n \in N_0, m > n$ and $z \in U$. We begin with a sufficient coefficient condition for functions in the class $E_H(m,n;\gamma,\rho,z_0)$.

Theorem 2.1 . Let $f = h + \bar{g}$ be such that h and g are given by 2. Furthermore,

$$\sum_{k=2}^{\infty}\left[\frac{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}}{1-\gamma}|a_k| + \frac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right] \leq 2a_1 \tag{7}$$

where $a_1 = 1$ Then $f \in E_H(m,n;\gamma,\rho,z_0)$. $0 \leq \gamma < 1, \rho \geq 0$

Proof. We need to show that if (7) holds the condition (4) is satised, then we want to prove that

$$R_e\left[\frac{(1+\rho e^{i\alpha})I^n f(z) - \rho e^{i\alpha} I^m f(z)}{I^m f(z)}\right] = R_e\frac{A(z)}{B(z)} \geq \gamma \tag{8}$$

Using the fact that $R_e(\omega) > \gamma$ if and only if $|1-\gamma+\omega| > |1+\gamma-\omega|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0, \tag{9}$$

Where $A(z) = (1+\rho e^{i\alpha})I^n f(z) - \rho e^{i\alpha} I^m f(z)$ and $B(z) = I^m f(z)$. Substituting for $A(z)$ and $B(z)$ in the left side of (9) we obtain,

$$|(1+\rho e^{i\alpha})I^n f(z) - \rho e^{i\alpha} I^m f(z) + (1-\gamma) I^m f(z)|$$

$$-\,|(1+\rho e^{i\alpha})I^n f(z) - \rho e^{i\alpha} I^m f(z) - (1+\gamma) I^m f(z)|$$

$$= |(1+\rho e^{i\alpha})(I^n h(z) + (-1)^n I^n g(z)) - \rho e^{i\alpha}(I^m h(z) + (-1)^n I^n g(z)) + (1-\gamma) I^m f(z)|$$

$$-\,|(1+\rho e^{i\alpha})(I^n h(z) + (-1)^n I^n g(z)) - \rho e^{i\alpha}(I^m h(z) + (-1)^n I^n g(z)) - (1+\gamma) I^m f(z)|$$

$$= |a_1(z) + \sum_{k=2}^{\infty} k^{-n} a_k z^k + \rho e^{i\alpha} a_1(z) + \rho e^{i\alpha} \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} k^{-n} b_k z^k + \rho e^{i\alpha}(-1)^n \sum_{k=1}^{\infty} k^{-n} b_k z^k - \rho e^{i\alpha} a_1(z)$$

$$-\,\rho e^{i\alpha} \sum_{k=2}^{\infty} k^{-m} a_k z^k - \rho e^{i\alpha}(-1)^m \sum_{k=1}^{\infty} k^{-m} b_k z^k + a_1(z) + \sum_{k=2}^{\infty} k^{-m} a_k z^k$$

$$+ (-1)^m \sum_{k=1}^{\infty} k^{-m} b_k z^k - \gamma a_1(z) - \gamma \sum_{k=2}^{\infty} k^{-m} a_k z^k - (-1)^m \gamma \sum_{k=1}^{\infty} k^{-m} b_k z^k|$$

$$-\,|a_1(z) + \sum_{k=2}^{\infty} k^{-n} a_k z^k + \rho e^{i\alpha} a_1(z) + \rho e^{i\alpha} \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} k^{-n} b_k z^k + \rho e^{i\alpha}(-1)^n \sum_{k=1}^{\infty} k^{-n} b_k z^k - \rho e^{i\alpha} a_1(z)$$

$$-\,\rho e^{i\alpha} \sum_{k=2}^{\infty} k^{-m} a_k z^k - \rho e^{i\alpha}(-1)^m \sum_{k=1}^{\infty} k^{-m} b_k z^k - a_1(z)$$

$$-\sum_{k=2}^{\infty} k^{-m} a_k z^k - (-1)^m \sum_{k=1}^{\infty} k^{-m} b_k z^k - \gamma a_1(z) - \gamma \sum_{k=2}^{\infty} k^{-m} a_k z^k - (-1)^m \gamma \sum_{k=1}^{\infty} k^{-m} b_k z^k|$$

$$= |(2-\gamma) a_1(z) + \sum_{k=2}^{\infty} [((1+\rho e^{i\alpha}) k^{-n} + (1-\gamma-\rho e^{i\alpha}) k^{-m})] a_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} [(1+\rho e^{i\alpha}) k^{-n} - (-1)^{m-n}(\rho e^{i\alpha} + \gamma - 1) k^{-m}] \times \overline{b_k z^k}|$$

$$-\,|\gamma a_1(z) - \sum_{k=2}^{\infty} [(1+\rho e^{i\alpha}) k^{-n} - (1+\rho e^{i\alpha}+\gamma) k^{-m}] a_k z^k$$

Notes

$$-(-1)^n \sum_{k=1}^{\infty}[(1+\rho e^{i\alpha})k^{-n}-(-1)^{m-n}(1+\rho e^{i\alpha}+\gamma)k^{-m}]\overline{b_k z^k}|$$

$$\geq 2(1-\gamma)a_1|z|-2\sum_{k=2}^{\infty}[(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}]|a_k||z|^k-2\sum_{k=1}^{\infty}[(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}]|b_k||z|^k$$

Notes

$$\geq 2(1-\gamma)|z|\left[a_1-\sum_{k=2}^{\infty}\frac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{(1-\gamma)}|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k||z|^{k-1}\right]$$

By using (7), then the last expression is non negative, then (9) is satised.

The harmonic function

$$f(z)=a_1(z)+\sum_{k=2}^{\infty}\frac{(1-\gamma)}{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}x_k z^k+\sum_{k=1}^{\infty}\frac{1-\gamma}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}\overline{y_k z^k} \qquad (10)$$

Where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (7) is sharp .

In the following theorem , it is shown that the condition (7) is also necessary for function $f_n=h+g_n$, where h and g_n are of the form(5) .

Theorem 2.2 . Let $f_n=h+g_n$, where h and g_n are given by (5). Then $f_n\in V_{\bar{H}}(m,n;\gamma,\rho,z_0)$, if and "only if" the coecient condition (7) holds .

Proof. Since $V_{\bar{H}}(m,n;\gamma,\rho,z_0)\subseteq E_{\bar{H}}(m,n;\gamma,\rho,z_0)$ we only need to prove the only if part of the theorem.For functions $f_n=h+g_n$, where h and g_n are given by (5), the inequality (4) with $f=f_n$ is equivalent to

$$R_e\left\{\frac{(1+\rho e^{i\alpha})\left(a_1 z+\sum_{k=2}^{\infty}k^{-n}a_k z^k+(-1)^n\sum_{k=1}^{\infty}k^{-n}\bar{b}_k\bar{z}^k\right)}{a_1 z+\sum_{k=2}^{\infty}k^{-m}a_k z^k+(-1)^m\sum_{k=1}^{\infty}k^{-m}\bar{b}_k\bar{z}^k}\right\}-$$

$$-R_e\left\{\frac{(\gamma+\rho e^{i\alpha})\left(a_1 z+\sum_{k=2}^{\infty}k^{-m}a_k z^k+(-1)^m\sum_{k=1}^{\infty}k^{-m}\bar{b}_k\bar{z}^k\right)}{a_1 z+\sum_{k=2}^{\infty}k^{-m}a_k z^k+(-1)^m\sum_{k=1}^{\infty}k^{-m}\bar{b}_k\bar{z}^k}\right\}>0.$$

The above condition holds for all values of $\alpha\in R$ and $z\in U$.Upon choosing ϕ according (6) and substituting $\alpha=0$ and $z=re^{i\phi}(0<r<1)$, we must have

$$\frac{E}{a_1-\left[\sum_{k=2}^{\infty}k^{-m}|a_k|-(-1)^{m+n-1}\sum_{k=1}^{\infty}k^{-m}|b_k|\right]r^{k-1}}>0, \qquad (11)$$

Where

$$E=(1-\gamma)-\left(\sum_{k=2}^{\infty}[(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}]|a_k|\right)r^{k-1}-$$

$$-\left(\sum_{k=1}^{\infty}[(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}]|b_k|\right)r^{k-1}.$$

If the inequality(7) does not hold , then E is negative for r sufficiently close to 1. Thus there exists $z_0=r_0$ in$(0,1)$ for which the quotient in (11) is negative. But this is a contradiction, the proof of Theorem (2.2) is completed.

We now obtain the distortion bounds for functions in $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$.

Theorem 2.3 . Let $f_n=h+g_n$, where h and g_n are given by (5) and $f_n\in V_{\bar{H}}(m,n;\gamma,\rho,z_0)$. Then for $|z|=r<1$, we have

$$|f_n(z)|\le(a_1+|b_1|)r+\left[\frac{(1-\gamma)a_1}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}|b_1|\right]r^2 \quad (12)$$

and

$$|f_n(z)|\ge(a_1+|b_1|)r+\left[\frac{(1-\gamma)a_1}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}|b_1|\right]r^2. \quad (13)$$

Proof. We prove the first inequality.

Let $f_n\in V_{\bar{H}}(m,n;\gamma,\rho,z_0)$, we have

$$(a)\ .\ |f_n(z)|\le(a_1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\le(a_1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$\le(a_1+|b_1|)r+\frac{(1-\gamma)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\sum_{k=2}^{\infty}\frac{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}{(1-\gamma)}(|a_k|+|b_k|)r^2$$

$$\le(a_1+|b_1|)r+\frac{(1-\gamma)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\times\sum_{k=2}^{\infty}\left[\frac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{(1-\gamma)}|a_k|+\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{(1-\gamma)}|b_k|\right]r^2$$

$$\le(a_1+|b_1|)r+\frac{(1-\gamma)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\left[a_1-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1-\gamma)}|b_1|\right]r^2$$

$$\le(a_1+|b_1|)r+\left[\frac{(1-\gamma)a_1}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}|b_1|\right]r^2.$$

The proof of the second inequality is similar, thus it is left.

The bounds given in Theorem (2.3) for functions $f_n=h+g_n$ such that h and g_n are given by (6) also hold for functions $f_n=h+g_n$ such that h and g_n are given by (2) if the coefficient condition (7) is satisfied .

Using the same technique used earlier by Aghalary [1] we introduce the extreme points of the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$.

Ref

1. Aghalary, R., (2007), Goodman- Salagian-type hormonic univalent function with varying arguments. Int. Journal of math. Analysis, 1 (22), pp.1051-1057.

Notes

Theorem 2.4 . The closed convex hull of the class $V_H(m,n;\gamma,\rho,z_0)$ (denoted by $clcoV_{\bar{H}}$ $(m,n;\gamma,\rho,z_0)$) is

$$\{f(z) = a_1(z) + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \in V_H(m,n;\gamma,\rho,z_0) :$$

$$\sum_{k=1}^{\infty} \left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{(1-\gamma)}|a_k| + \tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{(1-\gamma)}|b_k|\right] \leq 2a_1\}$$

where $a_1 = 1$ .Set $\lambda_k = \frac{(1-\gamma)a_1}{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}$ and $\mu_k = \frac{(1-\gamma)a_1}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}$

For b_1 fixed , $|b_1| \leq \frac{(1-\gamma)a_1}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}$,the extreme points of the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$ are

$$\left\{z + \lambda_k x z^k + \bar{b_1} z\right\} \cup \left\{\overline{z + \mu_k x z^k + b_1 z}\right\} \tag{14}$$

where k ≥ 2 and $|x| = 1 - \frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1-\gamma)}$.

Proof. Any function $f \in V_{\bar{H}}(m,n;\gamma,\rho,z_0)$ may be expressed as .

$$f(z) = a_1(z) + \sum_{k=2}^{\infty} |a_k| e^{i\beta_k} z^k + \overline{\sum_{k=2}^{\infty} |b_k| e^{i\delta_k} z^k} + \overline{b_1 z} \ ,$$

where the coefficients satisfy the inequality (7) . Set

$h_1(z) = z$,$g_1(z) = b_1 z$, $h_k(z) = z + \lambda_k e^{i\beta_k} z^k$, $g_k(z) = b_1 z + \mu_k e^{i\delta_k} z^k$,$k = 2,3,....$

Writing $X_k = \frac{|a_k|}{\lambda_k}$, $Y_k = \frac{|b_k|}{\mu_k}$, $k = 2,3,\ldots.$ and $X_1 = 1 - \sum_{k=2}^{\infty} X_k$, $Y_1 = 1 - \sum_{k=2}^{\infty} Y_k$, we have

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + \overline{Y_k g_k(z)})$$

In particular , setting $f_1(z) = z + \overline{b_1 z}$ and $f_k(z) = z + \lambda_k x z^k + \overline{b_1 z} + \overline{\mu_k y z^k}$,

$$\left(k \geq 2, |x| + |y| = 1 - \tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1-\gamma)}|b_1|\right)$$

we see that extreme points of the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$ are contained in $\{f_k(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \tfrac{1}{2}\left[f_1(z) + \lambda\left(1 - \tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{(1-\gamma)}|b_1|\right) z^2\right]$$

$$+ \tfrac{1}{2}\left[f_1(z) - \lambda\left(1 - \tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{(1-\gamma)}|b_1|\right) z^2\right] ,$$

a convex linear combination of functions in the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$. Next we will show if both $|x| \neq 0$ and $|y| \neq 0$, then f_k is not an extreme point .

Without loss of generality, assume $|x| \geq |y|$ choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$ Set $A = 1+\epsilon$ and $B = 1 - \left|\frac{\epsilon x}{y}\right|$, we then see that both

$$t_1(z) = z + \lambda_k x A z^k + \overline{b_1 z + \mu_k y B z^k}$$

and

$$t_2(z) = z + \lambda_k x(2-A)z^k + \overline{b_1 z + \mu_k y(2-B)z^k}$$

are in the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$ and note that

$$f_k(z) = \tfrac{1}{2}t_1(z) + t_2(z)\ .$$

The extremal coefficient bounds shows that functions of the form (14) are the extreme points for the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$, then the proof of Theorem (2.4) is completed.

Now we will examine the closure properties of the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$ under the generalized Bernardi - Libera - Livingston integral operator (see[2, 7])$L_c(f)$ which is defined by

$$L_c f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt (c > -1) \tag{15}$$

Theorem 2.5. Let $f_n = h + g_n \in V_{\bar{H}}(m,n;\gamma,\rho,z_0)$, where h and g_n are given by (5).Then $L_c(f_n(z))$ belongs to the class $V_{\bar{H}}(m,n;\gamma,\rho,z_0)$.

Proof. From the representation of $L_c\{f_n(z)\}$, it follows that

$$L_c(f_n(z)) = \tfrac{c+1}{z^c}\int_0^z t^{c-1}\left\{h(t) + \bar{g}_n(t)\right\}dt =$$

$$= \tfrac{c+1}{z^c}\int_0^z t^{c-1}\left[t + \sum_{k=2}^{\infty} a_k t^k + \overline{\sum_{k=1}^{\infty} b_k t^k}\right]dt =$$

$$= a_1(z) + \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k}\ ,$$

Where $A_k = \left(\frac{c+1}{c+k}\right)a_k$, $B_k = \left(\frac{c+1}{c+k}\right)b_k$. Therefore , we have ,

$$\sum_{k=2}^{\infty}\frac{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}}{(1-\gamma)}\frac{c+1}{c+k}|a_k| + \sum_{k=1}^{\infty}\frac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{(1-\gamma)}\frac{c+1}{c+k}|b_k| \leq$$

$$\leq \sum_{k=2}^{\infty}\frac{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}}{(1-\gamma)}|a_k| + \sum_{k=1}^{\infty}\frac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{(1-\gamma)}|b_k| \leq 2a_1\ ,$$

and the proof of Theorem (2.5) is completed.

Ref

2. Bernardi, S. D., (1969), convex and starlike univalent functions. Trans. Amer. Math. Soc., 135, pp.429-446.

Notes

References Références Referencias

1. Aghalary, R., (2007), Goodman- Salagian-type hormonic univalent function with varying arguments. Int. Journal of math. Analysis, 1 (22), pp.1051-1057.
2. Bernardi, S. D., (1969), convex and starlike univalent functions. Trans. Amer. Math. Soc., 135, pp.429-446.
3. Clunie, J., Sheil-Small, T.,(1984), Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A. I. Math., 9, pp.3-25.
4. Cotirla, L. I., (2009), Harmonic univalent functions defined by an integral operator, Acta Universities Apulensis, 17, pp.95-105.
5. Guney, H. O., Sakar, F.M.,(2011), On uniformaly starlike functions defined by an integral operator, Acta Universities Apulensis, 28, pp.293-301.
6. Guney, H. O., Sakar, F.M.,(2012), A new class of hormonic uniformly starlike functions defined by an integral operator, General Math., 20(1), pp.75-83.
7. Libera, R. J., (1965), Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16, pp.755-758.
8. R. M. EL-Ashwah, M. K. Aouf, A. A. M. Hassan, A. H. hassan, harmonic Univalent functions with varying arguments defined by using Salagean Integral operator with fixed point, TWMS J. Pure Appl. Math.V.4,N.1,2013,pp.95-102.
9. Salagean, G. S., (1983), Subclass of univalent functions, Lecture Notes in Math. Springer-Verlag, 1013, pp.362-372.

This page is intentionally left blank